# 12

# Burnside–Pólya Counting Methods

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**Prerequisites:** The prerequisites for this chapter are combinations, equivalence relations, and bijections on sets. See Sections 2.3, 5.3, and 8.5 of *Discrete Mathematics and Its Applications*.

## Introduction

If a game of tic-tac-toe is played out until all spaces are filled, then the resulting grid will have five crosses and four naughts (also called "oh"s). From counting combinations, it follows that the number of $3 \times 3$ grids with five crosses and four naughts is $C(9,5)$, which equals 126.

In playing tic-tac-toe, it is natural to regard two configurations as equivalent if one can be obtained from the other by rotating or reflecting the board. For instance, in Figure 1, configuration (B) can be obtained by rotating configuration (A) counterclockwise 90°, and (C) can be obtained by reflecting (A) horizontally. If one considers such configurations to be the "same", then how many "really different" configurations are there?

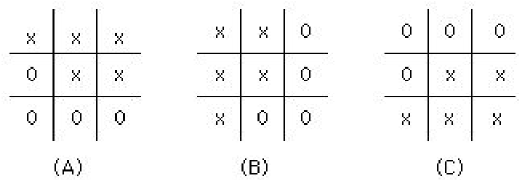


**Figure 1. Three equivalent tic-tac-toe configurations.**

We can formalize this intuitive concept of "sameness" by calling two tic-tac-toe configurations **congruent** if one can be obtained from the other by a rotation or by a reflection. Congruence is an equivalence relation, in the usual sense, and the equivalence classes into which it partitions the set of all configurations of the tic-tac-toe board are called **congruence classes**. Thus, we seek the solution to the following problem:

> **Congruence-Class Counting Problem (*CCCP*)**: Count the number of different congruence classes among the configurations with five crosses and four naughts.

If all the congruence classes were the same size, then we could solve *CCCP* by dividing the uniform class-size into $C(9,5)$, the total number of, individual configurations. Unfortunately, it is not quite this simple, as now illustrated.

**Example 1** Count the numbers of configurations that are equivalent to each of the configurations (D), (E), and (F) in Figure 2.

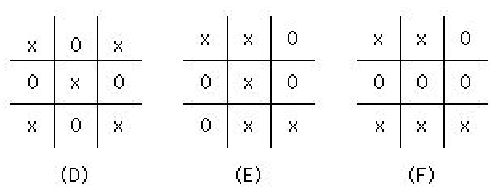


**Figure 2. Configurations with equivalence classes of different sizes.**

*Solution*: Every rotation and reflection of configuration (D) always yields (D) itself, so its congruence class has cardinality equal to one. Every configuration congruent to (E) is a pattern with one of its two diagonals filled with crosses and either its middle row or its middle column filled with crosses. Conversely, every such pattern is congruent to (E). Since there are four such patterns, it follows that the congruence class of (E) has cardinality equal to four.

The total number of rotations and reflections of the tic-tac-toe board onto itself is eight, that is, there are four rotations and four reflections. The four different rotations correspond to zero, one, two, or three quarter-turns of the tic-tac-toe board. We can reflect the board horizontally through its middle row, vertically through its middle column, through the "major" diagonal row that runs from the upper left to the lower right, or through the "minor" diagonal row that runs from the upper right to the lower left. Applying each of these eight rotations and reflections to configuration (F) yields a different configuration. □

As illustrated by the method we have just applied to configuration (F), we can easily calculate the size of the congruence class of any given configuration. That is, we simply apply all eight rotations to the given configuration, and we see how many different ones we obtain.

Our method for counting the congruence classes, known as **Burnside–Pólya enumeration theory**, also involves these rotations and reflections. This is an extremely important counting technique, since it can also be used to count graph isomorphism classes (see Section 7.3 of *Discrete Mathematics and Its Applications*), equivalence classes of graph colorings (see Section 9.8), equivalence classes of finite-state automata (see Section 12.2), and many other kinds of equivalence classes involving structural symmetries.

We shall ultimately solve *CCCP* by calculating that there are exactly 23 different congruence classes of configurations with five crosses and four naughts. A good exercise is to try to derive this number without using Burnside–Pólya theory, by systematically applying ad hoc methods. It is easy to make mistakes in such an endeavor. Burnside–Pólya theory offers two advantages: it reduces the chance of error inherent in ad hoc methods, and it is applicable in other calculations involving cardinalities so large that ad hoc counting is infeasible.

## Permutation Groups

A rotation or a reflection on a tic-tac-toe board is a special case of a bijection (i.e., a one-to-one, onto function) from a finite set to itself, which is commonly called a **permutation**. This relationship becomes clear if we label the nine squares of the tic-tac-toe board with the integers from 1 to 9, as illustrated in Figure 3.

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 4 | 5 | 6 |
| 7 | 8 | 9 |

**Figure 3. The labeling of tic-tac-toe squares with integers.**

Then rotating a quarter-turn counterclockwise takes square 1 to square 7, square 2 to square 4, square 3 to square 1, and so on. The change on every position of the tic-tac-toe board can be recorded in a $2 \times 9$ matrix, as follows:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 4 & 1 & 8 & 5 & 2 & 9 & 6 & 3 \end{pmatrix}.$$

In each of the nine columns, the entry in the top row is mapped to the entry immediately below it. In general, any permutation on a finite set of cardinality $n$ can be fully described by such a $2 \times n$ matrix.

All 9! permutations on the set $\{1, 2. \ldots, 9\}$ can be described by $2 \times 9$ matrices, but only four of them represent rotations on the tic-tac-toe board and only four others represent reflections. For the sake of simplification, we turn for the moment to $2 \times 2$ tic-tac-toe, whose squares are labeled as in Figure 4.

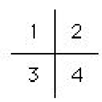


**Figure 4. A 2x2 tic-tac-toe board.**

All 4! permutations on the set $\{1, 2, 3, 4\}$ can be described by $2 \times 4$ matrices. The eight of them that represent rotations and reflections are as follows:

rotations:

$$\underset{90^\circ}{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}} \quad \underset{180^\circ}{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}} \quad \underset{270^\circ}{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}} \quad \underset{360^\circ}{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}}$$

reflections:

$$\underset{\text{horizontal}}{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}} \quad \underset{\text{vertical}}{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}} \quad \underset{\text{main diagonal}}{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}} \quad \underset{\text{minor diagonal}}{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1 \end{pmatrix}}$$

If the elements on the set being permuted are integers (or any other ordered set), then it is natural to arrange the columns of the matrix so that the elements of the first row are in ascending order. However, any ordering of the columns equally well expresses the action of the permutation on every element.

One instance in which it is convenient to consider other orderings is in calculating the composition $g \circ f$ of two permutations. To do this operation, write the $2 \times n$ matrix representation of $g$ immediately below the $2 \times n$ matrix representation of $f$. The columns of $f$ should be ordered as usual,

$$1, 2, \ldots, n.$$

The columns of $g$ should be ordered so that the top row is in the order

$$f(1), f(2), \ldots, f(n)$$

which matches the bottom row of the matrix for $f$. Then the bottom row of the matrix for $g$ will appear in the order

$$g(f(1)), g(f(2)), \ldots, g(f(n)).$$

It follows that the $2 \times n$ matrix representing composition $g \circ f$ is formed by writing the bottom row of the matrix for $g$ in that order immediately below the top row of the matrix for $f$.

For instance, suppose that we take the quarter-turn counterclockwise as $f$ and the horizontal reflection as $g$, for the $2 \times 2$ board. Then this process is as follows:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} \qquad \text{1/4 turn counterclockwise}$$

$$\begin{pmatrix} 3 & 1 & 4 & 2 \\ 1 & 3 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \qquad \text{horizontal reflection}$$

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$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix} \qquad \text{reflection through main diagonal}$$

In addition to reviewing the mechanics of the process on the left by which the matrix representation of the composition is calculated, you should also apply your spatial perceptions to the corresponding description at the right of the composition, as viewed geometrically.

The following properties of $2 \times 2$ tic-tac-toe boards can be proved either by applying the composition rule for $2 \times n$ matrix representations or by geometric reasoning:

(1) The composition of two rotations is a rotation.

(2) The composition of two reflections is a rotation.

(3) The composition of a rotation with a reflection or of a reflection with a rotation is a reflection.

From the combination of these three assertions, we immediately infer the following theorem.

**Theorem 1** The composition of a rotation or a reflection of a $2 \times 2$ tic-tac-toe board with another rotation or reflection is a rotation or a reflection. ■

A collection of permutations on a finite set is said to be **closed under composition** if whenever any two of them are composed, the resulting permutation is also in the collection.

**Example 2** Show that the collection of rotations and reflections on the $2 \times 2$ tic-tac-toe board is closed under composition.

*Solution*: We have established this with Theorem 1. □

**Example 3** Show that the collection of all four rotations on the $2 \times 2$ tic-tac-toe board is closed under composition.

*Solution*: This is obvious from geometry. □

**Example 4** Show that the collection of permutations on the $2 \times 2$ tic-tac-toe board consisting of the 360° rotation alone is closed under composition.

*Solution*: Its composition with itself is itself. □

**Example 5** Show that the collection of all four reflections on the $2 \times 2$ tic-tac-toe board is not closed under composition.

*Solution*: This follows from property (2) preceding Theorem 1. □

Since a permutation $\pi$ is a bijective function from a set to itself, it has an inverse, denoted $\text{inv}(\pi)$ or $\pi^{-1}$, which is also a bijective function from that set to itself. In other words, the inverse of a permutation on any set is a permutation on the same set. To calculate the inverse of a permutation in terms of its $2 \times n$ matrix representation, simply swap the two rows. If you want the top row of the result to be sorted, you can rearrange the columns accordingly.

For instance, it is clear from geometric reasoning that the inverse of a quarter-turn counterclockwise is a three-quarter turn counterclockwise. Here is how our calculation rule applies to this example.

$$\begin{aligned} \text{inv}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} &= \begin{pmatrix} 3 & 1 & 4 & 2 \\ 1 & 2 & 3 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}. \end{aligned}$$

These two additional properties of $2 \times 2$ tic-tac-toe boards both follow immediately from geometric reasoning:

(4) The inverse of a rotation is a rotation.

(5) The inverse of a reflection is that same reflection.

From the combination of these two properties, we immediately infer the following theorem.

**Theorem 2** The inverse of a rotation or a reflection of a $2 \times 2$ tic-tac-toe board is also a rotation or a reflection. ■

A collection of permutations on a finite set is said to be **closed under inversion** if the inverse of every permutation in the collection is also in the collection.

There is a special name for a collection of permutations that is closed both under composition and under inversion. It is called a **permutation group**.

**Example 6** Show that the collection of all rotations and reflections on the $2 \times 2$ tic-tac-toe board form a permutation group.

*Solution*: This follows from Theorems 1 and 2 together. □

**Example 7** Show that the collection of all rotations and reflections on the $3 \times 3$ tic-tac-toe board form a permutation group.

*Solution*: This follows from applying to the $3 \times 3$ board the same methods used for the $2 \times 2$ board. □

**Example 8** Show that the collection of all permutations on a finite set forms a permutation group.

*Solution*: Since the composition of any two permutations is a permutation and since the inverse of any permutation is a permutation, this follows immediately. □

**Example 9** Show that the singleton collection containing only the identity permutation (which fixes every element, just like any other identity function on a set) is a permutation group.

*Solution*: The composition of the identity with itself is the identity, and the inverse of the identity is the identity, so the collection is closed both under composition and under inversion. □

We conclude this section by placing the concept of congruence classes of tic-tac-toe boards into a context of greater generality, amenable to the Burnside–Pólya theory of counting.

When a permutation group $G$ acts on a set $Y$, we often call the elements of that set *objects*. In particular, the tic-tac-toe configurations with five crosses and four naughts are the objects, when the permutation group is the collection of rotations and reflections on the $3 \times 3$ tic-tac-toe board.

If $y$ is any object in the set $Y$, then for every permutation $\pi \in G$, the object $\pi(y)$ is considered to be related to $y$. We write $yRy'$ if $y$ and $y'$ are related this way, and we say that $R$ is the **relation induced by the action** of $G$.

**Theorem 3** The relation $R$ induced by the action of a permutation group $P$ on a set of objects $Y$ is an equivalence relation.

*Proof:* Given any object $y$, let $I$ denote the identity permutation. Then we have $y\,R\,I(y)$. In other words, every object $y$ is related to itself, from which it follows that the relation $R$ is reflexive.

Next, given any two objects, $y$ and $y'$, suppose that $yRy'$, which means that there exists a permutation $\pi$ in the group $G$ such that $\pi(y) = y'$. Since a permutation group is closed under inversion, it follows that $\pi^{-1} \in G$. Evidently, we have $\pi^{-1}(y') = y$, from which it follows that $y'Ry$. Thus, we have established that the relation $R$ is symmetric.

Finally, if $yRy'$ and $y'Ry''$, then there exist permutations $\pi$ and $\mu$ in $G$ such that $\pi(y) = y'$ and $\mu(y') = y''$. Obviously, $\mu \circ \pi(y) = y''$. Since the permutation group $G$ is closed under composition, it is clear that $\mu \circ \pi \in G$. Hence, we have $yRy''$, from which it follows that the relation $R$ is transitive.

Having established reflexivity, symmetry, and transitivity for relation $R$, we conclude that it is an equivalence relation. ■

The equivalence classes induced by the action of a permutation group on a set of objects are often called **orbits**.

**Example 10** What are the orbits induced by the action of the group of rotations and reflections on the configurations of a tic-tac-toe board?

*Solution*: They are simply the congruence classes. □

**Example 11** When the group of all possible permutations acts on a set of objects, how many orbits are formed?

*Solution*: There is only one orbit, which comprises all elements of the set. □

The introduction of these algebraic concepts enables us to formulate the objective of Burnside–Pólya enumeration theory most clearly, as a method for counting the orbits of a permutation group acting on a finite set. Actually, Frobenius invented the method for counting orbits, but its appearance in a classic tract of Burnside [1] led to its being called "Burnside's Lemma".

Pólya* [3] augmented this theory with a means to assign "weights" to orbits

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* George Pólya (1887–1985) was a Hungarian mathematician who received his Ph.D. in probability theory from the University of Budapest. He taught at the Swiss Federal Institute of Technology and Brown University before taking a position at Stanford University in 1942. He published over 250 articles in various areas of mathematics and wrote several books on problem-solving.

and to decompose the total number of orbits into a sum of numbers of orbits of each weight class. Actually Pólya rediscovered methods already known to Redfield [4], whose paper had escaped most attention. F. Harary (see [2]) developed the use of Burnside–Pólya enumeration theory for graphical enumeration, which has attracted many mathematicians to its further application. We will show how to count the 23 congruence classes of configurations cited in the introduction, and thereby solve *CCCP*, by a detailed application of Burnside–Pólya theory. Along the way, we shall strive for the intermediate goal of counting the congruence classes of the $2 \times 2$ tic-tac-toe configurations of naughts and crosses in which no square is left empty.

There are 16 such configurations, since the filling of each of four squares involves a binary choice. Burnside's methods alone enable us to establish that there are six orbits, and Pólya's augmentation provide an "inventory" of these six orbits into subclasses, as follows:

1 orbit with four crosses and zero naughts
1 orbit with three crosses and one naught
2 orbits with two crosses and two naughts
1 orbit with one cross and three naughts
1 orbit with zero crosses and four naughts.

Figure 5 provides one example configuration for each of these six orbits. You can easily demonstrate that every one of the 16 configurations can be rotated or reflected into one of the six examples shown.

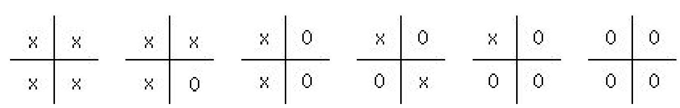

**Figure 5. One configuration from each of the six orbits of $2 \times 2$ tic-tac-toe under the group of rotations and reflections.**

## The Cycle Structure of Permutations

We now return to the $2 \times 9$ representation of the counterclockwise quarter-turn on the $3 \times 3$ tic-tac-toe configurations.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 4 & 1 & 8 & 5 & 2 & 9 & 6 & 3 \end{pmatrix}$$

When we apply this permutation, the content (cross or naught) of location 1 will move to location 7, as we see from column 1. Column 7 indicates that the content of location 7 moves to location 9. Column 9 says that the content of location 9 moves to location 3. We also note that column 3 says the content of location 3 moves to location 1. In other words, the contents of the four locations 1, 7, 9, 3 move in a closed cycle.

Similarly we note that the contents of locations 2, 4, 8, and 6 moves in another closed cycle of length four. We also note that the permutation fixes the content of location 5, the center of the board. We may regard this as a cycle of length one. The two cycles of length four together with the cycle of length one account for the content of all nine of the locations of the board.

This analysis enables us to represent the permutation as a composition

$$(1\ \ 7\ \ 9\ \ 3)(2\ \ 4\ \ 8\ \ 6)(5)$$

of cycles, from which we could easily reconstruct the $2\times n$ matrix representation. We can encode the fact that there are two 4-cycles and one 1-cycle as the multivariate monomial $t_1t_4^2$. (By a *monomial*, we mean a polynomial with only one term. By *multivariate*, we mean it has more than one variable. In this case, its variables are $t_1$ and $t_4$.) In general, such a monomial is called the **cycle structure** of the permutation.

We observe that the objects in any one cycle are disjoint from the objects in any of the other cycles. In general, any permutation can be represented as a composition of disjoint cycles, and we call such a representation the **disjoint cycle form** of the permutation. Table 1 indicates the disjoint cycle form and the cycle structure for all rotations and reflections on the $2 \times 2$ tic-tac-toe configurations.

We see from the table that there is one permutation with cycle structure $t_1^4$, two with structure $t_1^2t_2$, three with structure $t_2^2$, and two with structure $t_4$. When we add the cycle structures for all permutations in a group $G$ together and divide by the cardinality of the group, the resulting polynomial is called the **cycle index** of $G$, and is denoted $Z(G)$.

**Example 12** What is the cycle index of the group of rotations and reflections on the $2 \times 2$ configurations?

*Solution*: $\frac{1}{8}(t_1^4 + 2t_1^2t_2 + 3t_2^2 + 2t_4)$. □

**Example 13** Calculate the cycle index of the group of rotations and reflections on the $3 \times 3$ tic-tac-toe configurations.

*Solution*: The cycle index is

$$\tfrac{1}{8}(t_1^9 + 4t_1^3t_2^3 + t_1t_2^4 + 2t_1t_4^2).$$

Its first term corresponds to the null (i.e., 360°) rotation; its second term corresponds to the four reflections, since each of them fixes three squares and swaps three pairs; its third term corresponds to the 180° rotation; and its fourth term accounts for the quarter-turn and the three-quarter turn. □

| **rotations** | **cycle form** | **cycle structure** |
|---|---|---|
| $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$ | $(1\ \ 3\ \ 4\ \ 2)$ | $t_4$ |
| $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$ | $(1\ \ 4)(2\ \ 3)$ | $t_2^2$ |
| $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$ | $(1\ \ 2\ \ 4\ \ 3)$ | $t_4$ |
| $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$ | $(1)(2)(3)(4)$ | $t_1^4$ |
| **reflections** | | |
| $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$ | $(1\ \ 3)(2\ \ 4)$ | $t_2^2$ |
| $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$ | $(1\ \ 2)(3\ \ 4)$ | $t_2^2$ |
| $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}$ | $(1)(2\ \ 3)(4)$ | $t_1^2t_2$ |
| $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1 \end{pmatrix}$ | $(1\ \ 4)(2)(3)$ | $t_1^2t_2.$ |

**Table 1. Cycle forms and cycle structures for rotations and reflections on $2 \times 2$ tic-tac-toe configurations.**

## Burnside's Lemma

The Burnside–Pólya counting method is based upon substitutions into the cycle index. A simple illustration of how this works is to substitute the number 2 for each variable in the cycle index

$$\tfrac{1}{8}(t_1^4 + 2t_1^2t_2 + 3t_2^2 + 2t_4).$$

The result of the substitution is the arithmetic expression

$$\tfrac{1}{8}(2^4 + 2 \cdot 2^2 2 + 3 \cdot 2^2 + 2 \cdot 2).$$

This simplifies to

$$\tfrac{1}{8}(16 + 16 + 12 + 4) = 6,$$

exactly the number of configurations appearing in Figure 5.

The mathematical result that this process always yields the total number of orbits is called Burnside's Lemma. Its proof will be obtained here with the aid of three auxiliary lemmas.

Let $G$ be a permutation group acting on a set $Y$ of objects. By the **stabilizer** of an object $y$, denoted $\mathrm{Stab}(y)$, we mean the collection of permutations in $G$ that map $y$ to itself.

It is clear that the composition of two permutations that fix $y$ is also a permutation that fixes $y$. It is also clear that the inverse of a permutation that fixes $y$ is also a permutation that fixes $y$. Thus, $\mathrm{Stab}(y)$ is also a permutation group. Since it is wholly contained in the group $G$, we call $\mathrm{Stab}(y)$ a **subgroup** of $G$.

**Example 14** What is the stabilizer of the $2 \times 2$ tic-tac-toe configuration with crosses in locations 1 and 4 and naughts in locations 2 and 3?

*Solution*: Its stabilizer is the subgroup comprising these four permutations

$$(1)(2)(3)(4) \qquad (1\ \ 4)(2\ \ 3) \qquad (1\ \ 4)(2)(3) \qquad (1)(2\ \ 3)(4). \qquad \square$$

The necessary and sufficient condition on a permutation for fixing that configuration is that all the locations occurring within each cycle be marked the same, that is, all with crosses or all with naughts.

Just as we associate with each object $y$ the collection $\mathrm{Stab}(y)$ of permutations that fix $y$, we associate with each permutation $\pi$ the set of objects that it fixes. This set is called the **fixed-point set** of $\pi$ and it is denoted $\mathrm{Fix}(\pi)$, and it comprises the set of objects that occur in 1-cycles of the disjoint cycle form of $\pi$.

**Example 15** What $2 \times 2$ tic-tac-toe configurations are fixed by the permutation $(1)(2)(3)(4)$?

*Solution*: It fixes all 16 completely filled $2 \times 2$ tic-tac-toe configurations. $\square$

**Example 16** What $2 \times 2$ tic-tac-toe configurations are fixed by the permutation $(1\ \ 4)(2\ \ 3)$?

*Solution*: When the permutation $(1\ 4)(2\ 3)$ acts on the $2 \times 2$ tic-tac-toe configurations, it swaps the marks in locations 1 and 4, and it swaps the marks in locations 2 and 3. Therefore, it fixes a tic-tac-toe configuration if and only if the contents of locations 1 and 4 are the same and the contents of locations 2 and 3 are the same. Since each of the four locations is to be filled with either a cross or a naught, there are two choices for the kind of mark for locations 1 and 4 and two choices for the kind of mark for locations 2 and 3. Thus, $(1\ \ 4)(2\ \ 3)$ fixes four tic-tac-toe configurations. $\square$

**Example 17** What $2 \times 2$ tic-tac-toe configurations are fixed by the permutation $(1\ \ 4)(2)(3)$?

*Solution*: By the same analysis we just performed for Example 16, we see that $(1\ \ 4)(2)(3)$ fixes a configuration if and only if locations 1 and 4 are filled with the same mark; the marks in locations 2 and 3 are both arbitrary. Thus, we have three binary choices for marks: the mark for locations 1 and 4, the mark for location 2, and the mark for location 3. It follows that the permutation $(1\ \ 4)(2)(3)$ fixes eight (i.e., $2^3$) configurations. □

The first of the three lemmas needed for the proof of Burnside's Lemma concerns the effect of interchanging the indices for a double summation. Its proof involves the use of the *Iverson truth function* $\text{true}(p)$, whose value is 1 if $p$ is a true statement, and whose value is 0 otherwise.

**Lemma 1** The sum of the cardinalities of the stabilizers, taken over all objects of a permuted set $Y$, equals the sum of the cardinalities of the fixed-point sets, taken over all permutations in the group $G$ that acts on $Y$. That is,

$$\sum_{y \in Y} |\text{Stab}(y)| = \sum_{\pi \in G} |\text{Fix}(\pi)|.$$

*Proof:* Since $\text{Stab}(y) = \{\pi \colon \pi(y) = y\}$, it follows that

$$|\text{Stab}(y)| = \sum_{\pi \in G} \text{true}(\pi(y) = y).$$

Therefore,

$$\begin{aligned} \sum_{y \in Y} |\text{Stab}(y)| &= \sum_{y \in Y} \sum_{\pi \in G} \text{true}(\pi(y) = y) \\ &= \sum_{\pi \in G} \sum_{y \in Y} \text{true}(\pi(y) = y) \\ &\qquad \text{(by interchanging indices of summation)} \\ &= \sum_{\pi \in G} |\text{Fix}(\pi)|. \\ &\qquad \text{(because } \textstyle\sum_{y \in Y} \text{true}(\pi(y) = y) = |\text{Fix}(\pi)|) \end{aligned}$$

■

**Lemma 2** For any object $y$ in a set $Y$ of permuted objects under the action of a permutation group $G$,

$$|\text{Stab}(y)| = \frac{|G|}{|\text{orbit}(y)|}.$$

*Proof:* Suppose that $\text{orbit}(y) = \{y_1, y_2, \ldots, y_n\}$, where $y = y_1$. Then for $j = 1, 2, \ldots, n$, we define

$$G_j = \{\pi \in G: \pi(y) = y_j\}.$$

In other words, $G_j$ is the subset of permutations in $G$ that map $y$ to $y_j$. Clearly, $\{G_1, G_2, \ldots, G_n\}$ is a partition of the permutations in the group, and $G_1 = \text{Stab}(y)$.

For $j = 1, 2, \ldots, n$, let $\pi_j$ be any permutation such that $\pi_j(y) = y_j$. Then composition with $\pi_j$ maps every permutation in $G_1$ to a permutation in $G_j$, and composition with $\pi_j^{-1}$ maps every permutation in $G_j$ to a permutation in $G_1$. It follows that composition with $\pi_j$ is a bijection from $G_1$ to $G_j$, which implies that $|G_j| = |G_1| = |\text{Stab}(y)|$.

Since $\{G_1, G_2, \ldots, G_n\}$ is a partition of $G$ into subsets of size $|\text{Stab}(y)|$, it follows that the product of $|\text{Stab}(y)|$ with the number $n = |\text{orbit}(y)|$ of subsets equals $|G|$. Equivalently, $|\text{Stab}(y)| = |G|/|\text{orbit}(y)|$. ■

**Example 14, revisited.** We recall that the configuration with crosses on its main diagonal (i.e., in locations 1 and 4) and naughts on its minor diagonal (i.e., in locations 2 and 3) has four elements in its stabilizer. Since the only other configuration in its orbit is the configuration with crosses on its minor diagonal and naughts on its main diagonal, its orbit has cardinality two. Since there are eight rotations and reflections in the permutation group acting on the set of configurations, the equation

$$4 = 8/2$$

serves as empirical evidence of the correctness of Lemma 2. □

**Lemma 3** Let $G$ be a permutation group acting on a set of objects $Y$. Then

$$\sum_{y \in Y} \frac{1}{|\text{orbit}(y)|} = \#\ \text{orbits}.$$

*Proof:* Suppose that $\#\ \text{orbits} = k$, and that $Y_1, Y_2, \ldots, Y_k$ are the orbits. Then

$$\begin{aligned} \sum_{y \in Y} \frac{1}{|\text{orbit}(y)|} &= \sum_{j=1}^{k} \sum_{y \in Y_j} \frac{1}{|\text{orbit}(y)|} \\ &= \sum_{j=1}^{k} \sum_{y \in Y_j} \frac{1}{|Y_j|} \\ &= \sum_{j=1}^{k} \frac{1}{|Y_j|} \sum_{y \in Y_j} 1 \end{aligned}$$

$$= \sum_{j=1}^{k} \frac{1}{|Y_j|}|Y_j|$$
$$= \sum_{j=1}^{k} 1$$
$$= k$$
$$= \# \text{ orbits.}$$

(This lemma is really a fact about partitions of sets, and depends in no way upon the algebra of permutation groups.) ■

**Theorem 4 (Burnside's Lemma)** Let $G$ be a permutation group acting on a set of objects $Y$. Then

$$\# \text{ orbits} = \frac{1}{|G|} \sum_{\pi \in G} |\mathrm{Fix}(\pi)|.$$

*Proof:* Having already proved Lemmas 1, 2, and 3, we may prove Burnside's Lemma by a sequence of substitutions.

$$\begin{aligned}
\frac{1}{|G|} \sum_{\pi \in G} |\mathrm{Fix}(\pi)| &= \frac{1}{|G|} \sum_{y \in Y} |\mathrm{Stab}(y)| && \text{by Lemma 1} \\
&= \frac{1}{|G|} \sum_{y \in Y} \frac{|G|}{|\mathrm{orbit}(y)|} && \text{by Lemma 2} \\
&= \frac{1}{|G|} |G| \sum_{y \in Y} \frac{1}{|\mathrm{orbit}(y)|} \\
&= \sum_{y \in Y} \frac{1}{|\mathrm{orbit}(y)|} \\
&= \# \text{ orbits.} && \text{by Lemma 3} \quad \square
\end{aligned}$$

For a direct application of Burnside's Lemma, we add the values of $|\mathrm{Fix}(\pi)|$, the number of objects fixed, over all permutations $\pi$ in the group $G$, and then divide their sum by $|G|$, the number of elements of the group. Theorem 5 generalizes the analysis of Examples 14, 15, and 16 into a method for calculating the values of $|\mathrm{Fix}(\pi)|$, and Corollary 1 employs the result of Theorem 5 in establishing how to count the number of congruence classes of configurations.

**Theorem 5** Let $\pi$ be a permutation of the completely filled configurations of $2 \times 2$ tic-tac-toe boards, and let $\pi$ have exactly $r$ cycles in its disjoint cycle form. Then $|\mathrm{Fix}(\pi)| = 2^r$.

*Proof:* The permutation $\pi$ fixes a configuration if any only if, for each cycle in its disjoint cycle form, all the locations are marked the same. ■

**Corollary 1** The number of congruence classes of completely filled $2 \times 2$ tic-tac-toe configurations equals the value of the arithmetic expression obtained by substituting the number 2 into the cycle index for the action of the group of rotations and reflections.

*Proof:* Theorem 5 indicates that this is exactly how to apply Burnside's Lemma to this counting problem. ■

**Example 18** At the beginning of this section, we performed this calculation and confirmed that the result was 6, exactly the number of configurations appearing in Figure 5. A related question we might ask is this: if some of the locations of the $2 \times 2$ board remain unfilled, then how many congruence classes of configurations are there?

*Solution*: An analysis analogous to Examples 14, 15, and 16 and Corollary 1 indicates that the answer can be obtained by substituting the number 3 for each variable in the cycle index $\frac{1}{8}(t_1^4 + 2t_1^2t_2 + 3t_2^2 + 2t_4)$, an operation which we now perform.

$$\begin{aligned}\tfrac{1}{8}(3^4 + 2 \cdot 3^2 3 + 3 \cdot 3^2 + 2 \cdot 3) &= \tfrac{1}{8}(81 + 54 + 27 + 6)\\ &= \tfrac{1}{8}(168)\\ &= 21.\end{aligned}$$ □

One way to confirm the validity of this result is to draw representative configurations from each of the 21 congruence classes, and then to confirm that the list of drawings is complete and non-repetitive. However, there is an easier approach that still uses Figure 5, if we regard "empty" as just another kind of mark on a tic-tac-toe board.

First of all, only one congruence class of configurations uses exactly one mark. Since there are three possible marks, we have three orbits with exactly one mark. Figure 5 shows four congruence classes with exactly two different marks. Since we have three choices of two kinds of marks from the set {cross, naught, empty}, there are 12 orbits (i.e., $3 \cdot 4$) with exactly two marks. That makes 15 orbits so far, and it remains to count the orbits with three different kinds of marks.

Suppose that we intend to use two crosses, one naught, and one empty location. There are two different possible orbits, one with the two crosses in horizontally or vertically adjacent locations, and the other with two crosses juxtaposed diagonally. Since the same analysis applies to counting orbits with two naughts, one cross, and one empty or orbits with two empties, one cross, and one naught, it follows that there are six (i.e., $3 \cdot 2$) orbits with three different marks. Since $15 + 6 = 21$, we have another empirical verification of the validity of Burnside's Lemma and of our method of applying it to configuration counting problems.

## Pólya's Inventory Method

Pólya's augmentation of Burnside's Lemma involves the assignment of a monomial weight to each configuration so that any two configurations in the same orbit are assigned the same weight. When a weighting polynomial is substituted into the cycle index, the resulting polynomial provides an enumeration of the orbits according to weight.

As a first illustration of how Pólya's method is used and the inventoried information it provides, we continue the analysis of $2 \times 2$ tic-tac-toe configurations. The weight assigned to a configuration with $r$ crosses and $4 - r$ naughts is the monomial $x^r$. Observe carefully how we perform the substitution into the multivariate cycle index

$$\tfrac{1}{8}(t_1^4 + 2t_1^2 t_2 + 3t_2^2 + 2t_4).$$

For $r = 1, 2, 3, 4$, the binomial $1 + x^r$ is substituted for each instance of the variable $t_r$. The resulting expression is

$$\tfrac{1}{8}((1+x)^4 + 2(1+x)^2(1+x^2) + 3(1+x^2)^2 + 2(1+x^4)),$$

which we expand into

$$\begin{array}{llllll} \tfrac{1}{8}( & 1 + 4x + 6x^2 & + 4x^3 + & x^4 & \\ & + 2 + 4x + 4x^2 & + 4x^3 + & 2x^4 & \\ & + 3 \qquad + 6x^2 & & + 3x^4 & \\ & + 2 & & + 2x^4 & ). \end{array}$$

By collecting terms of like degree, we obtain the univariate polynomial

$$\tfrac{1}{8}( \quad 8 + 8x + 16x^2 + 8x^3 + \ 8x^4 \quad )$$

which simplifies to

$$1 + \ x + \ 2x^2 + \ x^3 + \ x^4.$$

Our interpretation of this simplified univariate polynomial is that there is one orbit with no crosses, one orbit with one cross, two orbits with two crosses, one orbit with three crosses, and one orbit with four crosses, exactly what we saw in Figure 5. That is, the coefficient of the term of degree $r$ is the number of orbits of weight $x^r$, which we designed to equal the number of orbits with $r$ crosses.

Theorem 6 formalizes this counting technique. Its proof involves the derivation of a "weighted" version of Burnside's Lemma.

**Theorem 6** Let $\pi$ be a permutation of the completely filled configurations of $2 \times 2$ tic-tac-toe boards, and let

$$t_1^{e_1} t_2^{e_2} t_3^{e_3} t_4^{e_4}$$

be the cycle structure of $\pi$. For $r = 1, 2, 3, 4$, substitute the binomial $1 + x^r$ for $t_r$ (and collect terms of like degree). Then the coefficient of the term of degree $r$ in the resulting polynomial equals the number of orbits of weight $x^r$. ■

**Solving CCCP** With the aid of Theorem 6, we are ready to solve *CCCP*. We begin by recalling from Example 13 that the cycle index for the action of the group of rotations and reflections on the $3 \times 3$ tic-tac-toe configurations is the multivariate polynomial

$$\tfrac{1}{8}(t_1^9 + 4t_1^3 t_2^3 + t_1 t_2^4 + 2t_1 t_4^2).$$

If we need to know the number of congruence classes for each possible number of crosses from zero to nine, with naughts in all other grid locations, then we substitute $(1+x^r)$ for each instance of the variable $t_r$ in the cycle index above and proceed as in our calculation for the $2 \times 2$ configurations.

Since *CCCP* restricts its concern to counting the configurations with five crosses, our task is somewhat reduced. That is, it is sufficient to calculate, for each of the four terms of the cycle index, the coefficient of $x^5$ that results when we substitute $(1 + x^r)$ for each instance of the variable $t_r$ in that term, for $r = 1, 2, \ldots, 9$. Here are the details of this step.

| **term** | **substitution** | **coefficient of $\mathbf{x^5}$** | **value** |
|---|---|---|---|
| $t_1^9$ | $(1+x)^9$ | $C(9,5)$ | 126 |
| $4t_1^3 t_2^3$ | $4(1+x)^3(1+x^2)^3$ | $4(C(3,1)C(3,2) + C(3,3)C(3,1))$ | 48 |
| $t_1 t_2^4$ | $(1+x)(1+x^2)^4$ | $C(1,1)C(4,2)$ | 6 |
| $2t_1 t_4^2$ | $2(1+x)(1+x^4)^2$ | $2C(1,1)C(2,1)$ | 4 |
| | | **sum** = | 184 |

The sum of the values in the rightmost column is 184. Division by 8, the cardinality of the permutation group, yields 23, which is the number of congruence classes of configurations with five crosses and four naughts that we previously promised. Figure 6 provides a complete list of representatives of the 23 congruence classes.

Boldface crosses are used to organize these 23 congruence classes into subsets that can be easily checked by ad hoc methods for completeness and non-duplication. For instance, there are nine classes in which an entire side contains crosses.

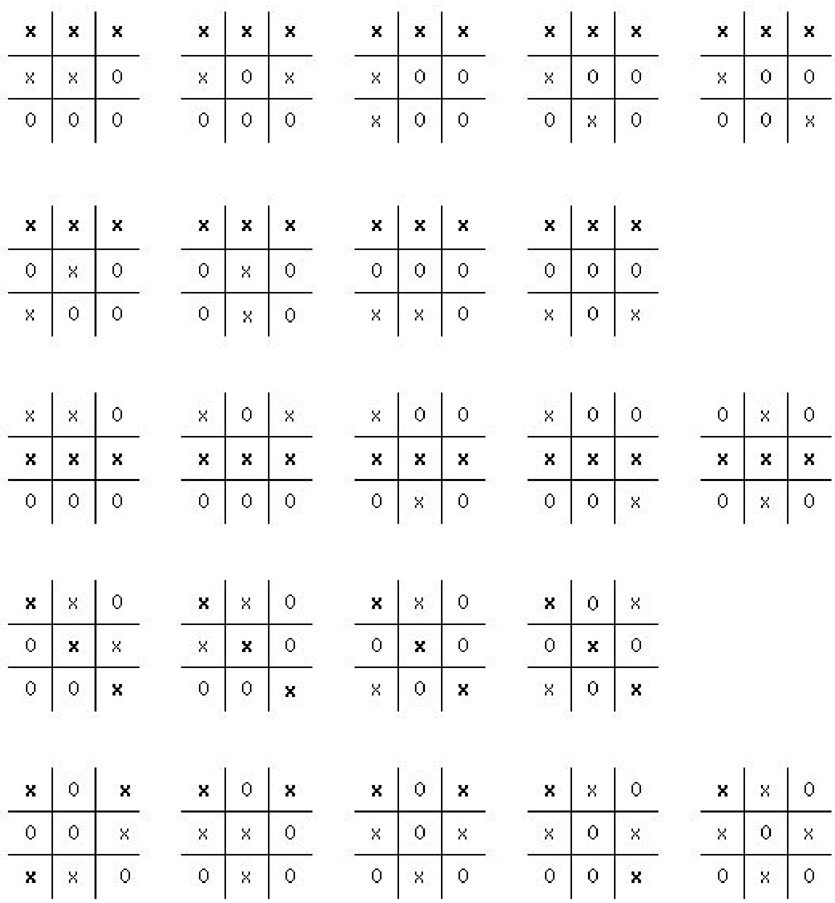

**Figure 6. The 23 different $3 \times 3$ tic-tac-toe configurations with five crosses and four naughts.**

## Suggested Readings


1. W. Burnside, *Theory of Groups of Finite Order*, Second Edition, Dover Publications, Mineola, N.Y., 2004.
2. F. Harary and E. Palmer, *Graphical Enumeration*, Academic Press, New York and London, 1973.
3. G. Pólya, "Kombinatorische Anzahlbestimmungen für Gruppen, Graphen, und chemische Verbindungen", *Acta Mathematica*, Vol. 68, 1937, pp. 344–51.
4. J. Redfield, "The theory of group-reduced distributions", *American Journal of Mathematics*, Vol. 49, 1927, pp. 433–55.


# Exercises

1. Draw a $3 \times 3$ tic-tac-toe configuration with two crosses and seven naughts such that the congruence class has cardinality equal to two.
2. Represent the four rotations on the $3 \times 3$ tic-tac-toe board of Figure 3 by $2 \times 9$ matrices.
3. Represent the four reflections on the $3 \times 3$ tic-tac-toe board of Figure 9 by $2 \times 9$ matrices.
4. Use the composition rule for $2 \times n$ matrix representations of permutations to show that the composition of the $90^\circ$ rotation and the $180^\circ$ rotation of the $2 \times 2$ tic-tac-toe board is the $270^\circ$ rotation.
5. Use $2 \times n$ matrix representations to show that the inverse of the $90^\circ$ rotation of the $2 \times 2$ tic-tac-toe board is the $270^\circ$ rotation.
6. Calculate the cycle index for the group of rotations and reflections on an equilateral triangle.
7. Calculate the cycle index for the group of rotations and reflections on a regular pentagon.
8. Calculate the cycle index for the group of rotations and reflections on a regular hexagon.
9. How many congruence classes are there for a $3 \times 3$ tic-tac-toe board with crosses and naughts everywhere but the center, which is left unfilled?

**10.** In Exercise 9, suppose we further require that there are four crosses and four naughts. How many congruence classes meet this additional restriction?

## Computer Projects

**1.** Write a computer program that takes as input a $3 \times 3$ array of 0s and 1s and prints all reflections and rotations of this array.

**2.** Write a computer program that accepts as input a completely filled-in $3 \times 3$ tic-tac-toe configuration and determines its orbit.

**3.** Write a computer program that takes a cycle index, substitutes $1 + x^r$, and evaluates and collects like terms to produce a polynomial in $x$.