Calculus Supplement to Accompany

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Microeconomics, 1st edition

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Part I: Introduction

Chapter 1: Preliminaries

Note to the Student

Your textbook is written to promote "thinking like an economist" in order to solve quantitative problems, relevant to real world economic decision making, whether or not you are fluent in advanced mathematical tools such as calculus. This supplement, however, extends what you have learned in your text by showing how you can solve economic problems using calculus. A theme of your book is the separability of the economics from the mathematics. This supplement continues that theme but presents the relationships between economics and calculus, instead of between economics and algebra. Although you might find the idea of applying calculus concepts to economic problems intimidating at first, be forewarned that you may find the calculus approaches outlined here to be easier than the algebraic approaches of your textbook in some cases. In addition, recognizing relationships between economics and calculus will be useful if you decide to study even more advanced microeconomics in the future. Before jumping into economic problem solving, however, let's introduce one calculus concept that you may not have studied in your calculus class, but which will be useful for this supplement.

Partial Derivatives

One of the central themes of microeconomics, as presented in your textbook, is that optimal choices are made at the margin. In other words, in microeconomics, we study whether or not small adjustments (marginal changes) in consumption or production make the agents in our models better off. We can use calculus to formalize these margins mathematically.

You should remember standard derivatives from your calculus class. However, if you took a course focusing on single variable calculus, you might not have learned about partial derivatives. Since partial derivatives are a mathematical tool commonly used in economics and will be used throughout this supplement, let's define a partial derivative now. When you studied standard one variable derivatives, you considered functions such as $f(X)$. You took the derivative of this function f with respect to its single argument X. For example, if $f(X) = X^2$, then you were able to calculate the derivative of f with respect to X as $\frac{df}{dX} = \frac{d(f(X))}{dX} = \frac{d(X^2)}{dX} = 2X$ using the common derivative rule that $\frac{d(X^n)}{dX} = nX^{n-1}$.

Partial derivatives are useful when you have a function of two or more variables and want to take a derivative with respect to only one variable while holding the other variable or other variables constant. Instead of dealing with $f(X)$, you might be dealing with $f(X,Y)$. Or better yet, you might be dealing with $f(X,Y,Z)$. For the time being though, let's consider the two variable case where the function is $f(X,Y)$. Your objective might be to take the partial derivative of f with respect to X, or alternately, to take the partial derivative of f with respect to Y. As you will see in a moment, these calculations are not equivalent.

The partial derivative of f with respect to X is written $\frac{\partial f}{\partial X}$. Similarly, the partial derivative of f with respect to Y is written $\frac{\partial f}{\partial Y}$. An easy way to think about calculating the partial derivative of $f(X,Y)$ with respect to X is to pretend that Y is a constant, not a variable. Recall the constant rule for derivatives that $\frac{d(cf(X))}{dX} = c\frac{df(X)}{dX}$. Using this fact, you can take the derivative of $f(X,Y)$ with respect to X just like you would take the derivative of $f(X)$ with respect to X, remembering that Y is just a constant. Likewise, if you wanted to take the partial derivative of $f(X,Y)$ with respect to Y, just pretend that X is a constant and take the derivative of $f(X,Y)$ with respect to Y just as you would solve for the derivative of $f(Y)$ with respect to Y. This time, just remember that for your purposes, X is treated as a constant. Let's look at a couple of examples.

WORKED-OUT PROBLEM

The Problem Consider the function $f(X,Y) = X^2 + Y^2$. What is $\frac{\partial f}{\partial X}$? What is $\frac{\partial f}{\partial Y}$?

The Solution The problem asks for both the partial derivative of f with respect to X, $\frac{\partial f}{\partial X}$, and for the partial derivative of f with respect to Y, $\frac{\partial f}{\partial Y}$. In this case, $\frac{\partial f}{\partial X} = \frac{\partial(f(X,Y))}{\partial X} = \frac{\partial(X^2+Y^2)}{\partial X} = \frac{d(X^2)}{dX} + \frac{d(Y^2)}{dX} = 2X + 0 = 2X$. Since we are treating Y as a constant, we also can treat Y^2 as a constant since a constant to a power is still a constant. Since the derivative of a constant is zero (here $\frac{d(Y^2)}{dX} = 0$), this term drops out. Similarly, we can calculate the partial derivative of $f(X,Y)$ with respect to Y. In that case, $\frac{\partial f}{\partial Y} = \frac{\partial(f(X,Y))}{\partial Y} = \frac{\partial(X^2+Y^2)}{\partial Y} = \frac{d(X^2)}{dY} + \frac{d(Y^2)}{dY} = 0 + 2Y = 2Y$. Notice that we are treating X as a constant for this calculation.

WORKED-OUT PROBLEM

The Problem Let $f(X,Y) = X^2Y^2$. What is $\frac{\partial f}{\partial X}$? What is $\frac{\partial f}{\partial Y}$?

The Solution In this example, the X and Y parts of the function are multiplied by each other. This is in contrast to the previous worked-out problem where they were added. The general procedure to calculate partial derivatives, however, stays the same. Specifically, when we are calculating the partial derivative of f with respect to X, we treat Y as if it were a constant, and vice versa when we are calculating the partial derivative of f with respect to Y. The partial derivative with respect to X therefore is $\frac{\partial f}{\partial X} = \frac{\partial(X^2Y^2)}{\partial X} = 2XY^2$. (Recall that our constant rule for derivatives is $\frac{d(cf(X))}{dX} = c\frac{df(X)}{dX}$.) The partial derivative of f with respect to Y is $\frac{\partial f}{\partial Y} = \frac{\partial(X^2Y^2)}{\partial Y} = 2X^2Y$.

ADDITIONAL EXERCISES

1. Consider the function $f(X,Y) = 2X + 3Y$. What is $\frac{\partial f}{\partial X}$? What is $\frac{\partial f}{\partial Y}$?

2. Consider the function $f(X,Y) = 2X^2 + 3Y^3$. What is $\frac{\partial f}{\partial X}$? What is $\frac{\partial f}{\partial Y}$?

3. Consider the function $f(X,Y) = 6X^2Y^3$. What is $\frac{\partial f}{\partial X}$? What is $\frac{\partial f}{\partial Y}$?

4. Consider the function $f(X,Y) = 9X^{1/9}Y^{1/3}$. What is $\frac{\partial f}{\partial X}$? What is $\frac{\partial f}{\partial Y}$?

5. Consider the function $f(X,Y) = X^{1/3}Y^{1/2}$. What is $\frac{\partial f}{\partial X}$? What is $\frac{\partial f}{\partial Y}$?

6. Consider the function $f(X,Y) = X^3Y^2 + 3Y$. What is $\frac{\partial f}{\partial X}$? What is $\frac{\partial f}{\partial Y}$?

7. Let $f(X,Y) = X^7Y^9 + 10X$. What is $\frac{\partial f}{\partial X}$? What is $\frac{\partial f}{\partial Y}$?

8. Let $f(X,Y) = 9X^{1/9}Y^{1/3} + 6X^{1/3}Y^{1/2}$. What is $\frac{\partial f}{\partial X}$? What is $\frac{\partial f}{\partial Y}$?

9. Let $f(X,Y) = X^4Y^2 + 8X^2Y + 4XY^2$. What is $\frac{\partial f}{\partial X}$? What is $\frac{\partial f}{\partial Y}$?

10. Let $f(X,Y) = X^3Y^3 + 12X^2Y^2 + 6XY + 10$. What is $\frac{\partial f}{\partial X}$? What is $\frac{\partial f}{\partial Y}$?

11. Let $f(X,Y) = X^5Y^6 + X^4Y^3 + X^{1/2}Y^{1/4} + 10Y$. What is $\frac{\partial f}{\partial X}$? What is $\frac{\partial f}{\partial Y}$?

12. Let $f(X,Y) = 7X^4Y^3 + 14X^2Y + 10X + 12Y + 6$. What is $\frac{\partial f}{\partial X}$? What is $\frac{\partial f}{\partial Y}$?

Chapter 2: Supply and Demand

Supply and Demand Functions

A demand function for a good describes mathematically how that good's quantity demanded changes for different combinations of its price and other related factors such as income or the price of a related product. A simple demand function might take the linear form $Q^d = A - BP$ where A is a constant and B is a slope parameter. In this simple case, Q^d is a function of only one variable, P, which is the price of the product. Note that we can interpret B in terms of calculus by noting that $\frac{\partial Q^d}{\partial P} = \frac{dQ^d}{dP} = -B$. Similarly, a supply function is a mathematical representation of how quantity supplied of a product varies for different price levels and levels of other relevant factors. A simple linear supply curve might take the form $Q^s = C + DP$ where C is a constant and D is a slope parameter. In this case, $\frac{\partial Q^s}{\partial P} = \frac{dQ^s}{dP} = D$. Similarly, we can consider a more complicated demand curve which is a function of additional parameters such as the price of another good (P_O) and income (M): $Q^d = \alpha - \beta P + \gamma P_O + \delta M$. We can again characterize the slope parameters given calculus. Here, $\frac{\partial Q^d}{\partial P} = -\beta$, $\frac{\partial Q^d}{\partial P_O} = \gamma$, and $\frac{\partial Q^d}{\partial M} = \delta$. Notice that you can extend this framework for more complicated forms of the demand and supply functions. The general method to find the slope of the demand or supply function in the direction of a specific variable is to take the first partial derivative of the relevant function with respect to the variable of interest.

Substitutes and Complements

Two goods are substitutes if an increase in the price of one causes buyers to demand more of the other product. Two goods are complements if an increase in the price of one causes buyers to demand less of the other product. The relationships of substitutes and of complements therefore are relationships between the price of one good and the quantity of another good. Considering the demand curve $Q^d = \alpha - \beta P + \gamma P_O + \delta M$, we can easily tell if products are substitutes or complements by examining $\frac{\partial Q^d}{\partial P_O}$, which equals the coefficient

γ. If $\gamma > 0$, then the two goods are substitutes indicating that an increase in the price of a related good, P_O, is associated with an increase in the quantity demanded of the good itself. If $\gamma < 0$, then the two goods are complements indicating that an increase in P_O is associated with a decrease in Q^d.

Elasticities and Calculus

Elasticities of demand and supply are used to examine the responsiveness of quantity demanded and quantity supplied respectively to changes in the price of a product, the prices of related products, income, or other factors. In Chapter 2, elasticities are defined mathematically as equaling the percentage change of one variable divided by the percentage change of another. This can be written algebraically as $E^d = \frac{(\Delta Q/Q)}{(\Delta P/P)}$. Elasticities also can be defined in terms of calculus. For example, the price elasticity of demand can be written $E^d = \frac{\partial Q^d}{\partial P}\frac{P}{Q}$ where P and Q represent a point on the demand curve, and $\frac{\partial Q^d}{\partial P}$ is a partial derivative like those calculated in Chapter 1 in this supplement and in the section above. The algebraic and calculus formulas are equivalent, and the principles of elasticities that you have learned in your textbook all extend here.

In Chapter 2, you learned that one important principle of demand elasticities is that total expenditure is largest at the price for which the demand elasticity equals -1. Given the case that quantity demanded takes the general linear form of $Q^d = A - BP$, you saw in that chapter that you could calculate the elasticity of demand as $E^d = -B(\frac{P}{Q})$, where Q is defined as above and is a function of P. Setting this equation equal to -1 and rearranging to solve for P, you were able to determine the price at which total expenditure is largest. This general approach remains valid when you calculate demand elasticities using calculus. Notice that B simply equals $\frac{\partial Q^d}{\partial P}$.

Similar to the price elasticity of demand formula, the price elasticity of supply can be written as $E^s = \frac{\partial Q^s}{\partial P}\frac{P}{Q}$ where P and Q now represent a point along the supply curve. Similarly still, you can calculate other elasticities using this general method. Your textbook for example also defines the income elasticity of demand and the cross-price elasticity of demand. Consider the more complicated version of the demand function written above where quantity demanded is a function of P, P_O, and M. The income elasticity of demand is $E^d_M = \frac{(\Delta Q/Q)}{(\Delta M/M)} = \frac{\Delta Q}{\Delta M}\frac{M}{Q}$ where M is income. Using calculus, the formula for the income elasticity of demand is $E^d_M = \frac{\partial Q^d}{\partial M}\frac{M}{Q}$.

Likewise, the formula for the cross-price elasticity of demand is $E^d_{P_O} = \frac{(\Delta Q/Q)}{(\Delta P_O/P_O)} = \frac{\Delta Q}{\Delta P_O}\frac{P_O}{Q} = \frac{\partial Q^d}{\partial P_O}\frac{P_O}{Q}$ where Q refers to the quantity of the good whose demand curve we are looking at and P_O refers to the price of another good that is somehow related. Given your understanding of partial derivatives from Chapter 1 of this supplement, you should be able to calculate these and other elasticities with ease using calculus.

Remember that you can also solve these types of questions using the algebraic methods outlined in your textbook. Both algebra and calculus are mathematical tools that you can use to solve economics problems. Here, the economic concept is that of demand and supply elasticities which measure changes in quantity demanded or supplied associated with changes in prices, income, or other relevant variables. Let's look at some examples.

WORKED-OUT PROBLEM

The Problem Consider the demand curve $Q^d = 8 - 3P$. What is the price elasticity of demand at $P = 2$?

The Solution To solve this problem, we need to know three things. Since the price elasticity of demand is different at each point along the demand curve, we first want to know at which point we should calculate this elasticity. This is given in the problem since we are told to calculate the price elasticity when $P = 2$. Secondly, we want to know the quantity that will be demanded at this price. We can calculate this by substituting $P = 2$ into the demand curve equation. For this problem, $Q^d = 8 - 3P = 8 - 3(2) = 8 - 6 = 2$. Thirdly, we want to know the partial derivative of Q^d with respect to P. Since the demand curve here is a function of only one variable, price, the partial derivative is equivalent to the standard single variable derivative. Therefore, $\frac{\partial Q^d}{\partial P} = \frac{dQ^d}{dP} = \frac{d(8-3P)}{dP} = -3$. Now, using our three pieces of information, we can calculate the price elasticity of demand by subbing into our formula.

Namely, $E^d = \frac{\partial Q^d}{\partial P}\frac{P}{Q} = -3(\frac{2}{2}) = -3$. The price elasticity of demand at $P = 2$ is -3.

WORKED-OUT PROBLEM

The Problem Consider the supply curve $Q^s = 2 + 3P$. What is the price elasticity of supply at $P = 2$?

The Solution To solve this problem, again we need to know three things. We know that $P = 2$ at the point along the supply curve at which we want to calculate this elasticity. The quantity supplied at this price is calculated by subbing $P = 2$ into the supply curve equation. Here, $Q^s = 2 + 3P = 2 + 3(2) = 8$. Now since we have the relevant price and quantity, all we need is the relevant slope or partial derivative of the quantity supplied with respect to price. In our case, $\frac{\partial Q^s}{\partial P} = \frac{dQ^s}{dP} = \frac{d(2+3P)}{dP} = 3$. We can calculate the price elasticity of supply by subbing into our formula. Here, $E^s = \frac{\partial Q^s}{\partial P}\frac{P}{Q} = 3(\frac{2}{8}) = \frac{3}{4}$. This is the price elasticity of supply at $P = 2$.

ADDITIONAL EXERCISES

1. Consider the demand curve $Q^d = 8 - 3P$. What is the price elasticity of demand at $P = 1$?

2. Consider the demand curve $Q^d = 10 - 5P$. What is the price elasticity of demand at $P = 1$?

3. Consider the demand curve $Q^d = 10 - 5P$. What is the price elasticity of demand at $P = 2$?

4. Consider the demand curve $Q^d = 10 - 5P$. What is the price elasticity of demand at $Q = 5$? How does the answer relate to question 2 above? Is total expenditure largest at this point?

5. Consider the supply curve $Q^s = 2 + 3P$. What is the price elasticity of supply at $P = 1$?

6. Consider the supply curve $Q^s = 10 + 5P$. What is the price elasticity of supply at $P = 2$?

7. Consider the supply curve $Q^s = 9 + 7P$. What is the price elasticity of supply at $P = 3$?

8. Consider the supply curve $Q^s = 9 + 7P$. What is the price elasticity of supply at $Q = 9$?

9. Consider the demand curve $Q^d = 10 - P + 5M$. What is the price elasticity of demand if $P = 9$ and $M = 2$?

10. Consider the demand curve $Q^d = 10 - P + 5M$. What is the income elasticity of demand if $P = 9$ and $M = 2$?

11. Consider the demand curve $Q^d = 10 - P + P_O$. What is the price elasticity of demand if $P = 9$ and $P_O = 5$?

12. Consider the demand curve $Q^d = 10 - P + P_O$. What is the cross-price elasticity of demand if $P = 9$ and $P_O = 5$? Are the two goods substitutes or complements?

Chapter 3: Balancing Benefits and Costs

Marginal Cost and Marginal Benefit

In Chapter 3 of your textbook, you are challenged to think on the margin. Think of the marginal unit as the smallest possible increment of a good that you can add or subtract. Marginal cost is defined as that cost incurred or associated with that last or marginal unit of production. Marginal benefit is defined analogously as the extra benefit gained by producing a marginal unit. Marginal costs and benefits can be defined using calculus. If our cost function is a function of some action X and is written $C(X)$, then marginal cost can be calculated as $\frac{\partial C}{\partial X}$. Similarly, if our benefits are a function of X and are written $B(X)$, then marginal benefit can be calculated as $\frac{\partial B}{\partial X}$. Marginal cost is the slope of the cost curve at the point associated with the marginal unit, and marginal benefit is the slope of the benefit curve at the point of the marginal unit. Note that although we are using the generic X in this section, we may think of this variable as representing quantity of a product. In fact, when we consider the firm's problem more specifically later, we will model total cost (and marginal cost) in term of the quantity variable Q. Alternately, we might consider a cost or benefit function in terms of multiple variables. For example, cost might be a function of capital, labor, and other inputs. Then, we could calculate the marginal cost with respect to each of these inputs by taking the first partial derivative with respect to the variable of interest. Benefit might also be a function of more than one variable. In any of these cases, however, the general mathematical methods are the same.

No Marginal Improvement Principle

The No Marginal Improvement Principle states that at an optimum, $MC = MB$. This principle states that the economic optimum is at the point where the cost of a marginal unit exactly equals the benefit of a marginal unit. Just as your book used algebra to express this point, we can also use calculus to formulate this principle.

We can rewrite the No Marginal Improvement Principle in terms of partial derivatives as $\frac{\partial C}{\partial X} = \frac{\partial B}{\partial X}$. Note that this is the first-order condition from the problem maximizing net benefits (the difference between B and C). If our objective is to maximize the function $B(X) - C(X)$ with respect to X, then the first derivative (or first-order condition) of this problem is $\frac{\partial B}{\partial X} - \frac{\partial C}{\partial X} = 0$. This reduces to our No Marginal Improvement Principle: $\frac{\partial C}{\partial X} = \frac{\partial B}{\partial X}$. Note that since this is a single variable equation (both costs and benefits are expressed in terms of only one variable X), the partial derivative solution ($\frac{\partial C}{\partial X} = \frac{\partial B}{\partial X}$) is equivalent to the standard derivative result ($\frac{dC}{dX} = \frac{dB}{dX}$). Economically, the No Marginal Improvement Principle means that economic decision makers should continue consuming or producing until the point where their marginal benefit from continued action exactly equals their marginal cost. This is the best that a consumer or a producer can do given his or her limited resources, and we see here that we can use calculus to express this condition.

WORKED-OUT PROBLEM

The Problem Consider the cost function $C(X) = 10X + 4X^2$. Write an expression for marginal cost in terms of X.

The Solution Marginal cost is the first derivative of the cost function with respect to X. Since cost is a function of only one variable here, marginal cost can be thought of as either a partial or standard derivative of the cost function. For this problem, marginal cost can be expressed $\frac{\partial C}{\partial X} = \frac{dC}{dX} = 10 + 8X$.

WORKED-OUT PROBLEM

The Problem If benefits are expressed $B(X) = 2X - X^2$ and costs are $C(X) = X^2$, what is the optimal amount of X?

The Solution We can apply our No Marginal Improvement Principle here. Since $MB = \frac{\partial B}{\partial X} = 2 - 2X$ and $MC = \frac{\partial C}{\partial X} = 2X$, then by the No Marginal Improvement Principle $2 = 4X$, or $X = \frac{1}{2}$.

ADDITIONAL EXERCISES

1. Consider the cost function $C(X) = 5X + 2X^2$. Write an expression for marginal cost in terms of X.

2. Consider the cost function $C(X) = 4X + 3X^2$. Write an expression for marginal cost in terms of X.

3. Consider the cost function $C(X) = 20X + 10X^2$. Write an expression for marginal cost in terms of X.

4. Consider the benefits function $B(X) = 10X - X^2$. Write an expression for marginal benefit in terms of X.

5. Consider the benefits function $B(X) = 200X - 3X^2$. Write an expression for marginal cost in terms of X.

6. Consider the benefits function $B(X) = 50X - 6X^2$. Write an expression for marginal cost in terms of X.

7. If benefits are expressed $B(X) = 100X - X^2$ and costs are $C(X) = 2X$, what is the optimal amount of X?

8. If benefits are expressed $B(X) = 85X - 3X^2$ and costs are $C(X) = 2X^2 + 5X$, what is the optimal amount of X?

9. Consider the benefit function $B(X, Y) = XY$. Write an expression for marginal benefit of X. Write an expression for the marginal benefit of Y.

10. Consider the benefit function $B(X, Y) = X^2Y^5$. Write an expression for marginal benefit of X. Write an expression for the marginal benefit of Y.

11. Consider the cost function $C(X, Y) = X^{1/2}Y^{1/2}$. Write an expression for marginal cost of X. Write an expression for the marginal cost of Y.

12. Consider the cost function $C(X, Y) = X^6Y^5$. Write an expression for marginal cost of X. Write an expression for the marginal cost of Y.

Part II: Economic Decision Making

Part IIA: Consumption Decisions

Chapter 4: Principles and Preferences

Indifference Curves and Utility Functions

The concepts of indifference curves and of utility functions are introduced in Chapter 4 of your textbook. A utility function is a mathematical representation or functional form of the indifference curve. Utility functions can be written as $U = f(X, Y)$ where U represents a utility level of the indifference curve and X and Y represent two goods. All combinations of X and Y that yield the same constant U are points along one indifference curve. A standard convex indifference curve has a utility functional form written as $U = X^{\alpha}Y^{\beta}$ where α and β are constants. This is called the Cobb-Douglas form of the utility function.

Two special cases of indifference curves are introduced in your textbook. These are the cases of perfect substitutes and of perfect complements. The utility function corresponding to a perfect substitutes indifference curve takes the general form $U = \alpha X + \beta Y$ where again α and β are constants. In the case of perfect substitutes, a consumer is willing to substitute one good for the other at a fixed rate. The utility function corresponding to a perfect complements indifference curve takes the general form $U = \min(\alpha X, \beta Y)$. In the case of perfect complements, a consumer is willing to substitute one good for the other only in fixed proportions.

Marginal Utility

Marginal utility is defined in your textbook as the change is a consumer's utility for a marginal change in the amount of some good divided by the amount added. For good X, algebraically, marginal utility is $MU_X = \frac{\Delta U}{\Delta X}$. In terms of calculus, marginal utility of X is $MU_X = \frac{\partial U}{\partial X}$. Similarly, marginal utility of Y is $MU_Y = \frac{\partial U}{\partial Y}$.

Marginal Rate of Substitution

A consumer's marginal rate of substitution between X and Y is an empirical, measurable concept. If you were to observe a consumer with a given budget choosing among various combinations of X and Y, then you could calculate his or her marginal rate of substitution between the two goods by observing the rate at which the consumer trades one good for the other. You can also obtain the consumer's marginal rate of substitution between the two goods, however, using calculus if you know his or her utility function or have a formula for an indifference curve. Specifically, the marginal rate of substitution is the slope of the indifference curve holding the level of utility constant. Since marginal rate of substitution is a slope, we can derive it in terms of calculus. We can also write it in terms of the marginal utilities that we defined above. Specifically, $MRS_{XY} = \frac{MU_X}{MU_Y} = \frac{\partial U/\partial X}{\partial U/\partial Y} = \frac{\partial Y}{\partial X}$. Utility level, U, is held constant in this final calculation. Note that is is unlikely that we would ever observe a utility function in actuality. Still, it is important to realize how what we do observe, the consumer's marginal rate of substitution, relates to the theoretical constructs in this chapter.

WORKED-OUT PROBLEM

The Problem Consider the utility function $U = X^2Y^2$. What is the marginal utility of good X? What is the marginal utility of good Y?

The Solution The marginal utility of good X is the partial derivative of U with respect to X. The marginal utility of good Y is the partial derivative of U with respect to Y. These calculations are as follows: $MU_X = \frac{\partial U}{\partial X} = 2XY^2$ and $MU_Y = \frac{\partial U}{\partial Y} = 2X^2Y$. Note that these quantities are not equivalent.

WORKED-OUT PROBLEM

The Problem As above, consider the utility function $U = X^2Y^2$. What is the marginal rate of substitution between goods X and Y?

The Solution The marginal rate of substitution between goods X and Y can be derived using the marginal utilities that we calculated above. For this problem, $MRS_{XY} = \frac{MU_X}{MU_Y} = \frac{2XY^2}{2X^2Y} = \frac{Y}{X}$. To see the marginal rate of substitution at a specific point along the indifference curve, simply plug in the relevant values of X and Y.

ADDITIONAL EXERCISES

1. Consider the utility function $U = XY$. What is the marginal utility of good X? What is the marginal utility of good Y?

2. Consider the utility function $U = XY^2$. What is the marginal utility of good X? What is the marginal utility of good Y?

3. Consider the utility function $U = X^3Y^2$. What is the marginal utility of good X? What is the marginal utility of good Y?

4. Consider the utility function $U = X^5Y^5$. What is the marginal utility of good X? What is the marginal utility of good Y?

5. Consider the utility function $U = X + Y$. What is the marginal utility of good X? What is the marginal utility of good Y?

6. Consider the utility function $U = 4X + 2Y$. What is the marginal utility of good X? What is the marginal utility of good Y?

7. Consider the utility function $U = XY$. What is the marginal rate of substitution between goods X and Y?

8. Consider the utility function $U = XY^2$. What is the marginal rate of substitution between goods X and Y?

9. Consider the utility function $U = X^3Y^2$. What is the marginal rate of substitution between goods X and Y?

10. Consider the utility function $U = X^5Y^5$. What is the marginal rate of substitution between goods X and Y?

11. Consider the utility function $U = X + Y$. What is the marginal rate of substitution between goods X and Y?

12. Consider the utility function $U = 4X + 2Y$. What is the marginal rate of substitution between goods X and Y?

Chapter 5: Constraints, Choices, and Demand

Maximizing a Utility Function

The consumer choice problem is formalized in Chapter 5. Specifically, budget constraints are defined in this chapter and combined with the indifference curves that we saw in Chapter 4. Ultimately, a consumer faces fixed resources and wants to maximize his or her well-being (which can be described using a utility function) subject to resource constraints. In other words, the consumer wants to be as happy as possible within the confines of his or her budget.

In Chapter 4 of this supplement, we saw how a consumer's marginal rate of substitution can be calculated from a consumer's utility function using calculus by calculating the marginal utilities associated with two goods and taking their ratio. We also learned that a consumer's optimal consumption bundle (where his or her marginal benefit from consumption exactly equals marginal cost) happens at the point where his or her marginal rate of substitution equals the going price ratio in the economy. For two goods, X and Y, this equation can be written $MRS_{XY} = \frac{MU_X}{MU_Y} = \frac{P_X}{P_Y}$. The consumer's budget constraint can be expressed as $P_X X + P_Y Y = M$, where P_X and P_Y are the prices of goods X and Y respectively and M is the income available to the consumer. Notice that the marginality condition and the budget constraint are simply two equations in two unknowns so we can easily rearrange and solve for the optimal amounts of X and Y. As we will see below, the Method of Lagrange Multipliers, which more intensively utilizes calculus, also can be used to solve these types of consumer problems.

Maximizing a Utility Function–Calculus

The first step in the Method of Lagrange Multipliers is to state the problem in terms of the Lagrangian. The Lagrangian is a mathematical tool for constrained optimization problems. We want to maximize $U(X, Y)$ subject to our budget constraint $P_X X + P_Y Y \leq M$ where

P_X and P_Y are the prices of goods X and Y respectively and M is the income available to the consumer. The budget constraint shows that total expenditure must be less than or equal to total income. In the simplified world of our problem where there are only two goods and no savings, it will be in the consumer's best interest to consume at the point that total expenditure on goods X and Y exactly equals total income. By doing this, our consumer will maximize his or her well-being and reach the highest possible indifference curve. Of course, we can imagine extending the budget constraint to include additional consumption goods or savings by adding additional choice variables. However, we will examine the simpler case here. In the two good world, we rewrite the budget constraint as $M-P_XX-P_YY=0$ and use this form when we write the Lagrangian for the optimization problem. The Lagrangian can be written as $\mathcal{L}=U(X,Y)+\lambda(M-P_XX-P_YY)$ where the new variable λ is the Lagrange multiplier. Note that the Lagrangian includes both the objective function, $U(X,Y)$, and the budget constraint.

The next step in the Method of Lagrange Multipliers is to differentiate the Lagrangian with respect to the three variables X, Y, and λ in turn. These equations are the first-order conditions of the constrained maximization problem. Let's write these out:

$\frac{\partial \mathcal{L}}{\partial X}=\frac{\partial U}{\partial X}-\lambda P_X=0$

$\frac{\partial \mathcal{L}}{\partial Y}=\frac{\partial U}{\partial Y}-\lambda P_Y=0$

$\frac{\partial \mathcal{L}}{\partial \lambda}=M-P_XX-P_YY=0$

Recall that marginal utility of X, MU_X, can be written as $\frac{\partial U}{\partial X}$ and that MU_Y can be written analogously. Substituting into the equations above, the three equations above can be rearranged and rewritten as:

$MU_X=\lambda P_X$

$MU_Y=\lambda P_Y$

$P_XX+P_YY=M$

Note that the first two equations can be expressed as marginal utilities per dollar spent on good X and good Y respectively. Specifically, we see that $\lambda=\frac{MU_X}{P_X}=\frac{MU_Y}{P_Y}$. Recall that this describes the point at which marginal benefit equals marginal cost. This occurs when the marginal utility of good X per dollar spent on X exactly equals the marginal utility of good Y per dollar spent on Y. Notice that we can also rewrite this as $\frac{MU_X}{MU_Y} = \frac{P_X}{P_Y}$, or as $MRS_{XY} = \frac{P_X}{P_Y}$, which is the equation describing the optimal combinations of X and Y. The third equation is simply the budget constraint.

The final step of the Method of Lagrange Multipliers is to solve these equations for X and Y, and λ. Let's examine a couple worked-out problems.

WORKED-OUT PROBLEM

The Problem An agent's utility function is written as $U = XY$, and his budget constraint is $X + Y = 100$. What are the optimal amounts of X and Y?

The Solution Two ways of approaching the problem are outlined above. First, let's consider the algebraic approach. Notice first that we have enough information in the problem to calculate the consumer's marginal rate of substitution. Here, $MRS_{XY} = \frac{MU_X}{MU_Y} = \frac{Y}{X}$. The price ratio here is $\frac{1}{1}$ since $P_X = P_Y = 1$ is implicitly written in the budget constraint. Therefore, at the optimal allocation $MRS_{XY} = \frac{Y}{X} = \frac{1}{1}$, or rearranging $X = Y$. This is one equation in two unknowns. The second equation in two unknowns is the budget constraint. Here, $X + Y = 100$. Subbing in the relationship that $X = Y$, we see that $X + X = 100$ or $X = 50$. Since $X = Y$, Y must also equal 50.

Let's work out the problem using the three step Lagrangian process outlined above to show that the answer is the same. The first step is to state the problem in terms of the Lagrangian equation. Here, $\mathcal{L} = XY + \lambda(100 - X - Y)$. Recall that λ is the Lagrange multiplier, an additional variable in the problem. The second step in the Method of Lagrange Multipliers is to take the derivative of the Lagrangian with respect to the three variables X, Y, and λ.

$\frac{\partial \mathcal{L}}{\partial X} = Y - \lambda = 0$

$\frac{\partial \mathcal{L}}{\partial Y} = X - \lambda = 0$

$\frac{\partial \mathcal{L}}{\partial \lambda} = 100 - X - Y = 0$

The final step is to solve these three equations for the unknowns X and Y. Combining the first two equations, we see that $\lambda = X = Y$. Since $X = Y$, we can plug this relationship into the third equation in order to solve for numerical quantities of X and Y. Subbing in X for Y, we see that $100 = X + X$, or $X = 50$. Since $X = Y$, we know that Y also equals 50. Therefore, it is optimal for our agent to divide his budget equally over the two goods. This is the same solution that we found above when we started with the marginal rate of substitution instead of with the utility function itself.

WORKED-OUT PROBLEM

The Problem An agent's utility function is written as $U = X^2Y$, and his budget constraint is $X + 2Y = 100$. What are the optimal amounts of X and Y?

The Solution Let's use the three step Lagrangian process here. First, write the Lagrangian as $\mathcal{L} = X^2Y + \lambda(100 - X - 2Y)$. Recall that λ is the Lagrange multiplier, an additional variable in the problem. Now, take the derivative of the Lagrangian with respect to the three variables X, Y, and λ.

$\frac{\partial \mathcal{L}}{\partial X} = 2XY - \lambda = 0$

$\frac{\partial \mathcal{L}}{\partial Y} = X^2 - 2\lambda = 0$

$\frac{\partial \mathcal{L}}{\partial \lambda} = 100 - X - 2Y = 0$

Finally, solve these three equations for the unknowns. The first two equations can be rearranged and written: $\lambda = 2XY$ and as $\lambda = \frac{X^2}{2}$. Combining the first two equations, we see that $2XY = \frac{X^2}{2}$, or $4Y = X$. We can plug this relationship into the third equation in order to solve for X and Y. Subbing in $4Y$ for X yields $100 - (4Y) - 2Y = 0$, or $Y = \frac{50}{3}$. Since $X = 4Y$, the optimal amount of X is $4(\frac{50}{3})$, or $\frac{200}{3}$. Check this solution starting with the marginal rate of substitution on your own for extra practice.

Note on the Substitution Method

In addition to the Method of Lagrange Multipliers illustrated above, Add-On 5A presents the Substitution Method of solving optimization problems using calculus. In that method, we rewrite the utility function in terms of one variable by rearranging the budget constraint so that it can be directly substituted into the utility function. We can then take the first derivative of the utility function with respect to the remaining variable and solve for the optimal quantity. While the Substitution Method is equivalent, the Lagrangian method is more powerful and can be used in a wider variety of cases. Therefore, try the Lagrangian method below as you work through the additional exercises.

Note on Duality of the Consumer Problem

We have modeled the consumer's objective as to maximize his or her utility subject to his or her budget constraint. However, we also could specify the consumer's problem as minimizing expenditure subject to a given level of utility. In that case, we would set up the Lagrangian as $\mathcal{L} = P_X X + P_Y Y + \lambda(U^* - U(X, Y))$ where U^* is the target utility level. The method to solve this formulation of the consumer's problem would be equivalent. Specifically, we could then take the partial derivative of this Lagrangian with respect to X, Y, and λ and set each of these first order conditions equal to zero. We would then rearrange these three equations in three unknowns until we have solved for the optimal combination of goods.

ADDITIONAL EXERCISES

1. An agent's utility function is written as $U = XY$, and his budget constraint is $5X + 2Y = 100$. What are the optimal amounts of X and Y?

2. An agent's utility function is written as $U = X^2Y^2$, and his budget constraint is $X + Y = 100$. What are the optimal amounts of X and Y?

3. An agent's utility function is written as $U = X^3Y$, and his budget constraint is $X + 3Y = 50$. What are the optimal amounts of X and Y?

4. An agent's utility function is written as $U = 5XY$, and his budget constraint is $X + 2Y = 200$. What are the optimal amounts of X and Y?

5. An agent's utility function is written as $U = 4X^2Y$, and his budget constraint is $2X + 3Y = 60$. What are the optimal amounts of X and Y?

6. An agent's utility function is written as $U = 6X^3Y^3$, and his budget constraint is $3X + 6Y = 120$. What are the optimal amounts of X and Y?

7. An agent's utility function is written as $U = 8XY^2$, and his budget constraint is $X + Y = 45$. What are the optimal amounts of X and Y?

8. An agent's utility function is written as $U = 10X^2Y^2$, and his budget constraint is $2X + 2Y = 300$. What are the optimal amounts of X and Y?

9. An agent's utility function is written as $U = X^4Y^2$, and his budget constraint is $8X + Y = 48$. What are the optimal amounts of X and Y?

10. An agent's utility function is written as $U = 12X^2Y^2$, and his budget constraint is $X + 3Y = 120$. What are the optimal amounts of X and Y?

11. An agent's utility function is written as $U = 3X^3Y$, and his budget constraint is $X + 2Y = 400$. What are the optimal amounts of X and Y?

12. An agent's utility function is written as $U = 4X^4Y^4$, and his budget constraint is $4X + Y = 88$. What are the optimal amounts of X and Y?

Chapter 6: From Demand to Welfare

Income and Substitution Effects

Chapter 6 considers the effect of a price change on a consumer's demand for a product, and decomposes this effect into an income effect and a substitution effect. The income effect of a price change is the portion of the price change effect due to the change in purchasing power associated with the price change. The substitution effect of a price change is the effect on demand of a compensated price change, or in other words, the effect on demand of the consumer's decision to substitute away from or toward a product as a result of the price change. The effect of an uncompensated price change is the summation of these two effects, and we can write this relationship as the Slutsky Equation.

Slutsky Equation

The Slutsky Equation is the mathematical representation of the relationship that the effect of an uncompensated price change equals the substitution effect plus the income effect. In words, the Slutsky Equation states that the effect of an uncompensated price change exactly equals the effect of a compensated price change plus the effect of removing the compensation. Until this point, we have been primarily concerned with uncompensated (or Marshallian) demand curves. Uncompensated demand curves show the relationship between price and quantity demanded while holding income constant and allowing utility to vary. Compensated (or Hicksian) demand curves, on the other hand, show the relationship between price and quantity demanded while holding utility constant and allowing income to vary.

Algebraically, the Slutsky Equation can be written as $(\frac{\Delta Q}{\Delta P})^{Uncomp} = (\frac{\Delta Q}{\Delta P})^{Comp} - Q(\frac{\Delta Q}{\Delta Y})$ where Q and P are quantity and price and Y is income. We also can write this equation in terms of calculus and partial derivatives where these partial derivatives are taken based on the uncompensated and compensated demand functions as $(\frac{\partial Q}{\partial P})^{Uncomp} = (\frac{\partial Q}{\partial P})^{Comp.} - Q(\frac{\partial Q}{\partial Y})$. Note that $\frac{\Delta Q}{\Delta Y}$ in the algebraic formulation and $\frac{\partial Q}{\partial Y}$ in the calculus formulation are implicitly defined based on the uncompensated demand function since the compensated demand function holds income constant.

From the discussion of elasticities in Chapter 2, recall that we can write the price elasticity of demand as $E_P = \frac{\partial Q^d}{\partial P}\frac{P}{Q}$. We can use this formula to rewrite the Slutsky equation in terms of price elasticities defined based on both the uncompensated or compensated demand functions as defined in Chapter 6 and in terms of the income elasticity of demand, which can be expressed $E_Y = \frac{\partial Q^d}{\partial Y}\frac{Y}{Q}$. Substituting these elasticity relationships into the Slutsky equation, we can rewrite the equation as $E_P^{Uncomp.} = E_P^{Comp.} - S * E_Y$ where S is the fraction of income spent on the good. Note that E_P and E_Y are equivalent to E^d and E_M^d in Chapter 2 of this supplement, but that the notation here follows that in the Appendix to Chapter 6 of your textbook.

Consumer Surplus

Consumer surplus is the net benefit (benefits minus costs) that a consumer receives from a market transaction. Consumer surplus can be measured as the area below the demand curve and above the equilibrium price level, P^*. Consumer surplus results from a consumer being willing to pay a price higher than the equilibrium price but only having to pay the equilibrium price. This measure therefore represents the extra well-being that the consumer receives from participating in market transactions. Graphically, the economic concept of consumer surplus can be calculated using geometry, or using calculus, as an area. Particularly, we can integrate the area under the demand curve and above the equilibrium price in order to quantify the concept of consumer surplus. Let $\bar{P}$ represent the price at which a consumer's demand would equal zero, or in other words, the price at which the demand curve intersects the y-axis or "choke price." Consumer surplus can be calculated as $\int_{P^*}^{\bar{P}} Q^d(P)dP$.

Equivalently, consumer surplus can be calculated by integrating with respect to quantity instead of with respect to price. In that case, consumer surplus is $\int_0^{Q^*}(P(Q^d) - P^*)dQ$. Instead of integrating over the function for quantity demanded as a function of price, we integrate over the difference between price as a function of quantity demanded and the equilibrium price. Note that price as a function of quantity demanded is the inverse demand function, or simply a rearrangement of the standard demand function. The two integral calculations presented above are equivalent. Prove this to yourself when you try some of the additional exercises below.

WORKED-OUT PROBLEM

The Problem If the uncompensated demand curve is written as $Q^d = 50 - 50P + \frac{1}{5}Y$ and price is 1 and income is 100, what is the uncompensated price elasticity of demand? What is the income elasticity of demand? What is the compensated price elasticity of demand?

The Solution Given an uncompensated demand curve, price, and income, we can solve for the uncompensated price elasticities of demand and of income in the same way that we solved for these elasticities in Chapter 2 of this supplement. If $P = 1$ and $Y = 100$, we can solve for the corresponding quantity by subbing into the demand curve. Namely, $Q = 50 - 50(1) + \frac{1}{5}100 = 20$. The uncompensated price elasticity of demand therefore is $E_P^{Uncomp.} = (\frac{\partial Q}{\partial P})^{Uncomp.}\frac{P}{Q} = -50(\frac{1}{20}) = -\frac{5}{2}$. The income elasticity of demand is $E_Y = \frac{\partial Q}{\partial Y}\frac{Y}{Q} = \frac{1}{5}(\frac{100}{20}) = 1$. The fraction of income spent on the good is calculated as total expenditure on the good divided by income. Total expenditure is price times quantity, or 20. The share of income spent on the good therefore is $\frac{20}{100} = \frac{1}{5}$. The compensated price elasticity of demand therefore is $E_P^{Comp.} = E_P^{Uncomp.} + S * E_Y = -\frac{5}{2} + (\frac{1}{5})(1) = -\frac{23}{10}$.

WORKED-OUT PROBLEM

The Problem Suppose that the demand curve is $Q^d = 50 - 5P$ and the equilibrium price is 4. What is the consumer surplus?

The Solution Referring to the equation for consumer surplus, we note that for the calculation we will need the choke price (price at which demand falls to zero) and the equilibrium price. Equilibrium price, P^*, is given at 4. For our demand curve equation, we see that $Q^d = 0$ when $P = 10$. This is the choke price, $\bar{P}$. Subbing the demand curve function and the prices, $\bar{P} = 10$ and $P^* = 4$, into our consumer surplus equation, we find that $\int_4^{10}(50-5P)dP = [50P-2.5P^2]_4^{10} = 50(10)-2.5(10^2)-(50(4)-2.5(4^2)) = 500-250-200+40 = 90$. Note that since the demand curve in this problem is linear, we can check our work by taking the area of the triangle defined by the demand curve and the equilibrium price level. Using the formula that the area of a triangle is one half of the length of the base of the triangle times the height of the triangle, we see that $\frac{1}{2}(30)(6) = 90$, which is the same consumer surplus figure calculated above using calculus. Note that the base of the triangle is the difference between the equilibrium quantity and zero production, and equilibrium quantity is $Q^* = 50-5(4) = 30$. The height of the triangle is the difference between the choke price and the equilibrium price $(10-4=6)$. In the linear demand curve case, solving for the consumer surplus algebraically is easy enough. However, as you might imagine, in cases where the demand curve is a more complicated function, integral calculus is the appropriate approach for finding the relevant areas.

ADDITIONAL EXERCISES

1. If the uncompensated demand curve is written as $Q^d = 100-100P+Y$ and price is 2 and income is 150, what is the uncompensated price elasticity of demand? What is the income elasticity of demand? What is the compensated price elasticity of demand?

2. If the uncompensated demand curve is written as $Q^d = 200 - 150P + \frac{1}{2}Y$ and price is 1 and income is 100, what is the uncompensated price elasticity of demand? What is the income elasticity of demand? What is the compensated price elasticity of demand?

3. If the uncompensated demand curve is written as $Q^d = 200 - 200P + \frac{1}{2}Y$ and price is 1 and income is 100, what is the uncompensated price elasticity of demand? What is the income elasticity of demand? What is the compensated price elasticity of demand?

4. If the uncompensated demand curve is written as $Q^d = 500 - 200P + \frac{1}{4}Y$ and price is 2 and income is $1,000$, what is the uncompensated price elasticity of demand? What is the income elasticity of demand? What is the compensated price elasticity of demand?

5. If the uncompensated demand curve is written as $Q^d = 1,000 - 200P + \frac{1}{5}Y$ and price is 5 and income is 500, what is the uncompensated price elasticity of demand? What is the income elasticity of demand? What is the compensated price elasticity of demand?

6. If the uncompensated demand curve is written as $Q^d = 480 - 80P + \frac{1}{2}Y$ and price is 1 and income is 2,000, what is the uncompensated price elasticity of demand? What is the income elasticity of demand? What is the compensated price elasticity of demand?

7. Suppose that the demand curve is $Q^d = 10 - 2P$ and the equilibrium price is 2. What is the consumer surplus?

8. Suppose that the demand curve is $Q^d = 50 - 5P$ and the equilibrium price is 6. What is the consumer surplus?

9. Suppose that the demand curve is $Q^d = 100 - 5P$ and the equilibrium price is 10. What is the consumer surplus?

10. Suppose that the demand curve is $Q^d = 40 - P$ and the equilibrium price is 10. What is the consumer surplus?

11. Suppose that the demand curve is $Q^d = 100 - 10P$ and the equilibrium price is 1. What is the consumer surplus?

12. Suppose that the demand curve is $Q^d = 2,700 - 300P$ and the equilibrium price is 2. What is the consumer surplus?

Part IIB: Production Decisions

Chapter 7: Technology and Production

Isoquants and Production Functions

Chapter 7 of your textbook begins to characterize the firm's problem. First, the production function is defined as a function relating the firm's output to its inputs. If the firm has only one variable input, we might write the production function as $Q = f(L)$ where L can be thought of as labor or as some other generic input. If the firm has two variable inputs, we might write the production function as $Q = f(L, K)$ where L could be thought of as labor input and K as capital input. Similarly, if the firm has more inputs, then the production function will have more arguments. If the production function takes the form $Q = F(L, K)$, then we can graph constant levels of Q in two dimensional space similarly to how we graph indifference curves for two goods. This graphical representation is called an isoquant. All combinations of L and K that yield the same constant Q are points along one isoquant.

A standard convex isoquant is associated with a production function functional form written $Q = L^\alpha K^\beta$ where α and β are constants. This is called the Cobb-Douglas form of the production function. Special cases of production functions are introduced in your textbook. Specifically, we can write the functional forms associated with the cases of perfect substitutes and of perfect complements. The production function corresponding to a perfect substitutes isoquant takes the general form $Q = \alpha L + \beta K$ where again α and β are constants. In this case, firms can exchange capital and labor for each other at a fixed rate and still produce the same amount of output. The production function corresponding to a perfect complements isoquants takes the general form $Q = \min(\alpha L, \beta K)$. For perfect complements, firms should maintain a fixed proportion relationship between capital and labor. Increases in either capital or labor input beyond the fixed proportion requirement result in zero additional production but additional cost. You will note the similarities between these production function functional forms and the utility function forms in Chapter 4.

Marginal Products

The marginal product of labor is defined as the extra output produced due to a marginal unit of labor per unit of labor added. The marginal product of capital is defined as the extra output produced due to a marginal unit of capital per unit of capital added. Both of these concepts can be easily defined using either algebra or calculus. Specifically, $MP_L = \frac{\Delta Q}{\Delta L} = \frac{\partial Q}{\partial L}$ and $MP_K = \frac{\Delta Q}{\Delta K} = \frac{\partial Q}{\partial K}$. Again, it is important to note that the algebraic and calculus methods are equivalent and that the applications presented in your textbook still hold. For example, a producer faced with the problem of allocating labor between two plants to maximize output should producer such that the marginal products for its two plants are equal. This holds whether we obtain our marginal products algebraically or using calculus.

Marginal Rate of Technical Substitution

The marginal rate of technical substitution between the two inputs L and K is calculated by taking the slope of the isoquant holding the level of Q constant. This can be written as $MRTS_{LK} = \frac{MP_L}{MP_K} = \frac{\partial Q/\partial L}{\partial Q/\partial K} = \frac{\partial K}{\partial L}$ where Q is held constant in this final calculation. The marginal rate of technical substitution between labor and capital inputs is an empirical, measurable concept just as the marginal rate of substitution for the consumer's problem is. If we were to observe a producer choosing among various combinations of L and K, we could calculate the producer's marginal rate of technical substitution by observing the rate at which the inputs are traded while holding quantity constant.

WORKED-OUT PROBLEM

The Problem Consider the production function $Q = \sqrt{LK}$. What is the marginal product of L? What is the marginal product of K?

The Solution The marginal product of labor L is the partial derivative of Q with respect to L. The marginal product of capital K is the partial derivative of Q with respect to K. Recall that $Q = \sqrt{LK} = L^{1/2}K^{1/2}$. The marginal product calculations are as follows: $MP_L = \frac{\partial Q}{\partial L} = \frac{1}{2}L^{-1/2}K^{1/2} = \frac{1}{2}\sqrt{\frac{K}{L}}$ and $MP_K = \frac{\partial Q}{\partial K} = \frac{1}{2}L^{1/2}K^{-1/2} = \frac{1}{2}\sqrt{\frac{L}{K}}$.

WORKED-OUT PROBLEM

The Problem As above, consider the production function $Q = \sqrt{LK}$. What is the marginal rate of technical substitution between L and K?

The Solution The marginal rate of technical substitution between goods L and K can be derived by taking the ratio of the marginal products that we calculated in the previous worked-out problem. For this production function therefore, $MRTS_{LK} = \frac{MP_L}{MP_K} = \frac{\frac{1}{2}\sqrt{\frac{K}{L}}}{\frac{1}{2}\sqrt{\frac{L}{K}}} = \frac{K}{L}$.

ADDITIONAL EXERCISES

1. Consider the production function $Q = 2\sqrt{LK}$. What is the marginal product of L? What is the marginal product of K?

2. Consider the production function $Q = \sqrt{2LK}$. What is the marginal product of L? What is the marginal product of K?

3. Consider the production function $Q = L + K$. What is the marginal product of L? What is the marginal product of K?

4. Consider the production function $Q = 4L + 2K$. What is the marginal product of L? What is the marginal product of K?

5. Consider the production function $Q = L^{1/5}K^{4/5}$. What is the marginal product of L? What is the marginal product of K?

6. Consider the production function $Q = 9L^{1/9}K^{8/9}$. What is the marginal product of L? What is the marginal product of K?

7. Consider the production function $Q = 2\sqrt{LK}$. What is the marginal rate of technical substitution between L and K?

8. Consider the production function $Q = \sqrt{2LK}$. What is the marginal rate of technical substitution between L and K?

9. Consider the production function $Q = L + K$. What is the marginal rate of technical substitution between L and K?

10. Consider the production function $Q = 4L + 2K$. What is the marginal rate of technical substitution between L and K?

11. Consider the production function $Q = L^{1/5}K^{4/5}$. What is the marginal rate of technical substitution between L and K?

12. Consider the production function $Q = 9L^{1/9}K^{8/9}$. What is the marginal rate of technical substitution between L and K?

Chapter 8: Cost

Finding a Least-Cost Input Combination

The producer choice problem is formalized in Chapter 8. This problem is similar to the consumer choice problem of Chapter 5, and therefore the same mathematical methods for constrained optimization problems apply here. Ultimately, a producer faces fixed resources and wants to minimize costs subject to a specified output level.

In the previous chapter of this supplement, we saw how a firm's marginal rate of technical substitution can be calculated from the production function using calculus by calculating the marginal productions of labor and capital and taking the ratio. Specifically, recall that $MRTS_{LK} = \frac{MP_L}{MP_K} = \frac{\partial Q/\partial L}{\partial Q/\partial K}$. The firm's optimal allocation between these inputs occurs at the point at which the marginal rate of technical substitution equals the price ratio associated with the inputs. This can be written $MRTS_{LK} = \frac{W}{R}$, where W and R are the wage rate and rental rate of capital respectively. The firm's cost constraint can be expressed as $C = WL + RK$. These two equations are functions of two unknowns. We can therefore easily rearrange and solve for the optimal amounts of L and K analogously to how we solved the consumer problem in Chapter 5.

Finding a Least-Cost Input Combination–Calculus

We can use the Method of Lagrange Multipliers to solve the producer problem, just as we used it to solve the consumer problem previously. Recall that the Method of Lagrange Multipliers has three main steps. The first step is to state the problem in terms of a Lagrangian. Here, we want to minimize costs $C = WL + RK$ subject to the production constraint that $Q = F(L, K)$. The equation $C = WL + RK$ is the general form of the isocost line which represents all potential input combinations (of labor L and capital K) with the same cost (C). We can write the Lagrangian for the optimization problem as $\mathcal{L} = WL + RK + \lambda(Q - F(L, K))$. As in Chapter 5, λ is the Lagrange multiplier.

The second step of the Method of Lagrange Multipliers is to differentiate the Lagrangian with respect to the three variables L, K, and λ in turn. Let's write out these three equations:

$\frac{\partial \mathcal{L}}{\partial L} = W - \lambda \frac{\partial Q}{\partial L} = 0$

$\frac{\partial \mathcal{L}}{\partial K} = R - \lambda \frac{\partial Q}{\partial K} = 0$

$\frac{\partial \mathcal{L}}{\partial \lambda} = Q - f(L, K) = 0$

Recall that the partial derivative $\frac{\partial Q}{\partial L}$ is the marginal product of labor and $\frac{\partial Q}{\partial K}$ is the marginal product of capital. Using this and rearranging, the three equations above are:

$W = \lambda MP_L$

$R = \lambda MP_K$

$Q = f(L, K)$

Note that $\lambda = \frac{W}{MP_L} = \frac{R}{MP_K}$. This implies that $\frac{MP_L}{W} = \frac{MP_K}{R}$, or $\frac{MP_L}{MP_K} = \frac{W}{R}$. This equation represents the tangency between the isoquant and isocost curve, and therefore mathematically expresses the condition for an input combination to be optimal. This equation is similar to the equation from Chapter 5 that the marginal utility of good X divided by its price P_X equals the marginal utility of good Y divided by P_Y (or equivalently, that the ratio of marginal utilities equals the price ratio) for the optimal combination of goods X and Y. Here, we see that the marginal product per dollar spent on labor should equal the marginal product per dollar spent on capital at the equilibrium allocation of inputs.

As before, the final step of the Method of Lagrange Multipliers is to solve the three equations in three unknowns for our three variables L, K, and λ. Let's examine a couple worked-out problems.

WORKED-OUT PROBLEM

The Problem A firm faces the production function $Q = \sqrt{LK}$. The wage rate is 1 and the rental rate of capital is 1. What are the optimal amounts of L and K if the firm's objective is to produce $Q = 10$? Use the Method of Lagrange Multipliers to solve for the optimal quantities.

The Solution Let's work out the problem using the three step process. First, let's state the problem in terms of the Lagrangian. This equation is written as $\mathcal{L} = L + K + \lambda(10 - \sqrt{LK})$, where λ is the Lagrange multiplier for the problem. Second, let's take the derivative of the Lagrangian with respect to the three variables L, K, and λ.

$\frac{\partial \mathcal{L}}{\partial L} = 1 - \frac{\lambda}{2}\sqrt{\frac{K}{L}} = 0$

$\frac{\partial \mathcal{L}}{\partial K} = 1 - \frac{\lambda}{2}\sqrt{\frac{L}{K}} = 0$

$\frac{\partial \mathcal{L}}{\partial \lambda} = 10 - \sqrt{LK} = 0$

The third step is to solve these three equations for the unknowns L and K. Combining the first two equations, we see that $\lambda = \frac{2L}{K} = \frac{2K}{L}$. Since this reduces to $L = K$, we can plug this relationship into the third equation in order to solve for numerical quantities of L and K. Subbing in L for K, we see that $10 = \sqrt{L^2}$, or $L = 10$. Since $L = K$, K also equals 10.

Note that we also can solve this problem by setting the marginal rate of technical substitution (derived using calculus) equal to the price ratio corresponding to the inputs, and combining this equation with the production function relationship. The marginal rate of technical substitution can be derived using calculus as $MRTS_{LK} = \frac{MP_L}{MP_K} = \frac{\partial Q/\partial L}{\partial Q/\partial K} = \frac{1/2\sqrt{K/L}}{1/2\sqrt{L/K}} = \frac{K}{L}$. The relevant price ratio is $\frac{W}{R} = \frac{1}{1}$. The optimality condition that describes the tangency between the isoquant and isocost line therefore is $\frac{K}{L} = \frac{1}{1}$, or $K = L$. Combining this equation with the production constraint that $\sqrt{LK} = 10$, we see that $\sqrt{L^2} = 10$, or $L = 10$. This implies that $K = 10$ as well. This is the same solution that we found using the Method of Lagrange Multipliers, again demonstrating the equivalence of the solution methods.

Note on the Substitution Method

In addition to the Method of Lagrange Multipliers illustrated above, Add-On 8A presents the Substitution Method of solving least-cost optimization problems using calculus. In that method, we rewrite the cost constraint in terms of one variable by rearranging the production function so that it can be directly substituted into the cost constraint. We can then take the first derivative of the cost constraint with respect to the remaining variable and solve for the optimal quantity of the input. This method is equivalent to the Method of Lagrange Multipliers. However, the Method of Lagrange Multipliers is more widely applicable.

Note on Duality of the Producer Problem

We have modeled the firm's objective above as to minimize total cost subject to a production function and a specified production level. However, we could equivalently specify the firm's problem as maximizing quantity produced subject to the total cost function. In that case, we would set up the Lagrangian as $\mathcal{L} = F(L, K) + \lambda(C - WL - RK)$. We could then take the partial derivative of the Lagrangian with respect to L, K, and λ and set each of these first order conditions equal to zero. The final step would be to rearrange these three equations in three unknowns until we have solved for the optimal input combination.

WORKED-OUT PROBLEM

The Problem A firm faces the production function $Q = \sqrt{LK}$. The wage rate is 1 and the rental rate of capital is 1. What are the optimal amounts of L and K if the firms total expenditure is limited at 20? Use the Method of Lagrange Multipliers to solve for the optimal quantities. Hint: Think about the duality of the producer problem.

The Solution Since this is a constrained optimization problem, we can set up a Lagrangian in order to solve for the optimal amounts of L and K. This problem is different from the previous worked-out example since we are given an expenditure limit (here 20) instead of a specific quantity to produce. Therefore, we should use the duality feature of the producer problem in order to solve this problem.

As usual, the first step is to state the problem in terms of the Lagrangian. Here, this equation is written as $\mathcal{L} = \sqrt{LK} + \lambda(20 - L - K)$. Notice how the form of the Lagrangian differs from the previous problem. The first part of the Lagrangian is the production function instead of the cost function. This is because we now want to maximize quantity subject to a given budget, instead of minimizing cost subject to a given production level as we did before. The parameter λ is still a Lagrange multiplier, but notice that the constraint portion of the Lagrangian (the part multiplied by λ) is now a rearranged expression of the cost relationship. Since wage and rental rate of capital are both equal to 1, the cost constraint for the producer can be written as $L + K = 20$. Since the producer will want to spend the entire budget in order to maximize production, we rewrite the cost constraint in a form such that it is equal to zero and substitute into the second part of the Lagrangian. This is just like what we did with the production function when the constraint was a production quantity.

Like before, the second step in the Method of Lagrange Multipliers is to take the derivative of the Lagrangian with respect to L, K, and λ.

$\frac{\partial \mathcal{L}}{\partial L} = \frac{1}{2}\sqrt{\frac{K}{L}} - \lambda = 0$

$\frac{\partial \mathcal{L}}{\partial K} = \frac{1}{2}\sqrt{\frac{L}{K}} - \lambda = 0$

$\frac{\partial \mathcal{L}}{\partial \lambda} = 20 - L - K = 0$

The final step is to solve for the unknowns L and K in order to determine the optimal input levels. Combining the first two equations, we see that $\lambda = \frac{1}{2}\sqrt{\frac{K}{L}} = \frac{1}{2}\sqrt{\frac{L}{K}}$, or $L = K$. We can plug this relationship into the third equation in order to solve for numerical quantities of L and K. Subbing in for L, we see that $20 = 2K$ or $K = 10$. Since $L = K$, we know that L is also equal to 10. These are the optimal input levels. Notice that this implies that the firm should produce 10 units since $Q = \sqrt{LK} = \sqrt{100} = 10$.

Now compare this solution to the previous worked-out problem. Notice that the optimal quantities of L and K were the same when we started with the constraint that $Q = 10$ and when we started with the alternative constraint that $C = 20$. This demonstrates the duality of the producer problem. You will practice setting up alternative producer problems like this one in some of the additional exercises below.

ADDITIONAL EXERCISES

1. A firm faces the production function $Q = LK$. The wage rate is 5 and the rental rate of capital is 3. What are the optimal amounts of L and K if the firm's objective is to produce $Q = 60$? Use the Method of Lagrange Multipliers to solve for the optimal quantities.

2. A firm faces the production function $Q = \sqrt{LK}$. The wage rate is 2 and the rental rate of capital is 2. What are the optimal amounts of L and K if the firm's objective is to produce $Q = 25$? Use the Method of Lagrange Multipliers to solve for the optimal quantities.

3. A firm faces the production function $Q = 3L^2K$. The wage rate is 1 and the rental rate of capital is 1. What are the optimal amounts of L and K if the firm's objective is to produce $Q = 96$? Use the Method of Lagrange Multipliers to solve for the optimal quantities.

4. A firm faces the production function $Q = 5LK$. The wage rate is 2 and the rental rate of capital is 3. What are the optimal amounts of L and K if the firm's objective is to produce $Q = 750$? Use the Method of Lagrange Multipliers to solve for the optimal quantities.

5. A firm faces the production function $Q = 10L^2K^2$. The wage rate is 2 and the rental rate of capital is 1. What are the optimal amounts of L and K if the firm's objective is to produce $Q = 25,000$? Use the Method of Lagrange Multipliers to solve for the optimal quantities.

6. A firm faces the production function $Q = 15LK$. The wage rate is 1 and the rental rate of capital is 5. What are the optimal amounts of L and K if the firm's objective is to produce $Q = 1,200$? Use the Method of Lagrange Multipliers to solve for the optimal quantities.

7. A firm faces the production function $Q = L^{1/3}K^{1/2}$. The wage rate is 1 and the rental rate of capital is 1. What are the optimal amounts of L and K if the firm's total expenditure is limited at 200? Use the Method of Lagrange Multipliers to solve for the optimal quantities. Hint: Think about the duality of the producer problem.

8. A firm faces the production function $Q = L^{1/4}K^{3/4}$. The wage rate is 2 and the rental rate of capital is 1. What are the optimal amounts of L and K if the firm's total expenditure is limited at 200? Use the Method of Lagrange Multipliers to solve for the optimal quantities. Hint: Think about the duality of the producer problem.

9. A firm faces the production function $Q = LK^2$. The wage rate is 5 and the rental rate of capital is 1. What are the optimal amounts of L and K if the firm's total expenditure is limited at 225? Use the Method of Lagrange Multipliers to solve for the optimal quantities. Hint: Think about the duality of the producer problem.

10. A firm faces the production function $Q = 8L^2K^3$. The wage rate is 1 and the rental rate of capital is 1. What are the optimal amounts of L and K if the firm's total expenditure is limited at 200? Use the Method of Lagrange Multipliers to solve for the optimal quantities. Hint: Think about the duality of the producer problem.

11. A firm faces the production function $Q = L^{2/5}K^{3/5}$. The wage rate is 3 and the rental rate of capital is 6. What are the optimal amounts of L and K if the firm's total expenditure is limited at 100? Use the Method of Lagrange Multipliers to solve for the optimal quantities. Hint: Think about the duality of the producer problem.

12. A firm faces the production function $Q = L^5K^5$. The wage rate is 8 and the rental rate of capital is 6. What are the optimal amounts of L and K if the firm's total expenditure is limited at 360? Use the Method of Lagrange Multipliers to solve for the optimal quantities. Hint: Think about the duality of the producer problem.

Chapter 9: Profit Maximization

Marginal Revenue

Competitive firms are profit maximizers. The profit function can be written as total revenue minus total cost where both revenue and cost are functions of quantity. This can be expressed as $\Pi = R(Q) - C(Q)$. We defined marginal cost in Chapter 3 as the first derivative of the cost function with respect to some generic variable X, $\frac{\partial C}{\partial X}$. Here, we will write both the cost function and the marginal cost as functions of quantity Q. Therefore, marginal cost corresponding to the cost function $C(Q)$ is $\frac{\partial C}{\partial Q}$.

Economically, marginal revenue is the extra revenue that a firm receives for the marginal units sold measured on a per unit basis. Mathematically, marginal revenue is calculated analogously to marginal cost. The revenue function is calculated as price times quantity where price is expressed as a function of quantity, and therefore revenue is a function of quantity. Marginal revenue is the first derivative of the revenue function with respect to quantity. Specifically, $MR = \frac{\Delta R}{\Delta Q} = \frac{\partial R}{\partial Q}$, where $R = P(Q)Q$.

Profit-Maximizing Sales Quantity

Profit-maximizing sales quantity can be calculated by finding the quantity at which marginal revenue exactly equals marginal cost, or $MR = MC$. This relationship is analogous to the marginal benefit equals marginal cost condition from Chapter 3. Considering the profit function and taking the first derivative and setting it equal to zero to find a maximum, we note that $\frac{\partial \Pi}{\partial Q} = \frac{\partial R}{\partial Q} - \frac{\partial C}{\partial Q} = 0$. This condition reduces to $MR = MC$. For the special case of a price-taking firm, the profit-maximizing sales quantity can be found by noting that marginal revenue for a price-taking firm exactly equals price. This is because the firm faces a horizontal demand curve instead of a downward-sloping demand curve. Therefore, the profit maximization condition for the case of a price-taking firm can be expressed as $P = MC$.

Producer Surplus

In Chapter 6, we examined how consumer surplus could be calculated from the demand curve using calculus. Producer surplus can be calculated in an analogous fashion from the supply curve. Producer surplus is the net benefit (benefits minus costs) that a producer receives from selling his or her product, and can be measured as the area above the supply curve and below the equilibrium price level. Producer surplus results from a producer being willing to accept a price lower than the equilibrium price which results from the interaction of demand and supply. Producer surplus therefore is a measure of the extra well-being that the producer receives from participating in market transactions. Producer surplus can be calculated using calculus by integrating the area above the supply curve and below the equilibrium price. Let $\underline{P}$ represent the price at which supply would equal zero. Producer surplus then is $\int_{\underline{P}}^{P^*} Q^s(P)dP$.

Equivalently, producer surplus can be calculated by integrating with respect to quantity instead of with respect to price. In that case, $\int_0^{Q^*}(P^* - P(Q^s))dQ$. Instead of integrating over the function for quantity demanded as a function of price, we integrate over the difference between the equilibrium price and price as a function of quantity demanded. The two calculations are equivalent. (Challenge: Draw a picture to prove this to yourself!) Note, however, that if the particular form of the supply function indicates that there will be positive quantity at $P = 0$ then the later integration method is appropriate in order to prevent including the area corresponding to prices less than zero. Alternately, we would have to set $\underline{P} = 0$ in this special case in order to use the integral formula where we integrate over price. Supply curves, for which these types of modifications would be necessary however, are uncommon.

WORKED-OUT PROBLEM

The Problem Consider the demand function $Q^d = 100 - P$. What is the marginal revenue function?

The Solution Marginal revenue is calculated as the first derivative of the revenue function. To find the revenue function, first solve for the inverse demand function by rearranging the demand function. The inverse demand function here is $P = 100 - Q$. Revenue is price times quantity, or $(100 - Q)Q = 100Q - Q^2$. Marginal revenue is the first derivative of this function, or $MR = \frac{\partial R}{\partial Q} = 100 - 2Q$.

WORKED-OUT PROBLEM

The Problem A firm faces the demand function $Q^d = 250 - 2P$ and the cost function $C = 5Q$. What is the optimal quantity that a price-taking firm should supply?

The Solution The firm should produce until the point that marginal revenue equals marginal cost. For a price-taking firm, marginal revenue is price, which can be expressed as the inverse demand function. Here, the inverse demand function is $P = 125 - \frac{1}{2}Q$. Marginal cost is $\frac{\partial C}{\partial Q} = 5$. Setting price equal to marginal cost, we see that $125 - \frac{1}{2}Q = 5$. The optimal quantity therefore is $Q = 240$.

WORKED-OUT PROBLEM

The Problem Suppose that the supply curve is $Q^s = -10 + 5P$ and the equilibrium price is 6. What is the producer surplus?

The Solution As in our calculation for consumer surplus in Chapter 6, we first need to know one extra price in addition to the given equilibrium price. Specifically, we need to know that price at which supply equals zero. From our supply curve equation, we see that $Q^s = 0$ when $P = 2$. Subbing the supply curve function, $P^* = 6$, and $\underline{P} = 2$ into the producer surplus equation, we find that $\int_2^6 (-10 + 5P)dP = [-10P + 2.5P^2]_2^6 = -10(6) + 2.5(6^2) - (-10(2) + 2.5(2^2)) = -60 + 90 + 20 - 10 = 40$. Note that since the supply curve in this problem is linear, we can check our work by taking the area of the triangle defined by the supply curve and the equilibrium price level. Using the formula that the area of a triangle is one half times the base times the height of the triangle, we see that $\frac{1}{2}(20)(4) = 40$. The base of the triangle is the difference between the equilibrium quantity and zero production. For our problem, equilibrium quantity, $Q^* = 50 - 5(6) = 20$. The height of the triangle is the difference between the equilibrium price and the minimum price supported in the market ($6 - 2 = 4$). Note that in cases where the supply curve is more complicated than the linear function presented in this problem, integral calculus is the best (and sometimes only) way to find the relevant areas.

ADDITIONAL EXERCISES

1. Consider the demand function $Q^d = 75 - 5P$. What is the marginal revenue function?

2. Consider the demand function $Q^d = 200 - 2P$. What is the marginal revenue function?

3. Consider the demand function $Q^d = 300 - 6P$. What is the marginal revenue function?

4. Consider the demand function $Q^d = 40 - \frac{1}{4}P$. What is the marginal revenue function?

5. A firm faces the demand function $Q^d = 100 - 5P$ and the cost function $C = 5Q + 10$. What is the optimal quantity that a price-taking firm should supply?

6. A firm faces the demand function $Q^d = 800 - P$ and the cost function $C = 100Q + 50$. What is the optimal quantity that a price-taking firm should supply?

7. A firm faces the demand function $Q^d = 60 - \frac{1}{2}P$ and the cost function $C = 11Q^2 + 5$. What is the optimal quantity that a price-taking firm should supply?

8. A firm faces the demand function $Q^d = 200 - \frac{1}{6}P$ and the cost function $C = 5Q^2$. What is the optimal quantity that a price-taking firm should supply?

9. Suppose that the supply curve is $Q^s = -20 + 10P$ and the equilibrium price is 4. What is the producer surplus?

10. Suppose that the supply curve is $Q^s = -15 + P$ and the equilibrium price is 25. What is the producer surplus?

11. Suppose that the supply curve is $Q^s = -30 + 3P$ and the equilibrium price is 20. What is the producer surplus?

12. Suppose that the supply curve is $Q^s = -400 + 8P$ and the equilibrium price is 100. What is the producer surplus?

Part IIC: Additional Topics Concerning Decisions

Chapters 10-13 provide additional extensions and topics concerning how we can model decision making by consumers and producers as microeconomic problems. The previous supplementary materials explain how calculus techniques may be applied to general consumer and producer theory problems. Although not formalized here, the methods learned in the previous chapters of this supplement all extend to the applications of the basic model presented in these chapters.

Part III: Markets

Part IIIA: Competitive Markets

Chapter 14: Equilibrium and Efficiency

Aggregate Surplus

Aggregate surplus is the summation of aggregate consumer and producer surpluses. When aggregate surplus is maximized in an equilibrium, we have achieved economic efficiency. Consumer surplus can be calculated either for an individual consumer, by integrating below his or her demand curve and above equilibrium price, or for the market as a whole, by performing the same calculation using the market demand curve. Specifically, we can calculate consumer surplus as $\int_{P*}^{\bar{P}} Q^d(P)dP$ or as $\int_0^{Q^*}(P(Q^d) - P^*)dQ$, where $Q^d(P)$ and $P(Q^d)$ are the market demand function and inverse market demand function respectively, and P^* and $\bar{P}$ are the equilibrium price and choke price. The equilibrium price is given by the intersection of the market demand and supply curves. The choke price is the price at which total market quantity demanded equals zero.

Analogously, producer surplus can be calculated either for an individual producer, by integrating above his or her individual supply curve and below equilibrium price, or for the market as a whole, by performing this calculation using the market supply curve. Market producer surplus is calculated as $\int_{\underline{P}}^{P^*} Q^s(P)dP$, or $\int_0^{Q^*}(P^* - P(Q^s))dQ$ where $Q^s(P)$ and $P(Q^s)$ are the market supply function and inverse market supply function respectively, and $\underline{P}$ is the price at which quantity supplied falls to zero.

WORKED-OUT PROBLEM

The Problem Suppose that the market demand curve is $Q^d = 30 - 5P$ and the market supply curve is $Q^s = -10 + 5P$. What is the aggregate surplus being generated in this market?

The Solution Aggregate surplus is the summation of aggregate consumer surplus and aggregate producer surplus. To solve for the consumer surplus, we first need to know what the equilibrium price is in this market. To solve for the equilibrium price, we can use the market clearing condition that $Q^d = Q^s$. For this problem, setting quantity demanded equal to quantity supplied means $30 - 5P = -10 + 5P$. Rearranging, we find that $10P = 40$, or $P^* = 4$ in equilibrium. The choke price is the price at which demand falls to zero. For our demand curve equation, we see that $Q^d = 0$ when $\bar{P} = 6$. Substituting the demand function, $\bar{P} = 6$, and $P^* = 4$ into our consumer surplus equation, we find that $\int_4^6 (30 - 5P)dP = [30P - 2.5P^2]_4^6 = 30(6) - 2.5(6^2) - (30(4) - 2.5(4^2)) = 180 - 90 - 120 + 40 = 10$. To solve for aggregate producer surplus, we need to know the price at which supply is zero. Here, $Q^s = 0$ when $\underline{P} = 2$. Producer surplus therefore is $\int_2^4 (-10 + 5P)dP = [-10P + 2.5P^2]_2^4 = -10(4) + 2.5(4^2) - (-10(2) + 2.5(2^2)) = -40 + 40 + 20 - 10 = 10$. Since aggregate surplus is the summation of consumer and producer surpluses, aggregate surplus is $10 + 10 = 20$. Recall that in cases such as this one where both the demand curve and the supply curve are linear, we can check our work by using the geometric formula that the area of a triangle is one half of the length of the base of the triangle times the height of the triangle. Although we will not check this here, you can solve this problem via geometric methods for extra practice and confirm that the answer is the same as the calculus solution above.

ADDITIONAL EXERCISES

1. Suppose that the market demand curve is $Q^d = 80 - 5P$ and the market supply curve is $Q^s = -10 + P$. What is the aggregate surplus being generated in this market?

2. Suppose that the market demand curve is $Q^d = 100 - P$ and the market supply curve is $Q^s = -90 + 9P$. What is the aggregate surplus being generated in this market?

3. Suppose that the market demand curve is $Q^d = 80 - 2P$ and the market supply curve is $Q^s = -30 + 3P$. What is the aggregate surplus being generated in this market?

4. Suppose that the market demand curve is $Q^d = 80 - 4P$ and the market supply curve is $Q^s = -30 + 6P$. What is the aggregate surplus being generated in this market?

5. Suppose that the market demand curve is $Q^d = 90 - 5P$ and the market supply curve is $Q^s = -30 + 15P$. What is the aggregate surplus being generated in this market?

6. Suppose that the market demand curve is $Q^d = 270 - 3P$ and the market supply curve is $Q^s = 6P$. What is the aggregate surplus being generated in this market?

7. Suppose that the market demand curve is $Q^d = 90 - 5P$ and the market supply curve is $Q^s = -90 + 15P$. What is the aggregate surplus being generated in this market?

8. Suppose that the market demand curve is $Q^d = 60 - 6P$ and the market supply curve is $Q^s = -20 + 4P$. What is the aggregate surplus being generated in this market?

9. Suppose that the market demand curve is $Q^d = 75 - 5P$ and the market supply curve is $Q^s = 10P$. What is the aggregate surplus being generated in this market?

10. Suppose that the market demand curve is $Q^d = 120 - 4P$ and the market supply curve is $Q^s = -40 + 4P$. What is the aggregate surplus being generated in this market?

11. Suppose that the market demand curve is $Q^d = 500 - 40P$ and the market supply curve is $Q^s = -50 + 10P$. What is the aggregate surplus being generated in this market?

12. Suppose that the market demand curve is $Q^d = 900 - 9P$ and the market supply curve is $Q^s = 36P$. What is the aggregate surplus being generated in this market?

Chapter 15: Market Interventions

Deadweight Loss

Deadweight loss as defined in Chapter 14 of your textbook as a reduction in aggregate surplus below its maximum possible value. Market interventions can affect the workings of the competitive market system and result in deadweight loss to the economy. Deadweight loss, however, can be calculated as the difference between aggregate surplus under competitive market assumptions and aggregate surplus after interventions such as taxes, subsidies, price floor or ceilings, quotas, tariffs, or other forms of intervention. Surplus areas can be calculated using integrals analogously to how we calculated consumer and producer surpluses in the previous section of this supplement. Whether or not there are justifications for intervention that outweigh these surplus losses are normative questions.

Tax Incidence

Taxes and subsidies are public policy instruments that affect the prices paid and received by consumers and producers. The incidence of a tax, or how price changes because of a tax specifically affect consumers and producers, is determined by the relative elasticities of demand and supply. The consumers' share of the tax incidence can be calculated as $\frac{E^s}{E^s - E^d}$. Similarly, the producers' share of tax incidence can be calculated as $-\frac{E^d}{E^s - E^d}$. Recall, that we can calculate price elasticities of demand and of supply using formulas based on partial derivatives. Specifically, $E^d = \frac{\partial Q^d}{\partial P}\frac{P}{Q}$ and $E^s = \frac{\partial Q^s}{\partial P}\frac{P}{Q}$. Subbing these elasticity definitions into the formulas above, we can find the tax shares for consumers and producers respectively. Note that the tax shares sum to exactly one.

WORKED-OUT PROBLEM

The Problem If the demand curve is given by $Q^d = 100 - 5P$ and the supply curve is $Q^s = -100 + 15P$, what percentage of the overall tax burden will fall on consumers?

The Solution The percentage of the tax burden that will fall on consumers is $\frac{E^s}{E^s - E^d}$. The price elasticity of demand at the equilibrium price and quantity is $\frac{\partial Q^d}{\partial P}\frac{P}{Q}$. The partial derivative of the demand function with respect to price is $\frac{\partial Q^d}{\partial P} = -5$. To solve for equilibrium price and quantity, set $Q^d = Q^s$, or $100 - 5P = -100 + 15P$. This reduces to $P = 10$. At this price, quantity is $Q = 100 - 5(10) = 50$. Therefore, the price elasticity of demand, E^d, equals $-5(\frac{10}{50}) = -1$. The price elasticity of supply is calculated as $\frac{\partial Q^s}{\partial P}\frac{P}{Q}$. For supply, $\frac{\partial Q^s}{\partial P} = 15$. The price elasticity of supply therefore is $15(\frac{10}{50}) = 3$. The tax burden on consumers is $\frac{E^s}{E^s - E^d} = \frac{3}{3-(-1)} = \frac{3}{4}$. Therefore, $\frac{3}{4}$ of the tax burden falls on consumers and the remaining $\frac{1}{4}$ of the tax burden falls on producers. This incidence is independent of whether the tax was initially imposed on consumers or on producers since consumers and producers will "pass-through" the tax onto each other based on the relative demand and supply elasticities.

ADDITIONAL EXERCISES

1. If the demand curve is given by $Q^d = 200 - 5P$ and the supply curve is $Q^s = -100 + 25P$, what percentage of the overall tax burden will fall on consumers?

2. If the demand curve is given by $Q^d = 200 - 8P$ and the supply curve is $Q^s = -25 + 7P$, what percentage of the overall tax burden will fall on consumers?

3. If the demand curve is given by $Q^d = 800 - P$ and the supply curve is $Q^s = -20 + P$, what percentage of the overall tax burden will fall on consumers?

4. If the demand curve is given by $Q^d = 40 - 5P$ and the supply curve is $Q^s = -60 + 15P$, what percentage of the overall tax burden will fall on consumers?

5. If the demand curve is given by $Q^d = 350 - P$ and the supply curve is $Q^s = -150 + 4P$, what percentage of the overall tax burden will fall on consumers?

6. If the demand curve is given by $Q^d = 60 - 6P$ and the supply curve is $Q^s = -30 + 9P$, what percentage of the overall tax burden will fall on consumers?

7. If the demand curve is given by $Q^d = 200 - 5P$ and the supply curve is $Q^s = -100 + 25P$, what percentage of the overall tax burden will fall on producers?

8. If the demand curve is given by $Q^d = 200 - 8P$ and the supply curve is $Q^s = -25 + 7P$, what percentage of the overall tax burden will fall on producers?

9. If the demand curve is given by $Q^d = 800 - P$ and the supply curve is $Q^s = -20 + P$, what percentage of the overall tax burden will fall on producers?

10. If the demand curve is given by $Q^d = 40 - 5P$ and the supply curve is $Q^s = -60 + 15P$, what percentage of the overall tax burden will fall on producers?

11. If the demand curve is given by $Q^d = 350 - P$ and the supply curve is $Q^s = -150 + 4P$, what percentage of the overall tax burden will fall on producers?

12. If the demand curve is given by $Q^d = 60 - 6P$ and the supply curve is $Q^s = -30 + 9P$, what percentage of the overall tax burden will fall on producers?

Chapter 16: General Equilibrium, Efficiency, and Equity

Competitive Equilibrium

In Chapter 4, we defined the marginal rate of substitution as the slope of the indifference curve. Mathematically, this is given by the equation $MRS_{XY} = \frac{MU_X}{MU_Y} = \frac{\partial U/\partial X}{\partial U/\partial Y} = \frac{\partial Y}{\partial X}$. We can use this definition to rewrite the exchange efficiency condition of Chapter 16 in terms of calculus. Specifically, we find that an efficient allocation in an exchange economy has the property that every pair of individuals shares the same marginal rate of substitution for every pair of goods. This is the exchange efficiency condition, and can be written as $MRS^A_{XY} = MRS^B_{XY}$ for all A and B, where A and B refer to individual consumers in the exchange economy. Furthermore, if these common marginal rates of substitution also equal the equilibrium price ratio then we have found a competitive equilibrium in addition to an efficient allocation. A competitive equilibrium in an exchange economy can be described by $MRS^A_{XY} = MRS^B_{XY} = \frac{P_X}{P_Y}$ for all A and B.

Similarly, an efficient allocation in an input market has the characteristic that every pair of producers shares the same marginal rate of technical substitution for every pair of inputs (for example between labor and capital). This can be expressed as $MRTS^X_{LK} = MRS^Y_{LK}$ for all X and Y, where X and Y denote two individual firms (and here also denote the product that they produce). This is the input efficiency condition. If these common marginal rates of technical substitution also equal the equilibrium price ratio for inputs, then we have found a competitive equilibrium in addition to an efficient allocation. In this case, $MRTS^X_{LK} = MRS^Y_{LK} = \frac{W}{R}$ for all X and Y.

Production Possibility Frontier

The production possibility frontier shows the combinations of aggregate outputs that firms can produce with their available inputs and technology when inputs are allocated efficiently among them. The opportunity cost of additional amounts of good X in terms of reductions in the amounts of good Y that can be produced, or additional Y from reductions in X, can be found mathematically by calculating the slope of the production possibility frontier at the relevant starting point. Since the production possibility frontier is a function of both X and Y, we can find the slope of this frontier by solving for the first derivative, $\frac{dY}{dX}$, or $\frac{\partial Y}{\partial X}$ in the case of more than two outputs.

Marginal Rate of Transformation

The marginal rate of transformation between goods X and Y is the additional amount of good Y that can be produced by giving up a marginal unit of good X. In other words, the marginal rate of transformation refers to the slope of the production possibilities frontier and gives the opportunity cost of Y in terms of foregone X. Output efficiency holds when the equilibrium marginal rate of substitution is equal to the marginal rate of transformation. This can be expressed as $MRS^A_{XY} = MRS^B_{XY} = MRT_{XY}$ for all A and B. Note that both marginal rates of substitution and marginal rates of transformation can be calculated using algebra as in your textbook or using calculus as in this supplement.

WORKED-OUT PROBLEM

The Problem There are 10 units of X and 10 units of Y available in a simple exchange economy. Consumer A's utility function can be expressed $U = XY$, and consumer B's utility function is $U = X + Y$. Consumer A's allocation is 2 units of X and 8 units of Y. Is this allocation economically efficient?

The Solution An allocation is economically efficient if the marginal rate of substitution between products is equalized for the consumers in the exchange economy. If there are 10 units of X and 10 units of Y and consumer A has 2 units of X and 8 units of Y, then consumer B must own the remaining 8 units of X and 2 units of Y. Consumer A's marginal rate of substitution between goods X and Y can be calculated as the ratio of the marginal utility that he or she derives from consumption of X relative to consumption of Y. Therefore,

$MRS^A_{XY} = \frac{MU_X}{MU_Y} = \frac{\partial U/\partial X}{\partial U/\partial Y} = \frac{Y}{X}$. At the current allocation, $MRS^A_{XY} = \frac{8}{2} = 4$. Consumer B has a perfect substitutes utility function with a constant marginal rate of substitution that equals exactly one at all allocations. Namely, $MRS^B_{XY} = 1$ since $\frac{\partial U}{\partial X} = 1$ and $\frac{\partial U}{\partial Y} = 1$. Since $4 \neq 1$, the current allocation is not economically efficient, and therefore there is room for efficiency enhancing trade.

WORKED-OUT PROBLEM

The Problem If the production possibility frontier for goods X and Y is written $X^2 + Y^2 = 100$, what is the opportunity cost of producing one more unit of X?

The Solution Note that the opportunity cost of producing one more unit of good X is also the marginal rate of transformation between goods X and Y or the slope of the production possibility frontier, $\frac{dY}{dX}$. To solve for the opportunity cost of producing one more unit of good X in terms of the amount of good Y which must be given up, we can first rewrite the production possibility frontier to isolate Y. Subtracting X^2 from both sides of the production possibility frontier equation and taking the square root yields $Y = \sqrt{100 - X^2}$. The opportunity cost of more X in terms of good Y is the first derivative, or slope, of this equation. Using the chain rule for derivatives and then rearranging, we see that $\frac{dY}{dX} = \frac{d(\sqrt{100-X^2})}{dX} = \frac{1}{2}(100 - X^2)^{1/2-1}(-2X) = -\frac{X}{\sqrt{100-X^2}} = -\frac{X}{Y}$. We can substitute various specific combinations of good X and Y that lie along the production possibility frontier into this solution in terms of X and Y in order to numerically calculate the opportunity cost at various points.

ADDITIONAL EXERCISES

1. There are 20 units of X and 20 units of Y available in a simple exchange economy. Consumer A's utility function can be expressed $U = XY$, and consumer B's utility function is $U = 2X + Y$. Consumer A's allocation is 5 units of X and 10 units of Y. Is this allocation economically efficient?

2. There are 10 units of X and 10 units of Y available in a simple exchange economy. Consumer A's utility function can be expressed $U = \sqrt{XY}$, and consumer B's utility function is $U = XY$. Consumer A's allocation is 5 units of X and 5 units of Y. Is this allocation economically efficient?

3. There are 20 units of X and 30 units of Y available in a simple exchange economy. Consumer A's utility function can be expressed $U = \sqrt{XY}$, and consumer B's utility function is $U = X + Y$. Consumer A's allocation is 8 units of X and 15 units of Y. Is this allocation economically efficient?

4. There are 100 units of X and 10 units of Y available in a simple exchange economy. Consumer A's utility function can be expressed $U = XY$, and consumer B's utility function is $U = 3X + 6Y$. Consumer A's allocation is 20 units of X and all 10 units of Y. Is this allocation economically efficient?

5. There are 50 units of X and 40 units of Y available in a simple exchange economy. Consumer A's utility function can be expressed $U = \sqrt{5XY}$, and consumer B's utility function is $U = 5X + Y$. Consumer A's allocation is 5 units of X and 10 units of Y. Is this allocation economically efficient?

6. There are 80 units of X and 40 units of Y available in a simple exchange economy. Consumer A's utility function can be expressed $U = 6\sqrt{XY}$, and consumer B's utility function is $U = 3XY$. Consumer A's allocation is 50 units of X and all 10 units of Y. Is this allocation economically efficient?

7. If the production possibility frontier for goods X and Y is written $X^2 + Y^2 = 200$, what is the opportunity cost of producing one more unit of X?

8. If the production possibility frontier for goods X and Y is written $2X^2 + 3Y^2 = 80$, what is the opportunity cost of producing one more unit of X?

9. If the production possibility frontier for goods X and Y is written $X^2 + 5Y^2 = 150$, what is the opportunity cost of producing one more unit of X?

10. If the production possibility frontier for goods X and Y is written $6X^2 + 3Y^2 = 300$, what is the opportunity cost of producing one more unit of X?

11. If the production possibility frontier for goods X and Y is written $2X^2 + 4Y^2 = 100$, what is the opportunity cost of producing one more unit of X?

12. If the production possibility frontier for goods X and Y is written $9X^2 + Y^2 = 900$, what is the opportunity cost of producing one more unit of X?

Part IIIB: Market Failures

Chapter 17: Monopoly

Monopoly

Recall that profit-maximizing firms set marginal revenue equal to marginal cost, $MR = MC$, in order to solve for the optimal amount of output that they should produce. Since firms are price takers in the perfect competition framework, in that case, we can use the identity that price equals marginal revenue to simplify the problem. Substituting in this relationship, we can set price equal to marginal cost to solve for the optimal quantity. Monopolists, however, are not price takers and therefore the simplification that $P = MR = MC$ cannot be made. Recall, however, that we can easily solve for marginal revenue by taking the first derivative of the revenue function as we did in Chapter 9 of this supplement. Since marginal cost is simply the first derivative of the cost function, we can get optimal quantity by taking the derivatives of both the revenue and cost functions, setting these derivatives equal, and then rearranging until we isolate Q. To get the monopolist's price, we can simply plug the optimal quantity into the demand function and solve for P.

Markup

A firm's markup, or Lerner's index, equals the amount by which its price exceeds its marginal cost expressed as a percentage of price. The Lerner's index can be written as $\frac{P-MC}{P} = -\frac{1}{E^d}$. Recall from Chapter 2 of this supplement that we can write E^d in terms of calculus as $E^d = \frac{\partial Q^d}{\partial P}\frac{P}{Q}$ and substitute into this formula.

Monopsony

Monopsony can be analyzed similarly to monopoly. In the case of monopsony, we have a single buyer instead of a single seller. The monopsonist wants to buy until the point that his or her marginal benefit equals marginal cost. Here, we will use "marginal expenditure" to distinguish from the marginal cost of the producer problem. Specifically, the objective of a monopsonist may be expressed by $MB = ME$, where MB is marginal benefit as previously defined and ME is marginal expenditure. Marginal expenditure is simply the first derivative of the expenditure function, and marginal benefit is defined as before. We can solve the equation $MB = ME$ for the optimal quantity and then substitute this quantity into the supply function to find the monopsony price. Therefore, the method to solve monopsony problems parallels that for monopoly. However, the monopsony price is read off of the supply curve while the monopoly price is read off of the demand curve. This is because, in this case, it is the consumer that has market power.

WORKED-OUT PROBLEM

The Problem A monopolist faces the demand curve $Q^d = 100 - P$ and a marginal cost of 10 for the units of product that she produces. What is the monopolist's profit-maximizing price and quantity?

The Solution The monopolist wants to set marginal revenue equal to marginal cost in order to determine the profit-maximizing quantity. Revenue is defined as price times quantity. Since we want to solve for the optimal quantity, we should write the revenue function in terms of Q. To do this, we can substitute the inverse demand function relationship, $P = 100 - Q$, in for price. Here, $R = PQ = (100 - Q)Q = 100Q - Q^2$. Marginal revenue is the first derivative of this function with respect to quantity. Namely, $MR = \frac{\partial R}{\partial Q} = 100 - 2Q$. To maximize profit, the monopolist should set $MR = MC$. Here, $100 - 2Q = 10$, or $Q = 45$. Subbing $Q = 45$ into the demand curve and solving for price, we find that $P = 100 - 45 = 55$. The profit-maximizing quantity therefore is 45 units, and the profit-maximizing price is 55.

WORKED-OUT PROBLEM

The Problem A monopolist faces the demand curve $Q^d = 48 - 6P$ and the cost curve $C = 100 + 2Q$. What is the monopolist's profit-maximizing price and quantity?

The Solution Again, the monopolist wants to set marginal revenue equal to marginal cost in order to determine the profit-maximizing quantity. Since revenue is defined as price times quantity, $R = (8 - \frac{1}{6}Q)Q = 8Q - \frac{1}{6}Q^2$. Since marginal revenue is the first derivative of this function with respect to quantity, $MR = \frac{\partial R}{\partial Q} = 8 - \frac{1}{3}Q$. Marginal cost is the first derivative of the cost function with respect to quantity. Here, $MC = \frac{\partial C}{\partial Q} = 2$. To maximize profit, the monopolist should set $MR = MC$, or $8 - \frac{1}{3}Q = 2$, or $Q = 18$. Subbing $Q = 18$ into the demand curve and solving for price, we find that $P = 8 - \frac{1}{6}(18) = 5$. The profit-maximizing price is 5, and the profit-maximizing quantity is 18 units.

ADDITIONAL EXERCISES

1. A monopolist faces the demand curve $Q^d = 80 - P$ and a marginal cost of 10 for the units of product that she produces. What is the monopolist's profit-maximizing price and quantity?

2. A monopolist faces the demand curve $Q^d = 120 - 4P$ and a marginal cost of 20 for the units of product that she produces. What is the monopolist's profit-maximizing price and quantity?

3. A monopolist faces the demand curve $Q^d = 90 - 9P$ and a marginal cost of 8 for the units of product that she produces. What is the monopolist's profit-maximizing price and quantity?

4. A monopolist faces the demand curve $Q^d = 11 - P$ and a marginal cost of 1 for the units of product that she produces. What is the monopolist's profit-maximizing price and quantity?

5. A monopolist faces the demand curve $Q^d = 18 - P$ and a marginal cost of 2 for the units of product that she produces. What is the monopolist's profit-maximizing price and quantity?

6. A monopolist faces the demand curve $Q^d = 80 - 4P$ and a marginal cost of 6 for the units of product that she produces. What is the monopolist's profit-maximizing price and quantity?

7. A monopolist faces the demand curve $Q^d = 180 - 4P$ and the cost curve $C = 11Q^2$. What is the monopolist's profit-maximizing price and quantity?

8. A monopolist faces the demand curve $Q^d = 24 - P$ and the cost curve $C = 6 + 5Q^2$. What is the monopolist's profit-maximizing price and quantity?

9. A monopolist faces the demand curve $Q^d = 120 - 6P$ and the cost curve $C = 10 + Q^2$. What is the monopolist's profit-maximizing price and quantity?

10. A monopolist faces the demand curve $Q^d = 100 - 2P$ and the cost curve $C = 4 + 2Q^2$. What is the monopolist's profit-maximizing price and quantity?

11. A monopolist faces the demand curve $Q^d = 100 - 2P$ and the cost curve $C = 5 + \frac{1}{2}Q^2$. What is the monopolist's profit-maximizing price and quantity?

12. A monopolist faces the demand curve $Q^d = 550 - 50P$ and the cost curve $C = 10 + \frac{1}{5}Q^2$. What is the monopolist's profit-maximizing price and quantity?

Chapter 18: Pricing Policies

Price Discrimination

Price discrimination is the practice of charging different prices to different consumers for the same product. In the case of perfect price discrimination, a monopolist charges a different price to each consumer based on his or her willingness to pay for the product. In the case of quantity-dependent pricing, a producer charges prices based on how many units the consumer buys. Bulk discounts are an example of this type of price discrimination. In the case of price discrimination based on observable customer characteristics, the producer charges differential prices to consumers based on demand elasticities. Student or senior discounts are examples of this.

Price Discrimination based on Observable Characteristics

When a monopolist can observe two or more identifiable groups, it can engage in price discrimination by setting multiple prices. This is the most common form of price discrimination. To find the profit-maximizing prices to offer, the monopolist should set $MR_g = MC$ for each group g separately. Since marginal revenue is calculated based on the demand curve, it differs for groups with different demand elasticities.

Markup over marginal cost as a percentage of price for each group can be written as $\frac{P_g - MC}{P_g} = -\frac{1}{E_g^d}$. Note that the formula is that of the Lerner's index for monopoly where now we are defining markup separately for each observable group. The price elasticity of demand for each group, E_g^d, can be written in terms of calculus as $\frac{\partial Q_g^d}{\partial P_g}\frac{P_g}{Q_g}$. This can be subbed into the formula above.

WORKED-OUT PROBLEM

The Problem A firm faces a marginal cost of 5 for the product that it produces and can observe two groups of consumers. Group A's demand function is $Q_A^d = 75 - P$. Group

B's demand function is $Q_B^d = 55 - 5P$. What are the profit-maximizing prices offered to the two groups?

The Solution Being a profit maximizer, the firm should set marginal revenue equal to marginal cost separately for each group. For Group A, the revenue function is $R_A = PQ_A = (75 - Q_A)Q_A = 75Q_A - Q_A^2$. Marginal revenue therefore is $\frac{\partial R_A}{\partial Q_A} = 75 - 2Q_A$. Since marginal cost is 5, we know that $75 - 2Q_A = 5$, or $Q_A = 35$. Substituting quantity into group A's demand function, we find that the optimal price to charge group A members is $P_A = 75 - 35 = 40$. To find the optimal price to charge group B, the firm should set marginal revenue from group B equal to marginal cost. The revenue function for group B is $R_B = PQ_B = (11 - \frac{1}{5}Q_B)Q_B = 11Q_B - \frac{1}{5}Q_B^2$. Marginal revenue is the derivative of this function, $11 - \frac{2}{5}Q_B$. The condition that marginal revenue equals marginal cost can be written as $11 - \frac{2}{5}Q_B = 5$, or $Q_B = 15$. The profit-maximizing price for group B is $P_B = 11 - \frac{1}{5}(15) = 8$.

ADDITIONAL EXERCISES

1. A firm faces a marginal cost of 2 for the product that it produces and can observe two groups of consumers. Group A's demand function is $Q_A^d = 100 - 2P$. Group B's demand function is $Q_B^d = 100 - 5P$. What are the profit-maximizing prices offered to the two groups?

2. A firm faces a marginal cost of 1 for the product that it produces and can observe two groups of consumers. Group A's demand function is $Q_A^d = 20 - P$. Group B's demand function is $Q_B^d = 75 - 3P$. What are the profit-maximizing prices offered to the two groups?

3. A firm faces a marginal cost of 10 for the product that it produces and can observe two groups of consumers. Group A's demand function is $Q_A^d = 180 - 3P$. Group B's demand function is $Q_B^d = 70 - 7P$. What are the profit-maximizing prices offered to the two groups? Will both types of consumers stay in the market?

4. A firm faces a marginal cost of 3 for the product that it produces and can observe two groups of consumers. Group A's demand function is $Q_A^d = 40 - 4P$. Group B's demand function is $Q_B^d = 90 - 6P$. What are the profit-maximizing prices offered to the two groups?

5. A firm faces a marginal cost of 6 for the product that it produces and can observe two groups of consumers. Group A's demand function is $Q_A^d = 80 - 2P$. Group B's demand function is $Q_B^d = 200 - 10P$. What are the profit-maximizing prices offered to the two groups?

6. A firm faces a marginal cost of 5 for the product that it produces and can observe two groups of consumers. Group A's demand function is $Q_A^d = 100 - 2P$. Group B's demand function is $Q_B^d = 50 - 5P$. What are the profit-maximizing prices offered to the two groups?

7. A firm's cost function is $C = 10Q$. The firm can observe two groups of consumers. Group A's demand function is $Q_A^d = 100 - P$. Group B's demand function is $Q_B^d = 200 - 5P$. What are the profit-maximizing prices offered to the two groups?

8. A firm's cost function is $C = 10Q$. The firm can observe two groups of consumers. Group A's demand function is $Q_A^d = 20 - P$. Group B's demand function is $Q_B^d = 75 - 3P$. What are the profit-maximizing prices offered to the two groups?

9. A firm's cost function is $C = 15 + 5Q$. The firm can observe two groups of consumers. Group A's demand function is $Q_A^d = 50 - P$. Group B's demand function is $Q_B^d = 160 - 8P$. What are the profit-maximizing prices offered to the two groups?

10. A firm's cost function is $C = 25 + Q$. The firm can observe two groups of consumers. Group A's demand function is $Q_A^d = 60 - 4P$. Group B's demand function is $Q_B^d = 90 - 5P$. What are the profit-maximizing prices offered to the two groups?

11. A firm's cost function is $C = \frac{1}{2}Q^2$. The firm can observe two groups of consumers. Group A's demand function is $Q_A^d = 300 - 4P$. Group B's demand function is $Q_B^d = 600 - 4P$. What are the profit-maximizing prices offered to the two groups?

12. A firm's cost function is $C = 30Q + Q^2$. The firm can observe two groups of consumers. Group A's demand function is $Q_A^d = 600 - 2P$. Group B's demand function is $Q_B^d = 50 - P$. What are the profit-maximizing prices offered to the two groups?

Chapter 19: Oligopoly

Chapter 19 introduces models of oligopoly in which firms make their profit-maximizing choices while taking into account the choices of their rivals. Resultant equilibria given this type of strategic behavior are known as Nash equilibria.

Cournot Model

In the Cournot model, firms choose their quantities of an identical good simultaneously and the price clears the market given the total quantity produced. To solve the Cournot model, we need to find the best response functions for the two firms. The best response function of a firm represents the best choice in response to each possible action by the other firm. The intersection of the best response functions of the two firms is the Cournot equilibrium.

To solve for the Cournot equilibrium, we first derive the marginal revenue function for each firm individually. Notice that marginal revenue is the first derivative of the revenue function. Revenue, as defined previously, is price times quantity. Therefore, in the case of two firms, the revenue function of firm A is $R_A = PQ_A$, and the revenue function of firm B is $R_B = PQ_B$. Notice that both firms A and B face the same price since the firms produce identical products. Price therefore is a function of both Q_A and Q_B since the inverse demand function is a function of total quantity Q, which is the summation of Q_A and Q_B. Using these revenue functions, we can calculate marginal revenue for each individually by taking the first partial derivative with respect to the relevant quantity. Specifically, $MR_A = \frac{\partial R_A}{\partial Q_A}$ and $MR_B = \frac{\partial R_B}{\partial Q_B}$.

A second step toward solving the Cournot model is to derive the profit-maximizing level of output for each firm for each possible output level of the other firm. This is the firm's best response function. Since both firms are profit-maximizers, they each individually set marginal revenue equal to marginal cost. Here, $MR_A = MC_A$ and $MR_B = MC_B$. We can then rewrite these solutions as $Q_A = f(Q_B)$ and $Q_B = g(Q_A)$ where f and g are functions. These are the best response functions for firms A and B. The intersection of these two best response functions is the Nash equilibrium. The intersection can be found by plugging one best response function into the other and solving for Q_A and Q_B.

Note that although the two firms face the same aggregate demand curve, they may optimally produce different quantities if they face different marginal costs of production. Recall that marginal cost is simply the first derivative of the cost function with respect to quantity. If the cost functions for the two firms differ, their marginal costs likely differ.

WORKED-OUT PROBLEM

The Problem Firms A and B are duopolists facing a demand curve $Q^d = 200 - 10P$. The marginal cost of production is 1 for firm A and 2 for firm B. Solve for the best response functions of firms A and B respectively.

The Solution To solve this problem, we should first find the inverse demand curve. The inverse demand curve for this problem is $P = 20 - \frac{1}{10}Q$. The aggregate quantity Q is the summation of the demands of the two firms Q_A and Q_B. Namely, $Q = Q_A + Q_B$. The inverse demand curve therefore can be rewritten as $P = 20 - \frac{1}{10}Q_A - \frac{1}{10}Q_B$. Firm A's revenue function is $R_A = PQ_A = (20 - \frac{1}{10}Q_A - \frac{1}{10}Q_B)Q_A = 20Q_A - \frac{1}{10}Q_A^2 - \frac{1}{10}Q_AQ_B$. Marginal revenue for firm A is the first partial derivative of firm A's revenue function. Here, $MR_A = \frac{\partial R_A}{\partial Q_A} = 20 - \frac{1}{5}Q_A - \frac{1}{10}Q_B$. Notice that MR_A is a function of both Q_A and Q_B. Firm A maximizes profit by setting $MR_A = MC_A$, or $20 - \frac{1}{5}Q_A - \frac{1}{10}Q_B = 1$. Rearranging, firm A's profit-maximizing quantity as a function of firm B's quantity is $Q_A = 95 - \frac{1}{2}Q_B$. This is firm A's best response function. Firm B's best response function is calculated analogously. Firm B's revenue function is $R_B = PQ_B = (20 - \frac{1}{10}Q_A - \frac{1}{10}Q_B)Q_B = 20Q_B - \frac{1}{10}Q_AQ_B - \frac{1}{10}Q_B^2$. Firm B's marginal revenue is $MR_B = \frac{\partial R_B}{\partial Q_B} = 20 - \frac{1}{10}Q_A - \frac{1}{5}Q_B$. Setting $MR_B = MC_B$, we see that $20 - \frac{1}{10}Q_A - \frac{1}{5}Q_B = 2$. Rearranging and solving for Q_B as a function of Q_A, firm B's best response function is $Q_B = 90 - \frac{1}{2}Q_A$.

WORKED-OUT PROBLEM

The Problem Firms A and B are duopolists facing a demand curve $Q^d = 200 - 10P$. The marginal cost of production is 1 for firm A and 2 for firm B. Solve for the Cournot equilibrium quantities.

The Solution The Cournot equilibrium is defined by the intersection between the best response functions of firms A and B. We solved for the best response functions for this specific example above in the previous worked-out problem. These best response functions are two equations in two unknowns. We can rewrite firm A's optimal quantity therefore as $Q_A = 95 - \frac{1}{2}(90 - \frac{1}{2}Q_A)$. Rearranging and solving, we find that $Q_A = \frac{200}{3}$, and therefore $Q_B = 90 - \frac{1}{2}(\frac{200}{3}) = \frac{170}{3}$.

ADDITIONAL EXERCISES

1. Firm A and B are duopolists facing a demand curve $Q^d = 100 - 5P$. The marginal cost of production is 1 for firm A and 2 for firm B. Solve for the best response functions of firms A and B respectively.

2. Firm A and B are duopolists facing a demand curve $Q^d = 250 - 5P$. The marginal cost of production is 5 for firm A and 5 for firm B. Solve for the best response functions of firms A and B respectively.

3. Firm A and B are duopolists facing a demand curve $Q^d = 40 - P$. The marginal cost of production is 1 for firm A and 2 for firm B. Solve for the best response functions of firms A and B respectively.

4. Firm A and B are duopolists facing a demand curve $Q^d = 250 - P$. The marginal cost of production is 20 for firm A and 30 for firm B. Solve for the best response functions of firms A and B respectively.

5. Firm A and B are duopolists facing a demand curve $Q^d = 10 - 10P$. The marginal cost of production is $\frac{3}{4}$ for firm A and $\frac{1}{2}$ for firm B. Solve for the best response functions of firms A and B respectively.

6. Firm A and B are duopolists facing a demand curve $Q^d = 810 - 9P$. The marginal cost of production is 9 for firm A and 9 for firm B. Solve for the best response functions of firms A and B respectively.

7. Firm A and B are duopolists facing a demand curve $Q^d = 100 - 5P$. The marginal cost of production is 1 for firm A and 2 for firm B. Solve for the Cournot equilibrium quantities.

8. Firm A and B are duopolists facing a demand curve $Q^d = 250 - 6P$. The marginal cost of production is 5 for firm A and 5 for firm B. Solve for the Cournot equilibrium quantities.

9. Firm A and B are duopolists facing a demand curve $Q^d = 40 - P$. The marginal cost of production is 1 for firm A and 2 for firm B. Solve for the Cournot equilibrium quantities.

10. Firm A and B are duopolists facing a demand curve $Q^d = 250 - P$. The marginal cost of production is 20 for firm A and 30 for firm B. Solve for the Cournot equilibrium quantities.

11. Firm A and B are duopolists facing a demand curve $Q^d = 10 - 10P$. The marginal cost of production is $\frac{3}{4}$ for firm A and $\frac{1}{2}$ for firm B. Solve for the Cournot equilibrium quantities.

12. Firm A and B are duopolists facing a demand curve $Q^d = 800 - 9P$. The marginal cost of production is 9 for firm A and 9 for firm B. Solve for the Cournot equilibrium quantities.

Chapter 20: Externalities and Public Goods

Externalities

An externality is an effect, either positive or negative, on an agent that was generated by another decision-maker and which is not reflected in equilibrium prices. A negative externality is a negative effect or harm on someone that has occurred outside of a market transaction, and a positive externality is a positive effect or benefit generated by another agent and not reflected in a market transaction. External cost, EC, is the economic harm that a negative externality imposes on others. External benefit, EB, is the economic gain that a positive externality provides to others. Externalities are reflected in the external cost function or external benefit function depending on whether they are negative or positive. The presence of externalities can result in inefficient levels of production and consumption if they are not taking into account by decision-makers.

Marginal External Cost

Marginal external cost of production is derived from the external cost function by taking the first partial derivative. Specifically, $MEC = \frac{\partial EC}{\partial Q}$. Marginal social cost is then the summation of marginal cost (as defined in Chapter 3) and marginal external cost. Alternately, marginal social cost is the first derivative of the total social cost function which is the summation of the firm's private cost function and the external cost function.

Marginal External Benefit

Marginal external benefit is derived from the external benefit function by taking the first partial derivative. Specifically, $MEB = \frac{\partial EB}{\partial Q}$. Marginal social benefit is then the summation of marginal benefit (as defined in Chapter 3) and marginal external benefit. Alternately, marginal social benefit is the first derivative of the total social benefit function which is the summation of total private benefit and external benefit.

WORKED-OUT PROBLEM

The Problem Suppose that a firm's total cost function is given by $C = 100Q + Q^2$ and its external cost is given by $EC = 10Q + Q^2$. What is the marginal external cost of production? What is the marginal social cost of production?

The Solution The marginal external cost of production is calculated as the first derivative of the external cost function with respect to quantity. Here, $MEC = \frac{\partial EC}{\partial Q} = 10{+}2Q$. The marginal social cost is marginal cost plus marginal external cost. We already have calculated marginal external cost. The firm's private marginal cost is the first derivative of the total cost function, the private cost function that the firm is facing. Here, $MC = \frac{\partial C}{\partial Q} = 100 + 2Q$. Marginal social cost therefore is $MSC = MC + MEC = 100 + 2Q + 10 + 2Q = 110 + 4Q$. Note that the total social cost function here is $110Q + 2Q^2$, and marginal social cost is the derivative of this function, or $110 + 4Q$.

WORKED-OUT PROBLEM

The Problem Suppose that an agent's total benefit function is given by $B = 100Q - Q^2$ and its external benefit is given by $EB = 10Q - Q^2$. What is the marginal external benefit of production? What is the marginal social benefit of production?

The Solution The marginal external benefit of production is calculated as the first derivative of the external benefit function with respect to quantity. Here, $MEB = \frac{\partial EB}{\partial Q} = 10 - 2Q$. The marginal social benefit is marginal benefit plus marginal external benefit. Marginal benefit is the first derivative of the total benefit function. Here, $MB = \frac{\partial B}{\partial Q} = 100 - 2Q$. Marginal social benefit therefore is $MSB = MB + MEB = 100 - 2Q + 10 - 2Q = 110 - 4Q$. Note that the total social benefit function here is $110Q - 2Q^2$, and marginal social benefit is the derivative, or $110 - 4Q$.

ADDITIONAL EXERCISES

1. Suppose that a firm's total cost function is given by $C = 100Q + 3Q^2$ and its external cost is given by $EC = Q + Q^2$. What is the marginal external cost of production? What is the marginal social cost of production?

2. Suppose that a firm's total cost function is given by $C = 8Q^2$ and its external cost is given by $EC = 10Q$. What is the marginal external cost of production? What is the marginal social cost of production?

3. Suppose that a firm's total cost function is given by $C = 20 + 5Q + Q^2$ and its external cost is given by $EC = 4 + Q + Q^2$. What is the marginal external cost of production? What is the marginal social cost of production?

4. Suppose that a firm's total cost function is given by $C = 200 + 30Q + Q^2$ and its external cost is given by $EC = 20Q^2$. What is the marginal external cost of production? What is the marginal social cost of production?

5. Suppose that a firm's total cost function is given by $C = Q + Q^2 + Q^3$ and its external cost is given by $EC = 1 + Q$. What is the marginal external cost of production? What is the marginal social cost of production?

6. Suppose that a firm's total cost function is given by $C = 10 + 7Q + 2Q^2 + 3Q^3$ and its external cost is given by $EC = 8 + 4Q + 6Q^2$. What is the marginal external cost of production? What is the marginal social cost of production?

7. Suppose that an agent's total benefit function is given by $B = 100Q - 4Q^2$ and its external benefit is given by $EB = Q - Q^2$. What is the marginal external benefit of production? What is the marginal social benefit of production?

8. Suppose that an agent's total benefit function is given by $B = 200Q - Q^2$ and its external benefit is given by $EB = 20Q$. What is the marginal external benefit of production? What is the marginal social benefit of production?

9. Suppose that an agent's total benefit function is given by $B = 300Q - 3Q^2$ and its external benefit is given by $EB = Q - 5Q^2$. What is the marginal external benefit of production? What is the marginal social benefit of production?

10. Suppose that an agent's total benefit function is given by $B = 80Q - Q^2$ and its external benefit is given by $EB = 40Q - 6Q^2$. What is the marginal external benefit of production? What is the marginal social benefit of production?

11. Suppose that an agent's total benefit function is given by $B = 900Q - 9Q^2$ and its external benefit is given by $EB = 80Q - 8Q^2$. What is the marginal external benefit of production? What is the marginal social benefit of production?

12. Suppose that an agent's total benefit function is given by $B = 100Q - 10Q^2$ and its external benefit is given by $EB = 10Q - 10Q^2$. What is the marginal external benefit of production? What is the marginal social benefit of production?

Chapter 21: Asymmetric Information

Chapter 21 provides a final extension to the models presented in your textbook. As with the other application chapters, the methods learned throughout the supplement extend to this final chapter though are not formalized here.

Solutions to Additional Exercises

Chapter 1

1. $\frac{\partial f}{\partial X} = 2; \frac{\partial f}{\partial Y} = 3$

2. $\frac{\partial f}{\partial X} = 4X; \frac{\partial f}{\partial Y} = 9Y^2$

3. $\frac{\partial f}{\partial X} = 12XY^3; \frac{\partial f}{\partial Y} = 18X^2Y^2$

4. $\frac{\partial f}{\partial X} = X^{-8/9}Y^{1/3}; \frac{\partial f}{\partial Y} = 3X^{1/9}Y^{-2/3}$

5. $\frac{\partial f}{\partial X} = \frac{1}{3}X^{-2/3}Y^{1/2}; \frac{\partial f}{\partial Y} = \frac{1}{2}X^{1/3}Y^{-1/2}$

6. $\frac{\partial f}{\partial X} = 3X^2Y^2; \frac{\partial f}{\partial Y} = 2X^3Y + 3$

7. $\frac{\partial f}{\partial X} = 7X^6Y^9 + 10; \frac{\partial f}{\partial Y} = 9X^7Y^8$

8. $\frac{\partial f}{\partial X} = X^{-8/9}Y^{1/3} + 2X^{-2/3}Y^{1/2}; \frac{\partial f}{\partial Y} = 3X^{1/9}Y^{-2/3} + 3X^{1/3}Y^{-1/2}$

9. $\frac{\partial f}{\partial X} = 4X^3Y^2 + 16XY + 4Y^2; \frac{\partial f}{\partial Y} = 2X^4Y + 8X^2 + 8XY$

10. $\frac{\partial f}{\partial X} = 3X^2Y^3 + 24XY^2 + 6Y; \frac{\partial f}{\partial Y} = 3X^3Y^2 + 24X^2Y + 6X$

11. $\frac{\partial f}{\partial X} = 5X^4Y^6 + 4X^3Y^3 + \frac{1}{2}X^{-1/2}Y^{1/4}; \frac{\partial f}{\partial Y} = 6X^5Y^5 + 3X^4Y^2 + \frac{1}{4}X^{1/2}Y^{-3/4} + 10$

12. $\frac{\partial f}{\partial X} = 28X^3Y^3 + 28XY + 10; \frac{\partial f}{\partial Y} = 21X^4Y^2 + 14X^2 + 12$

Chapter 2

1. $-\frac{3}{5}$

2. -1

3. ∞

4. -1; the elasticities are the same since $P = 1$ and $Q = 5$ refer to the same point on the demand curve; yes, total expenditure is largest at this point since $E^d = -1$

5. $\frac{3}{5}$

6. $\frac{1}{2}$

7. $\frac{7}{10}$

8. 0

9. $-\frac{9}{11}$

10. $\frac{10}{11}$

11. $-\frac{3}{2}$

12. $\frac{5}{6}$; substitutes

Chapter 3

1. $5 + 4X$

2. $4 + 6X$

3. $20 + 20X$

4. $10 - 2X$

5. $200 - 6X$

6. $50 - 12X$

7. $X = 49$

8. $X = 8$

9. $MB_X = Y; MB_Y = X$

10. $MB_X = 2XY^5; MB_Y = 5X^2Y^4$

11. $MC_X = \frac{1}{2}X^{-1/2}Y^{1/2}; MC_Y = \frac{1}{2}X^{1/2}Y^{-1/2}$

12. $MC_X = 6X^5Y^5; MC_Y = 5X^6Y^4$

Chapter 4

1. $MU_X = Y; MU_Y = X$

2. $MU_X = Y^2; MU_Y = 2XY$

3. $MU_X = 3X^2Y^2; MU_Y = 2X^3Y$

4. $MU_X = 5X^4Y^5; MU_Y = 5X^5Y^4$

5. $MU_X = 1; MU_Y = 1$

6. $MU_X = 4; MU_Y = 2$

7. $MRS_{XY} = \frac{Y}{X}$

8. $MRS_{XY} = \frac{Y}{2X}$

9. $MRS_{XY} = \frac{3Y}{2X}$

10. $MRS_{XY} = \frac{Y}{X}$

11. $MRS_{XY} = 1$

12. $MRS_{XY} = 2$

Chapter 5

1. $X = 10; Y = 25$

2. $X = 50; Y = 50$

3. $X = \frac{75}{2}; Y = \frac{25}{6}$

4. $X = 100; Y = 50$

5. $X = 20; Y = \frac{20}{3}$

6. $X = 20; Y = 10$

7. $X = 15; Y = 30$

8. $X = 75; Y = 75$

9. $X = 4; Y = 16$

10. $X = 60; Y = 20$

11. $X = 300; Y = 50$

12. $X = 11; Y = 44$

Chapter 6

1. $E_P^{Uncomp.} = -4; E_Y = 3; E_P^{Comp.} = -2$

2. $E_P^{Uncomp.} = -\frac{3}{2}; E_Y = \frac{1}{2}; E_P^{Comp.} = -1$

3. $E_P^{Uncomp.} = -4; E_Y = 1; E_P^{Comp.} = -\frac{7}{2}$

4. $E_P^{Uncomp.} = -\frac{8}{7}; E_Y = \frac{5}{7}; E_P^{Comp.} = -\frac{9}{14}$

5. $E_P^{Uncomp.} = -10; E_Y = 1; E_P^{Comp.} = -9$

6. $E_P^{Uncomp.} = -\frac{2}{35}; E_Y = \frac{5}{7}; E_P^{Comp.} = \frac{31}{70}$

7. 9

8. 40

9. 250

10. 450

11. 405

12. $7,350$

Chapter 7

1. $MP_L = \sqrt{\frac{K}{L}}; MP_K = \sqrt{\frac{L}{K}}$

2. $MP_L = \sqrt{\frac{K}{2L}}; MP_K = \sqrt{\frac{L}{2K}}$

3. $MP_L = 1; MP_K = 1$

4. $MP_L = 4; MP_K = 2$

5. $MP_L = \frac{1}{5}L^{-4/5}K^{4/5}; MP_K = \frac{4}{5}L^{1/5}K^{-1/5}$

6. $MP_L = L^{-8/9}K^{8/9}; MP_K = 8L^{1/9}K^{-1/9}$

7. $MRTS_{LK} = \frac{K}{L}$

8. $MRTS_{LK} = \frac{K}{L}$

9. $MRTS_{LK} = 1$

10. $MRTS_{LK} = 2$

11. $MRTS_{LK} = \frac{K}{4L}$

12. $MRTS_{LK} = \frac{K}{8L}$

Chapter 8

1. $L = 6; K = 10$

2. $L = 25; K = 25$

3. $L = 4; K = 2$

4. $L = 15; K = 10$

5. $L = 5; K = 10$

6. $L = 20; K = 4$

7. $L = 80; K = 120$

8. $L = 25; K = 150$

9. $L = 15; K = 150$

10. $L = 80; K = 120$

11. $L = \frac{40}{3}; K = 10$

12. $L = \frac{45}{2}; K = 30$

Chapter 9

1. $MR = 15 - \frac{2}{5}Q$
2. $MR = 100 - Q$
3. $MR = 50 - \frac{1}{3}Q$
4. $MR = 160 - 8Q$
5. 75
6. 700
7. 5
8. 75
9. 20
10. 50
11. 150
12. 10,000

Chapter 14

1. 15
2. 3,645
3. 540
4. 270
5. 480
6. 8,100
7. 270
8. 30
9. 375
10. 400
11. 225
12. 36,000

Chapter 15

1. $\frac{5}{6}$
2. $\frac{7}{15}$
3. $\frac{1}{2}$
4. $\frac{3}{4}$
5. $\frac{4}{5}$
6. $\frac{3}{5}$
7. $\frac{1}{6}$
8. $\frac{8}{15}$
9. $\frac{1}{2}$
10. $\frac{1}{4}$
11. $\frac{1}{5}$
12. $\frac{2}{5}$

Chapter 16

1. $MRS^A_{XY} = 2 = MRS^B_{XY}$; yes, economically efficient
2. $MRS^A_{XY} = 1 = MRS^B_{XY}$; yes, economically efficient
3. $MRS^A_{XY} = \frac{15}{8}; MRS^B_{XY} = 1$; no, not economically efficient
4. $MRS^A_{XY} = \frac{1}{2} = MRS^B_{XY}$; yes, economically efficient
5. $MRS^A_{XY} = 2; MRS^B_{XY} = 5$; no, not economically efficient
6. $MRS^A_{XY} = \frac{1}{5}; MRS^B_{XY} = 1$; no, not economically efficient
7. $-\frac{X}{Y}$
8. $-\frac{2X}{3Y}$
9. $-\frac{X}{5Y}$
10. $-\frac{2X}{Y}$
11. $-\frac{X}{2Y}$
12. $-\frac{9X}{Y}$

Chapter 17

1. $Q = 35; P = 45$

2. $Q = 20; P = 25$

3. $Q = 9; P = 9$

4. $Q = 5; P = 6$

5. $Q = 8; P = 10$

6. $Q = 28; P = 13$

7. $Q = 2; P = 44.5$

8. $Q = 2; P = 22$

9. $Q = \frac{60}{7}; P = \frac{130}{7}$

10. $Q = 10; P = 45$

11. $Q = 25; P = 37.5$

12. $Q = 25; P = 10.5$

Chapter 18

1. $P_A = 26; P_B = 11$

2. $P_A = 10.5; P_B = 13$

3. $P_A = 35; P_B = 10$; $Q_B = 0$ when $P_B = 10$, therefore group B exits the market

4. $P_A = 6.5; P_B = 9$

5. $P_A = 23; P_B = 13$

6. $P_A = 27.5; P_B = 7.5$

7. $P_A = 55; P_B = 25$

8. $P_A = 15; P_B = 17.5$

9. $P_A = 27.5; P_B = 12.5$

10. $P_A = 8; P_B = 9.5$

11. $P_A = 62.5; P_B = 125$

12. $P_A = 255; P_B = 45$

Chapter 19

1. $Q_A = \frac{95}{2} - \frac{1}{2}Q_B; Q_B = 45 - \frac{1}{2}Q_A$

2. $Q_A = \frac{225}{2} - \frac{1}{2}Q_B; Q_B = \frac{225}{2} - \frac{1}{2}Q_A$

3. $Q_A = \frac{39}{2} - \frac{1}{2}Q_B; Q_B = 19 - \frac{1}{2}Q_A$

4. $Q_A = 115 - \frac{1}{2}Q_B; Q_B = 110 - \frac{1}{2}Q_A$

5. $Q_A = \frac{5}{4} - \frac{1}{2}Q_B; Q_B = \frac{5}{2} - \frac{1}{2}Q_A$

6. $Q_A = \frac{729}{2} - \frac{1}{2}Q_B; Q_B = \frac{729}{2} - \frac{1}{2}Q_A$

7. $Q_A = \frac{100}{3}; Q_B = \frac{85}{3}$

8. $Q_A = 75; Q_B = 75$

9. $Q_A = \frac{40}{3}; Q_B = \frac{37}{3}$

10. $Q_A = 80; Q_B = 70$

11. $Q_A = 0; Q_B = \frac{5}{2}$

12. $Q_A = 243; Q_B = 243$

Chapter 20

1. $MEC = 1 + 2Q; MSC = 101 + 8Q$

2. $MEC = 10; MSC = 10 + 16Q$

3. $MEC = 1 + 2Q; MSC = 6 + 4Q$

4. $MEC = 40Q; MSC = 30 + 42Q$

5. $MEC = 1; MSC = 2 + 2Q + 3Q^2$

6. $MEC = 4 + 12Q; MSC = 11 + 16Q + 9Q^2$

7. $MEB = 1 - 2Q; MSB = 101 - 10Q$

8. $MEB = 20; MSB = 220 - 2Q$

9. $MEB = 1 - 10Q; MSB = 301 - 16Q$

10. $MEB = 40 - 12Q; MSB = 120 - 14Q$

11. $MEB = 80 - 16Q; MSB = 980 - 34Q$

12. $MEB = 10 - 20Q; MSB = 110 - 40Q$