

Rosen, Discrete Mathematics and Its Applications, 6th edition
Extra Examples

Section 1.2—Propositional Equivalences



— Page references correspond to locations of Extra Examples icons in the textbook.

p.22, icon below Definition 2

#1. Prove that $\neg[r \vee (q \wedge (\neg r \rightarrow \neg p))] \equiv \neg r \wedge (p \vee \neg q)$ by using a truth table.

Solution:

We construct the truth tables for $\neg[r \vee (q \wedge (\neg r \rightarrow \neg p))]$ and for $\neg r \wedge (p \vee \neg q)$, and show that they are identical. We insert “intermediate” columns as we build each of the propositions.

| p | q | r | $\neg p$ | $\neg r$ | $\neg r \rightarrow \neg p$ | $q \wedge (\neg r \rightarrow \neg p)$ | $r \vee (q \wedge (\neg r \rightarrow \neg p))$ | $\neg(r \vee (q \wedge (\neg r \rightarrow \neg p)))$ | $p \vee \neg q$ | $\neg r \wedge (p \vee \neg q)$ |
|-----|-----|-----|----------|----------|-----------------------------|--|---|---|-----------------|---------------------------------|
| T | T | T | F | F | T | T | T | F | T | F |
| T | T | F | F | T | F | F | F | T | T | T |
| T | F | T | F | F | T | F | T | F | T | F |
| T | F | F | F | T | F | F | F | T | T | T |
| F | T | T | T | F | T | T | T | F | F | F |
| F | T | F | T | T | T | T | T | F | F | F |
| F | F | T | T | F | T | F | T | F | T | F |
| F | F | F | T | T | T | F | F | T | T | T |

Note that the ninth and eleventh columns are identical. Therefore the two propositions are equivalent.

p.22, icon below Definition 2

#2. Show that $\neg(p \vee q) \not\equiv \neg p \vee \neg q$.

Solution:

To show that these two propositions are not equivalent, we need to find values for p and q such that $\neg(p \vee q)$ and $\neg p \vee \neg q$ have different truth values. One way to do this is to construct the truth table for each proposition and show that their truth values are different in at least one case. In this case we obtain

| p | q | $\neg p$ | $\neg q$ | $p \vee q$ | $\neg(p \vee q)$ | $\neg p \vee \neg q$ |
|-----|-----|----------|----------|------------|------------------|----------------------|
| T | T | F | F | T | F | F |
| T | F | F | T | T | F | T |
| F | T | T | F | T | F | T |
| F | F | T | T | F | T | T |

In the second row of the truth table (the case where p is true and q is false), the proposition $\neg(p \vee q)$ is false, but the proposition $\neg p \vee \neg q$ is true. Therefore, the two propositions are not equivalent. (Note that the truth values for the two propositions also differ in the third row.)

p.26, icon at Example 6

#1. Prove that $\neg[r \vee (q \wedge (\neg r \rightarrow \neg p))] \equiv \neg r \wedge (p \vee \neg q)$ by using a series of logical equivalences.

Solution:

We will begin with $\neg[r \vee (q \wedge (\neg r \rightarrow \neg p))]$ and use rules of logic to show that this is equivalent to $\neg r \wedge (p \vee \neg q)$. Here is one possible proof:

| | | |
|--|--|--------------------------------------|
| $\neg[r \vee (q \wedge (\neg r \rightarrow \neg p))]$ | $\equiv \neg r \wedge \neg(q \wedge (\neg r \rightarrow \neg p))$ | De Morgan's law |
| $\equiv \neg r \wedge \neg(q \wedge (\neg \neg r \vee \neg p))$ | $\equiv \neg r \wedge \neg(q \wedge (r \vee \neg p))$ | conditional rewritten as disjunction |
| $\equiv \neg r \wedge \neg(\neg q \vee \neg(r \vee \neg p))$ | $\equiv \neg r \wedge (\neg q \vee \neg(\neg r \wedge p))$ | double negation law |
| $\equiv \neg r \wedge (\neg q \vee (\neg r \wedge \neg p))$ | $\equiv (\neg r \wedge \neg q) \vee (\neg r \wedge (\neg r \wedge p))$ | De Morgan's law |
| $\equiv (\neg r \wedge \neg q) \vee ((\neg r \wedge \neg r) \wedge p)$ | $\equiv (\neg r \wedge \neg q) \vee (\neg r \wedge p)$ | De Morgan's law and double negation |
| $\equiv (\neg r \wedge \neg q) \vee (\neg r \wedge p)$ | $\equiv (\neg r \wedge \neg q) \vee (\neg r \wedge p)$ | distributive law |
| $\equiv \neg r \wedge (\neg q \vee p)$ | $\equiv \neg r \wedge (p \vee \neg q)$ | associative law |
| $\equiv \neg r \wedge (p \vee \neg q)$ | | idempotent law |
| | | distributive law |
| | | commutative law |

p.26, icon at Example 6

#2. Here is a newspaper headline:

“Legislature Fails to Override Governor’s Veto of Bill to Cancel Sales Tax Reform.”

Did the legislature vote in favor of or against sales tax reform?

Solution:

The issue is sales tax reform, so we let s stand for “sales tax reform is supported”. We unravel the negatives one at a time. The bill to cancel sales tax reform is $\neg s$, and the governor’s veto of this bill is $\neg \neg s$. Overriding this would be $\neg \neg \neg s$, and failing to override is $\neg \neg \neg \neg s$. Therefore, using the double negation law twice, the headline $\neg \neg \neg \neg s$ is equivalent to s , and hence the Legislature supports sales tax reform.

p.26, icon at Example 6

#3. Suppose you want to prove a theorem of the form $p \rightarrow (q \vee r)$. Prove that this is equivalent to showing that $(p \wedge \neg q) \rightarrow r$.

Solution:

We will give a proof by replacing the first statement by equivalent statements until finally the second statement is obtained. (We will use the equivalence $a \rightarrow b \equiv \neg a \vee b$ twice.)

$$\begin{aligned}
 p \rightarrow (q \vee r) &\equiv \neg p \vee (q \vee r) \\
 &\equiv (\neg p \vee q) \vee r \\
 &\equiv \neg(p \wedge \neg q) \vee r \\
 &\equiv (p \wedge \neg q) \rightarrow r.
 \end{aligned}$$

We could also give a proof by constructing the truth table for each statement and showing that the two statements have the same truth values.

p.26, icon at Example 6

#4. Write the statement $p \rightarrow (\neg q \wedge r)$ using only the connectives \neg and \wedge .

Solution:

We need to remove the implication sign. Note that an implication $A \rightarrow B$ is equivalent to $\neg A \vee B$. Therefore, we can rewrite $p \rightarrow (\neg q \wedge r)$ as

$$\neg p \vee (\neg q \wedge r).$$

De Morgan's law will allow us to change this to an equivalent form without using \vee . Take De Morgan's law $\neg(A \vee B) \equiv \neg A \wedge \neg B$ and negate both sides to obtain

$$A \vee B \equiv \neg(\neg A \wedge \neg B).$$

Using $\neg p$ in place of A and $\neg q \wedge r$ in place of B , we have

$$p \rightarrow (\neg q \wedge r) \equiv \neg p \vee (\neg q \wedge r) \equiv \neg(p \wedge \neg(\neg q \wedge r)).$$
