Rosen, Discrete Mathematics and Its Applications, 6th edition
Extra Examples

Section 9.5—Euler and Hamilton Paths

Extra Examples 🖱 — *Page references correspond to locations of Extra Examples icons in the textbook.*

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p.634, icon at Example 2

#1. Determine whether the following graph has an Euler circuit or Euler path.
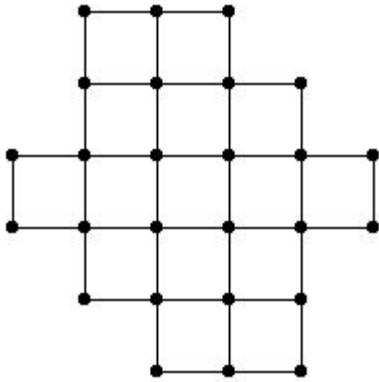
Solution:
The graph has no Euler circuit or Euler path because it has four vertices of odd degree.

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p.634, icon at Example 2

#2. Determine whether the following graph has an Euler circuit or Euler path.
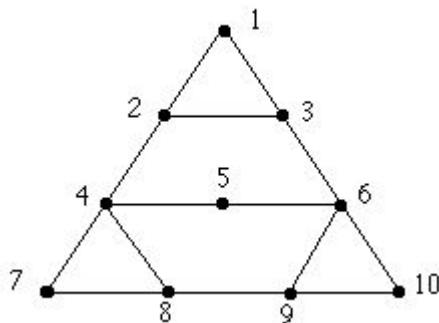
Solution:
The graph has no Euler circuit or path because it has 4 vertices of odd degree.

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p.634, icon at Example 2

#3. *The Chinese Postman Problem:* The following graph shows the streets along which a mail carrier

must deliver mail. Each street segment has a label consisting of two numbers: the first number gives the time (in minutes) that it takes for the mail carrier to deliver mail along that street; the second number gives the time (in minutes) that it takes the mail carrier to walk along that street without delivering mail. What is the minimum total length of time required to start from point A, complete mail delivery along all the streets in the map, and return to A?

Note: A further discussion of this problem and similar ones can be found on this website in Chapter 20 of *Applications of Discrete Mathematics*.
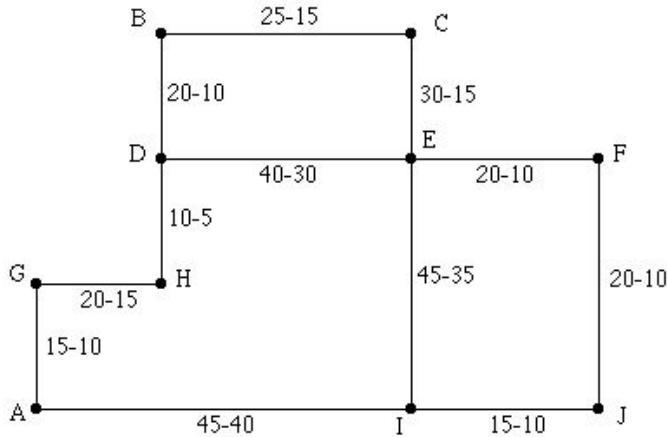
Solution:

Ideally, an Euler circuit would give the best solution, but one does not exist in the graph because vertices D and I have odd degree. Thus, in order for the mail carrier to start at A, deliver mail along the entire route, and return to A, the mail carrier will need to retrace steps along some blocks without delivering mail (called *deadheading*).

In particular, the mail carrier must somehow retrace steps between the odd vertices D and I. (This deadheading may occur in pieces; i.e. the mail carrier might deliver mail along a block, deadhead along a couple of blocks, then deliver mail along another block, etc.) Therefore, we need to find the shortest path between D and I — these edges will minimize the deadheading time. Here are the possible simple paths joining D and I with the length of time it takes to walk along each path:

> *D-B-C-E-F-J-I*: 70 min
> *D-B-C-E-I*: 75 min
> *D-E-F-J-I*: 60 min
> *D-E-I*: 65 min
> *D-H-G-A-I*: 70 min.

The path that takes the least amount of time to follow is the path *D-E-F-J-I*, which means that the mail carrier will spend 60 minutes in deadheading time. Therefore, the total length of time it takes the mail carrier is the time delivering mail (305 minutes) plus the deadheading time (60 minutes), or 6 hours 5 minutes. An example of a route that uses this amount of time is

$$A - G - H - D - B - C - E - F - J - I \stackrel{*}{-} J \stackrel{*}{-} F \stackrel{*}{-} E - D \stackrel{*}{-} E - I - A.$$

(An asterisk indicates a deadheading segment.)

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p.639, icon at Example 5

**#1.** Determine whether the following graph has a Hamilton circuit or Hamilton path.
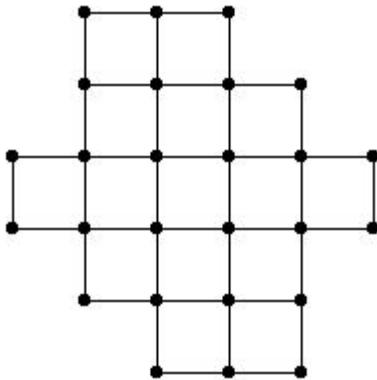
Solution:

The graph has no Hamilton circuit or Hamilton path. To see this, note that the graph is bipartite. If the vertices are labeled A (starting with the left vertex at the top of the figure) and B so that no adjacent vertices have the same label, there are 14 vertices labeled A and 12 labeled B. Any Hamilton circuit or path must consist of an alternating sequence of A's and B's, which is not possible with two more A's than B's.


#2. Determine whether the following graph has a Hamilton circuit or Hamilton path.
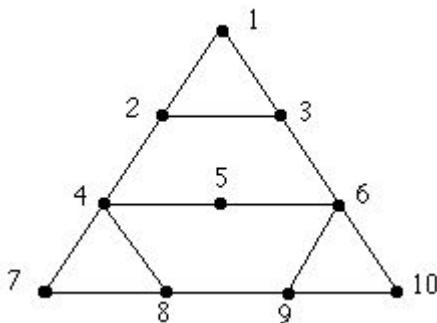
Solution:

The graph does not have a Hamilton circuit. Suppose the graph did have a Hamilton circuit. Then the following edges must all be used in such a circuit: $\{1,2\}$, $\{1,3\}$, $\{4,5\}$, $\{5,6\}$, $\{4,7\}$, $\{7,8\}$, $\{6,10\}$, $\{9,10\}$. If these edges must be used, then the following edges cannot be used: $\{2,4\}$, $\{3,6\}$, $\{4,8\}$, and $\{6,9\}$. If it impossible to use these four edges, they can be removed from the graph, yielding

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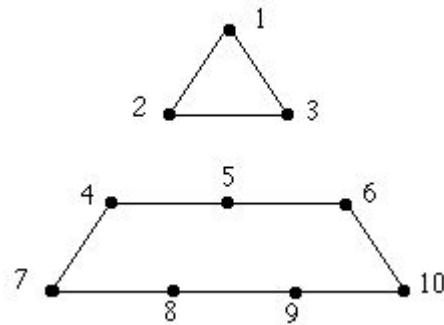
But this graph has no Hamilton circuit because it is disconnected. Therefore the original graph has no Hamilton circuit.

The graph does have a Hamilton path — for example, $1, 2, 3, 6, 5, 4, 7, 8, 9, 10$.

#3. Find a Hamilton circuit in the graph at the right, called the Grötzsch graph.
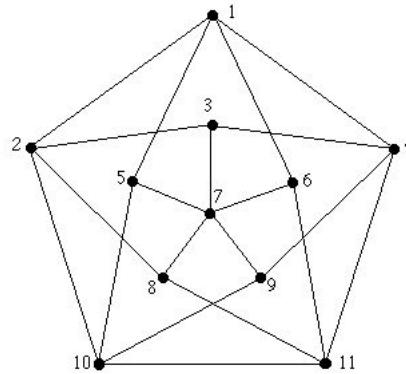
Solution:
Here is one line of reasoning. We know that any Hamilton circuit must pass through vertex 7. There are two ways in which a circuit can pass through this vertex:

  (1) by using two edges that are not "next to each other" (such as $\{7, 8\}$ and $\{7, 6\}$), or

  (2) by using two edges that are "next to each other" (such as $\{7, 8\}$ and $\{7, 9\}$).
We will consider these two cases separately.

(1) Suppose we choose edges $\{7, 8\}$ and $\{7, 6\}$. If we do this, then edges $\{3, 7\}$, $\{5, 7\}$, and $\{9, 7\}$ cannot be used (because a Hamilton circuit only uses two edges incident with a vertex).
Erasing these three edges makes vertices 3, 5, and 9 of degree 2. This is shown in the following graph on the left. But if a vertex has degree 2, both edges incident with that vertex must be used in forming any Hamilton

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circuit. Therefore, if we use edges $\{7, 8\}$ and $\{7, 6\}$, we must use edges $\{2, 3\}$, $\{3, 4\}$, $\{1, 5\}$, $\{5, 10\}$, $\{4, 9\}$, and $\{9, 10\}$. We have made these eight edges dark in the following graph on the right.
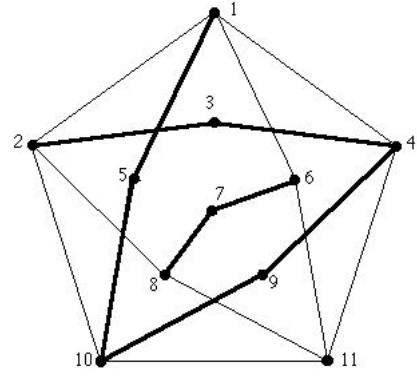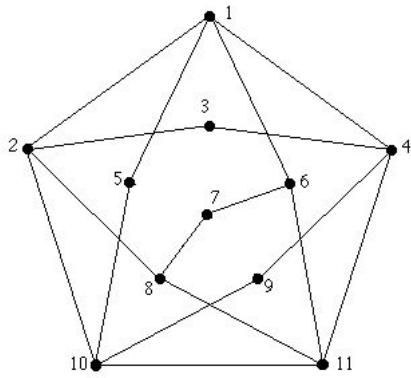
We have used edges that pass through vertices 4 and 10, so we can now remove edges $\{1, 4\}$, $\{4, 11\}$, $\{2, 10\}$, and $\{10, 11\}$ because they are no longer needed. This is illustrated in the following graph on the left. But this forces us to use edges $\{6, 11\}$ and $\{8, 11\}$, as shown in the following graph on the right. But this gives a circuit — 6, 7, 8, 11, 6 — and this cannot be part of a Hamilton circuit. Therefore, taking two edges that are not "next to each other" does not lead to a Hamilton circuit.
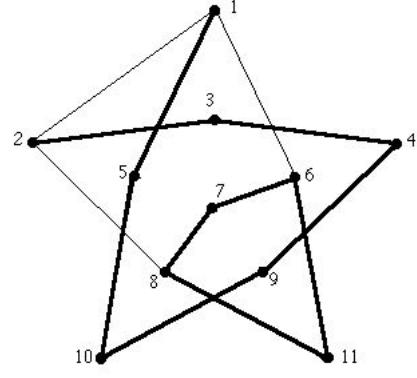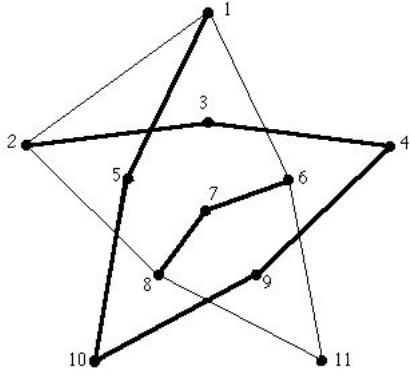
(2) We now consider the case where we use two edges that are "next to each other", such as $\{7, 8\}$ and $\{7, 9\}$. If we use these two edges, we cannot use edges $\{3, 7\}$, $\{5, 7\}$, or $\{6, 7\}$. We erase them, and obtain the following graph on the left. Vertices 3, 5, and 6 now have degree 2, so the remaining edges incident with them must be used. This gives the following graph on the right.
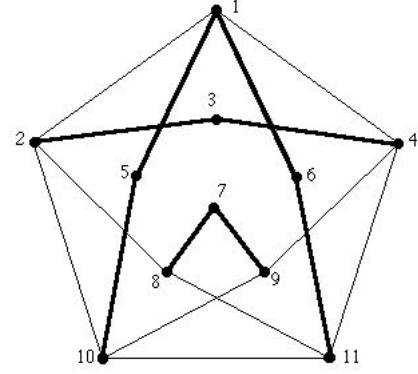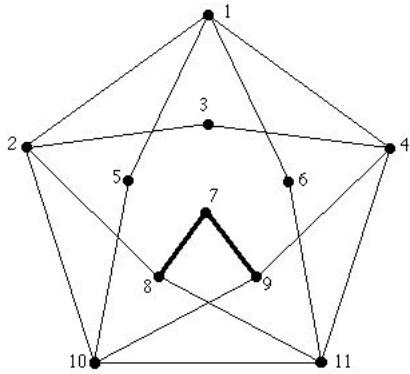
We can no longer use edges $\{1, 2\}$ or $\{1, 4\}$ (because we have already passed through vertex 1), so we erase them, obtaining the following graph on the left. Examining this graph, we see that we need to try to link the three shaded paths into one circuit. It is not difficult to see that we can do this by adding edges $\{4, 9\}$, $\{8, 11\}$, and $\{2, 10\}$.

Thus, an example of a Hamilton circuit in the Grötzsch graph is 1–6–11–8–7–9–4–3–2–10–5–1.