Rosen, Discrete Mathematics and Its Applications, 6th edition
Extra Examples

Section 1.4—Nested Quantifiers

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p.51, icon at Example 1

#1. Write the following statements in English, using the predicate $S(x, y)$: "x shops in y", where x represents people and y represents stores:

(a) $\forall y \, S(Margaret, y)$.

(b) $\exists x \forall y \, S(x, y)$.

Solution:

(a) The predicate states that if y is a store, then Margaret shops there. That is, "Margaret shops in every store."

(b) The predicate states that there is a person x with the property that x shops in every store y. That is, "There is a person who shops in every store." [Note that part (a) is obtained from part (b) by taking a particular value, *Margaret*, for the variable x. If we do this, we do not need to quantify x.]

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#2. Write in symbols using predicates and quantifiers: "Every Junior in this class scored above 90 on the first exam."

Solution:
The solution depends on what we take for the universe for the variable. If we take all Juniors in this class as the universe, we can write the proposition as

$$\forall x \, S(x)$$

where $S(x)$ is the predicate "x scored above 90 on the first exam."

However, if we take all students in this class as the universe, then we can write the proposition as

$$\forall x \, (J(x) \rightarrow S(x))$$

where $J(x)$ is the predicate "x is a Junior."

We can extend the universe still further. Suppose we take all students as the universe. Then we need to introduce a third predicate $C(x)$ to mean "x is in this class." In this case, the proposition becomes

$$\forall x \, ((C(x) \wedge J(x)) \rightarrow S(x)).$$

If we also wish to distinguish among possible scores on the first exam, we can use "nested quantifiers", discussed later in this section of the book. We can replace $S(x)$ by $S(x, y)$ where $S(x, y)$ means "x received a score of y on the first exam" and the universe for y is the set of all possible exam scores. In this case the proposition becomes

$$\forall x \exists y (C(x) \land J(x) \rightarrow (y > 90) \land S(x, y)).$$

Note that we used a predicate, $y > 90$, without giving it a name.

#3. Write the following statement in English, using the predicates

$$S(x, y): \text{``}x \text{ shops in } y\text{''}$$
$$T(x): \text{``}x \text{ is a student''}$$

where x represents people and y represents stores:

$$\exists y \, \forall x \, (T(x) \rightarrow \neg S(x, y)).$$

Solution:
The statement $\exists y \, \forall x \, (T(x) \rightarrow \neg S(x, y))$ says that "there is a store y with a certain property, namely, if x is any student whatever, then x does not shop in y." We have "There is a store in which no student shops."

#4. Write the following statement in English, using the predicates

$$S(x, y): \text{``}x \text{ shops in } y\text{''}$$
$$T(x): \text{``}x \text{ is a student''}$$

where x represents people and y represents stores:

$$\forall y \, \exists x \, (T(x) \land S(x, y)).$$

Solution:
The statement $\forall y \, \exists x \, (T(x) \land S(x, y))$ asserts that for every store y that can be chosen, there is a person x who is a student and who shops in y. Therefore: "Every store has at least one student who shops in it."

#5. Write the following statement in English, using the predicate $S(x, y)$ for "x shops in y", where x represents people and y represents stores:

$$\exists x_1 \, \exists y \, \forall x_2 \, [S(x_1, y) \land (x_1 \neq x_2 \rightarrow \neg S(x_2, y))].$$

Solution:
The statement $S(x_1, y) \land (x_1 \neq x_2 \rightarrow \neg S(x_2, y))$ tells us two things: person x_1 shops in store y, and if x_2 is any other person then x_2 does not shop in y. Therefore, we have "There is a store in which exactly one person shops."

#6. Write the following statement in English, using the predicates

$$C(x): \text{"}x \text{ is a Computer Science major"}$$
$$M(y): \text{"}y \text{ is a math course"}$$
$$T(x,y): \text{"}x \text{ is taking } y\text{"}$$

where x represents students and y represents courses:

$$\forall x\, \exists y\, (C(x) \rightarrow M(y) \wedge T(x,y)).$$

Solution:
The statement $\forall x\, \exists y\, (C(x) \rightarrow M(y) \wedge T(x,y))$ asserts that for every student x there is a course y such that if x is a major in Computer Science then x is taking y and y is a math course. Therefore, "Every Computer Science major is taking at least one math course."

#7. Write the following statement in English, using the predicates

$$C(x): \text{"}x \text{ is a Computer Science major"}$$
$$T(x,y): \text{"}x \text{ is taking } y\text{"}$$

where x represents students and y represents courses:

$$\forall y\, \exists x\, (\neg C(x) \wedge T(x,y)).$$

Solution:
The statement $\forall y\, \exists x\, (\neg C(x) \wedge T(x,y))$ says that for every course y there is a student x such that x is not a Computer Science major and x is taking y. That is, "Every course has a student in it who is not a Computer Science major."

#8. Write the following statement in English, using the predicates

$$F(x): \text{"}x \text{ is a Freshman"}$$
$$M(y): \text{"}y \text{ is a math course"}$$
$$T(x,y): \text{"}x \text{ is taking } y\text{"}$$

where x represents students and y represents courses:

$$\neg \exists x\, [F(x) \wedge \forall y\, (M(y) \rightarrow T(x,y))].$$

Solution:
First examine part of the statement, $\forall y\, (M(y) \rightarrow T(x,y))$. This says that "if y is a math course, then x is taking y", or, equivalently, "x is taking every math course". The given statement says that there is no

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student with this property: $F(x) \land \forall y \, (M(y) \to T(x,y))$; that is, there is no student who is both a freshman and who is taking every math course. Therefore, we have "No Freshman is taking every math course."

#9. Write the following statement using quantifiers and the predicate $S(x,y)$ for "x shops in y", where the universe for x consists of people and the universe for y consists of stores:

"Will shops in Al's Record Shoppe."

Solution:
Using "Will" for x and "Al's Record Shoppe" for y, we have

$$S(\text{Will, Al's Record Shoppe}).$$

No quantifiers are needed.

#10. Write the following statement using quantifiers and the predicates

$$S(x,y): \text{"x shops in y"}$$
$$T(x): \text{"x is a student"}$$

where the universe for x consists of people and the universe for y consists of stores:

"There is no store that has no students who shop there."

Solution:
We can begin by stating that "It is false that there exists a store y with the property that no students shop in y." Saying that "no students shop in y" is saying $\forall x \, (T(x) \to \neg S(x,y))$. Completely written in symbols, we have

$$\neg \exists y \, \forall x \, (T(x) \to \neg S(x,y)).$$

#11. Write the following statement using quantifiers and the predicates

$$S(x,y): \text{"x shops in y"}$$
$$T(x): \text{"x is a student"}$$

where the universe for x consists of people and the universe for y consists of stores:

"The only shoppers in some stores are students."

Solution:
The given statement asserts that "There is at least one store, y, such that only students shop there." Saying

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that "only students shop in y" means that $\forall x\,(S(x,y) \to T(x))$. Putting these together gives

$$\exists y\,\forall x\,(S(x,y) \to T(x)).$$

#12. Suppose that the universe for x and y is $\{1, 2, 3\}$. Also, assume that $P(x, y)$ is a predicate that is true in the following cases, and false otherwise: $P(1, 3), P(2, 1), P(2, 2), P(3, 1), P(3, 2), P(3, 3)$. Determine whether each of the following is true or false:
(a) $\forall y \exists x\,(x \neq y \wedge P(x, y))$.
(b) $\forall x \exists y\,(x \neq y \wedge \neg P(x, y))$.
(c) $\forall y \exists x\,(x \neq y \wedge \neg P(x, y))$.

Solution:
(a) True. We need to consider three cases: $y = 1$, $y = 2$, $y = 3$.

If $y = 1$, we can take $x = 2$, obtaining the true statement $2 \neq 1 \wedge P(2, 1)$.

If $y = 2$, we can take $x = 3$, obtaining the true statement $3 \neq 2 \wedge P(3, 2)$.

If $y = 3$, we can take $x = 1$, obtaining the true statement $1 \neq 3 \wedge P(1, 3)$.

Therefore, the statement $\exists x\,(x \neq y \wedge P(x, y))$ is true for all possible choices of y. Hence, $\forall y \exists x\,(x \neq y \wedge P(x, y))$ is true.

(b) False. Take $x = 3$. The statements $P(3, 1)$, $P(3, 2)$, and $P(3, 3)$ are true; that is, the statements $\neg P(3, 1)$, $\neg P(3, 2)$, and $\neg P(3, 3)$ are false. Therefore, there is no value y such that $3 \neq y \wedge \neg P(3, y)$ is true.

(c) False. Take $y = 1$. We need to consider $x = 1$, $x = 2$, and $x = 3$. The conjunctions $1 \neq 1 \wedge \neg P(1, 1)$, $2 \neq 1 \wedge \neg P(2, 1)$, and $3 \neq 1 \wedge \neg P(3, 1)$ are all false.

#13. Suppose that the universe for x and y is $\{1, 2, 3, 4\}$. Assume that $P(x, y)$ is a predicate that is true in the following cases and false otherwise: $P(1, 4), P(2, 1), P(2, 2), P(3, 4), P(4, 1), P(4, 4)$. Determine whether each of the following is true or false:
(a) $\exists y \forall x P(x, y)$.
(b) $\forall x P(x, x)$.
(c) $\forall x \exists y\,(x \neq y \wedge P(x, y))$.

Solution:
(a) False. If we take $y = 1$, not all four statements $P(x, 1)$ are true. (Take $x = 1$ for example.) If we take $y = 2$, not all four statements $P(x, 2)$ are true. (Take $x = 1$ for example.) If we take $y = 3$, not all four statements $P(x, 3)$ are true. (Take $x = 1$ for example.) If we take $y = 4$, not all four statements $P(x, 4)$ is true. (Take $y = 2$.)

(b) False. $P(1, 1)$ is false.

(c) True. For every x we can find a value $y \neq x$ such that $P(x, y)$ is true: $P(1, 4)$, $P(2, 1)$, $P(3, 4)$, and $P(4, 1)$.

#14. Consider this sentence, which is Amendment 3 to the U.S. Constitution: "No soldier shall, in time of peace, be quartered in any house, without the consent of the owner, nor in time of war, but in a manner to be prescribed by law."
(a) The sentence has the form of a conjunction of two conditional sentences. Write the given sentence in this form.

(b) Using the six predicates, $S(x)$: "x is a soldier," $P(t)$ "t is a peaceful time," $Q(x, y, h)$: "x is required to allow y to be quartered in h," $O(x, h)$: "x owns h," $C(x, y, h)$: "x consents to quarter y in h,', $A(x, h)$: "the law allows x to be quartered in h," where the universe for x and y consists of all people, the universe for t consists of all points in time, and the universe for h consists of all houses, rewrite the sentence using quantifiers and predicates.

Solution:
(a) The sentence has the form "If it is a time of peace, then ..., and, if it is a time of war, then ... ." Written in full, the sentence is "If it is a time of peace, then no soldier shall be quartered in any house without the consent of the owner, and, if it is a time of war, then no soldier shall be quartered in any house except in a manner to be prescribed by law."
(b) The statement is a conjunction; it has the form "(if $P(t)$, then ...) $\wedge$ (if $\neg P(t)$, then ...)."

Let us examine the case when it is a time of peace. The statement says that "if the owner of a house does not give consent, then no soldier shall be quartered in that house." That is, if person x owns house h and does not consent to quarter soldier y in h, then x is not required to quarter y in h. In symbols, we have

$$((O(x, h) \wedge S(y) \wedge \neg C(x, y, h)) \rightarrow \neg Q(x, y, h).$$

Similarly, in the case when it is not a time of peace, we have

$$((O(x, h) \wedge S(y) \wedge A(y, h)) \rightarrow Q(x, y, h)).$$

Completely written in symbols we have

$$\forall t \, \forall x \, \forall y \, \forall h \, \left\{ \left[P(t) \rightarrow \left((O(x, h) \wedge S(y) \wedge \neg C(x, y, h)) \rightarrow \neg Q(x, y, h) \right) \right] \wedge \right.$$
$$\left. \left[\neg P(t) \rightarrow \left((O(x, h) \wedge S(y) \wedge A(y, h)) \rightarrow Q(x, y, h) \right) \right] \right\}.$$

#15. Consider these lines of code from a C++ program:

```
if (!(x!=0 && y/x < 1) || x==0)
   cout << "True";
else
   cout << "False"
```

(a) Express the code in this statement as a compound statement using the logical connectives $\neg, \vee, \wedge, \rightarrow$, and these predicates

$$E(x): x = 0$$
$$L(x, y): y/x < 1$$
$$A(z): \text{"z is assigned to \texttt{cout}"}$$

where x and y are integers and z is a Boolean variable (with values True and False).

(b) Use the laws of propositional logic to simplify the statement by expressing it in a simpler form.

(c) Translate the answer in part (b) back into C++.

Solution:

(a) First we insert the predicates into the code, obtaining

```
if  (!(!E(x) && L(x,y)) || E(x))
    A(True)
else
    A(False).
```

Next change to the usual logical connective symbols, keeping in mind that C++ code of the form "if p then q else r" is really a statement of the form $(p \to q) \land (\neg p \to r)$:

$$[\neg(\neg E(x) \land L(x,y)) \lor E(x)] \to$$
$$A(\text{True})$$
$$\land$$
$$\neg[\neg(\neg E(x) \land L(x,y)) \lor E(x)] \to$$
$$A(\text{False}), \text{ or}$$
$$\Big([\neg(\neg E(x) \land L(x,y)) \lor E(x)] \to A(\text{True})\Big) \land \Big(\neg[\neg(\neg E(x) \land L(x,y)) \lor E(x)] \to A(\text{False})\Big).$$

Because this statement applies to all numbers x and y, we have

$$\forall x\, \forall y\, \Big(([\neg(\neg E(x) \land L(x,y)) \lor E(x)] \to A(\text{True})) \land (\neg[\neg(\neg E(x) \land L(x,y)) \lor E(x)] \to A(\text{False}))\Big).$$

(b) Using one of De Morgan's laws on the negation of the conjunction, the statement becomes

$$\forall x\, \forall y\, \Big(([(E(x) \lor \neg L(x,y)) \lor E(x)] \to A(\text{True})) \land (\neg[(E(x) \lor \neg L(x,y)) \lor E(x)] \to A(\text{False}))\Big),$$

which can be simplified to give

$$\forall x\, \forall y\, \Big(((E(x) \lor \neg L(x,y)) \to A(\text{True})) \land (\neg(E(x) \lor \neg L(x,y)) \to A(\text{False}))\Big).$$

(c) Translating the statement in (b) into C++ yields

```
if (x==0 || y/x >= 1)
    cout << "True"
else
    cout << "False".
```

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p.52, icon at Example 3

#1. What are the truth values of each of these? Assume that in each case the universe consists of all real numbers.

  (a) $\exists x\, \exists y\, (xy = 2)$

  (b) $\exists x\, \forall y\, (xy = 2)$

  (c) $\forall x\, \exists y\, (xy = 2)$

  (d) $\forall x\, \forall y\, (xy = 2)$

Solution:

(a) This statement asserts that there are numbers x and y such that $xy = 2$. This is true because we can take $x = 2$ and $y = 1$, for example.

(b) This statement asserts that there is a number x such that when we multiply this particular x by every possible number y we obtain $xy = 2$. There is no such number x. (If there were such a number x, then $xy = 2$ for all y. If we take $y = 0$, the product xy cannot equal 2.) Therefore the statement is false.

(c) This statement asserts that for every number x we choose, we can find a number y such that the $xy = 2$. This is almost always the case, except if we choose $x = 0$. If we take $x = 0$, there is no number y such that $xy = 2$. Therefore the statement is false. (Note that the statement would be true if the universe for x consisted of all *nonzero* real numbers.)

(d) This statement claims that no matter what numbers x and y we choose, we obtain $xy = 2$. Clearly, this is false, because we could choose $x = y = 1$.

#2. Write the following statements in English, using the predicate $S(x, y)$: "x shops in y", where x represents people and y represents stores:

$$(a) \; \exists y \, \forall x \, S(x, y).$$

$$(b) \; \forall x \, \exists y \, S(x, y).$$

Solution:
(a) The sentence states that there is a store y such that every person x shops there. Thus, "There is a store in which everyone shops."
(b) The sentence states that for every person x there is a store y in which x shops. Therefore, we have "Everyone shops somewhere."

#3. Suppose that the universe for x and y is $\{1, 2, 3\}$. Also, assume that $P(x, y)$ is a predicate that is true in the following cases, and false otherwise: $P(1, 3), P(2, 1), P(2, 2), P(3, 1), P(3, 2), P(3, 3)$. Determine whether each of the following is true or false:
(a) $\exists x \forall y \, (y < x \rightarrow P(x, y))$.
(b) $\forall y \exists x \, (y < x \lor P(x, y))$.
(c) $\exists x \exists y \, (P(x, y) \land P(y, x))$.
(d) $\forall y \exists x \, (P(x, y) \rightarrow \neg P(y, x))$.

Solution:
(a) True. We can take $x = 1$. Because there is no y such that $y < 1$, the hypothesis of the implication $y < x \rightarrow P(x, y)$ is false, making the implication true.

(b) True. We need to consider the cases $y = 1$, $y = 2$, and $y = 3$.

If $y = 1$, then the statement $\exists x \, (y < x \lor P(x, y))$ is true for $x = 2$ (because $1 < 2$).

If $y = 2$, then the statement $\exists x \, (y < x \lor P(x, y))$ is true for $x = 3$ (because $2 < 3$).

If $y = 3$, then the statement $\exists x \, (y < x \lor P(x, y))$ is true for $x = 1$ (because $P(1, 3)$ is true).

(c) True. Take $x = y = 2$, for example.

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(d) True. We need to consider the cases $y = 1$, $y = 2$, and $y = 3$. This means that we must examine the three statements

$$\exists x\,(P(x,1) \to \neg P(1,x)) \quad (\text{true for } x = 2 \text{ because } P(2,1) \to \neg P(1,2) \text{ is true})$$
$$\exists x\,(P(x,2) \to \neg P(2,x)) \quad (\text{true for } x = 1 \text{ because } P(1,2) \to \neg P(2,1) \text{ is true})$$
$$\exists x\,(P(x,3) \to \neg P(3,x)) \quad (\text{true for } x = 2 \text{ because } P(2,3) \to \neg P(3,2) \text{ is true}).$$

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#4. Suppose $P(x,y,z)$ is a predicate where the universe for x, y, and z is $\{1,2\}$. Also suppose that the predicate is true in the following cases $P(1,1,1), P(1,2,1), P(1,2,2), P(2,1,1), P(2,2,2)$, and false otherwise. Determine the truth value of each of the following quantified statements:

(a) $\forall x\,\exists y\,\exists z\,P(x,y,z)$.

(b) $\forall x\,\forall y\,\exists z\,P(x,y,z)$.

(c) $\forall y\,\forall z\,\exists x\,P(x,y,z)$.

(d) $\forall x\,\exists y\,\forall z\,P(x,y,z)$.

Solution:
(a) True. For every value of x ($x = 1$ and $x = 2$) there are y and z such that $P(x,y,z)$ is true. In both cases we can choose both $y = z = 2$.

(b) True. For each choice of values for x and y, we can find z such that $P(x,y,z)$ is true. We need to consider four cases.

    (1) $x = y = 1$: we take $z = 1$,

    (2) $x = 1$ and $y = 2$: we can take z to be 1 or 2,

    (3) $x = 2$, $y = 1$: we take $z = 1$,

    (4) $x = y = 2$: we take $z = 2$.

(c) False. If we take $y = 1$ and $z = 2$, there is no value of x such that $P(x,1,2)$ is true.

(d) False. Take $x = 2$. There is no value of y such that $\forall z\,P(2,y,z)$ is true.

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p.52, icon at Example 3

#5. Suppose $P(x,y,z)$ is a predicate where the universe for x, y, and z is $\{1,2\}$. Also suppose that the predicate is true in the following cases $P(1,1,1), P(1,2,1), P(1,2,2), P(2,1,1), P(2,2,2)$, and false otherwise. Determine the truth value of each of the following quantified statements:

(a) $\exists x\,\forall y\,\forall z\,P(x,y,z)$.

(b) $\forall x\,\exists z\,\forall y\,P(x,y,z)$.

(c) $\forall y\,\exists x\,\exists z\,\neg P(x,y,z)$.

(d) $\exists x\,\forall z\,\neg\forall y\,P(x,y,z)$.

Solution:
(a) False. If we take $x = 1$, we do not have $P(1,y,z)$ true for all possible values of y and z — $P(1,1,2)$ is false. If we take $x = 2$, we do not have $P(2,y,z)$ true for all possible values of y and z — $P(2,1,2)$ and $P(2,2,1)$ are both false.

(b) False. Take $x = 2$. Then $\exists z\,\forall y\,P(2,y,z)$ is false. To see this, suppose we try $z = 1$; then $P(2,y,1)$ is false for $y = 1$. If we try $z = 2$, $P(2,y,2)$ is false for $y = 1$.

(c) True. We must consider the cases where $y = 1$ and $y = 2$. If we take $y = 1$. Then $\exists x\,\exists z\,\neg P(x,1,z)$ is true if $x = z = 2$, that is, $\neg P(2,1,2)$ is true. If we take $y = 2$. Then $\exists x\,\exists z\,\neg P(x,2,z)$ is true if $x = 2$ and

$z = 1$, that is, $\neg P(2,2,1)$ is true.

(d) True. The given statement is equivalent to $\neg \forall x \exists z \forall y \, P(x,y,z)$, which is the negation of the statement in part (b). Because the statement in part (b) is false, this statement must be true.

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p.52, icon at Example 3

#6. Suppose that the universe for x and y is $\{1,2,3,4\}$. Assume that $P(x,y)$ is a predicate that is true in the following cases and false otherwise: $P(1,4), P(2,1), P(2,2), P(3,4), P(4,1), P(4,4)$. Determine whether each of the following is true or false:
(a) $\forall x \exists y P(x,y)$.
(b) $\forall y \exists x P(x,y)$.
(c) $\exists x \forall y P(x,y)$.

Solution:
(a) True. For every value of x taken from the universe, there is a value y such that $P(x,y)$ is true: $P(1,4)$, $P(2,1)$, $P(3,4)$, and $P(4,1)$ are all true.

(b) False. If $y = 3$, there is no value of x such that $P(x,3)$ is true.

(c) False. If we take $x = 1$, not all four statements $P(1,y)$ are true. (Take $y = 1$ for example.) If we take $x = 2$, not all four statements $P(2,y)$ are true. (Take $y = 3$ for example.) If we take $x = 3$, not all four statements $P(3,y)$ are true. (Take $y = 1$ for example.) If we take $x = 4$, not all four statements $P(4,y)$ is true. (Take $y = 2$ for example.)

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p.54, icon at Example 6

#1. Write this fact about numbers using predicates and quantifiers: "Given a number, there is a number greater than it."

Solution:
The statement says that "For every number x we choose, there is a number y such that $y > x$." That is,

$$\forall x \, \exists y \, (y > x)$$

where the universe for x and y consists of all numbers.

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p.54, icon at Example 6

#2. Express the following statement using predicates and quantifiers: "The product of two positive numbers is positive."

Solution:
Using the universe consisting of all real numbers for x and y, we are saying that "If x and y are greater than zero, then xy is greater than zero. That is,

$$\forall x \, \forall y \, [(x > 0 \, \wedge \, y > 0) \rightarrow (xy > 0)].$$

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If we use all *positive* real numbers as the universe for x and y, we can write the statement more simply:

$$\forall x \, \forall y \, (xy > 0).$$

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p.54, icon at Example 6

#3. Write these statements in symbols using the predicates:

$$S(x): x \text{ is a perfect square}; \quad N(x): x \text{ is negative}.$$

Assume that the variable x is an integer.

(a) No perfect squares are negative.

(b) No negative numbers are perfect squares.

Solution:
(a) We are saying that it is not possible to have a perfect square that is negative. That is, $\neg \, \exists x \, (S(x) \wedge N(x))$. Equivalently, we could say that if x is a perfect square, then x is not negative. That is,

$$\forall x \, (S(x) \rightarrow \neg \, N(x)).$$

We could rewrite this as its contrapositive: If x is negative, then x is not a perfect square. That is,

$$\forall x \, (N(x) \rightarrow \neg S(x)).$$

(b) This statement is equivalent to (a). This statement says that it is not possible to have a negative number that is a perfect square. That is,

$$\neg \, \exists x \, (N(x) \wedge S(x)).$$

You should use the various laws of logic to show that $\neg \, \exists x \, (N(x) \wedge S(x))$ is indeed equivalent to $\forall x \, (S(x) \rightarrow \neg \, N(x))$

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p.54, icon at Example 6

#4. Write the following statement in symbols using the predicates

$$S(x): x \text{ is a perfect square} \quad P(x): x \text{ is positive}$$

where the universe for x is the set of all integers:

"Perfect squares are positive."

Solution:
Note that "for all" is implied. When we say "Perfect squares are positive" we are really saying that "For all integers x we choose, if x is a perfect square, then x is positive." In symbols we have

$$\forall x \, (S(x) \rightarrow P(x)).$$

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#5. Write the following statement in symbols using the predicate $P(x)$ to mean "x is positive", where the universe for x is the set of all integers.

<center>"Exactly one number is positive."</center>

Solution:
We are making a two-part statement:

(1) there is a number x that is positive, that is, $\exists x\, P(x)$; and

(2) x is the only number with this property; that is, if y is any number different from x, then y is not positive. This can be written as $\forall y\, (y \neq x \rightarrow \neg P(y))$.

Forming the conjunction of these two statements, we have

$$\exists x\, [P(x) \wedge \forall y\, (y \neq x \rightarrow \neg P(y))],$$

or

$$\exists x\, \forall y\, [P(x) \wedge (y \neq x \rightarrow \neg P(y))].$$

#6. Write the following statements in symbols, using $P(x)$ to mean "x is positive" and $F(x)$ to mean "x ends in the digit 5". Assume that the universe for x is the set of all integers.

(a) Some positive integers end in the digit 5.

(b) Some positive integers end in the digit 5, while others do not.

Solution:
(a) We are asserting that there is an integer x that has two properties: (1) it is positive, (2) it ends in the digit 5. That is, $\exists x\, (P(x) \wedge F(x))$.

(b) This statement begins with the statement for (a) and then asserts that there is a different positive integer that does not end in the digit 5. That is,

$$\exists x\, (P(x) \wedge F(x)) \wedge \exists y\, ((y \neq x) \wedge P(x) \wedge \neg F(x)).$$

Equivalently, we could write

$$\exists x\, \exists y\, [(x \neq y) \wedge P(x) \wedge P(y) \wedge F(x) \wedge \neg F(y)].$$

#7. Write in symbols: There is no smallest positive number.

Solution:

Using all positive real numbers as the universe for x and y, we are saying that "For every number x we can choose, there is a number y that is smaller than x." In symbols,

$$\forall x \, \exists y \, (y < x).$$

If we use all real numbers as the universe for x and y, we are saying that "For every positive real number x we can choose, there is a real number y that is positive and smaller than x." In symbols,

$$\forall x \, (x > 0 \rightarrow \exists y \, (0 < y < x)).$$

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p.54, icon at Example 6

#8. Write in symbols: If $a < b$, then $\dfrac{a+b}{2}$ lies between a and b.

Solution:
Note that it is understood that the predicate applies to all a and b chosen from some universe. Using all real numbers as the universe for a and b, we have

$$\forall a \, \forall b \left(a < b \;\rightarrow\; a < \frac{a+b}{2} < b \right).$$

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p.54, icon at Example 6

#9. Write in symbols: For all choices of a and b, $\dfrac{a+b}{2}$ lies between a and b.

Solution:
Note that we cannot write $\forall a \forall b \left(a < \dfrac{a+b}{2} < b \right)$ because we do not know that $a < b$. (It may be the case that $a = b$ or that $a > b$.) We can write

$$\forall a \, \forall b \left(\left(a \leq \frac{a+b}{2} \leq b \right) \;\vee\; \left(b \leq \frac{a+b}{2} \leq a \right) \right).$$

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p.57, icon at Example 14

#1. Write the negation of the statement $\exists x \, \forall y \, (xy = 0)$ in symbols and in English. Determine the truth or falsity of the statement and its negation. Assume that the universe for x and y is the set of all real numbers.

Solution:
We take the negation and then move the negation sign inside:

$$\neg(\exists x \, \forall y \, (xy = 0)) \;\equiv\; \forall x \, (\neg \forall y \, (xy = 0)) \;\equiv\; \forall x \, \exists y \, \neg(xy = 0) \;\equiv\; \forall x \, \exists y \, (xy \neq 0).$$

The original statement says that "There is a number with the property that no matter what number we multiply it by, we obtain 0." (The statement is true because the number 0 is such a number x.) The negation

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states that "No matter what number is chosen, there is a number such that the product is nonzero." (As expected, the negation is false because it is the negation of a true statement. To see that the negation is false, take x to be 0. Then no matter what value we take for y, the product $xy = 0$.)

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p.57, icon at Example 14

#2. Write the statement "There is a largest number" using predicates and quantifiers. Then give its negation in symbols.

Solution:
Taking the universe for x and y to consist of all real numbers, we are stating that there is a number x such that, no matter what number y is chosen, we have $x \geq y$. Therefore.

$$\exists x \, \forall y \, (x \geq y).$$

Its negation can be formed using these steps:

$$\neg(\exists x \, \forall y \, (x \geq y)) \;\equiv\; \forall x \, \exists y \, \neg(x \geq y) \;\equiv\; \forall x \, \exists y \, (x < y).$$

(This says that there is no largest number.)