

The Assignment Problem

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Prerequisites: The prerequisites for this chapter are matrices, permutations, and basic concepts of graphs. See Sections 3.8, 5.3, 9.1, and 9.2 of *Discrete Mathematics and Its Applications*.

Introduction

Consider the following problem: Given n workers ($i = 1, 2, \dots, n$) and n jobs ($j = 1, 2, \dots, n$), and the cost c_{ij} to train the i th worker for the j th job, find an assignment of one worker to each job which minimizes the total training cost. This problem is an example of an assignment problem, which we will define shortly. The assignment problem is particularly interesting because many seemingly different problems may be solved as assignment problems. Moreover, it is an important example of a combinatorial optimization problem; that is, a problem that seeks the best possible arrangement of objects among a specified set of many possible arrangements. Typically, the best possible arrangement means an arrangement with the smallest total cost possible.

Let us begin by looking at some examples.

Example 1 Job Assignment Suppose three workers must be assigned to

three jobs. The following matrix shows the cost of training each worker for each job.

		Job Number		
		1	2	3
Worker Number	1	5	7	9
	2	14	10	12
	3	15	13	16

One possible assignment is to assign worker 1 to job 2, worker 2 to job 1, and worker 3 to job 3. This assignment has a total cost of $7 + 14 + 16 = 37$. Is this an assignment with minimal total cost? We will discover the answer later in this chapter. □

Example 2 The Marriage Problem A pioneering colony of 10 bachelors is joined by 10 prospective brides. After a short period of courting, it is decided to have an immediate ceremony. Each bride is given a list of 10 names on which she is to rank her preferences from 1 to 10; that is, she assigns 1 to her first choice, 2 to her second choice, and so on. Let c_{ij} be the rank bride i gives to bachelor j , and let $M = \{(1, j_1), (2, j_2), \dots, (10, j_{10})\}$ be a set of 10 marriages where (i, j_i) pairs bride i with bachelor j_i . Then we assume that $\sum_{i=1}^n c_{ij_i}$ constitutes a valid measure of the anticipated “happiness” of the colony under the set of marriages M , in the sense that the smaller the sum, the happier the colony. What set of 10 marriages maximizes the happiness of the colony? □

Example 3 The Shortest Path Problem Cargo must be delivered by train from New York to Los Angeles. The train routes available are shown in Figure 1, along with the time (in hours) required for each route. Notice that the time depends on the direction since some routes are express routes while others are not. What path from New York to Los Angeles gives the smallest total delivery time? □

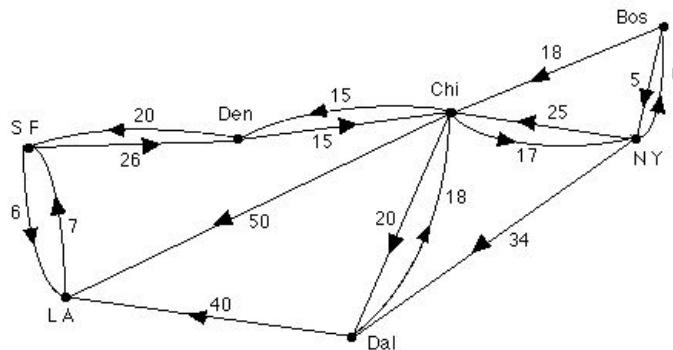


Figure 1. Train routes and times.

These problems are all examples of problems which may be solved as assignment problems. In this chapter we will derive an efficient algorithm for solving assignment problems, and then discuss several problems which may be solved using this algorithm. The assignment problem will then be described in terms of graphs.

Solving Assignment Problems

Recall that a permutation of a set $N = \{1, 2, \dots, n\}$ is a function $\sigma : N \rightarrow N$ which is one-to-one and onto. For example, the function from $\{1, 2, 3, 4, 5\}$ to itself where $\sigma(1) = 5$, $\sigma(2) = 4$, $\sigma(3) = 2$, $\sigma(4) = 1$, and $\sigma(5) = 3$, is a permutation which we denote 54213.

Definition 1 Given any $n \times n$ matrix $C = [c_{ij}]$, the *assignment problem* specified by C is the problem of finding a permutation σ of $\{1, 2, \dots, n\}$ that minimizes

$$z = \sum_{i=1}^n c_{i\sigma(i)}. \quad \square$$

One method for solving assignment problems is to generate all $n!$ permutations of $\{1, 2, \dots, n\}$ (a method for doing this is given in Section 5.6 of *Discrete Mathematics and Its Applications*), compute

$$z = \sum_{i=1}^n c_{i\sigma(i)}$$

for each permutation σ , and then find a permutation on which the minimum of z is attained.

Example 4 In the job assignment problem described in Example 1 of the Introduction, there are $3! = 6$ permutations:

$$\begin{array}{llll} \sigma_1 = 123 & \text{cost: } 31 & \sigma_2 = 132 & \text{cost: } 30 & \sigma_3 = 213 & \text{cost: } 37 \\ \sigma_4 = 312 & \text{cost: } 36 & \sigma_5 = 231 & \text{cost: } 34 & \sigma_6 = 321 & \text{cost: } 34. \end{array}$$

Thus, σ_2 solves the problem and indicates that the best assignment is to assign worker 1 to job 1, worker 2 to job 3, and worker 3 to job 2. \square

The above method is helpful only for n quite small, since one must check all $n!$ possibilities. In practice assignment problems often have $n \geq 30$. If each

permutation can be generated in just 10^{-9} seconds, an assignment problem with $n = 30$ would require at least $8 \cdot 10^{15}$ years of computer time to solve by generating all $30!$ permutations. Therefore a better method is needed.

Before developing a better algorithm, we need to set up a model for the assignment problem. Let $\mathbf{C} = [c_{ij}]$ be any $n \times n$ matrix in which c_{ij} is the cost of assigning worker i to job j . Let $\mathbf{X} = [x_{ij}]$ be the $n \times n$ matrix where

$$x_{ij} = \begin{cases} 1 & \text{if row } i \text{ is assigned to column } j \text{ (that is,} \\ & \text{worker } i \text{ is assigned to job } j) \\ 0 & \text{otherwise} \end{cases}$$

The assignment problem can then be expressed in terms of a function z as:

$$\text{minimize } z(\mathbf{X}) = \sum_{i=1}^n \sum_{j=1}^n c_{ij}x_{ij},$$

subject to the constraints

$$\sum_{j=1}^n x_{ij} = 1, \quad \text{for } i = 1, 2, \dots, n \quad (1)$$

$$\sum_{i=1}^n x_{ij} = 1, \quad \text{for } j = 1, 2, \dots, n \quad (2)$$

Notice that condition (1) guarantees that each row subscript i is assigned to exactly one column subscript. Condition (2) guarantees that each column subscript j is assigned to exactly one row subscript. Hence any matrix \mathbf{X} satisfying conditions (1) and (2) is called a solution and corresponds to a permutation σ of N obtained by setting $\sigma(i) = j$ if and only if $x_{ij} = 1$. Furthermore, if \mathbf{X} is a solution corresponding to σ , then

$$\sum_{j=1}^n c_{ij}x_{ij} = c_{i\sigma(i)}.$$

Summing over i from 1 to n , we obtain

$$\sum_{i=1}^n c_{i\sigma(i)} = \sum_{i=1}^n \sum_{j=1}^n c_{ij}x_{ij}.$$

Thus, any solution \mathbf{X} on which $z(\mathbf{X})$ is minimum is called an **optimal solution**. For instance, in Example 1 it was noted that the permutation σ_2 given by 132 gives the best possible assignment of workers to jobs for that assignment problem. The permutation σ_2 corresponds to the matrix

$$\mathbf{X}^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Since

$$\mathbf{C} = \begin{pmatrix} 5 & 7 & 9 \\ 14 & 10 & 12 \\ 15 & 13 & 16 \end{pmatrix},$$

we have

$$z(\mathbf{X}^*) = \sum_{i=1}^3 \sum_{j=1}^3 c_{ij} x_{ij}^* = 5 + 12 + 13 = 30.$$

This model allows for the derivation of an efficient algorithm known as the **Hungarian method**. The idea behind the Hungarian method is to try to transform a given assignment problem specified by \mathbf{C} into another one specified by a matrix $\widehat{\mathbf{C}} = [\widehat{c}_{ij}]$, such that $\widehat{c}_{ij} \geq 0$, for all pairs i, j , where both problems have the same set of optimal solutions. We then find a solution \mathbf{X}^* for which

$$\widehat{z}(\mathbf{X}^*) = \sum_{i=1}^n \sum_{j=1}^n \widehat{c}_{ij} x_{ij}^* = 0.$$

Since $\widehat{c}_{ij} \geq 0$ (and hence $\widehat{z}(\mathbf{X}) \geq 0$ for all \mathbf{X}), \mathbf{X}^* must be an optimal solution to the problem specified by $\widehat{\mathbf{C}}$, and hence must also be an optimal solution to the one specified by \mathbf{C} . Theorem 1 describes how we can transform a matrix into another one which has the same set of optimal solutions.

Theorem 1 A solution \mathbf{X} is an optimal solution for

$$z(\mathbf{X}) = \sum_{i=1}^n \sum_{j=1}^n c_{ij} x_{ij}$$

if and only if it is an optimal solution for

$$\widehat{z}(\mathbf{X}) = \sum_{i=1}^n \sum_{j=1}^n \widehat{c}_{ij} x_{ij}$$

where $\widehat{c}_{ij} = c_{ij} - u_i - v_j$ for any choice of (u_1, \dots, u_n) and (v_1, \dots, v_n) where u_i and v_j are real numbers for all i and j .

Proof: We will show that the functions $z(\mathbf{X})$ and $\widehat{z}(\mathbf{X})$ differ only by the constant $\sum_{i=1}^n u_i + \sum_{j=1}^n v_j$.

$$\begin{aligned}
\widehat{z}(\mathbf{x}) &= \sum_{i=1}^n \sum_{j=1}^n \widehat{c}_{ij} x_{ij} \\
&= \sum_{i=1}^n \sum_{j=1}^n (c_{ij} - u_i - v_j) x_{ij} \\
&= \sum_{i=1}^n \sum_{j=1}^n c_{ij} x_{ij} - \sum_{i=1}^n \sum_{j=1}^n u_i x_{ij} - \sum_{i=1}^n \sum_{j=1}^n v_j x_{ij} \\
&= \sum_{i=1}^n \sum_{j=1}^n c_{ij} x_{ij} - \sum_{i=1}^n \sum_{j=1}^n u_i x_{ij} - \sum_{j=1}^n \sum_{i=1}^n v_j x_{ij} \\
&= z(\mathbf{x}) - \sum_{i=1}^n u_i \sum_{j=1}^n x_{ij} - \sum_{j=1}^n v_j \sum_{i=1}^n x_{ij} \\
&= z(\mathbf{x}) - \sum_{i=1}^n u_i - \sum_{j=1}^n v_j.
\end{aligned}$$

The last equation follows from conditions (1) and (2). This shows that

$$z(\mathbf{X}) - \widehat{z}(\mathbf{X}) = \sum_{i=1}^n u_i + \sum_{j=1}^n v_j.$$

Thus, a solution \mathbf{X} minimizes $z(\mathbf{X})$ if and only if it minimizes $\widehat{z}(\mathbf{X})$. ■

To describe an optimal solution in terms of entries of a matrix the following notion of *independent* entries is needed. A set of entries of any matrix \mathbf{A} is said to be **independent** if no two of them are in the same row or column.

Example 5 Apply Theorem 1 to the matrix given in Example 1 to obtain a new matrix with all nonnegative entries, which contains an independent set of three zeros and has the same set of optimal solutions as the original matrix.

Solution: Let

$$\mathbf{C} = \begin{pmatrix} 5 & 7 & 9 \\ 14 & 10 & 12 \\ 15 & 13 & 16 \end{pmatrix}.$$

Subtract from each entry in each row the smallest entry in that row; that is, let $u_1 = 5$, $u_2 = 10$, $u_3 = 13$, and $v_1 = v_2 = v_3 = 0$. This gives the matrix

$$\begin{pmatrix} 0 & 2 & 4 \\ 4 & 0 & 2 \\ 2 & 0 & 3 \end{pmatrix}.$$

The new matrix has an independent set of two zeros, but we need three. Next, subtract from each entry in each column the smallest entry in that column, that is, let $u_1 = u_2 = u_3 = 0$, $v_1 = v_2 = 0$, and $v_3 = 2$. This gives

$$\widehat{\mathbf{C}} = \begin{pmatrix} 0^* & 2 & 2 \\ 4 & 0 & 0^* \\ 2 & 0^* & 1 \end{pmatrix}.$$

The starred entries in $\widehat{\mathbf{C}}$ form an independent set of three zeros. By Theorem 1 applied twice, $\widehat{\mathbf{C}}$ and \mathbf{C} have the same set of optimal solutions. \square

We are interested in obtaining a matrix with nonnegative entries with an independent set of three zeros because it is easy to obtain an optimal solution from such a matrix, as Example 6 illustrates.

Example 6 Solve the assignment problem specified by

$$\mathbf{C} = \begin{pmatrix} 5 & 7 & 9 \\ 14 & 10 & 12 \\ 15 & 13 & 16 \end{pmatrix},$$

by obtaining an optimal solution to the problem specified by

$$\widehat{\mathbf{C}} = \begin{pmatrix} 0^* & 2 & 2 \\ 4 & 0 & 0^* \\ 2 & 0^* & 1 \end{pmatrix}.$$

Solution: Define the solution

$$\mathbf{X}^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then \mathbf{X}^* solves the assignment problem specified by $\widehat{\mathbf{C}}$ since $\widehat{z}(\mathbf{X}^*) = 0$ and $\widehat{z}(\mathbf{X}) \geq 0$ for any other solution \mathbf{X} . By Example 5, \mathbf{X}^* is also an optimal solution to the assignment problem specified by \mathbf{C} . Note that \mathbf{X}^* corresponds to the permutation 132. \square

The method used to obtain an optimal solution to the assignment problem specified by $\widehat{\mathbf{C}}$ in Example 6 is generalized in Theorem 2.

Theorem 2 If $\mathbf{C} = [c_{ij}]$ satisfies $c_{ij} \geq 0$ for all i and j ($1 \leq i \leq n$, $1 \leq j \leq n$), and $\{c_{1j_1}, c_{2j_2}, \dots, c_{nj_n}\}$ is an independent set of n zeros in \mathbf{C} , then $\mathbf{X}^* = [x_{ij}^*]$ where $x_{1j_1}^* = 1, x_{2j_2}^* = 1, \dots, x_{nj_n}^* = 1$, and $x_{ij}^* = 0$ for any other i and j , is an optimal solution to the assignment problem specified by \mathbf{C} .

Proof: We must show that for any solution \mathbf{X} , $z(\mathbf{X}) \geq z(\mathbf{X}^*)$. To see this, observe that

$$\begin{aligned} z(\mathbf{X}^*) &= \sum_{i=1}^n \sum_{j=1}^n c_{ij} x_{ij}^* \\ &= \sum_{i=1}^n c_{ij_i} x_{ij_i}^* \\ &= 0. \end{aligned}$$

The second step follows from the definition of \mathbf{X}^* , and the last step follows since $c_{ij_i} = 0$ for all i . Since $z(\mathbf{X}) \geq 0$ for all solutions \mathbf{X} , $z(\mathbf{X}) \geq z(\mathbf{X}^*)$, so \mathbf{X}^* is an optimal solution. ■

The objective of the Hungarian method is to use Theorem 1 to transform a matrix \mathbf{C} into another matrix $\widehat{\mathbf{C}}$, having the same set of optimal solutions as \mathbf{C} , such that $\widehat{\mathbf{C}}$ contains an independent set of n zeros. Then, using Theorem 2, an optimal solution to both problems can be obtained.

Example 7 Use Theorem 1 to transform

$$\mathbf{C} = \begin{pmatrix} 0 & 1 & 6 \\ 2 & -7 & 3 \\ -5 & 3 & 4 \end{pmatrix}$$

into a matrix with nonnegative entries containing an independent set of 3 zeros. Then use Theorem 2 to obtain an optimal solution to the assignment problem specified by \mathbf{C} .

Solution: Let $u_1 = 0$, $u_2 = -7$, $u_3 = -5$, and $v_1 = v_2 = v_3 = 0$. Then apply Theorem 1 to obtain the matrix

$$\begin{pmatrix} 0 & 1 & 6 \\ 9 & 0 & 10 \\ 0 & 8 & 9 \end{pmatrix},$$

which has the same set of optimal solutions as \mathbf{C} . This matrix does not have an independent set of 3 zeros. Letting $u_1 = u_2 = u_3 = v_1 = v_2 = 0$ and $v_3 = 6$ gives

$$\widehat{\mathbf{C}} = \begin{pmatrix} 0 & 1 & 0^* \\ 9 & 0^* & 4 \\ 0^* & 8 & 3 \end{pmatrix},$$

which also has the same set of optimal solutions as \mathbf{C} . By Theorem 2,

$$\mathbf{X}^* = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

is an optimal solution to the problem $\widehat{\mathbf{C}}$, hence to the problem \mathbf{C} . Note that \mathbf{X}^* corresponds to the permutation 321. \square

The matrix $\widehat{\mathbf{C}}$ in the solution of Example 7 is called a *reduced matrix* for \mathbf{C} , which we shall now define.

Definition 2 Given any $n \times n$ matrix $\mathbf{C} = [c_{ij}]$, let

$$\begin{aligned} u_i &= \text{minimum } \{c_{i1}, c_{i2}, \dots, c_{in}\}, \text{ for } i = 1, 2, \dots, n, \\ v_j &= \text{minimum } \{c_{1j} - u_1, c_{2j} - u_2, \dots, c_{nj} - u_n\}, \text{ for } j = 1, 2, \dots, n. \end{aligned}$$

The $n \times n$ matrix $\widehat{\mathbf{C}} = [\widehat{c}_{ij}]$ given by $\widehat{c}_{ij} = c_{ij} - u_i - v_j$ for all pairs i and j is called the *reduced matrix* for \mathbf{C} . \square

In words, a reduced matrix is obtained by first subtracting from each row the smallest entry in each row and then subtracting from each column the smallest entry in each column. By Theorem 1, the assignment problems specified by a matrix \mathbf{C} and by its reduced matrix $\widehat{\mathbf{C}}$ both have the same set of optimal solutions. Observe that all entries in the reduced matrix are nonnegative. However, the reduced matrix may not contain an independent set of n zeros.

Example 8 Determine the reduced matrix for

$$\mathbf{C} = \begin{pmatrix} 2 & 5 & 7 \\ 4 & 2 & 1 \\ 2 & 6 & 5 \end{pmatrix}.$$

Solution: The values of $u_1, u_2, u_3, v_1, v_2,$ and v_3 , as defined in the definition of the reduced matrix are $u_1 = 2, u_2 = 1, u_3 = 2$, and $v_1 = 0, v_2 = 1,$ and $v_3 = 0$. Therefore, the reduced matrix is

$$\widehat{\mathbf{C}} = \begin{pmatrix} 0 & 2 & 5 \\ 3 & 0 & 0 \\ 0 & 3 & 3 \end{pmatrix}. \quad \square$$

The matrix \widehat{C} in the solution of Example 8 does not contain an independent set of three zeros. To obtain a new matrix having the same optimal solutions as \widehat{C} , but containing more zeros, draw a set of lines through the rows and columns of \widehat{C} using as few lines as possible so that there is at least one line through every zero. This gives

$$\widehat{C} = \begin{pmatrix} 0 & 2 & 5 \\ 3 & 0 & 0 \\ 0 & 3 & 3 \end{pmatrix}.$$

Observe that the minimum number of lines needed to cover all zeros is equal to the maximum number of independent zeros. Theorem 3, which we will need in order to show that the Hungarian method terminates after a finite number of steps, shows that this fact is true in general.

Theorem 3 The maximum number of independent zeros in a matrix is equal to the minimum number of lines needed to cover all zeros in the matrix. ■

The proof is beyond the scope of the book. For a proof of Theorem 3, see [4] in the suggested readings.

Example 9 Use Theorem 1 to transform the reduced matrix

$$\widehat{C} = \begin{pmatrix} 0 & 2 & 5 \\ 3 & 0 & 0 \\ 0 & 3 & 3 \end{pmatrix}$$

from Example 8 into a matrix with nonnegative entries containing an independent set of three zeros.

Solution: First, subtract from every entry in \widehat{C} the smallest entry not covered by any line (which is 2). This is the transformation with $u_1 = u_2 = u_3 = 2$ and $v_1 = v_2 = v_3 = 0$, and gives the matrix

$$\begin{pmatrix} -2 & 0 & 3 \\ 1 & -2 & -2 \\ -2 & 1 & 1 \end{pmatrix}.$$

Next, add 2 to every entry in every row and column covered by one line, and add 2 twice to any entry covered by two lines (note that the (2,1) entry has 2 added to it twice, since it was covered twice). This is the transformation $u_1 = u_3 = v_2 = v_3 = 0$ and $u_2 = v_1 = 2$, and gives the matrix

$$\begin{pmatrix} 0 & 0^* & 3 \\ 5 & 0 & 0^* \\ 0^* & 1 & 1 \end{pmatrix}$$

which contains an independent set of three zeros. □

The Hungarian Method

We will now display the algorithm called the Hungarian method. We shall assume that the costs c_{ij} are integers. To begin the algorithm, given an $n \times n$ matrix \mathbf{C} , first obtain the reduced matrix $\widehat{\mathbf{C}}$ for \mathbf{C} ; that is, subtract from each entry of each row the smallest entry in that row. Then do the same for columns. Next perform the following two steps:

- (i) Find a maximal independent set of zeros. If this set has n elements, an optimal solution is available. Otherwise go to step (ii).
- (ii) Find a set of lines that cover all zeros using the smallest possible number of lines. Let k be the smallest entry not covered. Subtract k from each entry not covered by any line, and add k to each entry covered twice. Repeat step (i).

Algorithm 1 gives the pseudocode description of the Hungarian method.

Applying step (ii) of the algorithm produces a new matrix with the same set of optimal solutions as the original matrix, since step (ii) is equivalent to first subtracting k from every entry in the matrix, and then adding k to every entry covered by a line. Subtracting k from every entry is the transformation $u_i = k$, for all i , and $v_j = 0$, for all j . By Theorem 1, this does not change the set of optimal solutions. Adding k to every entry covered by a line is the transformation

$$u_i = \begin{cases} -k & \text{if there is a line through row } i \\ 0 & \text{otherwise} \end{cases}$$

$$v_j = \begin{cases} -k & \text{if there is a line through column } j \\ 0 & \text{otherwise.} \end{cases}$$

Again by Theorem 1, this does not change the set of optimal solutions.

We now show that the algorithm must terminate after a finite number of steps. (We leave it as Exercise 13 to show that, after performing the initial step, all entries in the reduced matrix are nonnegative.) We will show that the sum of all entries in the matrix decreases by at least 1 whenever step (ii) is performed. Clearly, if the sum of all entries is zero, then all entries in the matrix are zero and an independent set of n zeros exists. Thus, if the algorithm did not terminate, the sums of all matrix entries would give an infinite decreasing sequence of positive integers, which is impossible.

Step (ii) is performed only when no independent set of n zeros exists. Thus, if q is the minimum number of lines needed to cover all zeros, then Theorem 3 implies that $q < n$. Subtracting k from each entry subtracts kn^2 from the sum of entries, since there are n^2 entries in the matrix. Adding k to each covered entry adds qkn to the sum of entries, since there are q lines and n entries on each line. Therefore the net change in the sum of entries is $-kn^2 + qkn$. But $-kn^2 + qkn = kn(-n + q) < 0$, since $q < n$. Since k , q , and n are all integers, the sum of all entries must decrease by at least 1.

ALGORITHM 1. Hungarian Method

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procedure Hungarian (C:  $n \times n$  matrix of integers) for  $i := 1$  to  $n$ 
begin
   $u_i :=$  smallest integer in row  $i$  of C
  for  $j := 1$  to  $n$ 
     $\hat{c}_{ij} := c_{ij} - u_i$ 
  end
  for  $j := 1$  to  $n$ 
begin
   $v_j :=$  smallest integer in column  $j$  of  $\hat{\mathbf{C}}$ 
  for  $i := 1$  to  $n$ 
     $\hat{c}_{ij} := \hat{c}_{ij} - v_j$ 
  end
  { $\hat{\mathbf{C}}$  is now the reduced matrix}
   $S :=$  an independent set of zeros of maximal size in  $\hat{\mathbf{C}}$ 
   $q := |S|$ 
  while  $q < n$ 
begin
     $cover(\hat{\mathbf{C}})$ 
     $k :=$  smallest entry in  $\hat{\mathbf{C}}$  not covered by a line
    for  $i := 1$  to  $n$ 
      for  $j := 1$  to  $n$ 
begin
        if  $\hat{c}_{ij}$  is not covered then  $\hat{c}_{ij} := \hat{c}_{ij} - k$            if  $\hat{c}_{ij}$ 
        is covered twice then  $\hat{c}_{ij} := \hat{c}_{ij} + k$ 
      end
     $S :=$  an independent set of zeros of maximal size in  $\hat{\mathbf{C}}$     $q := |S|$ 
  end for  $i := 1$  to  $n$ 
  for  $j := 1$  to  $n$ 
    if  $\hat{c}_{ij} \in S$  then  $x_{ij}^* := 1$  else  $x_{ij}^* := 0$ 
  { $\mathbf{X}^* = [x_{ij}^*]$  is an optimal solution}

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Exercise 17 shows that the number of iterations is $O(n^2)$. To compare the Hungarian method to the exhaustive search method mentioned above, suppose that each iteration can be performed in one second. Then an assignment problem with $n = 30$ can be solved in at most $30^2 = 900$ seconds, or 15 minutes of

computer time.

The following example illustrates how the Hungarian method works.

Example 10 A foreman has five workers and five jobs to complete. The time in hours each worker needs to complete each job is shown in the following table.

	Job 1	Job 2	Job 3	Job 4	Job 5
Worker 1	3	4	8	7	8
Worker 2	2	5	3	2	6
Worker 3	7	9	1	8	3
Worker 4	5	3	4	6	6
Worker 5	8	9	7	5	8

How should the foreman make an assignment of one worker to each job so that the total time is minimized?

Solution: Let

$$C = \begin{pmatrix} 3 & 4 & 8 & 7 & 8 \\ 2 & 5 & 3 & 2 & 6 \\ 7 & 9 & 1 & 8 & 3 \\ 5 & 3 & 4 & 6 & 6 \\ 8 & 9 & 7 & 5 & 8 \end{pmatrix}.$$

The reduced matrix is

$$\begin{pmatrix} 0 & 1 & 5 & 4 & 3 \\ 0 & 3 & 1 & 0 & 2 \\ 6 & 8 & 0 & 7 & 0 \\ 2 & 0 & 1 & 3 & 1 \\ 3 & 4 & 2 & 0 & 1 \end{pmatrix}.$$

The minimum number of lines needed to cover all zeros of this matrix is four. For example, lines through row 3 and through columns 1, 2 and 4 will cover all zeros. Since we need an independent set of 5 zeros, step (ii) must be performed. This gives the matrix

$$\begin{pmatrix} 0^* & 1 & 4 & 4 & 2 \\ 0 & 3 & 0 & 0^* & 1 \\ 7 & 9 & 0^* & 8 & 0 \\ 2 & 0^* & 0 & 3 & 0 \\ 3 & 4 & 1 & 0 & 0^* \end{pmatrix}.$$

The starred entries indicate one possible independent set of five zeros. Thus an

optimal solution is

$$\mathbf{X}^* = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

which corresponds to the permutation 14325. An optimal job assignment is to assign Worker 1 to Job 1, Worker 2 to Job 4, Worker 3 to Job 3, Worker 4 to Job 2, and Worker 5 to Job 5. \square

Example 11 Find a path which minimizes total delivery time from New York to Los Angeles in the shortest path problem given in Figure 1.

Solution: We solve the assignment problem specified by the following matrix. Exercise 16 shows that any optimal solution of this assignment problem will find a path which minimizes total delivery time. We assign a time of 100 hours to any route joining a pair of cities for which there is no train route. The choice of 100 prohibits such a route from occurring as part of an optimal solution. This is because the path

New York \rightarrow Dallas \rightarrow Los Angeles

is a path with total time 74 hours. Hence, any path which minimizes total delivery time can have a total time of at most 74 hours, so any route requiring more than 74 hours can not be part of an optimal solution.

	Bos	Chi	Dal	Den	SF	LA
NY	6	25	34	100	100	100
Bos	0	18	100	100	100	100
Chi	100	0	20	15	100	50
Dal	100	18	0	100	100	40
Den	100	15	100	0	20	100
SF	100	100	100	26	0	6

By applying the Hungarian method, the optimal solution

$$\mathbf{X}^* = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

is obtained (details are left as Exercise 5). This matrix shows that the shortest path is

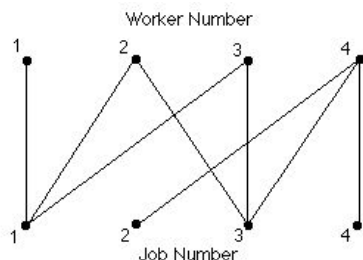


Figure 2. Job assignments.

NY \rightarrow Bos \rightarrow Chi \rightarrow Den \rightarrow SF \rightarrow L A.

(Note that the path does not pass through Dallas.)

□

The Hungarian method was discovered in 1955 by Harold Kuhn* of the Mathematics and Economics Departments of Princeton University (see [3] in the suggested readings for the original paper). The algorithm is called the “Hungarian” method in honor of Denes König**, who, in 1931, discovered Theorem 3. There are other efficient algorithms for solving the assignment problem. For example, [1] in the suggested readings describes an algorithm based on the degrees of vertices of certain spanning trees of $K_{n,n}$, the complete bipartite graph.

Perfect Matching Problems

Suppose four workers must be assigned to four jobs. The jobs that each worker can perform are indicated by an edge of the graph given in Figure 2. Is it possible to assign workers to jobs so that each worker is assigned one job and each job is assigned to one worker?

This problem may also be stated in terms of graphs. We need the following definition.

* Harold Kuhn (1925–) is an American mathematician who obtained his Ph.D. from Princeton University. Kuhn has made many fundamental contributions to the field of linear programming and is most famous for the Kuhn-Tucker conditions which characterize the set of optimal solutions of a linear program.

** Denes König (1884–1944) was a Hungarian mathematician. He is known as one of the pioneers of graph theory and is most famous for his work in matching theory.

Definition 3 Let $G = (V, E)$ be any bipartite graph with $V = V_1 \cup V_2$. A subset of edges M contained in E is called a *perfect matching* if every vertex in V is contained in exactly one edge of M . \square

For example, the bipartite graph given in Figure 3a contains the perfect matching given in Figure 3b.

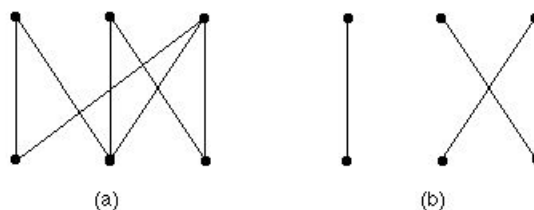


Figure 3. A bipartite graph and a perfect matching.

The problem stated in the beginning of this section may now be stated as follows. Does there exist a perfect matching in the graph given in Figure 2? The problem of determining if a given bipartite graph contains a perfect matching was solved by Georg Frobenius* who proved the well known “Marriage Theorem”. To state this theorem we will use the following notation: for any subset W contained in V_1 let $R(W)$ denote the set of vertices in V_2 adjacent to a vertex in W .

Theorem 4 Frobenius Marriage Theorem A bipartite graph $G = (V, E)$ with $V = V_1 \cup V_2$ has a perfect matching if and only if $|V_1| = |V_2|$ and for every subset W contained in V_1 , $|R(W)| \geq |W|$. \blacksquare

The proof of Theorem 4 may be obtained by applying Theorem 3. The details are left for Exercises 18 and 19.

Example 12 Determine if the graph given in Figure 2 contains a perfect matching.

Solution: Let $W = \{\text{Worker 1, Worker 2, Worker 3}\}$. Then $R(W) = \{\text{Job 1, Job 3}\}$. Since $|R(W)| = 2$ and $|W| = 3$, Theorem 4 implies that the graph given in Figure 2 does not contain a perfect matching. \square

* Georg Frobenius (1848–1917) was a German mathematician who attended schools in Göttingen and Berlin. He was a professor of mathematics at both the University of Berlin and the Eidgenössische Polytechnikum in Zurich. Frobenius is best known for his contributions to algebra, particularly in group theory.

The **weight** of any perfect matching M is the sum of the weights of the edges in M . Many matching problems involve searching a weighted bipartite graph for a perfect matching which has the smallest possible weight.

Example 13 Suppose three workers must be assigned to three jobs. The graph given in Figure 4 indicates the cost of training each worker for each job. How should the workers be assigned to jobs so that each worker is assigned one job, each job is assigned one worker, and the total training cost is minimized?

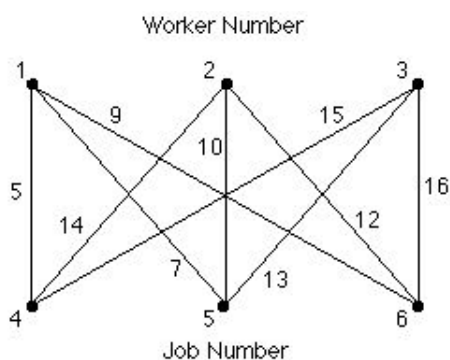


Figure 4. Training costs.

Solution: We solve the problem by listing the set of all perfect matches along with their weights:

$$\begin{aligned}
 M_1 &= \{\{1, 1\}, \{2, 2\}, \{3, 3\}\} && \text{weight: } 31 \\
 M_2 &= \{\{1, 1\}, \{2, 3\}, \{3, 2\}\} && \text{weight: } 30 \\
 M_3 &= \{\{1, 2\}, \{2, 1\}, \{3, 3\}\} && \text{weight: } 37 \\
 M_4 &= \{\{1, 3\}, \{2, 1\}, \{3, 2\}\} && \text{weight: } 36 \\
 M_5 &= \{\{1, 2\}, \{2, 3\}, \{3, 1\}\} && \text{weight: } 34 \\
 M_6 &= \{\{1, 3\}, \{2, 2\}, \{3, 1\}\} && \text{weight: } 34
 \end{aligned}$$

This shows that M_2 is the perfect matching with the smallest possible weight. Thus Worker 1 should be assigned to Job 1, Worker 2 should be assigned to Job 3, and Worker 3 should be assigned to Job 2. \square

The reader should compare the perfect matches listed in the solution to Example 13 and the permutations $\sigma_1, \dots, \sigma_6$ listed in Example 4. Note that the permutation σ_1 given by $\sigma_1(1) = 1$, $\sigma_1(2) = 2$, and $\sigma_1(3) = 3$, corresponds to the perfect matching M_1 , which matches 1 to 1, 2 to 2, and 3 to 3. The permutation σ_2 given by $\sigma_2(1) = 1$, $\sigma_2(2) = 3$, and $\sigma_2(3) = 2$, corresponds to

the perfect matching M_2 which matches 1 to 1, 2 to 3, and 3 to 2. Similarly the permutations $\sigma_3, \sigma_4, \sigma_5,$ and σ_6 correspond to the perfect matches $M_3, M_4, M_5,$ and M_6 respectively. Therefore, solving Example 13 is equivalent to solving the assignment problem specified by

$$C = \begin{pmatrix} 5 & 7 & 9 \\ 14 & 10 & 12 \\ 15 & 13 & 16 \end{pmatrix}$$

solved earlier. This is not a coincidence. Given any weighted complete bipartite graph $K_{n,n}$, any perfect matching M corresponds to a permutation σ of $\{1, 2, \dots, n\}$ defined by $\sigma(i) = j$ if and only if edge $\{i, j\} \in M$. Moreover, if the weight of edge $\{i, j\}$ is c_{ij} , then

$$\text{weight of } M = \sum_{i=1}^n c_{i\sigma(i)}.$$

Thus, finding a perfect matching of minimum weight in a complete bipartite graph is equivalent to solving the assignment problem specified by $C = [c_{ij}]$. In fact, finding a perfect matching with the smallest possible weight is simply a search for the best possible assignment of the vertices in V_1 to those in V_2 .

Example 14 Find a perfect matching with the smallest possible weight for the graph given in Figure 5.

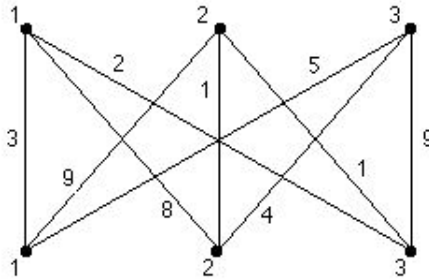


Figure 5. A weighted graph.

Solution: The problem is solved by solving the assignment problem specified by

$$\begin{pmatrix} 3 & 8 & 2 \\ 9 & 1 & 1 \\ 5 & 4 & 9 \end{pmatrix}.$$

Applying the Hungarian method gives the reduced matrix

$$\begin{pmatrix} 0^* & 6 & 0 \\ 7 & 0 & 0^* \\ 0 & 0^* & 5 \end{pmatrix}$$

which contains an independent set of 3 zeros. An optimal solution is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

which corresponds to the perfect matching $M = \{\{1, 1\}, \{2, 3\}, \{3, 2\}\}$. \square

Suggested Readings

1. M. Balinski, "Signature Methods for the Assignment Problem", *Operations Research*, Vol. 33, 1985, pp. 527–536.
2. D. Gale and L. Shapley, "College Admissions and the Stability of Marriage", *American Mathematical Monthly* 69, 1962, pp. 9–15.
3. H. Kuhn, "The Hungarian Method for the Assignment Problem" *Naval Res. Logist. Quart.*, Vol. 2, 1955, pp. 83–97.
4. M. Hall, *Combinatorial Theory*, Second Edition, John Wiley & Sons, Hoboken, N.J., 1998.

Exercises

In Exercises 1–4, solve the assignment problem specified by the given matrix.

1.

$$\begin{pmatrix} 5 & 8 & 3 & 9 \\ 2 & 6 & 1 & 9 \\ 3 & 9 & 4 & 8 \\ 4 & 7 & 2 & 9 \end{pmatrix}.$$

2.

$$\begin{pmatrix} 6 & 2 & 5 & 8 \\ 6 & 7 & 1 & 6 \\ 6 & 3 & 4 & 7 \\ 5 & 4 & 3 & 5 \end{pmatrix}.$$

3.

$$\begin{pmatrix} 7 & 5 & 9 & 7 & 8 & 6 \\ 1 & 2 & 4 & 1 & 3 & 1 \\ 9 & 9 & 8 & 8 & 9 & 3 \\ 4 & 7 & 5 & 6 & 3 & 4 \\ 5 & 3 & 7 & 4 & 3 & 5 \\ 3 & 2 & 2 & 1 & 2 & 1 \end{pmatrix}.$$

4.

$$\begin{pmatrix} -6 & 3 & 1 & 0 & 4 & 6 \\ 5 & -3 & 8 & 4 & 5 & 3 \\ -5 & 4 & 9 & 8 & 9 & 3 \\ 3 & 7 & 5 & -8 & 0 & 9 \\ 7 & 2 & 6 & 5 & 7 & 6 \\ -3 & 0 & 2 & -1 & 3 & 4 \end{pmatrix}.$$

5. Solve the assignment problem specified by the matrix in the solution to Example 11.
6. The coach of a certain swim team needs to assign swimmers to a 200-yard medley relay team . Most of his swimmers are quite fast in more than one stroke, so it is not clear which swimmer should be assigned to each of the four strokes. The four fastest swimmers and the best times they have achieved in each of the strokes, for 50 yards, are

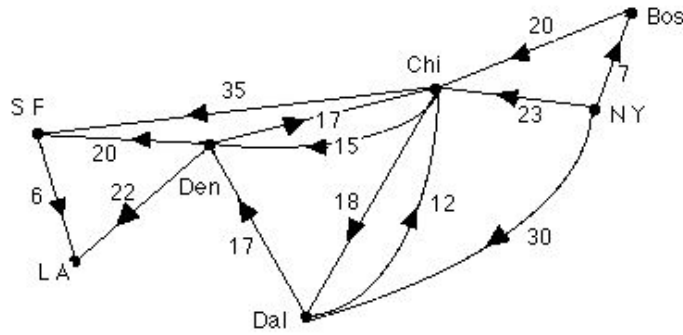
Stroke	Ken	Rob	Mark	David
Backstroke	39.3	33.6	34.0	35.6
Breaststroke	34.2	41.8	38.7	33.7
Butterfly	29.5	30.6	33.1	31.8
Freestyle	28.7	27.6	30.3	29.3

How should the coach assign the four swimmers to the four different strokes to minimize the sum of the corresponding best times?

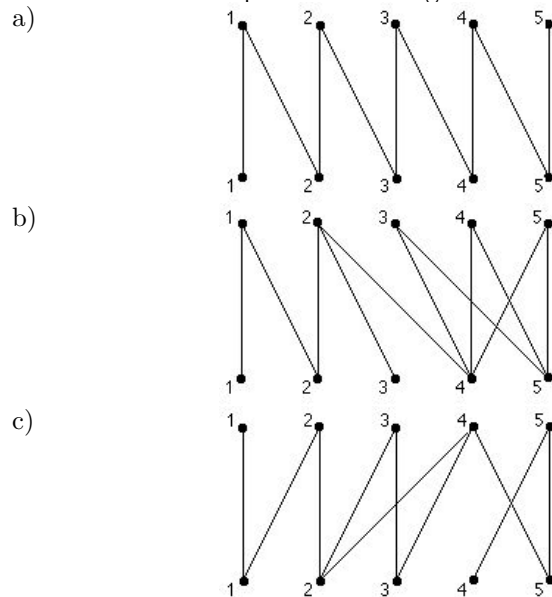
7. Find a set of marriages which maximizes the “happiness” of the following colony of 6 prospective brides and 6 bachelors, where the rating of the bachelors by the brides is given in the following table.

	John	Bill	Joe	Al	Bud	Hal
Jane	3	4	1	2	5	6
Mary	2	5	3	6	4	1
Carol	2	3	1	5	3	6
Jessica	1	2	5	6	3	4
Dawn	5	4	2	1	6	3
Lisa	4	3	6	2	1	5

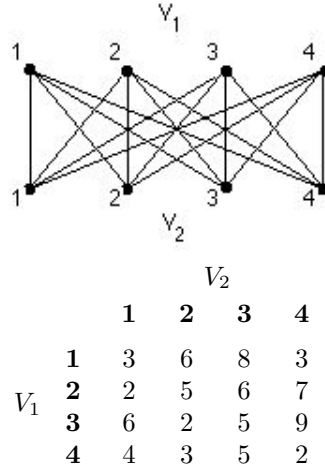
8. The following figure indicates available train routes along with the time required to travel the route. Find a route from Boston to San Francisco which minimizes total time required.



9. Determine which of the following bipartite graphs contain a perfect matching. List the edges in a perfect matching for those graphs that contain one, and show that Frobenius' Marriage Theorem does not hold for those graphs that do not contain a perfect matching.



10. Find a perfect matching with the smallest possible weight in the following weighted graph. The weights of the edges are given in the following table.



11. a) Construct a 5×5 matrix \mathbf{C} such that the assignment problem specified by \mathbf{C} has more than one optimal solution.
 b) Construct a 5×5 matrix \mathbf{C} such that the assignment problem specified by \mathbf{C} has a unique optimal solution.
12. Show that any permutation of $\{1, 2, 3, 4\}$ solves the assignment problem specified by

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix}.$$

13. Explain why the entries in the reduced matrix are nonnegative, even if the original costs are negative.
14. Given any $n \times n$ matrix \mathbf{C} , describe how the Hungarian Method may be used to solve an assignment problem in which we seek a permutation σ of $\{1, 2, \dots, n\}$ such that $\sum_{i=1}^n c_{i\sigma(i)}$ is *maximum*.
15. Use Exercise 14 to find a permutation which maximizes $\sum_{i=1}^n c_{i\sigma(i)}$, where \mathbf{C} is the matrix given in Exercise 3.
16. Show that any optimal solution to the assignment problem specified by the matrix given in Example 11 finds a path from New York to Los Angeles which minimizes total delivery time. *Hint:* Suppose σ^* is an optimal solution and P is a path shorter than $\sum_{i=1}^n c_{i\sigma^*(i)}$.
17. Show that the Hungarian Method uses $O(n^2)$ iterations to find an optimal solution of an assignment problem.

A **vertex cover** Q of the edges of a graph G is a set of vertices such that each edge of G contains at least one vertex in Q .

18. Show that in a bipartite graph $G = (V, E)$ the maximum number of edges in a matching is equal to the minimum number of vertices in a vertex cover.
Hint: Use Theorem 3.

19. Use Exercise 18 to prove Theorem 4.

Suppose that in a community of n bachelors together with n prospective brides each person ranks those of the opposite sex in accordance with his or her preferences for a marriage partner. A set of n marriages is called **unstable** if among the n marriages there are a man and a woman who are not married to each other, but prefer each other to the person they are married to; otherwise the set of marriages is called **stable**. (For a complete discussion of stable marriages see [2] in the suggested readings.)

20. If the first number of the following table gives the preferences of women by the men and the second number gives the preferences of men by the women, find a stable set of 4 marriages.

	Susan	Colleen	Nancy	Dawn
David	1,1	2,3	3,2	4,1
Richard	2,3	1,4	3,1	4,4
Paul	1,4	4,1	2,3	3,3
Kevin	3,2	1,2	4,4	2,2

- *21. Prove that in any community of n men and n women there always exists a stable set of n marriages. *Hint:* Construct an iterative procedure for finding a stable set of n marriages.

Computer Projects

1. Write a program that inputs a 5×5 matrix C and solves the assignment problem specified by C by generating all the permutations of $\{1, 2, 3, 4, 5\}$.
2. Write a program that inputs a 5×5 matrix C and finds an independent set of zeros of maximum size in C .