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REVISED PAGES

Chapter 5

Exponential and Logarithmic Functions

Chapter Outline

Preview

So far we've investigated a sizable number of functions with a large variety of applications. Still, many situations arising in business, science, and industry require the development of two additional tools—the exponential and logarithmic functions. Their applications are virtually limitless, and offer an additional glimpse of the true power and potential of mathematics. This chapter marks another significant advance in your mathematical studies, as we begin using the ideas underlying an inverse function to lay the groundwork needed for the introduction of these new function families.

5.1 Exponential Functions

LEARNING OBJECTIVES

In Section 5.1 you will learn how to:

A. Evaluate an exponential function
B. Graph exponential functions
C. Solve certain exponential equations
D. Solve applications of exponential functions

INTRODUCTION

Demographics is the statistical study of human populations. Perhaps surprisingly, an accurate study of how populations grow or decline cannot be achieved using only the functions discussed previously. In this section, we introduce the family of *exponential functions,* which are widely used to model population growth, with additional applications in finance, science, and engineering as well. As with other functions, we begin with a study of the graph and its characteristics, to aid in understanding the function and its applications.

POINT OF INTEREST

When a fair coin is flipped, we expect that roughly half the time it will turn up heads. In other words, the probability of a head showing on the first flip is one out of two or $\frac{1}{2}$. The probability that heads shows up two times in succession is $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$, and for three heads in a row the probability is $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$. As you can see, the probability of the coin continuing to turn up heads gets smaller and smaller and can be modeled by the function $f(x) = \left(\frac{1}{2}\right)^x$, where x represents the number of times the coin is flipped. This is one example from the family of exponential functions, which are used extensively in many different areas of scientific endeavor.

A. Evaluating Exponential Functions

In the boomtowns of the Old West, it was not uncommon for a town to double in size every year (at least for a time) as the lure of gold drew more and more people westward. When this type of growth is modeled using mathematics, exponents play a leading role. Suppose the town of Goldsboro had 1000 residents just before gold was discovered. After 1 yr the population doubled and the town had **2**000 residents. The next year it doubled again to **4**000, then again to **8**000, then to **16,**000 and so on. You probably recognize the digits in bold as powers of two (indicating the population is doubling), with each one multiplied by 1000 (the initial population). This suggests we can model the relationship using $P(x) = 1000 \cdot 2^x$, where $P(x)$ is the total population after x years. Sure enough, $P(4) = 1000 \cdot 2^4 = 16{,}000$. Further, we can also evaluate this function, called an **exponential function,** for *fractional parts of a year* using rational exponents. Using the ideas from Section R.6, the population of Goldsboro one-and-a-half years after the gold rush $(t = \frac{3}{2})$ was: $P(\frac{3}{2}) = 1000 \cdot 2^{\frac{3}{2}} = 1000 \cdot (\sqrt{2})^3 \approx 2828$ people. To actually *graph the function* using real numbers (see Exercise 55) requires that we define the expression 2^x when x is irrational. For example, what does $2^{\sqrt{5}}$ mean? We suspect it represents a real number since $2 < \sqrt{5} < 3$ seems to imply that $2^2 < 2^{\sqrt{5}} < 2^3$. While the technical details require calculus, it can be shown that successive approximations of $2^{\sqrt{5}}$ as in $2^{2.2360}, 2^{2.23606}\, 2^{2.236067}, \ldots$ approach a unique real number. This means we can approximate its value to any desired level of accuracy: $2^{\sqrt{5}} \approx 4.71111$ to five decimal places. In general, as long as b is a positive real number, b^x *is a real number for all real numbers* x and we have the following:

EXPONENTIAL FUNCTIONS

For $b > 0$, $b \neq 1$, $f(x) = b^x$ defines the base b exponential function. The domain of f is all real numbers.

Limiting b to positive values ensures outputs will be real numbers (if $b = -4$ and $x = \frac{1}{2}$, we have $(-4)^{\frac{1}{2}} = \sqrt{-4}$, which is not a real number). The restriction $b \neq 1$ is needed because $y = 1^x$ is a constant function (1 raised to *any* power is still 1).

WORTHY OF NOTE

Exponential functions are very different from the power functions studied earlier. For power functions, the base is variable and the exponent is constant: $y = x^b$, while for exponential functions the *exponent is a variable* and the *base is constant*: $y = b^x$.

EXAMPLE 1 Evaluate the expressions given, rounding each result to five decimal places.

a. $2^{3.14}$ **b.** $2^{3.1416}$ **c.** $2^{3.141592}$ **d.** 2^{π}

Solution: **a.** $2^{3.14} \approx 8.81524$ **b.** $2^{3.1416} \approx 8.82502$

c. $2^{3.141592} \approx 8.82497$ **d.** $2^{\pi} \approx 8.82498$

NOW TRY EXERCISES 7 THROUGH 12

Note the domain of the exponential function includes negative numbers as well, and expressions such as 2^{-3} and $2^{-\sqrt{5}}$ can easily be calculated: $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$, and $2^{-\sqrt{5}} = \frac{1}{2^{\sqrt{5}}} \approx 0.21226$. In fact, all of the familiar properties of exponents continue to hold for irrational exponents.

EXPONENTIAL PROPERTIES

Given a, b, x, and t are real numbers, with $b, c > 0$,

$$b^x b^t = b^{x+t} \qquad \frac{b^x}{b^t} = b^{x-t} \qquad (b^x)^t = b^{xt}$$

$$(bc)^x = b^x c^x \qquad b^{-x} = \frac{1}{b^x} \qquad \left(\frac{b}{a}\right)^{-x} = \left(\frac{a}{b}\right)^x$$

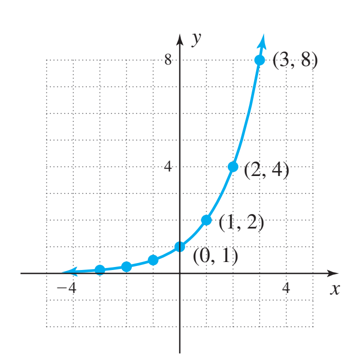
B. Graphing Exponential Functions

To gain a better understanding of exponential functions, we'll graph $y = b^x$ on a coordinate grid and note some of its important features. Since the base b cannot be equal to 1, it seems reasonable that we graph one exponential function where $b > 1$ and one where $0 < b < 1$.

EXAMPLE 2 Graph $y = 2^x$ using a table of values.

Solution: To get an idea of the graph's shape we'll use integer values from -3 to 3 in our table, then draw the graph as a continuous curve, knowing the function is defined for all real numbers.

x	$y = 2^x$
-3	$2^{-3} = \frac{1}{8}$
-2	$2^{-2} = \frac{1}{4}$
-1	$2^{-1} = \frac{1}{2}$
0	$2^0 = 1$
1	$2^1 = 2$
2	$2^2 = 4$
3	$2^3 = 8$



NOW TRY EXERCISES 13 AND 14

WORTHY OF NOTE

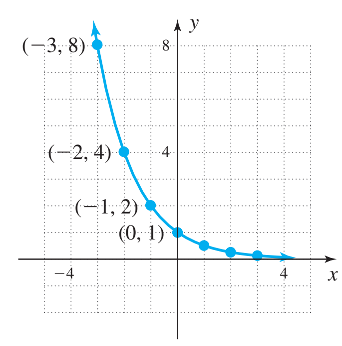
Functions that are increasing for all $x \in D$ are said to be **monotonically increasing** or simply **monotonic functions.**

Several important observations can now be made. First note the x-axis (the line $y = 0$) is a horizontal asymptote for the function, because as $x \to -\infty$, $y \to 0$. Second, it is evident that the function is increasing over its entire domain, giving the function a range of $y \in (0, \infty)$.

EXAMPLE 3 Graph $y = \left(\frac{1}{2}\right)^x$ using a table of values.

Solution: Using properties of exponents, we can write $\left(\frac{1}{2}\right)^x$ as $\left(\frac{2}{1}\right)^{-x} = 2^{-x}$. Again using integers from -3 to 3, we plot the ordered pairs and draw a continuous curve.

x	$y = 2^{-x}$
-3	$2^{-(-3)} = 2^3 = 8$
-2	$2^{-(-2)} = 2^2 = 4$
-1	$2^{-(-1)} = 2^1 = 2$
0	$2^0 = 1$
1	$2^{-1} = \frac{1}{2}$
2	$2^{-2} = \frac{1}{4}$
3	$2^{-3} = \frac{1}{8}$



NOW TRY EXERCISES 15 AND 16

We note this graph is also asymptotic to the x-axis, but *decreasing on its domain.* In addition, we see that both $y = 2^x$ and $y = 2^{-x} = \left(\frac{1}{2}\right)^x$ are one-to-one, and have a y-intercept of (0, 1)—which we expect since any base to the zero power is 1. Finally, observe that $y = b^{-x}$ is *a reflection of* $y = b^x$ *across the y-axis,* a property that suggests these basic graphs might also be transformed in other ways, as the toolbox functions were in Section 3.3. The most important characteristics of exponential functions are summarized here:

$$f(x) = b^x,\ b > 0 \text{ and } b \neq 1$$

- one-to-one function
- domain: $x \in \mathbb{R}$
- increasing if $b > 1$
- asymptotic to the x-axis
- y-intercept (0, 1)
- range: $y \in (0, \infty)$
- decreasing if $0 < b < 1$

WORTHY OF NOTE

When an exponential function is increasing, it can be referred to as a "growth function." When decreasing, it is often called a "decay function." Each of the graphs shown in Figures 5.1 and 5.2 should now be added to your repertoire of basic functions, to be sketched from memory and analyzed or used as needed.

Figure 5.1

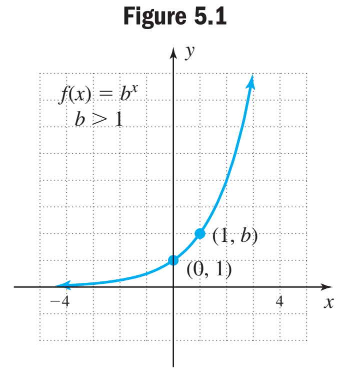
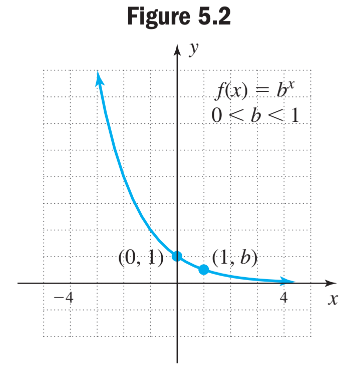


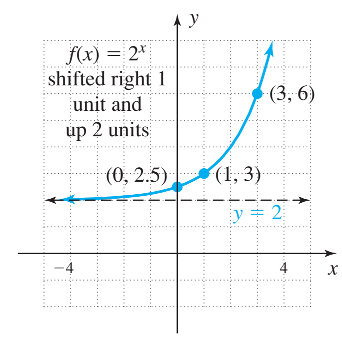
Figure 5.2



Just as a quadratic function maintains its parabolic shape regardless of the transformations applied, exponential functions will also maintain their general shape and features. Any sum or difference applied to the basic function will cause a vertical shift in the same direction as the sign, and any change to input values ($y = b^{x+h}$ versus $y = b^x$) will cause a horizontal shift in a direction opposite the sign.

EXAMPLE 4 Graph $F(x) = 2^{x-1} + 2$ using transformations of the basic function $f(x) = 2^x$ (not by simply plotting points). Clearly state what transformations are applied.

Solution: The graph of F is that of the basic function shifted 1 unit right and 2 units up. With this in mind we can sketch the horizontal asymptote at $y = 2$ and plot the point $(1, 3)$. The y-intercept of F is at $(0, 2.5)$ and to help sketch a fairly accurate graph, the additional point $(3, 6)$ is used: $F(3) = 6$.



NOW TRY EXERCISES 17 THROUGH 38

C. Solving Exponential Equations

The fact that exponential functions are one-to-one enables us to solve equations where each side is an exponential term with the identical base. This is because one-to-oneness guarantees a unique solution to the equation.

EXPONENTIAL EQUATIONS WITH LIKE BASES: THE UNIQUENESS PROPERTY

For all real numbers b, m, and n, where $b > 0$ and $b \neq 1$,

If $b^m = b^n$, then $m = n$.	If $m = n$, then $b^m = b^n$.

Equal bases imply exponents are equal.

The equation $2^x = 32$ can be written as $2^x = 2^5$, and we note $x = 5$ is a solution. Although $3^x = 32$ can be written as $3^x = 2^5$, the bases are not alike and the solution to this equation must wait until additional tools are developed in Section 5.4.

EXAMPLE 5 Solve the exponential equations using the uniqueness property.

a. $3^{2x-1} = 81$ **b.** $25^{-2x} = 125^{x+7}$ **c.** $\left(\frac{1}{6}\right)^{-3x-2} = 36^{x+1}$

Solution: **a.**

$$3^{2x-1} = 81 \quad \text{given}$$
$$3^{2x-1} = 3^4 \quad \text{rewrite using base 3}$$
$$\Rightarrow 2x - 1 = 4 \quad \text{uniqueness property}$$
$$x = \frac{5}{2} \quad \text{solve for } x$$

Check:

$3^{2x-1} = 81$ given

$3^{2\left(\frac{5}{2}\right)-1} = 81$ substitute $\frac{5}{2}$ for x

$3^{5-1} = 81$ simplify

$3^4 = 81$✓ result checks

The remaining checks are left to the student.

b.

$25^{-2x} = 125^{x+7}$ given

$(5^2)^{-2x} = (5^3)^{x+7}$ rewrite using base 5

$5^{-4x} = 5^{3x+21}$ power property of exponents

$\Rightarrow -4x = 3x + 21$ uniqueness property

$x = -3$ solve for x

c.

$\left(\frac{1}{6}\right)^{-3x-2} = 36^{x+1}$ given

$(6^{-1})^{-3x-2} = (6^2)^{x+1}$ rewrite using base 5

$6^{3x+2} = 6^{2x+2}$ power property of exponents

$\Rightarrow 3x + 2 = 2x + 2$ uniqueness property

$x = 0$ solve for x

NOW TRY EXERCISES 39 THROUGH 54

D. Applications of Exponential Functions

One application of exponential functions involves money invested at a fixed rate of return. If interest is compounded n times per year, the amount of money in an account after t years is modeled by the function $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$, where P is the principal amount invested, r is the annual interest rate, and $A(t)$ represents the amount of money in the account after t years. The result is called the **compound interest formula** (for a full development of this formula, see Section 5.5). After substituting given values, this formula simplifies to a more recognizable form, $y = ab^x$.

EXAMPLE 6 When she was 8 yr old, Marcy invested \$1000 of the money she earned working a paper route in an account paying 8% interest compounded quarterly (four times per year). At 18 yr old, Marcy now wants to withdraw the money to help with college expenses. Determine how much money is in the account.

Solution: For this exercise, $P = 1000$, $r = 8\%$, $n = 4$, and $t = 10$. The formula yields

$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$ given

$= 1000\left(1 + \frac{0.08}{4}\right)^{4\cdot 10}$ substitute given values

$= 1000(1.02)^{40}$ simplify, note ab^x form

$\approx 1000(2.20804)$ evaluate: $1.02^{40} \approx 2.20804$

≈ 2208.04 result

After 10 yr, there is approximately \$2208.04 in the account.

NOW TRY EXERCISES 57 AND 58

EXAMPLE 7 For insurance purposes, it is estimated that large household appliances lose $\frac{1}{4}$ of their value each year. The current value can then be modeled by the function $V(t) = V_0\left(\frac{3}{4}\right)^t$, where V_0 is the initial value and $V(t)$ represents the value after t years. How many years does it take a washing machine that cost \$256 new, to depreciate to a value of \$81?

Solution: For this exercise, $V_0 = \$256$ and $V(t) = \$81$. The formula yields

$$V(t) = V_0\left(\frac{3}{4}\right)^t \qquad \text{given}$$

$$81 = 256\left(\frac{3}{4}\right)^t \qquad \text{substitute known values}$$

$$\frac{81}{256} = \left(\frac{3}{4}\right)^t \qquad \text{divide by 256}$$

$$\left(\frac{3}{4}\right)^4 = \left(\frac{3}{4}\right)^t \qquad \text{equate bases } (3^4 = 81 \text{ and } 4^4 = 256)$$

$$\Rightarrow 4 = t \qquad \text{Uniqueness Property}$$

After 4 yr, the washing machine's value has dropped to \$81.

NOW TRY EXERCISES 59 THROUGH 64

TECHNOLOGY HIGHLIGHT

Solving Exponential Equations Graphically

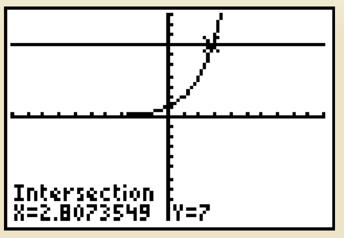
The keystrokes shown apply to a TI-84 Plus model. Please consult our Internet site or your manual for other models.

In this section, we showed that the exponential function $f(x) = b^x$ was defined for all real numbers. This is important because it establishes that equations like $2^x = 7$ must have a solution, even if x is not rational. In fact, since $2^2 = 4$ and $2^3 = 8$, the solution must be between 2 and 3:

$$2^2 = 4 < 2^x = 7 < 2^3 = 8$$
$$\text{and}$$
$$2 < x < 3$$

Until we develop an inverse for exponential functions, we are unable to solve many of these equations in exact form. We can, however, get a very close approximation using a graphing calculator. For the equation $2^x = 7$, enter $Y_1 = 2^x$ and $Y_2 = 7$ on the Y= screen. Then press ZOOM 6 to graph both functions (see Figure 5.3). To have the calculator compute the point of intersection, press 2nd CALC and select option **5: intersect** and press ENTER *three* times. The x- and y-coordinates of the point of intersection will appear at the bottom of the screen, with the x-coordinate being the solution. As you can see, x is indeed between 2 and 3. Solve the following equations. First estimate the answer by bounding it between two integers, then solve the equation graphically. Adjust the viewing window as needed.

Figure 5.3



Exercise 1: $3^x = 22$ $x = 2.8$ Exercise 2: $2^x = 0.125$ $x = -3$

Exercise 3: $5^{x-1} = 61$ $x = 3.6$

5.1 EXERCISES

CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. An exponential function is one of the form $y =$ ___ab^x___, where ___b___ > 0, ___b___ $\neq 1$, and ___x___ is any real number.

2. The domain of $y = b^x$ is all ___real numbers___, and the range is $y \in$ ___$(0, \infty)$___. Further, as $x \to -\infty$, y ___$\to 0$___.

3. For exponential functions of the form $y = ab^x$, the y-intercept is $(0,$ ___a___$)$, since $b^0 =$ ___1___ for any real number b.

4. If each side of an equation can be written as an exponential term with the same base, the equation can be solved using the ___uniqueness___ ___property___.

5. State true or false and explain why: $y = b^x$ is always decreasing if $0 < b < 1$. False; x may be less than zero.

6. Discuss/explain the statement, "For $k > 0$, the y intercept of $y = ab^x + k$ is $(0, a + k)$." Answers will vary.

DEVELOPING YOUR SKILLS

Use a calculator (as needed) to evaluate each function as indicated. Round answers to thousandths.

7. $P(t) = 2500 \cdot 4^t$; $t = 2$, $t = \frac{1}{2}$, $t = \frac{3}{2}$, $t = \sqrt{3}$

8. $Q(t) = 5000 \cdot 8^t$; $t = 2$, $t = \frac{1}{3}$, $t = \frac{5}{3}$, $t = 5$

9. $f(x) = 0.5 \cdot 10^x$; $x = 3$, $x = \frac{1}{2}$, $x = \frac{2}{3}$, $x = \sqrt{7}$

10. $g(x) = 0.8 \cdot 5^x$; $x = 4$, $x = \frac{1}{4}$, $x = \frac{4}{5}$, $x = \pi$

11. $V(n) = 10{,}000\left(\frac{2}{3}\right)^n$; $n = 0$, $n = 4$, $n = 4.7$, $n = 5$

12. $W(m) = 3300\left(\frac{4}{5}\right)^m$; $m = 0$, $m = 5$, $m = 7.2$, $m = 10$

Graph each function using a table of values and integer inputs between -3 and 3. Clearly label all important features and indicate whether the function is increasing or decreasing.

13. $y = 3^x$ increasing
14. $y = 4^x$ increasing
15. $y = \left(\frac{1}{3}\right)^x$ decreasing
16. $y = \left(\frac{1}{4}\right)^x$ decreasing

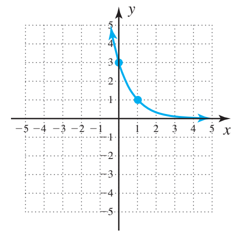
Graph each of the following functions by *translating the basic function* $y = b^x$ and strategically plotting a few points to round out the graph. Clearly state what shifts are applied.

17. $y = 3^x + 2$
18. $y = 3^x - 3$
19. $y = 3^{x+3}$
20. $y = 3^{x-2}$
21. $y = 2^{-x}$
22. $y = 3^{-x}$
23. $y = 2^{-x} + 3$
24. $y = 3^{-x} - 2$
25. $y = 2^{x+1} - 3$
26. $y = 3^{x-2} + 1$
27. $y = \left(\frac{1}{3}\right)^x + 1$
28. $y = \left(\frac{1}{3}\right)^x - 4$
29. $y = \left(\frac{1}{3}\right)^{x-2}$
30. $y = \left(\frac{1}{3}\right)^{x+2}$
31. $f(x) = \left(\frac{1}{3}\right)^x - 2$
32. $g(x) = \left(\frac{1}{3}\right)^x + 2$

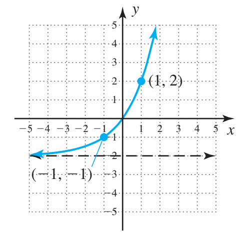
Match each graph to the correct exponential equation.

33. $f(x) = 5^{-x}$ e
34. $f(x) = 4^{-x}$ c
35. $f(x) = 3^{-x+1}$ a
36. $f(x) = 3^{-x} + 1$ f
37. $y = 2^{x+1} - 2$ b
38. $y = 2^{x+2} - 1$ d

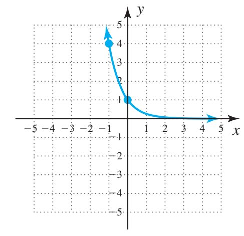
a.



b.

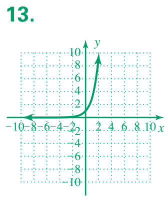


c.

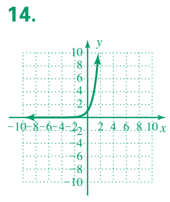


7. 40,000; 5000; 20,000; 27,589.162
8. 320,000; 10,000; 160,000; 163,840,000
9. 500; 1.581; 2.321; 221.168
10. 500; 1.196; 2.899; 125.594
11. 10,000; 1975.309; 1487.206; 1316.872
12. 3300; 1081.344; 661.853; 354.335

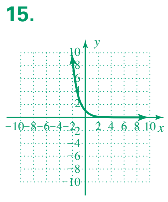
13.



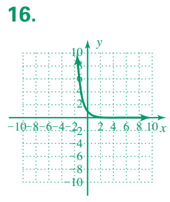
14.



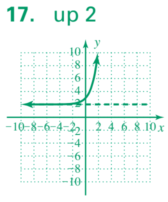
15.



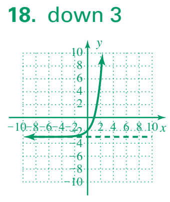
16.



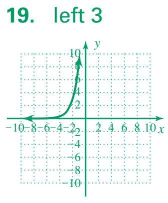
17. up 2



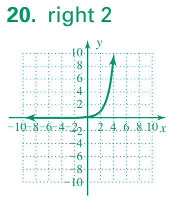
18. down 3



19. left 3

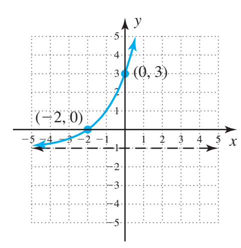


20. right 2

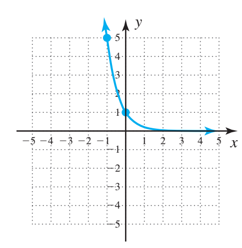


Additional answers can be found in the Instructor Answer Appendix.

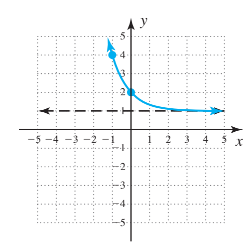
d.



e.



f.



Solve each exponential equation and check your answer by substituting into the original equation.

39. $10^x = 1000$ 3 **40.** $144 = 12^x$ 2 **41.** $25^x = 125$ $\frac{3}{2}$ **42.** $81 = 27^x$ $\frac{4}{3}$

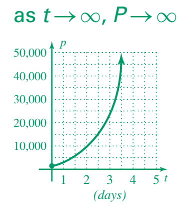
43. $8^{x+2} = 32$ $\frac{-1}{3}$ **44.** $9^{x-1} = 27$ $\frac{5}{2}$ **45.** $32^x = 16^{x+1}$ 4 **46.** $100^{x+2} = 1000^x$ 4

47. $\left(\frac{1}{5}\right)^x = 125$ -3 **48.** $\left(\frac{1}{4}\right)^x = 64$ -3 **49.** $\left(\frac{1}{3}\right)^{2x} = 9^{x-6}$ 3 **50.** $\left(\frac{1}{2}\right)^{3x} = 8^{x-2}$ 1

51. $\left(\frac{1}{9}\right)^{x-5} = 3^{3x}$ 2 **52.** $2^{-2x} = \left(\frac{1}{32}\right)^{x-3}$ 5 **53.** $25^{3x} = 125^{x-2}$ -2 **54.** $27^{2x+4} = 9^{4x}$ 6

WORKING WITH FORMULAS

55. **a.** 1732, 3000, 5196, 9000
b. yes
c. as $t \to \infty$, $P \to \infty$
d.



56. **a.** $\frac{1}{4}$
b. $\frac{1}{256}$
c. as $x \to \infty$, $P \to 0$

55. The growth of a bacteria population: $P(t) = 1000 \cdot 3^t$

If the initial population of a common bacterium is 1000 and the population triples every day, its population is given by the formula shown, where $P(t)$ is the total population after t days. (a) Find the total population 12 hours, 1 day, $1\frac{1}{2}$ days, and 2 days later. (b) Do the outputs show the population is tripling every 24 hr (1 day)? (c) Explain why this is an increasing function. (d) Graph the function using an appropriate scale.

56. Games involving a spinner with numbers 1 through 4: $P(x) = \left(\frac{1}{4}\right)^x$

Games that involve moving pieces around a board using a fair spinner are fairly common. If the spinner has the numbers 1 through 4, the probability that any one number is spun repeatedly is given by the formula shown, where x represents the number of spins and $P(x)$ represents the probability the same number results x times. (a) What is the probability that the first player spins a 2? (b) What is the probability that all four players spin a 2? (c) Explain why this is a decreasing function.

APPLICATIONS

Use the compound interest formula to work Exercises 57 and 58: $A = P\left(1 + \frac{r}{n}\right)^{nt}$. Round all answers to the nearest penny.

57. **a.** \$12,875.41
b. \$151.87

57. Investment savings: Janielle decides to begin saving for her 2-yr-old son's college education by depositing her \$5000 bonus check in an account paying 6% interest compounded semiannually. (a) How much will the account be worth when her son enters college (16 yr later)? (b) How much more would be in the account if the interest were compounded monthly?

58. **a.** \$17,046.32
b. \$113.18

58. Investment savings: To be more competitive, a local bank notifies its customers that all accounts earning 9% interest compounded *quarterly* will now earn 9% interest compounded *monthly.* (a) If \$7000 is invested for 10 yr in the account that paid 9% compounded quarterly, what would the account be worth? (b) How much additional interest will the bank have to pay if the same \$7000 were invested for 10 yr in an account that paid 9% interest compounded monthly?

59. **a.** \$100,000
b. 3

59. Depreciation: The financial analyst for a large construction firm estimates that its heavy equipment loses one-fifth of its value each year. The current value of the equipment is then modeled by the function $V(t) = V_0\left(\frac{4}{5}\right)^t$, where V_0 represents the initial value, t is in years, and $V(t)$ represents the value after t years. (a) How much is a large earthmover worth after 1 yr if it cost \$125 thousand new? (b) How many years does it take for the earthmover to depreciate to a value of \$64 thousand?

60. a. \$40,000
b. 2 yr

60. Depreciation: Photocopies have become a critical part of the operation of many businesses, and due to their heavy use they can depreciate in value very quickly. If a copier loses $\frac{3}{8}$ of its value each year, the current value of the copier can be modeled by the function $V(t) = V_0\left(\frac{5}{8}\right)^t$, where V_0 represents the initial value, t is in years, and $V(t)$ represents the value after t years. (a) How much is this copier worth after one year if it cost \$64 thousand new? (b) How many years does it take for the copier to depreciate to a value of \$25 thousand?

61. a. \$70,328.58
b. ≈9.7 yr

61. Depreciation: Margaret Madison, DDS, estimates that her dental equipment loses one-sixth of its value each year. (a) determine the value of an x-ray machine after 5 yr if it cost \$175 thousand new, and (b) determine how long until the machine is worth less than \$30 thousand.

62. a. 10.05 in.
b. ≈5.7 months

62. Exponential decay: The groundskeeper of a local high school estimates that due to heavy usage by the baseball and softball teams, the pitcher's mound loses one-eighth of its height every month. (a) determine the height of the mound after 3 months if it was 15 in. to begin, and (b) determine how long until the pitcher's mound is less than 7 in. high (meaning it must be rebuilt).

63. a. \$40 million
b. 7 yr

63. Exponential growth: Similar to a small town doubling in size after a discovery of gold, a business that develops a product in high demand has the potential for doubling its revenue each year for a number of years. The revenue would be modeled by the function $R(t) = R_0 2^t$, where R_0 represents the initial revenue, and $R(t)$ represents the revenue after t years. (a) How much revenue is being generated after 4 yr, if the company's initial revenue was \$2.5 million? (b) How many years does it take for the business to be generating \$320 million in revenue?

64. a. \$864 thousand
b. 5 yr

64. Exponential growth: If a company's revenue grows at a rate of 150% per year (rather than doubling as in Exercise 63), the revenue would be modeled by the function $R(t) = R_0\left(\frac{3}{2}\right)^t$, where R_0 represents the initial revenue, and $R(t)$ represents the revenue after t years. (a) How much revenue is being generated after 3 yr, if the company's initial revenue was \$256 thousand? (b) How long until the business is generating \$1944 thousand in revenue? (*Hint:* Reduce the fraction.)

Modeling inflation: Assuming the rate of inflation is 5% per year, the predicted price of an item can be modeled by the function $P(t) = P_0(1.05)^t$, where P_0 represents is the initial price of the item and t is in years. Use this information to solve Exercises 65 and 66.

65. \$32,577.89
66. \$4.81

65. What will the price of a new car be in the year 2010, if it cost \$20,000 in the year 2000?

66. What will the price of a gallon of milk be in the year 2010, if it cost \$2.95 in the year 2000?

Modeling radioactive decay: The half-life of a radioactive substance is the time required for half an initial amount of the substance to disappear through decay. The amount of the substance remaining is given by the formula $Q(t) = Q_0\left(\frac{1}{2}\right)^{\frac{t}{h}}$, where h is the half-life, t represents the elapsed time, and $Q(t)$ represents the amount that remains (t and h must have the *same unit of time*). Use this information to solve Exercises 67 and 68.

67. a. 12.5 g
b. 53.15 min

67. Some isotopes of the substance known as thorium have a half-life of only 8 min. (a) If 100 grams are initially present, how many gram (g) of the substance remain after 24 min? (b) How many minutes until only 1 gram (g) of the substance remains?

68. a. 16 g
b. 112 hr

68. Some isotopes of sodium have a half-life of about 16 hr. (a) If 128 g are initially present, how many grams of the substance remain after 2 days (48 hr)? (b) How many hours until only 1 g of the substance remains?

WRITING, RESEARCH, AND DECISION MAKING

69. 9.5×10^{-7}; answers will vary.

69. In the *Point of Interest* of this section, the formula $f(x) = \left(\frac{1}{2}\right)^x$ was given to determine the probability that "x" number of flips all resulted in heads (or tails). First determine the probability that 20 flips results in *20 heads in a row.* Then use the Internet or some other resource to determine the probability of winning a state lottery (expressed as a decimal). Which has the greater probability? Were you surprised?

70. Answers will vary.

70. In mathematics, it is generally true that for any function $f(x)$, we solve the equation $af(x + h) + k = c$ by isolating the function $f(x + h)$ on one side before the inverse function or procedure can be applied. Solve the equations $118 = 12 \cdot \sqrt{x} + 10$ and $118 = 12 \cdot 3^x + 10$ side-by-side to see how this idea applies to each. Carefully state which operations are used in each step. Comment on how this might apply to a new function or the general function $f(x)$: $118 = 12 \cdot f(x) + 10$.

EXTENDING THE CONCEPT

71. $\frac{1}{5}$

72. 2.4 hr

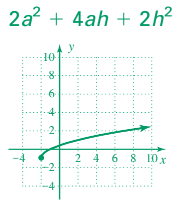
71. If $10^{2x} = 25$, what is the value of 10^{-x}?

72. Two candles have the same height, but different diameters. Both are lit at the same time and burn at a constant rate. If the first is consumed in 4 hr and the second in 3 hr, how long after being lit was the first candle twice the height of the second?

MAINTAINING YOUR SKILLS

73. 5
-7
9
$2a^2 - 3a$
$2a^2 + 4ah + 2h^2 - 3a - 3h$

74.



$x \in [-2, \infty), y \in [-1, \infty)$

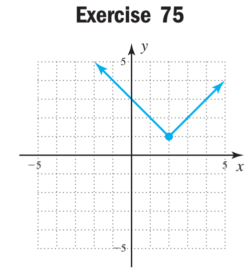
78. **a.** volume of a sphere
b. area of a triangle
c. volume of a rectangular prism
d. Pythagorean theorem

73. (2.2) Given $f(x) = 2x^2 - 3x$, determine: $f(-1)$, $f\left(\frac{1}{3}\right)$, $f(a)$, $f(a + h)$

74. (3.3) Graph $g(x) = \sqrt{x + 2} - 1$ using a shift of the parent function. Then state the domain and range of g.

75. (3.3) Given the parent function is $y = |x|$, what is the equation of the function shown? $y = 1|x - 2| + 1$

Exercise 75



76. (4.3) Factor $P(x)$ completely using the RRT and synthetic division:
$P(x) = 3x^4 - 19x^3 + 15x^2 + 27x - 10.$
$P(x) = (x - 2)(x - 5)(x + 1)(3x - 1)$

77. (1.3) Solve the following equations:

a. $-2\sqrt{x - 3} + 7 = 21$ $\emptyset$

b. $\dfrac{9}{x + 3} + 3 = \dfrac{12}{x - 3}$ $\{-5, 6\}$

78. (R.7) Identify each formula:

a. $\frac{4}{3}\pi r^3$ **b.** $\frac{1}{2}bh$

c. lwh **d.** $a^2 + b^2 = c^2$

5.2 Logarithms and Logarithmic Functions

LEARNING OBJECTIVES

In Section 5.2 you will learn how to:

A. Write exponential equations in logarithmic form

B. Graph logarithmic functions and find their domains

C. Use a calculator to find common logarithms

D. Solve applications of logarithmic functions

INTRODUCTION

A **transcendental function** is one whose solutions are beyond or *transcend* the methods applied to polynomial functions. The exponential function and its inverse, called the logarithmic function, are of this type. In this section, we'll use the concept of an inverse to develop an understanding of the logarithmic function, which has numerous applications that include measuring pH levels, sound and earthquake intensities, barometric pressure, and other natural phenomena.

POINT OF INTEREST

It appears that logarithms were invented by the Scottish mathematician John Napier (1550–1617). The word **logarithm** has its origins in the Greek work *logos,* which means "to reckon," and *arithmos,* which simply means "number." Until calculators and computers became commonplace, logarithms had been used for centuries to manually "reckon" or calculate with numbers, particularly if products, quotients, or powers were involved (see Exercise 96 in *Writing, Research, and Decision Making*). Still, technology has by no means diminished the usefulness of logarithms, since the logarithmic function is required to solve applications involving exponential functions and both exist in abundance.

A. Exponential Equations and Logarithmic Form

While exponential functions have a large number of significant applications, we can't appreciate their full value until we develop the inverse function. Without it, we're unable to solve all but the simplest equations, of the type encountered in Section 5.1. Using the fact that $f(x) = b^x$ is one-to-one, we have the following:

1. The inverse function $f^{-1}(x)$ must exist.
2. We can graph $f^{-1}(x)$ by interchanging the x- and y-coordinates of points from $f(x)$.
3. The domain of $f(x)$ will become the range of $f^{-1}(x)$.
4. The range of $f(x)$ will become the domain of $f^{-1}(x)$.
5. The graph of $f^{-1}(x)$ will be a reflection of $f(x)$ across the line $y = x$.

A table of selected values for $f(x) = 2^x$ is shown in Table 5.1. The points for $f^{-1}(x)$ in Table 5.2 were found by interchanging x- and y-coordinates. Both functions were then graphed using these points.

Table 5.1

$f(x)$: $y = 2^x$

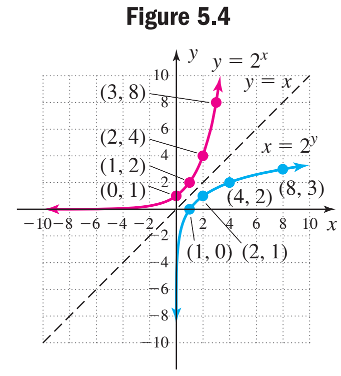
x	y
-3	$\frac{1}{8}$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8

Table 5.2

$f^{-1}(x)$: $x = 2^y$

x	$y = f^{-1}(x)$
$\frac{1}{8}$	-3
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2
8	3

Figure 5.4



The interchange of x and y and the graphs in Figure 5.4 show that $f^{-1}(x)$ has a y-intercept of (1, 0), a vertical asymptote at $x = 0$, a domain of $x \in (0, \infty)$, and a range of $y \in (-\infty, \infty)$, giving us a great deal of information about the function. To find *an equation* for $f^{-1}(x)$, we'll attempt to use the algebraic approach employed previously. For $f(x) = 2^x$,

1. use y instead of $f(x)$: $y = 2^x$.
2. interchange x and y: $x = 2^y$.

At this point we have an implicit equation for the inverse function, but no algebraic operations that enable us to solve explicitly for y in terms of x. Instead, we write $x = 2^y$ in function form by noting that "y is the exponent that goes on base 2 to obtain x."

In the language of mathematics, this phrase is represented by $y = \log_2 x$ and is called a base b **logarithmic function.** For $y = b^x$, $y = \log_b x$ is the inverse function and is read "y is the base-b logarithm of x." For this new function, always keep in mind what y represents—y is an exponent. In fact, *y is the exponent that goes on base b to obtain x:* $y = \log_b x$.

LOGARITHMIC FUNCTIONS

For $b > 0$, $b \neq 1$, the base-b logarithmic function is defined as

$$y = \log_b x \text{ if and only if } x = b^y$$

Domain: $x \in (0, \infty)$ Range: $y \in \mathbb{R}$

EXAMPLE 1 Write each equation in exponential form.

a. $3 = \log_2 8$ **b.** $1 = \log_2 2$ **c.** $0 = \log_2 1$ **d.** $-2 = \log_2 \frac{1}{4}$

Solution: **a.** $2^3 = 8$ **b.** $2^1 = 2$ **c.** $2^0 = 1$ **d.** $2^{-2} = \frac{1}{4}$

NOW TRY EXERCISES 7 THROUGH 22

This relationship can be applied to *any* base b, where $b > 0$ and $b \neq 1$. When given the exponential form, as in $5^3 = 125$, note the exponent on the base and begin there: 3 is the exponent that goes on base 5 to obtain 125, or more exactly, *3 is the base-5 logarithm of 125*: $3 = \log_5 125$.

EXAMPLE 2 Write each equation in logarithmic form.

a. $6^3 = 216$ **b.** $2^{-1} = \frac{1}{2}$ **c.** $b^0 = 1$ **d.** $9^{\frac{3}{2}} = 27$

Solution:

a. $6^3 = 216$

3 is the base-6 logarithm of 216

$3 = \log_6 216$

b. $2^{-1} = \frac{1}{2}$

-1 is the base-2 logarithm of $\frac{1}{2}$

$-1 = \log_2 \frac{1}{2}$

c. $b^0 = 1$

0 is the base-b logarithm of 1

$0 = \log_b 1$

d. $9^{\frac{3}{2}} = 27$

$\frac{3}{2}$ is the base-9 logarithm of 27

$\frac{3}{2} = \log_9 27$

NOW TRY EXERCISES 23 THROUGH 38

EXAMPLE 3 Determine the value of each expression without using a calculator:

a. $\log_2 8$ **b.** $\log_5 \frac{1}{25}$ **c.** $\log_b b$ **d.** $\log_{10} 100$

Solution:

a. $\log_2 8 = 3$, since $2^3 = 8$ **b.** $\log_5 \frac{1}{25} = -2$, since $5^{-2} = \frac{1}{25}$

c. $\log_b b = 1$, since $b^1 = b$ **d.** $\log_{10} 100 = 2$, since $10^2 = 100$

NOW TRY EXERCISES 39 THROUGH 62

Alternating between logarithmic form and exponential form reveals the following four properties, which we'll use in Section 5.3 to simplify expressions and in Section 5.4 to solve equations involving logarithms. Also see the *Technology Highlight* on page 491.

LOGARITHMIC PROPERTIES

For any base b, where $b > 0$ and $b \neq 1$,

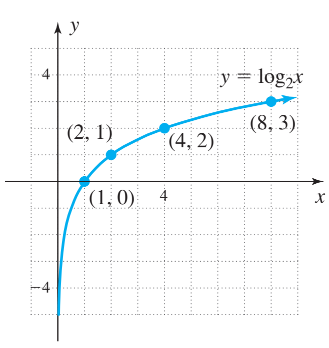
I. $\log_b b = 1$, since $b^1 = b$

II. $\log_b 1 = 0$, since $b^0 = 1$

III. $\log_b b^x = x$, since the exponential form gives $b^x = b^x$

IV. $b^{\log_b x} = x$, since $\log_b x$ is "the exponent on b" for x

Figure 5.5

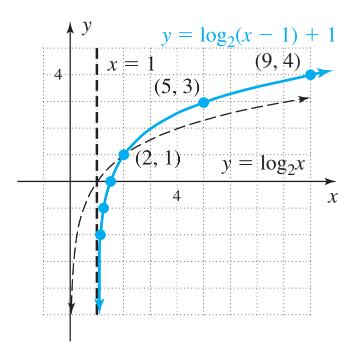


B. Graphing Logarithmic Functions

As with the other basic graphs we've encountered, logarithmic graphs maintain the same characteristics when transformations are applied, and its graph *should be added to your collection of basic functions,* ready for recall and analysis as the situation requires. We've already graphed the function $y = \log_2 x$ $(b > 1)$ earlier in this section, using $x = 2^y$ as the inverse function for $y = 2^x$. The graph is repeated in Figure 5.5.

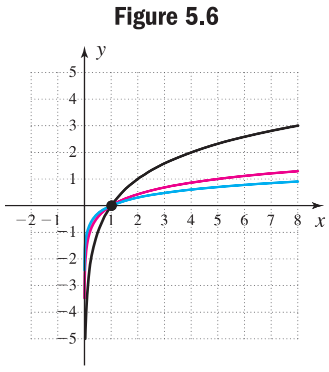
EXAMPLE 4 Graph $Y = \log_2(x - 1) + 1$ using *transformations of* $y = \log_2 x$ (not by simply plotting points). Clearly state what transformations are applied.

Solution: The graph of Y is the same as that of $y = \log_2 x$, shifted 1 unit right and 1 unit up. With this in mind we can sketch the vertical asymptote at $x = 1$ and plot the point (2, 1). Knowing the general shape of the graph, we need only one or two additional points to complete the graph. For $x = 5$ and $x = 9$ we find $Y = \log_2 4 + 1 = 2 + 1 = 3$, and $Y = \log_2 8 + 1 = 3 + 1 = 4$, respectively. The graph is shown in the figure. Note the domain of this function is $x \in (1, \infty)$.



NOW TRY EXERCISES 63 THROUGH 70

Figure 5.6



For convenience and ease of calculation, examples of graphing have been done using base-2 logarithms. However, the basic shape of a logarithmic graph remains unchanged regardless of the base used and transformations can be applied to $y = \log_b x$ for any value of b. The illustration in Figure 5.6 shows representative graphs for $y = \log_2 x$, $y = \log_5 x$, and $y = \log_{10} x$.

Example 5 illustrates how the domain of a logarithmic function can change when certain transformations are applied. Since the domain consists of *positive* real numbers, the argument of a logarithmic function must be greater than zero. This means finding the domain often consists of solving various inequalities, which can be done using the skills acquired in Sections 2.5 and 4.7.

EXAMPLE 5 Determine the domain of each function.

a. $p(x) = \log_2(2x + 3)$

b. $q(x) = \log_5(x^2 - 2x)$

c. $r(x) = \log_{10}\left(\dfrac{3 - x}{x + 3}\right)$

Solution: **a.** Solving $2x + 3 > 0$ gives $x > -\frac{3}{2}$. D: $x \in [-\frac{3}{2}, \infty)$.

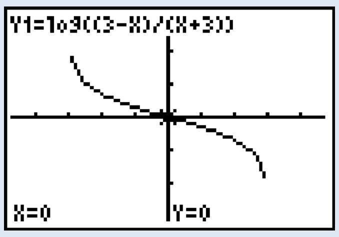
b. For $x^2 - 2x > 0$, we note $y = x^2 - 2x$ is a parabola, concave up, with roots at $x = 0$ and $x = 2$. This means $x^2 - 2x$ will be positive for $x < 0$ and $x > 2$. D: $x \in (-\infty, 0) \cup (2, \infty)$.

c. For $\frac{3 - x}{x + 3} > 0$, both factors have odd multiplicity, with r having a zero at 3 and a vertical asymptote at $x = -3$. Outputs are positive in the interval containing $x = 0$, so y is positive in the interval $(-3, 3)$. D: $x \in (-3, 3)$.

NOW TRY EXERCISES 71 THROUGH 76

GRAPHICAL SUPPORT

The domain for $r(x) = \log_{10}\left(\frac{3 - x}{x + 3}\right)$ from Example 5(c) can be confirmed using the LOG key on a graphing calculator. Use the key to enter the equation as Y_1 on the Y= screen, then graph the function using the ZOOM **4:ZDecimal** option of the TI-84Plus. Both the graph and TABLE feature help to confirm the domain is $x \in (-3, 3)$.



WORTHY OF NOTE

We do something similar with square roots. Technically, the "square root of x" should be written $\sqrt[2]{x}$. However, square roots are so common we often omit the 2, assuming that if no index is written, an index of 2 is intended.

C. Calculators and Common Logarithms

Of all possible bases for $\log_b x$, one of the most common is base 10, presumably due to our base-10 numeration system. In fact, $\log_{10} x$ is called a **common logarithm** and we most often do not write the "10." Instead, we simply write $\log x$ for $\log_{10} x$.

Some base-10 logarithms are easy to evaluate: $\log 1000 = 3$ since $10^3 = 1000$; and $\log \frac{1}{100} = -2$ since $10^{-2} = \frac{1}{100}$. But what about expressions like $\log 850$ and $\log \frac{1}{4}$? Due to the relationship between exponential and logarithmic functions, values must exist for these expressions, as well as for expressions like $\log \sqrt{5}$. Further, the inequality $100 < 850 < 1000$ implies $\log 100 = 2 < \log 850 < \log 1000 = 3$, telling us that $\log 850$ is **bounded** between 2 and 3. A similar inequality can be constructed for $\log \frac{1}{4}$: $\frac{1}{10} < \frac{1}{4} < 1$ implies that $\log \frac{1}{10} < \log \frac{1}{4} < \log 1$, and the value of $\log \frac{1}{4}$ is bounded between -1 and 0. Fortunately, modern calculators have been programmed to compute base-10 logarithms with great speed, and often with 10-decimal-place accuracy. For $\log 850$, press the LOG key, input 850 and press ENTER. The display should read 2.929418926. For $\log \frac{1}{4}$ we can use the decimal form: LOG 0.25 ENTER $\approx -.6020599913$.

WORTHY OF NOTE

After their invention in about 1614, logarithms became widely used as a tool for performing difficult computations. In 1624, Henry Briggs (1561–1630), a professor of geometry at Gresham College, London, published tables that included the base-10 logarithm (the exponent on base 10) for the numbers 1 to 20,000 to *14 decimal places*. This made it possible to compute many products, quotients, and powers by doing little more than looking up the needed values in a table and using the properties of exponents. Logarithmic tables remained in common use for nearly 400 years, right up until the invention of modern calculating technology.

EXAMPLE 6 Estimate the value of each expression by writing an inequality that bounds it between two integers. Then use a calculator to find an approximate value rounded to four decimal places. Note all examples use base 10.

a. $\log 492$ **b.** $\log 2.7$ **c.** $\log 0.009$

Solution: **a.** Since $100 < 492 < 1000$, $\log 492$ is bounded between 2 and 3: $\log 100 = 2 < \log 492 < \log 1000 = 3$. Using a calculator, $\log 492 \approx 2.6920$.

b. Since $1 < 2.7 < 10$, log 2.7 is bounded between 0 and 1: $\log 1 = 0 < \log 2.7 < \log 10 = 1$. Using a calculator, $\log 2.7 \approx .4314$.

c. Since $0.001 < 0.009 < 0.010$, log 0.009 is bounded between -3 and -2: $\log 0.001 = -3 < \log 0.009 < \log 0.010 = -2$. Using a calculator we find $\log 0.009 \approx -2.0458$.

NOW TRY EXERCISES 77 THROUGH 82

D. Applications of Common Logarithms

One application of common logarithms involves the measurement of earthquake intensities, in units called **magnitudes** (also called **Richter values**). The most damaging earthquakes have magnitudes of 6.5 and higher, while the slightest earthquakes are of magnitude 1 and are barely perceptible. The magnitude of the intensity $M(I)$ is given by $M(I) = \log\left(\frac{I}{I_0}\right)$, where I is the measured intensity and I_0 represents the smallest earth movement that can be recorded on a seismograph, called the **reference intensity.** The value of I is often given as a multiple of this reference intensity.

EXAMPLE 7A Find the magnitude of an earthquake (round to hundredths), given (a) $I = 4000I_0$ and (b) $I = 8{,}252{,}000I_0$.

Solution: **a.**

$$M(I) = \log\left(\frac{I}{I_0}\right) \quad \text{given}$$

$$M(4000I_0) = \log\left(\frac{4000I_0}{I_0}\right) \quad \text{substitute } 4000I_0 \text{ for } I$$

$$= \log 4000 \quad \text{simplify}$$

$$\approx 3.60 \quad \text{result}$$

The earthquake had a magnitude of 3.6.

b.

$$M(I) = \log\left(\frac{I}{I_0}\right) \quad \text{given}$$

$$M(8{,}252{,}000I_0) = \log\left(\frac{8{,}252{,}000I_0}{I_0}\right) \quad \text{substitute } 8{,}252{,}000I_0 \text{ for } I$$

$$= \log 8{,}252{,}000 \quad \text{simplify}$$

$$\approx 6.92 \quad \text{result}$$

The earthquake had a magnitude of 6.92.

EXAMPLE 7B How many times more intense than the reference intensity I_0 is an earthquake with a magnitude of 6.7?

Solution:

$$M(I) = \log\left(\frac{I}{I_0}\right) \quad \text{given}$$

$$6.7 = \log\left(\frac{I}{I_0}\right) \quad \text{substitute } M(I) = 6.7$$

$$10^{6.7} = \frac{I}{I_0} \quad \text{write in exponential form}$$

$$I = 10^{6.7}I_0 \qquad \text{solve for } I$$
$$= 5{,}011{,}872I_0 \qquad 10^{6.7} \approx 5{,}011{,}872$$

An earthquake of magnitude 6.7 is over 5 million times more intense than the reference intensity.

NOW TRY EXERCISES 85 THROUGH 90

TECHNOLOGY HIGHLIGHT

Logarithms and Exponents

The keystrokes shown apply to a TI-84 Plus model. Please consult our Internet site or your manual for other models.

Although a calculator can rarely be the final word in a mathematical demonstration, it can be used to verify certain ideas. In particular, we'll look at the properties given on page 488, which show an inverse function relationship exists between $y = \log_b x$ and $y = b^x$. Here, we'll simply demonstrate the inverse relation using a graphing calculator, although we'll limit ourselves to base-10 logarithms. Figure 5.7 shows how the base-10 logarithm "undoes" the base-10 exponential. Figure 5.8 shows how the base-10 exponential "undoes" the base-10 logarithm. Use these examples to work the following exercises.

Figure 5.7

```
log(10^3)
                3
log(10^-2)
               -2
log(10)
                1
log(10^-π)
```

Figure 5.8

```
10^log(50)
               50
10^log(.01)
              .01
10^log(1)
                1
10^log(π)
```

Exercise 1: As you can see, the display screen in Figure 5.10 could not display the result of the final calculation without scrolling. What will the display show after ENTER pressed? $-3.141592654 = -\pi$

Exercise 2: As in Exercise 1, the display screen in Figure 5.11 did not have enough room to display the result of the final calculation. What will the result be? $3.141592654 = \pi$

Exercise 3: First simplify each expression by hand, then use a graphing calculator to verify each result: (a) $\log_{10}10^{2.5}$ 2.5, (b) $\log_{10}10^{\pi}$ π, (c) $10^{\log_{10}2.5}$ 2.5, (d) $10^{\log_{10}\pi}$ π

5.2 EXERCISES

CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. A logarithmic function is of the form $y =$ __$\log_b x$__, where __b__ > 0, __b__ $\neq 1$ and inputs are __greater__ than zero.

2. The range of $y = \log_b x$ is all __real numbers__ and the domain is $x \in$ __$(0, \infty)$__. Further, as $x \to 0$, $y \to$ __$-\infty$__.

3. For logarithmic functions of the form $y = \log_b x$, the x-intercept is __(1, 0)__, since $\log_b 1 =$ __0__.

4. The function $y = \log_b x$ is an increasing function if __$b > 1$__, and a decreasing function if __$0 < b < 1$__.

5. What number does the expression $\log_2 32$ represent? Discuss/explain how $\log_2 32 = \log_2 2^5$ justifies this fact. 5; answers will vary.

6. Explain how the graph of $Y = \log_b(x - 3)$ can be obtained from $y = \log_b x$. Where is the "new" x-intercept? Where is the new asymptote? Shifts right 3; (4, 0); $x = 3$

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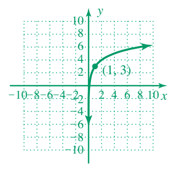
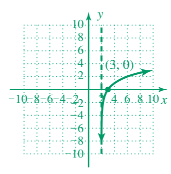
REVISED PAGES

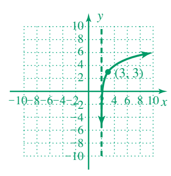
29. $\log_{\frac{1}{3}} 27 = -3$
30. $\log_{\frac{1}{5}}(25) = -2$
31. $\log 1000 = 3$
32. $\log 10 = 1$
33. $\log \frac{1}{100} = -2$
34. $\log \frac{1}{100{,}000} = -5$

63. 

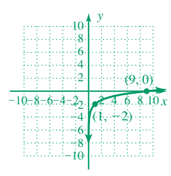
shift up 3

64. 

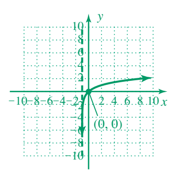
shift right 2

65. 

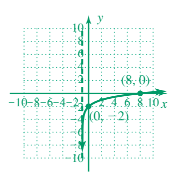
shift right 2, up 3

66. 

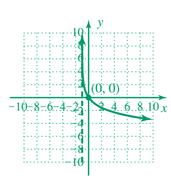
shift down 2

67. 

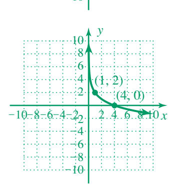
shift left 1

68. 

shift left 1, down 2

69. 

reflect across x-axis, shift left 1

70. 

reflect across x-axis, shift up 2

71. $x \in (-\infty, -1) \cup (3, \infty)$
72. $x \in (-\infty, -3) \cup (2, \infty)$
73. $x \in \left(\frac{3}{2}, \infty\right)$

DEVELOPING YOUR SKILLS

Write each equation in exponential form.

7. $3 = \log_2 8$ $2^3 = 8$
8. $2 = \log_3 9$ $3^2 = 9$
9. $-1 = \log_7 \frac{1}{7}$ $7^{-1} = \frac{1}{7}$
10. $-3 = \log_5 \frac{1}{125}$ $5^{-3} = \frac{1}{125}$
11. $0 = \log_9 1$ $9^0 = 1$
12. $0 = \log_4 1$ $4^0 = 1$
13. $\frac{1}{3} = \log_8 2$ $8^{\frac{1}{3}} = 2$
14. $\frac{1}{2} = \log_{81} 9$ $81^{\frac{1}{2}} = 9$
15. $1 = \log_2 2$ $2^1 = 2$
16. $1 = \log_{20} 20$ $20^1 = 20$
17. $\log_7 49 = 2$ $7^2 = 49$
18. $\log_4 16 = 2$ $4^2 = 16$
19. $\log_{10} 100 = 2$ $10^2 = 100$
20. $\log_{10} 10{,}000 = 4$ $10^4 = 10{,}000$
21. $\log_{10} 0.1 = -1$ $10^{-1} = 0.1$
22. $\log_{10} 0.001 = -3$ $10^{-3} = 0.001$

Write each equation in logarithmic form.

23. $4^3 = 64$ $\log_4 64 = 3$
24. $2^5 = 32$ $\log_2 32 = 5$
25. $3^{-2} = \frac{1}{9}$ $\log_3 \frac{1}{9} = -2$
26. $2^{-3} = \frac{1}{8}$ $\log_2 \frac{1}{8} = -3$
27. $9^0 = 1$ $\log_9 1 = 0$
28. $8^0 = 1$ $\log_8 1 = 0$
29. $\left(\frac{1}{3}\right)^{-3} = 27$
30. $\left(\frac{1}{5}\right)^{-2} = 25$
31. $10^3 = 1000$
32. $10^1 = 10$
33. $10^{-2} = \frac{1}{100}$
34. $10^{-5} = \frac{1}{100{,}000}$
35. $4^{\frac{3}{2}} = 8$ $\log_4 8 = \frac{3}{2}$
36. $216^{\frac{2}{3}} = 36$ $\log_{216} 36 = \frac{2}{3}$
37. $4^{\frac{-3}{2}} = \frac{1}{8}$ $\log_4 \frac{1}{8} = \frac{-3}{2}$
38. $27^{\frac{-2}{3}} = \frac{1}{9}$ $\log_{27} \frac{1}{9} = \frac{-2}{3}$

Determine the value of each expression without using a calculator.

39. $\log_{11} 121$ 2
40. $\log_{12} 144$ 2
41. $\log_3 243$ 5
42. $\log_6 216$ 3
43. $\log_7 \frac{1}{49}$ -2
44. $\log_9 \frac{1}{81}$ -2
45. $\log_4 4$ 1
46. $\log_9 9$ 1
47. $\log_{10} 10$ 1
48. $\log_8 8$ 1
49. $\log_4 2$ $\frac{1}{2}$
50. $\log_{81} 9$ $\frac{1}{2}$

Determine the value of x by writing the equation in exponential form.

51. $\log_5 x = 2$ 25
52. $\log_3 x = 3$ 27
53. $\log_{36} x = \frac{1}{2}$ 6
54. $\log_{64} x = \frac{1}{2}$ 8
55. $\log_x 36 = 2$ 6
56. $\log_x 64 = 2$ 8
57. $\log_x \frac{1}{4} = -2$ 2
58. $\log_x \frac{1}{3} = -1$ 3
59. $\log_{25} x = -\frac{3}{2}$ $\frac{1}{125}$
60. $\log_{16} x = -\frac{5}{4}$ $\frac{1}{32}$
61. $\log_8 32 = x$ $\frac{5}{3}$
62. $\log_9 27 = x$ $\frac{3}{2}$

Graph each function *using transformations* of $y = \log_b x$ and strategically plotting a few points. Clearly state the transformations applied.

63. $f(x) = \log_2 x + 3$
64. $g(x) = \log_2(x - 2)$
65. $h(x) = \log_2(x - 2) + 3$
66. $p(x) = \log_3 x - 2$
67. $q(x) = \log_3(x + 1)$
68. $r(x) = \log_3(x + 1) - 2$
69. $Y_1 = -\log_2(x + 1)$
70. $Y_2 = -\log_2 x + 2$

Determine the domain of the following functions.

71. $y = \log_6\left(\dfrac{x + 1}{x - 3}\right)$
72. $y = \log_3\left(\dfrac{x - 2}{x + 3}\right)$
73. $y = \log_5 \sqrt{2x - 3}$
74. $y = \log_4 \sqrt{5 - 3x}$ $x \in \left(-\infty, \frac{5}{3}\right)$
75. $y = \log(9 - x^2)$ $x \in (-3, 3)$
76. $y = \log(9x - x^2)$ $x \in (0, 9)$

Without using a calculator, write an inequality that bounds each expression between two integers. Then use a calculator to find an approximate value, rounded to four decimal places.

77. $\log 175$ ≈ 2.2430
78. $\log 8.2$ ≈ 0.9138
79. $\log 127{,}962$ ≈ 5.1071
80. $\log 9871$ ≈ 3.9944
81. $\log \frac{1}{5}$ ≈ -0.6990
82. $\log 0.075$ ≈ -1.1249

WORKING WITH FORMULAS

83. pH level: $f(x) = -\log_{10} x$

The pH level of a solution indicates the concentration of hydrogen (H^+) ions in a unit called *moles per liter.* The pH level $f(x)$ is given by the formula shown, where x is the ion concentration (given in scientific notation). A solution with pH < 7 is called an acid (lemon juice: pH ≈ 2), and a solution with pH > 7 is called a base (household ammonia:

pH $\approx$ 11). Use the formula to determine the pH level of tomato juice if $x = 7.94 \times 10^{-5}$ moles per liter. Is this an acid or base solution? pH $\approx$ 4.1; acid

84. Time required for an investment to double: $T(r) = \dfrac{\log 2}{\log(1 + r)}$

The time required for an investment to double in value is given by the formula shown, where r represents the interest rate (expressed as a decimal) and $T(r)$ gives the years required. How long would it take an investment to double if the interest rate were (a) 5%, (b) 8%, (c) 12%? $\approx$14.2 yr $\approx$9.0 yr $\approx$6.1 yr

APPLICATIONS

Brightness of a star: The brightness or intensity I of a star as perceived by the naked eye is measured in units called *magnitudes*. The brightest stars have magnitude 1 $[M(I) = 1]$ and the dimmest have magnitude 6 $[M(I) = 6]$. The magnitude of a star is given by the equation $M(I) = 6 - 2.5 \cdot \log\left(\frac{I}{I_0}\right)$, where I is the actual intensity of light from the star and I_0 is the faintest light visible to the human eye, called the reference intensity. The intensity I is often given as a multiple of this reference intensity.

85. Find the value of $M(I)$ given
a. $I = 27I_0$ $\approx$2.4 and **b.** $I = 85I_0$. $\approx$1.2

86. Find the intensity I of a star given
a. $M(I) = 1.6$ $\approx 57.5I_0$ and **b.** $M(I) = 5.2$. $\approx 2.1I_0$

Earthquake intensity: The intensity of an earthquake is also measured in units called *magnitudes*. The most damaging quakes have magnitudes of 6.5 or greater $[M(I) > 6.5]$ and the slightest are of magnitude 1 $[M(I) = 1]$ and are barely felt. The magnitude of an earthquake is given by the equation $M(I) = \log\left(\frac{I}{I_0}\right)$, where I is the actual intensity of the earthquake and I_0 is the smallest earth movement that can be recorded on a seismograph—called the reference intensity. The intensity I is often given as a multiple of I_0.

87. Find the value of $M(I)$ given
a. $I = 50{,}000I_0$ $\approx$4.7 and **b.** $I = 75{,}000I_0$. $\approx$4.9

88. Find the intensity I of the earthquake given
a. $M(I) = 3.2$ $\approx 1584.9I_0$ and **b.** $M(I) = 8.1$. $\approx 125{,}892{,}541.2I_0$

Intensity of sound: The intensity of sound as perceived by the human ear is measured in units called decibels (dB). The loudest sounds that can be withstood without damage to the eardrum are in the 120- to 130-dB range, while a whisper may measure in the 15- to 20-dB range. Decibel measure is given by the equation $D(I) = 10 \log\left(\frac{I}{I_0}\right)$, where I is the actual intensity of the sound and I_0 is the faintest sound perceptible by the human ear—called the reference intensity. The intensity I is often given as a multiple of this reference intensity, but often the constant 10^{-16} (watts per cm^2; W/cm^2) is used as the threshold of audibility.

89. Find the value of $D(I)$ given
a. $I = 10^{-14}$ 20 dB and **b.** $I = 10^{-4}$. 120 dB

90. Find the intensity I of the sound given
a. $D(I) = 83$ $199{,}526{,}231.5I_0$ and **b.** $D(I) = 125$. $\approx 3.2 \times 10^{12} I_0$

Memory retention: Under certain conditions, a person's retention of random facts can be modeled by the equation $P(x) = 95 - 14 \log_2 x$, where $P(x)$ is the percentage of those facts retained after x number of days. Find the percentage of facts a person might retain after:

91. **a.** 1 day **b.** 4 days **c.** 16 days **a.** 95% **b.** $\approx$67% **c.** $\approx$39%

92. **a.** 32 days **b.** 64 days **c.** 78 days **a.** 25% **b.** $\approx$11% **c.** $\approx$7%

93. Use the formula given in Exercise 93 to determine the pH level of black coffee if $x = 5.1 \times 10^{-5}$ moles per liter. Is black coffee considered an acid or base solution? $\approx$4.3; acid

94. The length of time required for an amount of money to *triple* is given by the formula $T(r) = \dfrac{\log 3}{\log(1 + r)}$ (refer to Exercise 94). Construct a table of values to help estimate what interest rate is needed for an investment to triple in nine years. ≈13%

WRITING, RESEARCH, AND DECISION MAKING

95. The decibel is a unit based on the faintest sound a person can hear, called the threshold of audibility. It is a base-10 logarithmic scale, meaning a sound 10 times more intense is 1-dB louder. Many texts and reference books give estimates of the noise level (in decibels) of common sounds. Through reading and research, try to locate or approximate where the following sounds would fall along this scale. In addition, determine at what point pain or ear damage begins to occur. Answers will vary.

a. threshold of audibility 0 dB **b.** lawn mower 90 dB **c.** whisper 15 dB

d. loud rock concert 120 dB **e.** lively party 100 dB **f.** jet engine 120 dB

96. Tables of base-10 logarithms are still readily available over the Internet, at a library, and in other places. Locate such a table and take an excursion back in time. After reading the example that follows, compute the value of $\frac{2843}{981}$ *using the table* and properties of exponents (verify the result on a calculator). Consider the product 93×207. If each number is rewritten using a base-10 exponential the calculation becomes $10^{1.968482949} \times 10^{2.315970345}$. Using the properties of exponents yields $10^{1.96848294 + 2.31597034} = 10^{4.284453294}$. From the entries within the log table, we find that the number whose log is 4.28445329 is 19,251—which *is* the product of 93×207. 2.898063201

97. Base-10 logarithms are sometimes called *Briggsian* logarithms due to the work of Henry Briggs. See the *Worthy of Note* preceding Example 6. Using resources available to you, locate some additional information on Henry Briggs and write up a short summary. Include information on his contributions to other areas of mathematics. Answers will vary.

EXTENDING THE CONCEPT

98. Determine the value of x that makes the equation true: $\log_3[\log_3(\log_3 x)] = 0$. 27

99. Use properties of exponents *and* logarithms to show $y = \log_{\frac{1}{2}} x$ is equivalent to $y = -\log_2 x$. $\left(\frac{1}{2}\right)^y = x$, $2^{-y} = x$, $-y = \log_2 x$, $y = -\log_2 x$

100. Find the value of each expression.

a. $\log_{64}\frac{1}{16}$ $\frac{-2}{3}$ **b.** $\log_{\frac{4}{9}}\frac{27}{8}$ $\frac{-3}{2}$ **c.** $\log_{0.25}32$ $\frac{-5}{2}$

MAINTAINING YOUR SKILLS

101. (1.2) Graph the solution set for (a) $x < 3$ and $x > -1$ and (b) $x < 3$ or $x > -1$.

102. (3.3) Graph $g(x) = \sqrt[3]{x + 2} - 1$ by shifting the parent function. Then state the domain and range of g.

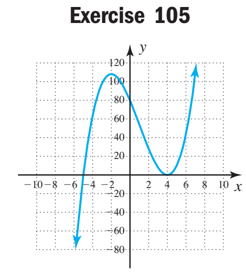
103. (R.4) Factor the following expressions:

a. $x^3 - 8$ **b.** $a^2 - 49$ **c.** $n^2 - 10n + 25$ **d.** $2b^2 - 7b + 6$

104. (1.4) Find the sum, difference, product, and quotient of the complex numbers $1 + 3i$ and $1 - 3i$. 2; $6i$; 10; $\frac{-4}{5} + \frac{3}{5}i$

105. (4.4) For the graph shown, write the solution set for $f(x) < 0$. Then write the equation of the graph in factored form and in polynomial form. $x \in (-\infty, -5)$; $f(x) = (x + 5)(x - 4)^2 = x^3 - 3x^2 - 24x + 80$

Exercise 105

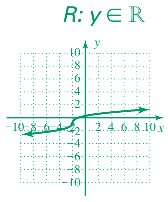


106. (2.2) A function $f(x)$ is defined by the ordered pairs shown in the table. Is the function (a) linear? (b) increasing? Justify your answers. **a.** No **b.** No, Answers will vary.

101. a.

b.

102. $D: x \in \mathbb{R}$
$R: y \in \mathbb{R}$



103. a. $(x - 2)(x^2 + 2x + 4)$
b. $(a + 7)(a - 7)$
c. $(n - 5)(n - 5)$
d. $(2b - 3)(b - 2)$

Exercise 106

x	y
−10	0
−9	−2
−8	−8
−6	−18
−5	−50
−4	−72

5.3 The Exponential Function and Natural Logarithms

LEARNING OBJECTIVES

In Section 5.3 you will learn how to:

A. Evaluate and graph base *e* exponential functions

B. Evaluate and graph base *e* logarithmic functions

C. Apply the properties of logarithms

D. Use the change-of-base formula

E. Solve applications of natural logarithms

F. Compute average rates of change for $y = e^x$ and $y = \ln(x)$

INTRODUCTION

Up to this point we've seen a large number of base-2 exponential and logarithmic functions. This is because they're convenient for most calculations and enable a study of the basic graphs without the outputs getting too large. We've also used a number of base-10 functions, primarily for traditional reasons and their connections with our base-10 numeration system. However, in virtually all future course work, base $e \approx 2.718$ will be more common by far. So much so, the base-*e* exponential function is referred to as *the* exponential function, as in the title of this section. We explore the reasons why here.

POINT OF INTEREST

In addition to the "rate of growth" advantage it offers, using base *e* simplifies a number of important calculations in future courses (some bases are easier to work with than others); adds a great deal of understanding to the study of complex numbers; and is extensively used in science, engineering, business, and finance applications.

A. *The* Exponential Function $y = e^x$

In Section 5.1, we discussed the city of Goldsboro—a hypothetical boomtown from the Old West whose population at time t in years was $P(t) = 1000 \cdot 2^t$. Suppose we were more interested in the *rate of growth* of this town, or more specifically, its rate of growth expressed as a percentage of the current population. In Section 5.6 we'll show that $P(t)$ can be equivalently written as $P(t) = 1000e^{kt}$, where k is a constant that gives this growth rate *exactly,* and e is an irrational number whose approximate value is 2.718281828 (to nine decimal places). Knowing this rate of growth offers a huge advantage in applications of the exponential function $y = e^x$, also known as the **natural exponential function.**

WORTHY OF NOTE

For the Goldsboro example, k is approximately 0.693147. Using a graphing calculator, compare the functions $Y_1 = 1000 \cdot 2^t$ and $Y_2 = 1000(2.7182818)^{0.693147t}$ using the TABLE feature. What do you notice? Soon we'll introduce a much more convenient way to write this exponential base.

In Section 5.5 the value of e is developed in the context of compound interest. For now we'll define it as the number that $\left(1 + \frac{1}{x}\right)^x$ approaches as x becomes very large. It can be shown that as $x \to \infty$, $\left(1 + \frac{1}{x}\right)^x$ approaches the unique, irrational number e we approximated earlier. Table 5.3 gives values of the expression for selected inputs x. Just as we use the symbol π to represent the irrational number 3.141592654 . . . (the ratio of a circle's circumference to its diameter), we use the symbol e to represent the irrational number 2.718281828 . . . . This symbol is used in honor of the famous mathematician Leonhard Euler (1707–1783), who studied the number extensively.

Table 5.3

x	$\left(1 + \frac{1}{x}\right)^x$
1	2
10	2.59
100	2.705
1000	2.7169
10,000	2.71815
100,000	2.718268
1,000,000	2.7182805
10,000,000	2.71828169

Instead of having to enter a decimal approximation when computing with e, most calculators have an e^x key, usually as the 2nd function for the key marked IN. To find the value of e^2, use the keystrokes 2nd LN 2 ENTER, and the calculator display should read 7.389056099.

EXAMPLE 1 Use a calculator to evaluate each expression, rounded to six decimal places.

a. e^3 **b.** e^1 **c.** e^0 **d.** $e^{\frac{1}{2}}$

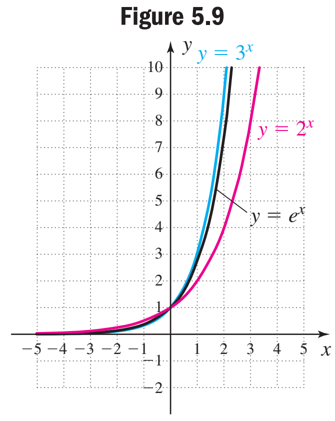
Solution: **a.** $e^3 \approx 20.085537$ **b.** $e^1 \approx 2.718282$

c. $e^0 = 1$ (exactly) **d.** $e^{\frac{1}{2}} \approx 1.648721$

NOW TRY EXERCISES 7 THROUGH 14

Although e is an irrational number, the graph of $y = e^x$ behaves in exactly the same way and has the same characteristics as other exponential graphs. Figure 5.9 shows the graph on the same grid as $y = 2^x$ and $y = 3^x$. As we might expect, all three graphs are increasing, have an asymptote at $y = 0$, and contain the point (0, 1), with the graph of $y = e^x$ "between" the other two. The domain for all three functions, as with all basic exponential functions, is $x \in (0, \infty)$ with a range of $y \in (-\infty, \infty)$. The same transformations applied earlier can also be applied to the graph of $y = e^x$. See Exercises 15 through 20.

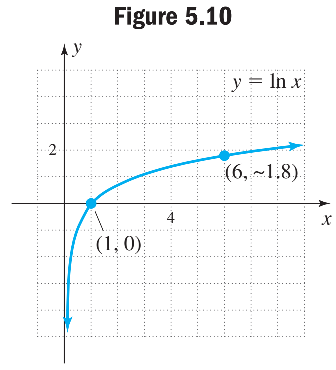
Figure 5.9



B. The Natural Log Function $y = \ln x$

In Section 5.2 we introduced the *common* logarithmic function or logarithms that use base 10: $y = \log x$. The **natural logarithmic function** uses base e, the irrational number just introduced: $y = \log_e x$. Partly due to its widespread use, the notation $y = \log_e x$ is abbreviated $y = \ln x$, and read "y is equal to the natural log of x." At this point you might realize that $y = \ln x$ is the inverse function for $y = e^x$, just as $y = \log_2 x$ was the inverse function for $y = 2^x$. Also, it's important to remember that *regardless of the base used,* a logarithm represents the exponent that goes on base b to obtain x. In other words, $y = \ln x$ can be written in exponential form as $e^y = x$. The graph of $y = \ln x$ is shown in Figure 5.10. Note it has the same shape and characteristics as other logarithmic graphs.

Figure 5.10



NATURAL LOGARITHMIC FUNCTION

$$y = \log_e x = \ln x$$

- domain: $x \in (0, \infty)$
- range: $y \in \mathbb{R}$
- one-to-one function
- x-intercept (1, 0)
- increasing
- asymptotic to the y-axis

To evaluate $y = \ln x$, the LN key is used in exactly the same way the LOG key was used for $y = \log x$. Refer to the Technology Highlight on page 491 if needed.

EXAMPLE 2 Use a calculator to evaluate each expression and round to six decimal places.

a. $\ln 22$ **b.** $\ln 0.28$ **c.** $\ln 7$ **d.** $\ln\left(-\frac{5}{2}\right)$

Solution: **a.** $\ln 22 \approx 3.091042$ **b.** $\ln 0.28 \approx -1.272966$

c. $\ln 7 \approx 1.945910$ **d.** $\ln\left(-\frac{5}{2}\right)$ is not a real number

($-\frac{5}{2}$ is not in the domain)

NOW TRY EXERCISES 21 THROUGH 28

EXAMPLE 3 Solve each equation by writing it in exponential form. Answer in exact form and approximate form using a calculator (round to thousandths).

a. $\ln x = 2$ **b.** $\ln x = -2.8$

Solution: **a.** $\ln x = 2$, $e^2 = x$, $x \approx 7.389$ **b.** $\ln x = -2.8$, $e^{-2.8} = x$, $x \approx 0.061$

NOW TRY EXERCISES 29 THROUGH 36

EXAMPLE 4 Solve each equation by writing it in logarithmic form. Answer in exact form and approximate form using a calculator (round to five decimal places).

a. $e^x = 120$ **b.** $e^x = 0.043214$

Solution: **a.** $e^x = 120$, $x = \ln 120$, ≈ 4.78749 **b.** $e^x = 0.043214$, $x = \ln 0.043214$, ≈ -3.14159

NOW TRY EXERCISES 37 THROUGH 44

C. The Product, Quotient, and Power Properties

To solve the equation $2\sqrt{x} + 5\sqrt{x} = 35$, we first combine like terms on the left and isolate the radical, before squaring both sides: $7\sqrt{x} = 35 \rightarrow \sqrt{x} = 5$, giving the solution $x = 25$. The same principle holds for equations that involve logarithms. Before we can solve $\log_4 x + \log_4(x + 6) = 2$, we must find a way to combine terms on the left. Due to the close connection between exponents and logarithms, their properties are very similar. For the product of exponential terms with like bases, we add the exponents: $x^m x^n = x^{m+n}$. This is reflected in the product property of logarithms, where the logarithm of a product also results in a sum of exponents, *which is exactly what* "$\log_b M + \log_b N$" *represents.*

PROPERTIES OF LOGARITHMS

Given M, N, and b are *positive* real numbers, where $b \neq 1$, and *any* real number x.

Product Property: $\log_b(MN) = \log_b M + \log_b N$

In words: "The log of a product is equal to a sum of logarithms."

Quotient Property: $\log_b \frac{M}{N} = \log_b M - \log_b N$

In words: "The log of a quotient is equal to a difference of logarithms."

Power Property: $\log_b(M)^x = x \log_b M$

In words: "The log of a number to a power is equal to the power times the log of the number."

Proof of the Product Property:
Given $P = \log_b M$ and $Q = \log_b N$, we have $b^P = M$ and $b^Q = N$ in exponential form. It follows that

$$\begin{aligned}\log_b(MN) &= \log_b(b^P b^Q) && \text{substitute } b^P \text{ for } M \text{ and } b^Q \text{ for } N\\ &= \log_b(b^{P+Q}) && \text{properties of exponents}\\ &= P + Q && \text{log property III (page 488)}\\ &= \log_b M + \log_b N && \text{substitute } \log_b M \text{ for } P \text{ and } \log_b N \text{ for } Q\end{aligned}$$

The proof of the quotient property is very similar to that of the product property, and is left as an exercise (see Exercise 118).

Proof of the Power Property:
Given $P = \log_b M$, we have $b^P = M$ in exponential form. It follows that

$$\begin{aligned}\log_b(M)^x &= \log_b(b^P)^x && \text{substitute } b^P \text{ for } M\\ &= \log_b(b^{Px}) && \text{properties of exponents}\\ &= Px && \text{log property III}\\ &= x\log_b M && \text{substitute } \log_b M \text{ for } P\end{aligned}$$

EXAMPLE 5 Use properties of logarithms to write each expression as a single term.

a. $\log_2 7 + \log_2 5$ **b.** $\ln x + \ln(x + 6)$

c. $\log_6 30 - \log_6 10$ **d.** $\ln(x + 2) - \ln x$

Solution:

a. $\log_2 7 + \log_2 5 = \log_2(7 \cdot 5)$ product property
$= \log_2 35$ simplify

b. $\ln x + \ln(x + 6) = \ln[x(x + 6)]$ product property
$= \ln[x^2 + 6x]$ simplify

c. $\log_6 30 - \log_6 10 = \log_6 \frac{30}{10}$ quotient property
$= \log_6 3$ simplify

d. $\ln(x + 2) - \ln x = \ln\left(\frac{x + 2}{x}\right)$ quotient property

NOW TRY EXERCISES 45 THROUGH 60

EXAMPLE 6 Use the power property to rewrite each term a product.

a. $\ln 5^x$ **b.** $\log 32^{x+2}$ **c.** $\log \sqrt{x}$ **d.** $\log_2 125$

Solution:

a. $\ln 5^x = x\ln 5$ power property

b. $\log 32^{x+2} = (x + 2)\log 32$ power property (note the use of parentheses)

c. $\log \sqrt{x} = \log x^{\frac{1}{2}}$ rewrite argument using a rational exponent
$= \frac{1}{2}\log x$ power property

d. $\log_2 125 = \log_2 5^3$ rewrite argument as an exponential term
$= 3\log_2 5$ power property

NOW TRY EXERCISES 61 THROUGH 68

CAUTION

Note from Example 6(b) that parentheses *must be used* whenever the exponent is a sum or difference. There is a huge difference between $(x + 2)\log 32$ and $x + 2\log 32$.

In some cases, applying these properties can help to rewrite an expression in a form that enables certain techniques to be applied more easily. Example 7 actually lays the foundation for more advanced work.

EXAMPLE 7 Use the properties of logarithms to write the following expressions as a sum or difference of simple logarithmic terms.

a. $\log(x^2y)$ **b.** $\log\left(\sqrt{\dfrac{x}{x+5}}\right)$

Solution:

a. $\log(x^2y) = \log x^2 + \log y$ product property

$= 2\log x + \log y$ power property

b. $\log\left(\sqrt{\dfrac{x}{x+5}}\right) = \log\left(\dfrac{x}{x+5}\right)^{\frac{1}{2}}$ write radicals in exponential form

$= \dfrac{1}{2}\log\left(\dfrac{x}{x+5}\right)$ power property

$= \dfrac{1}{2}[\log x - \log(x+5)]$ quotient property

NOW TRY EXERCISES 69 THROUGH 78

D. The Change-of-Base Formula

Although base-10 and base-e logarithms dominate the mathematical landscape, there are many practical applications that use other bases. Fortunately, a formula exists that will convert any given base into either base 10 or base e. It's called the **change-of-base formula.**

CHANGE-OF-BASE FORMULA

Given the positive real numbers M, b, and d, where $b \neq 1$ and $d \neq 1$,

$$\log_b M = \frac{\log M}{\log b} \qquad \log_b M = \frac{\ln M}{\ln b} \qquad \log_b M = \frac{\log_d M}{\log_d b}$$

base 10 base e arbitrary base d

Proof of the Change-of-Base Formula:

Given $P = \log_b M$, we have $b^P = M$ in exponential form. It follows that

$\log_d(b^P) = \log_d M$ take the logarithm of both sides

$P\log_d b = \log_d M$ apply power property of logarithms

$P = \dfrac{\log_d M}{\log_d b}$ divide by $\log_d b$

$\log_b M = \dfrac{\log_d M}{\log_d b}$ substitute $\log_b M$ for P

EXAMPLE 8 Find the value of each expression using the change-of-base formula. Answer in exact form and approximate form using nine digits, *then verify the result* using the original base.

a. $\log_3 29$ **b.** $\log_5 3.6$

Solution: **a.** $\log_3 29 = \dfrac{\log 29}{\log 3}$ $= 3.065044752$ **b.** $\log_5 3.6 = \dfrac{\log 3.6}{\log 5}$ $= 0.795888947$

Check: $3^{3.065044752} = 29$ ✓ **Check:** $5^{0.795888947} = 3.6$ ✓

NOW TRY EXERCISES 79 THROUGH 86

The change-of-base formula can also be used to study and graph logarithmic functions of *any* base. For $f(x) = \log_b x$, the right-hand expression is simply rewritten using the formula and the equivalent function is $f(x) = \dfrac{\log x}{\log b}$. The new function can then be evaluated in this form, or used to study the graph as in the *Technology Highlight* following Example 11. Also see Exercises 87 through 90.

E. Applications of Natural Logarithms

One application of the natural log function involves the relationship between altitude and barometric pressure. The altitude or height above sea level can be determined by the formula $h(T) = (30T + 8000)\ln\dfrac{P_0}{P}$, where $h(T)$ represents the height in feet at temperature T in degrees Celsius, P is the barometric pressure in **centimeters of mercury** (cmHg), P_0 is barometric pressure at sea level (76 cmHg).

EXAMPLE 9 Hikers climbing Mt. Everest take a reading of 6.4 cmHg at a temperature of 5°C. How far up the mountain are they?

Solution: For this exercise, $P_0 = 76$, $P = 6.4$, and $T = 5$. The formula yields

$$h(T) = (30T + 8000)\ln\frac{P_0}{P} \quad \text{given function}$$

$$h(5) = [30(5) + 8000]\ln\frac{76}{6.4} \quad \text{substitute given values}$$

$$= 8150\ln 11.875 \quad \text{simplify}$$

$$\approx 20{,}167 \quad \text{result}$$

The hikers are approximately 20,167 ft above sea level.

NOW TRY EXERCISES 93 AND 94

A second application of natural logarithms involves interest compounded continuously. The formula $T = \dfrac{1}{r}\ln\dfrac{A}{P}$ gives the length of time T (in years) required for an initial principal P to grow to an amount A at a given interest rate.

EXAMPLE 10 If Shelley deposits $2000 at 6% compounded continuously, how many years will it take the money to grow to $3500?

Solution: For this exercise, $A = 3500$, $P = 2000$, and $r = 6\%$.

$$T = \frac{1}{r}\ln\frac{A}{P} \qquad \text{given formula}$$

$$= \frac{1}{0.06}\ln\frac{3500}{2000} \qquad \text{substitute given values}$$

$$= \frac{\ln 1.75}{0.06} \qquad \text{simplify}$$

$$\approx \frac{0.55961579}{0.06} \qquad \text{find value of ln 9.6}$$

$$\approx 9.33 \qquad \text{result}$$

Under these conditions, \$2000 grows to \$3500 in about 9 yr and 4 months.

NOW TRY EXERCISES 95 AND 96

F. Rates of Change

As with the functions previously introduced, we are very interested in the concept of average rates of change due to the important role it plays in applications of mathematics. From the graph of $y = \ln x$, we note the function is "very steep" (increases very quickly) for $x \in (0, 0.25)$, with the secant line having a large and positive slope. The secant lines then become much less steep as $x \to \infty$, with very small (but always positive/increasing) slopes. We can quantify these descriptions using the rate of change formula from Section 2.4.

EXAMPLE 11 Use the rate-of-change formula to find the average rate of change of $y = \ln x$ in these intervals:

a. [0.20, 0.21], **b.** [0.99, 1.00], and **c.** [4.99, 5.00].

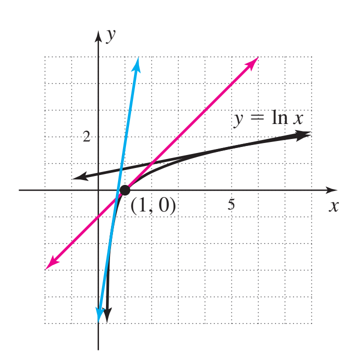
Solution: Apply the formula for average rates of change: $\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$.

a. $\frac{\Delta y}{\Delta x} = \frac{\ln 0.21 - \ln 0.20}{0.21 - 0.20}$
≈ 4.9

b. $\frac{\Delta y}{\Delta x} = \frac{\ln 1.00 - \ln 0.99}{1.00 - 0.99}$
≈ 1

c. $\frac{\Delta y}{\Delta x} = \frac{\ln 5.00 - \ln 4.99}{5.00 - 4.99}$
≈ 0.2

The corresponding secant lines are drawn on the graph shown here. As you can see, the secant line through [0.20, 0.21] (in blue) is much steeper than the secant line through [4.99, 5.00] (in black).



NOW TRY EXERCISES 103 AND 104

One of the many fascinating things about the exponential function involves the relationship between its rate of change in a small interval, and the value of the function in that interval. You are asked to explore this relationship in Exercises 105 and 106.

TECHNOLOGY HIGHLIGHT

Using the Change-of-Base Formula to Study Logarithms

The keystrokes shown apply to a TI-84 Plus model. Please consult your manual or our Internet site for other models.

Using the change-of-base formula, we can study logarithmic functions of any base. Many times we find base 2 or $y = \log_2 x$ more convenient to study than $y = \log_{10} x$, since the related exponential function $y = 10^x$ grows very large, very fast; or $y = \ln x$, where so many of the results are irrational numbers. Let's verify many of the things we've learned about logarithms using $y = \log_2 x$ (actually its equivalent equation $y = \frac{\log x}{\log 2}$) by using the change-of-base formula. Enter this expression as Y_1 on the Y= screen as shown in Figure 5.11. Since we know the general shape of the function and that the domain is $x \in (0, \infty)$, we can preset the viewing window before pressing GRAPH (see WINDOW screen in Figure 5.12).

Figure 5.11

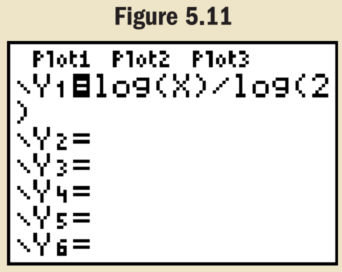
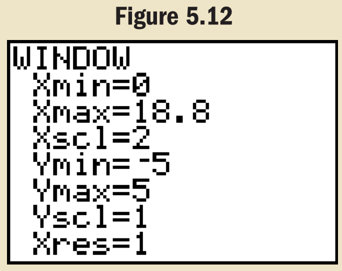
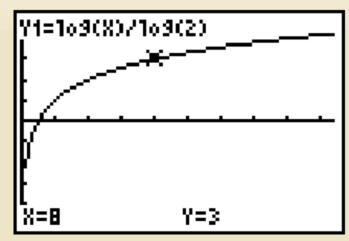


Figure 5.12



Set Xmax = 18.8 to ensure a friendly window as we TRACE through values on the graph. Now press GRAPH and the graph of $y = \log_2 x$ appears as seen in Figure 5.13. Use GRAPH, TABLE, TRACE or 2nd TRACE (CALC) features of your calculator to work the following exercises.

Figure 5.13



Exercise 1: Verify that the x-intercept of the function is (1, 0). $\frac{\log 1}{\log 2} = \frac{0}{\log 2} = 0$

Exercise 2: Solve by inspection, then verify using TRACE or TABLE: $\log_2 16 = k$. 4

Exercise 3: Find the value of $\sqrt{2}$ on your home screen, then find the value of $\log_2\sqrt{2}$. 0.5

Exercise 4: Solve by inspection, then verify using TABLE: $\log_2 x = -1$. What is output for an input of $x = 0.25$? -2

Exercise 5: How would you find the value of $\log_2\sqrt{3}$? $\frac{\log\sqrt{3}}{\log 2}$

Exercise 6: How would you solve the equation $\log_2 x = \sqrt{3}$? $2^{\sqrt{3}}$

5.3 EXERCISES

CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. The number e is defined as the value $\underline{(1 + \frac{1}{x})^x}$ approaches as $x \underline{\to \infty}$.

2. The statement $\log_e 10 = \frac{\log 10}{\log e}$ is an example of the __change__-of-__base__ property.

3. The expression $\log_e(x)$ is commonly written __ln x__, but still represents the __exponent__ that goes on base __e__ to obtain x.

4. If $\ln 12 \approx 2.485$, then $e^{2.485} \approx$ __12__. If (1.4, 4.055) is a point on the graph of $y = e^x$, then (__4.055__, __1.4__) is a point on the graph of $y = \ln x$.

5. Without using the change-of-base formula, which of the following represents a larger number: $\log_2 9$ or $\log_3 26$? Explain your reasons and justify your response. $\log_2 9 > 3$; $\log_3 26 < 3$

6. Compare/contrast the graphs of each function including a discussion of their domains, intercepts, and whether each is increasing or decreasing: $f(x) = \ln x$, $g(x) = -\ln x$, $p(x) = \ln(-x)$, $q(x) = -\ln(-x)$.
Answers will vary.

DEVELOPING YOUR SKILLS

Use a calculator to evaluate each expression, rounded to six decimal places.

7. e^1 2.718282
8. e^0 1
9. e^2 7.389056
10. e^5 148.413159
11. $e^{1.5}$ 4.481689
12. $e^{-3.2}$ 0.040762
13. $e^{\sqrt{2}}$ 4.113250
14. e^{π} 23.140693

Graph each exponential function.

15. $f(x) = e^{x+3} - 2$
16. $g(x) = e^{x-2} + 1$
17. $r(t) = -e^t + 2$
18. $s(t) = -e^{t+2}$
19. $p(x) = e^{-x+2} - 1$
20. $q(x) = e^{-x-1} + 2$

Use a calculator to evaluate each expression, rounded to six decimal places.

21. $\ln 50$ 3.912023
22. $\ln 28$ 3.332205
23. $\ln 0.5$ −0.693147
24. $\ln 0.75$ −0.287682
25. $\ln 225$ 5.416100
26. $\ln 382$ 5.945421
27. $\ln \sqrt{2}$ 0.346574
28. $\ln \pi$ 1.144730

Solve each equation by writing it in exponential form. Answer in exact form and approximate form using a calculator (round to thousandths).

29. $\ln x = 1$ $x = e$; $x \approx 2.718$
30. $\ln x = 0$ $x = 1$; $x = 1$
31. $\ln x = -1.961$ $x = e^{-1.961}$; $x \approx 0.141$
32. $\ln x = 2.485$ $x = e^{2.485}$; $x \approx 12.001$
33. $-2.4 = \ln\left(\frac{1}{x^2}\right)$ $x = \pm\sqrt{e^{2.4}}$; $x \approx \pm 3.320$
34. $-0.345 = \ln\left(\frac{1}{x^3}\right)$ $x = \sqrt[3]{e^{0.345}}$; $x \approx 1.122$
35. $\ln e^{2x} = -8.4$ $x = -4.2$; $x = -4.2$
36. $\ln e^{3x} = -9.6$ $x = -3.2$; $x = -3.2$

Solve each equation by writing it in logarithmic form. Answer in exact form and approximate form using a calculator (round to five decimal places).

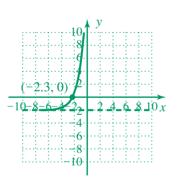
37. $e^x = 1$
38. $e^{3x} = \sqrt{2}$
39. $e^x = 7.389$
40. $e^x = 54.598$
41. $e^{\frac{2x}{5}} = 1.396$
42. $e^{\frac{3x}{2}} = 4.482$
43. $e^x = -0.30103$
44. $e^x = -23.14069$

Use properties of logarithms to write each expression as a single term.

45. $\ln(2x) + \ln(x - 7)$
46. $\ln(x + 2) + \ln(3x)$
47. $\log(x + 1) + \log(x - 1)$
48. $\log(x - 3) + \log(x + 3)$
49. $\log_3 28 - \log_3 7$
50. $\log_6 30 - \log_6 10$
51. $\log x - \log(x + 1)$
52. $\log(x - 2) - \log x$
53. $\ln(x - 5) - \ln x$
54. $\ln(x + 3) - \ln(x - 1)$
55. $\ln(x^2 - 4) - \ln(x + 2)$
56. $\ln(x^2 - 25) - \ln(x + 5)$
57. $\log_2 7 + \log_2 6$
58. $\log_9 2 + \log_9 15$
59. $\log_5(x^2 - 2x) + \log_5 x^{-1}$
60. $\log_3(-1) + \log_3(3x^2 + 5x)$

Use the power property of logarithms to rewrite each term as the product of a constant and a logarithmic term.

61. $\log 8^{x+2}$ $(x + 2)\log 8$
62. $\log 15^{x-3}$ $(x - 3)\log 15$
63. $\ln 5^{2x-1}$ $(2x - 1)\ln 5$
64. $\ln 10^{3x+2}$ $(3x + 2)\ln 10$
65. $\log \sqrt{22}$ $\frac{1}{2}\log 22$
66. $\log \sqrt[3]{34}$ $\frac{1}{3}\log 34$
67. $\log_5 81$ $4\log_5 3$
68. $\log_7 121$ $2\log_7 11$

15.


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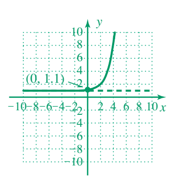

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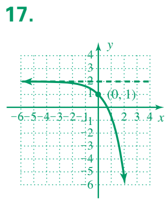

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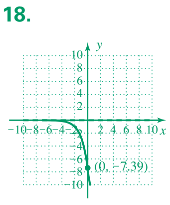

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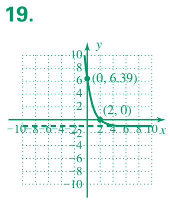
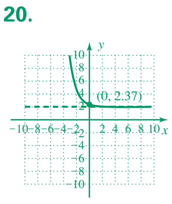


20.


37. $x = 0$; $x = 0$
38. $x = \frac{\ln\sqrt{2}}{3}$; $x \approx 0.11552$
39. $x = \ln 7.389$; $x \approx 1.99999$
40. $x = \ln 54.598$; $x \approx 4.00000$
41. $x = \frac{5\ln 1.396}{2}$; $x \approx 0.83403$
42. $x = \frac{2\ln 4.482}{3}$; $x \approx 1.00005$
43. not a real number
44. not a real number
45. $\ln(2x^2 - 14x)$
46. $\ln(3x^2 + 6x)$
47. $\log(x^2 - 1)$
48. $\log(x^2 - 9)$
49. $\log_3 4$

Additional answers can be found in the Instructor Answer Appendix.

Use the properties of logarithms to write the following expressions as a sum or difference of simple logarithmic terms.

69. $\log(a^3b)$ $3\log a + \log b$ **70.** $\log(m^2n)$ $2\log m + \log n$ **71.** $\ln(x\sqrt[4]{y})$ $\ln x + \frac{1}{4}\ln y$

72. $\ln(\sqrt[3]{p}q)$ $\frac{1}{3}\ln p + \ln q$ **73.** $\ln\left(\frac{x^2}{y}\right)$ $2\ln x - \ln y$ **74.** $\ln\left(\frac{m^2}{n^3}\right)$ $2\ln m - 3\ln n$

75. $\log\left(\sqrt{\frac{x-2}{x}}\right)$ $\frac{1}{2}[\log(x-2) - \log x]$ **76.** $\log\left(\sqrt[3]{\frac{3-v}{2v}}\right)$ $\frac{1}{3}[\log(3-v) - \log 2v]$ **77.** $\ln\left(\frac{7x\sqrt{3-4x}}{2(x-1)^3}\right)$ $\ln 7 + \ln x + \frac{1}{2}\ln(3-4x) - \ln 2 - 3\ln(x-1)$

78. $\ln\left(\frac{x^4\sqrt{x^2-4}}{\sqrt[3]{x^2+5}}\right)$ $4\ln x + \frac{1}{2}\ln(x^2-4) - \frac{1}{3}\ln(x^2+5)$

Evaluate each expression using the change-of-base formula and either base 10 or base *e*. Answer in exact form and in approximate form using nine digits, then verify the result using the original base.

79. $\log_7 60$ **80.** $\log_8 92$ **81.** $\log_5 152$ **82.** $\log_6 200$

83. $\log_3 1.73205$ **84.** $\log_2 1.41421$ **85.** $\log_{0.5} 0.125$ **86.** $\log_{0.2} 0.008$

Use the change-of-base formula to write an equivalent function, then evaluate the function as indicated (round to four decimal places). Investigate and discuss any patterns you notice in the output values, then determine the next input that will continue the pattern.

87. $f(x) = \log_3 x;\ f(5),\ f(15),\ f(45)$ **88.** $g(x) = \log_2 x;\ g(5),\ g(10),\ g(20)$

89. $h(x) = \log_9 x;\ h(2),\ h(4),\ h(8)$ **90.** $H(x) = \log_\pi x;\ H(\sqrt{2}),\ H(2),\ H(\sqrt{2^3})$

WORKING WITH FORMULAS

91. The altitude of an airplane: $A(P) = -4.762\ln(0.068P)$

The altitude of an airplane is given by the formula shown, where *P* represents the air pressure (in pounds per square inch) and $A(P)$ represents the altitude in miles. As a plane is flying, the following pressure readings are obtained: (a) 3.2 lb/in^2, (b) 7.1 lb/in^2, and (c) 10.2 lb/in^2. Use the formula to determine the altitude of the plane at each reading (rounded to the nearest quarter mile). Is the plane gaining or losing altitude? **a.** 7.25 mi **b.** 3.50 mi **c.** 1.75 mi; losing altitude

92. Time required for a population to double: $T(r) = \dfrac{\ln(2)}{r}$

The time required for a population to double is given by the formula shown, where *r* represents the growth rate of the population (expressed as a decimal) and $T(r)$ gives the years required. How long would it take a population to double (rounded to the nearest whole number of years) if the growth rate were (a) 5%; (b) 10%; and (c) 23%? **a.** 14 yr **b.** 7 yr **c.** 3 yr

APPLICATIONS

93. Altitude: Refering to Example 9, hikers on Mt. Everest take successive readings of 42 cm of mercury at 5°C, 30 cm of mercury at 2°C, and 12 cm of mercury at −6°C. (a) How far up the mountain are they at each reading? (b) Approximate the height of Mt. Everest if the temperature at the summit is −12°C and the barometric pressure is 1.7 cm of mercury. **a.** 4833.5 ft, 7492.1 ft 14,434.4 ft; **b.** 29,032.8 ft

94. Business: An advertising agency has determined that the number of items sold is related to the amount *A* spent on advertising by the equation $N(A) = 1500 + 315\ \ln(A)$, where *A* represents the amount spent on advertising and $N(A)$ gives the number of sales. Determine the approximate number of items that will be sold if (a) \$10,000 is spent on advertising and (b) \$50,000 is spent on advertising. (c) Use the TABLE feature of a calculator to estimate how large an advertising budget is needed (to the nearest \$500) to sell 5000 items. **a.** 4401 **b.** 4908 **c.** \$67,000

79. $\frac{\ln 60}{\ln 7}$; 2.104076884

80. $\frac{\ln 92}{\ln 8}$; 2.174520652

81. $\frac{\ln 152}{\ln 5}$; 3.121512475

82. $\frac{\ln 200}{\ln 6}$; 2.957047225

83. $\frac{\log 1.73205}{\log 3}$; 0.499999576

84. $\frac{\log 1.41421}{\log 2}$; 0.499996366

85. $\frac{\log 0.125}{\log 0.5}$; 3

86. $\frac{\log 0.008}{\log 0.2}$; 3

87. $f(x) = \frac{\log(x)}{\log(3)}$; $f(5) \approx 1.4650$; $f(15) \approx 2.4650$; $f(45) \approx 3.4650$; outputs increase by 1; $f(3^3 \cdot 5) = 4.465$

88. $g(x) = \frac{\log(x)}{\log(2)}$; $g(5) \approx 2.3219$; $g(10) \approx 3.3219$; $g(20) \approx 4.3219$; outputs increase by 1; $g(2^3 \cdot 5) \approx 5.3219$

89. $h(x) = \frac{\log(x)}{\log(9)}$; $h(2) \approx 0.3155$; $h(4) \approx 0.6309$; $h(8) \approx 0.9464$; outputs are multiples of 0.3155; $h(2^4) = 4(0.3155) \approx 1.2619$

90. $H(x) = \frac{\log(x)}{\log(\pi)}$; $H(\sqrt{2}) \approx 0.3028$; $H(2) \approx 0.6055$; $H(\sqrt{2^3}) \approx 0.9083$; outputs are multiples of 0.3028; $H(\sqrt{2^4}) = H(4) \approx 1.2110$

Exercise 93

Use the formula from Example 10 $\left[T = \frac{1}{r}\ln\left(\frac{A}{P}\right)\right]$ for the length of time T (in years) required for an initial principal P to grow to an amount A at a given interest rate r.

95. **Investment growth:** A small business is planning to build a new $350,000 facility in 8 yr. If they deposit $200,000 in an account that pays 5% interest compounded continuously, will they have enough for the new facility in 8 yr? If not, what amount should be invested on these terms to meet the goal? No; $234,612.01

96. **Investment growth:** After the twins were born, Sasan deposited $25,000 in an account paying 7.5% compounded continuously, with the goal of having $120,000 available for their college education 20 yr later. Will Sasan meet the 20-yr goal? If not, what amount should be invested on these terms to meet the goal? No; $26,775.62

97. **Population:** The time required for a population to *triple* is given by $T(r) = \frac{\ln 3}{r}$, where r represents the growth rate (expressed as a decimal) and $T(r)$ gives the years required. How long would it take a population to triple if the growth rate were (a) 3%, (b) 5.5%, and (c) 8%? (d) Use the TABLE feature of a calculator to estimate what growth rate will cause a population to triple in 10 years. **a.** ≈36.6 yr **b.** ≈20 yr **c.** ≈13.7 yr **d.** ≈11%

98. **Radioactive decay:** The rate of decay for radioactive material is related to the half-life of the substance by the formula $R(h) = \frac{\ln 2}{h}$, where h represents the half-life of the material and $R(h)$ is the rate of decay expressed as a decimal. An element known as potassium-42 is often used in biological studies and has a half-life of approximately 12.5 hr. (a) Find its rate of decay to the nearest hundredth of a percent. (b) Find the half-life of a given substance (to the nearest whole number) whose rate of decay is 2.89% per hour. **a.** 5.55% **b.** 24 hr

99. **Drug absorption:** The time required for a certain percentage of a drug to be absorbed by the body depends on the absorption rate of the drug. This can be modeled by the function $T(p) = \frac{-\ln p}{k}$, where p represents the percent of the drug that *remains* (expressed as a decimal), k is the absorption rate of the drug, and $T(p)$ represents the elapsed time. (a) Find the time required (to the nearest hour) for the body to *absorb* 35% of a drug that has an absorption rate of 7.2%. (b) Use the TABLE feature of a calculator to estimate the percent of this drug (to the nearest whole percent) that remains unabsorbed after 24 hr. **a.** 6 hr **b.** 18%

100. **Depreciation:** As time passes, the value of an automobile tends to depreciate. The amount of time required for a certain new car to depreciate to a given value can be determined using the formula $T(v_c) = 5\ln\left(\frac{25{,}000}{v_c}\right)$, where v_c represents the current value of the car and $T(v_c)$ gives the elapsed time in years. (a) Determine how many years (to the nearest one-half year) it will take for this car's value to drop to $10,000. (b) Use the TABLE feature of a calculator to estimate the value of the car after 2 yr (to the nearest $250). **a.** $4\frac{1}{2}$ yr **b.** $16,750

Carbon-14 dating: All living organisms (plants, animals, humans, and so on) contain trace amounts of the radioactive element known as carbon-14. Through normal metabolic activity, the ratio of carbon-14 to non-radioactive carbon remains constant throughout the life of the organism. After death, the carbon-14 begins disintegrating at a known rate, and no further replenishment of the element can take place. By measuring the percentage p that remains, as compared to other stable elements, the formula $T = -8266\ln p$ can be used to estimate the number of years since the organism died, where p is the percentage of carbon-14 that remains (expressed as a decimal) and T is the time in years since the organism died.

101. Bits of charcoal from Lascaux Cave (home of the prehistoric Lascaux Cave Paintings) were found to contain approximately 12.4% of their original amount of carbon-14. Approximately how many years ago did the fire burn in Lascaux Cave? ≈17,255 yr

102. Organic fragments found near Stonehenge were found to contain approximately 62.2% of their original amount of carbon-14. Approximately how many years ago did the organism live? $\approx$3925 yr

AVERAGE RATES OF CHANGE

103. Compute the average rate of change of $y = \ln x$ in the intervals (a) [1, 1.001], (b) [2, 2.001], and (c) [3, 3.001]. Based on what you observe, (d) estimate the rate of change for $y = \ln(x)$ in the interval [4.0, 4.001], then check your answer using the formula.

103. **a.** ~1 **b.** ~0.5 **c.** ~0.33 **d.** ~0.25

104. Compute the average rate of change of $y = \ln x$ in the intervals (a) [0.1, 0.1001], (b) [0.2, 0.2001], and (c) [0.3, 0.3001]. Based on what you observe, (d) estimate the rate of change for $y = \ln(x)$ in the interval [0.4, 0.4001], then check your answer using the formula.

104. **a.** ~10 **b.** ~5 **c.** ~3.33 **d.** ~2.5

105. Compute the average rate of change of $f(x) = e^x$ in the interval [3, 3.0001], then evaluate the function at $f(3)$. Repeat for the interval [2, 2.0001] and the value $f(2)$. What do you notice? Based on this observation, estimate the rate of change in the interval [4, 4.0001], then check your estimate with the value given by the formula for this interval.

105. $\frac{\Delta y}{\Delta x} \approx 20$, $f(3) \approx 20$; $\frac{\Delta y}{\Delta x} \approx 7.39$; $f(2) \approx 7.39$; apparently $\frac{\Delta y}{\Delta x} = f(x)$; $f(4) \approx 54.6$; $\frac{\Delta y}{\Delta x} \approx 54.6$

106. Compute the average rate of change of $f(x) = 2.5^x$ and $g(x) = 3^x$ in the interval [0.0, 0.0001]. Based on the fact that $2 < e < 3$, make a conjecture about the average rate of change for $y = e^x$ in this interval, then check your estimate with the value given by the formula for this interval.

106. $f(x) = 2.5^x$, $\frac{\Delta y}{\Delta x} \approx 0.916$; $g(x) = 3^x$, $\frac{\Delta y}{\Delta x} \approx 1.099$; $y = e^x$, $\frac{\Delta y}{\Delta x} \approx 1.000$

WRITING, RESEARCH, AND DECISION MAKING

107. Although the graphs of $f(x) = \ln x$ and $g(x) = \sqrt{x} - 1$ appear similar in many respects, each function serves a very different purpose and is used to model a very different phenomenon. Research and investigate why by carefully graphing both functions for $x \in (0, 20)$. (a) State the domain and range of each function. (b) Find or estimate the location of any points of intersection. (c) Compute the value of $f(x) - g(x)$ for $x = 15$, $x = 1500$, and $x = 150{,}000$, and discuss why they cannot be used interchangeably as mathematical models.

107. **a.** D_f: $x \in (0, \infty)$; R_f: $y \in \mathbb{R}$; D_g: $x \in [0, \infty)$; R_y: $y \in [-1, \infty)$ **b.** (1, 0) **c.** −0.16, −30.42, −374.38; as $x \to \infty$, the difference between the two functions increases

108. Until calculators and computers became commonplace, logarithms had been used for centuries to manually "reckon" or calculate with numbers, particularly if products, quotients, or powers were involved. Do some reading and research into the history of logarithms and how they were used to do difficult computations without the aid of a calculator. Prepare a short report that includes some sample computations.

108. Answers will vary.

109. Use test values to demonstrate that the following relationships are *false.*

$$\ln(p \cdot q) = \ln p \ln q \qquad \ln\left(\frac{p}{q}\right) = \frac{\ln(p)}{\ln(q)} \qquad \ln p + \ln q = \ln(p + q)$$

109. Answers will vary.

110. Prove the quotient property of logarithms using the proof of the product property as a model.

110. Answers will vary.

EXTENDING THE CONCEPT

111. Verify that $\ln(x) = \ln(10) \cdot \log(x)$, then use the relationship to find the value of $\ln(e)$, $\ln(10)$, and $\ln(2)$ to three decimal places. verified; 1,000, 2.303, 0.693

112. Suppose you and I represent two different numbers. Is the following cryptogram true or false? *The log of me to base me is one and the log of you to base you is one, but the log of you to base me is equal to the log of me to base you turned upside down.* True

Use prime factors, properties of logs, and the values given to evaluate each expression without a calculator. Check each result using the change-of-base formula.

113. $\log_3 4 \approx 1.2619$ and $\log_3 5 \approx 1.4649$: (a) $\log_3 20$, (b) $\log_3 \frac{4}{5}$, and (c) $\log_3 25$

113. **a.** $\log_3 4 + \log_3 5 = 2.7268$ **b.** $\log_3 4 - \log_3 5 = -0.2030$ **c.** $2 \log_3 5 = 2.9298$

114. $\log_5 2 \approx 0.4307$ and $\log_5 3 \approx 0.6826$: (a) $\log_5 \frac{9}{2}$, (b) $\log_5 216$, and (c) $\log_5 \sqrt[3]{6}$.

114. **a.** $2 \log_5 3 - \log_5 2 = 0.9345$ **b.** $3 \log_5 3 + 3 \log_5 2 = 3.3399$ **c.** $\frac{1}{3}[\log_5 3 + \log_5 2] = 0.3711$

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REVISED PAGES

MAINTAINING YOUR SKILLS

115. (2.4/3.5) Name all eight basic toolbox functions and draw a quick sketch of each.

116. (3.4) Graph the function by completing the square and label all important features: $f(x) = -2x^2 + 12x - 9$.

117. (4.5) Use polynomial long division to sketch the graph using shifts of a basic toolbox function:
$r(x) = \frac{x^2 + 4x + 3}{x^2 + 4x + 4}$. Label all asymptotes and intercepts.

118. (3.7) Graph the piecewise-defined function and state its domain and range.

$$p(x) = \begin{cases} -2 & -4 \le x < 0 \\ 4x - x^2 & 0 \le x < 4 \\ 8 - 2|x - 8| & 4 \le x \le 10 \end{cases}$$

119. (R.7/3.3) Sketch the graph using transformations of a basic function, then use basic geometry to compute the area in Quadrant I that is under the graph: $y = -|x - 3| + 6$. 31.5

120. (2.6) The data given tracks the total amount of debt carried by a family over a six-month period. Draw a scatter-plot of the data, decide on an appropriate form of regression, and use a graphing calculator to determine a regression, equation. (a) How fast is the debt load growing each month? (b) How much debt will the family have accumulated at the end of 12 months?

Exercise 120

Month (*x*)	Debt (*y*)
(start)	$0
Jan.	471
Feb.	1105
March	1513
April	1921
May	2498
June	3129

MID-CHAPTER CHECK

1. Write the following in logarithmic form.

a. $27^{\frac{2}{3}} = 9$ $\quad \frac{2}{3} = \log_{27} 9$ **b.** $81^{\frac{5}{4}} = 243$ $\quad \frac{5}{4} = \log_{81} 243$

2. Write the following in exponential form.

a. $\log_8 32 = \frac{5}{3}$ $\quad 8^{\frac{5}{3}} = 32$ **b.** $\log_{1296} 6 = 0.25$ $\quad 1296^{0.25} = 6$

3. Solve each equation for the unknown:

a. $4^{2x} = 32^{x-1}$ $\quad x = 5$ **b.** $\left(\frac{1}{3}\right)^{4b} = 9^{2b-5}$ $\quad b = \frac{5}{4}$

4. Solve each equation for the unknown:

a. $\log_{27} x = \frac{1}{3}$ $\quad x = 3$ **b.** $\log_b 125 = 3$ $\quad b = 5$

5. The homes in a popular neighborhood are growing in value according to the formula $V(t) = V_0\left(\frac{9}{8}\right)^t$, where t is the time in years, V_0 is the purchase price of the home, and $V(t)$ is the current value of the home. (a) In 3 yr, how much will a $50,000 home be worth? (b) Use the TABLE feature of your calculator to estimate how many years (to the nearest year) until the home doubles in value. **a.** $71,191.41 **b.** 6 yr

6. Estimate the value of each expression by bounding it between two integers. Then use a calculator to find the exact value.

a. log 243 $\quad$ 2.385606274 **b.** log 85,678 $\quad$ 4.93286932

7. Estimate the value of each expression by bounding it between two integers. Then use the change-of-base formula to find the exact value.

a. $\log_3 19$ $\quad \frac{\log 19}{\log 3} \approx 2.68$ **b.** $\log_2 60$ $\quad \frac{\log 60}{\log 2} \approx 5.91$

8. The rate of decay for radioactive material is related to the half-life of the substance by the formula $R(h) = \frac{\ln(2)}{h}$, where h represents the half-life of the material and $R(h)$ is the rate of decay expressed as a decimal. The strontium isotope ${}^{90}_{38}$Sr-14 has a half-life of approximately 28 yr. (a) Find its rate of decay to the nearest hundredth of a percent. (b) Find the half-life of a given substance (to the nearest whole number) whose rate of decay is 5.78% per year. **a.** 2.48% **b.** 12 yr

9. Use properties of logarithms to write each expression as a single logarithmic term:

a. $\log(2x - 3) + \log(2x + 3)$ $\quad \log(4x^2 - 9)$

b. $\log(x + 5) - \log(x^2 - 25)$ $\quad \log\left(\frac{1}{x - 5}\right)$

10. Given $\log_7 5 \approx 0.8271$ and $\log_7 10 \approx 1.1833$, use properties of logarithms to estimate the value of

a. $\log_7 50$ **b.** $\log_7 2$

c. $\log_7 25$ **d.** $\log_7 500$

a. 2.0104 **b.** 0.3562 **c.** 1.6542 **d.** 3.1937

Additional answers can be found in the Instructor Answer Appendix.

Reinforcing Basic Concepts

Understanding Properties of Logarithms

To effectively use the properties of logarithms as a mathematical tool, a student must attain some degree of comfort and fluency in their application. Otherwise we are resigned to using them as a template or formula, leaving little room for growth or insight. This *Reinforcing Basic Concepts* is divided into two parts. The first is designed to promote an understanding of the product and quotient properties of logarithms, which play a role in the solution of logarithmic and exponential equations.

We begin by looking at some logarithmic expressions that are obviously true:

$$\log_2 2 = 1 \qquad \log_2 4 = 2 \qquad \log_2 8 = 3 \qquad \log_2 16 = 4 \qquad \log_2 32 = 5 \qquad \log_2 64 = 6$$

Next, we view the same expressions with their value *understood mentally,* illustrated by the numbers in the background, rather than expressly written.

$$\log_2 2 \qquad \log_2 4 \qquad \log_2 8 \qquad \log_2 16 \qquad \log_2 32 \qquad \log_2 64$$

This will make the product and quotient properties of equality much easier to "see." Recall the product property states: $\log_b M + \log_b N = \log_b(MN)$ and the quotient property states: $\log_b M - \log_b N = \log_b\left(\frac{M}{N}\right)$. Consider the following.

$\log_2 4 + \log_2 8 = \log_2 32$

which is the same as saying

$\log_2 4 + \log_2 8 = \log_2(4 \cdot 8)$ (since $4 \cdot 8 = 32$)

$\log_b M + \log_b N = \log_b(MN)$

$\log_2 64 - \log_2 32 = \log_2 2$

which is the same as saying

$\log_2 64 - \log_2 32 = \log_2\left(\frac{64}{32}\right)$ (since $\frac{64}{32} = 2$)

$\log_b M - \log_b N = \log_b\left(\frac{M}{N}\right)$

Exercise 1: Repeat this exercise using logarithms of base 3 and various sums and differences. Answers will vary.

Exercise 2: Use the basic concept behind these exercises to combine these expressions: (a) $\log(x) + \log(x + 3)$, (b) $\ln(x + 2) + \ln(x - 2)$, and (c) $\log(x) - \log(x + 3)$. **a.** $\log(x^2 + 3x)$ **b.** $\ln(x^2 - 4)$ **c.** $\log\frac{x}{x+3}$

The second part is similar to the first, but highlights the power property: $\log_b M^x = x\log_b M$. For instance, knowing that $\log_2 64 = 6$, $\log_2 8 = 3$, and $\log_2 2 = 1$, consider the following:

$\log_2 8$ can be written as $\log_2 2^3$ (since $2^3 = 8$). Applying the power property gives $3 \cdot \log_2 2 = 3$.

$\log_2 64$ can be written as $\log_2 2^6$ (since $2^6 = 64$). Applying the power property gives $6 \cdot \log_2 2 = 6$.

$$\log_b M^x = x \log_b M$$

Exercise 3: Repeat this exercise using logarithms of base 3 and various powers. Answers will vary.

Exercise 4: Use the basic concept behind these exercises to rewrite each expression as a product: (a) $\log 3^x$, (b) $\ln x^5$, and (c) $\ln 2^{3x-1}$. **a.** $x\log 3$ **b.** $5\ln x$ **c.** $(3x - 1)\ln 2$

5.4 Exponential/Logarithmic Equations and Applications

LEARNING OBJECTIVES

In Section 5.4 you will learn how to:

- **A.** Write logarithmic and exponential equations in simplified form
- **B.** Solve exponential equations
- **C.** Solve logarithmic equations
- **D.** Solve applications involving exponential and logarithmic equations

INTRODUCTION

In this section, we'll use the relationships between $f(x) = b^x$ and $g(x) = \log_b x$ to solve equations that arise in applications of exponential and logarithmic functions. As you'll see, these functions have a large variety of significant uses.

POINT OF INTEREST

Using a calculator, we find $\log 6 = 0.7781512504$, meaning $10^{0.7781512504} = 6$. In years past, the number 6 was commonly called the *antilogarithm* of 0.7781512504, or the number whose (base-10) log is 0.7781512504. Prior to the widespread use of calculators, *tables* were used to compute with logarithms, and "finding an antilogarithm" simply meant we were searching the entries of a table for the number whose base b exponent was given. For example, the base-10 antilogarithm of 3 is 1000 since $10^3 = 1000$ and the base-e antilogarithm of 3.36729583 is 29 since $e^{3.36729583} = 29$. Now that calculators can easily produce logarithms and exponentials of any base (to over nine decimal places), we find the term "antilogarithm" gradually fading from common use.

A. Writing Logarithmic and Exponential Equations in Simplified Form

As we noted in Section 5.3, sums and differences of logarithmic terms (with like bases) are combined using the product and quotient properties, respectively. This is a fundamental step in equation solving, as it helps to simplify the equation and assist the solution process.

EXAMPLE 1 Rewrite each equation with a single logarithmic term on one side, as in $\log_b x = k$. Do not attempt to solve.

a. $\log_2 x + \log_2(x + 3) = 4$ **b.** $-\ln 2x = \ln x - \ln(x + 1)$

Solution:

a.
$$\log_2 x + \log_2(x + 3) = 4 \quad \text{given}$$
$$\log_2[x(x + 3)] = 4 \quad \text{product property}$$
$$\log_2[x^2 + 3x] = 4 \quad \text{result}$$

b.
$$-\ln 2x = \ln x - \ln(x + 1) \quad \text{given}$$
$$-\ln 2x = \ln \frac{x}{x + 1} \quad \text{quotient property}$$
$$0 = \ln \frac{x}{x + 1} + \ln 2x \quad \text{set equal to zero}$$
$$0 = \ln\left[\left(\frac{x}{x + 1}\right)\left(\frac{2x}{1}\right)\right] \quad \text{product property}$$
$$0 = \ln\left(\frac{2x^2}{x + 1}\right) \quad \text{result}$$

NOW TRY EXERCISES 7 THROUGH 12

EXAMPLE 2 Rewrite each equation with a single exponential term on one side, as in $b^x = k$. Do not solve.

a. $400e^{0.21x} + 325 = 1225$ **b.** $e^{x-1}(e^{3x}) = e^{2x}$

Solution:

a.
$$\begin{aligned} 400e^{0.21x} + 325 &= 1225 && \text{given} \\ 400e^{0.21x} &= 900 && \text{subtract 325} \\ e^{0.21x} &= 2.25 && \text{divide by 400} \end{aligned}$$

b.
$$\begin{aligned} e^{x-1}(e^{3x}) &= e^{2x} && \text{given} \\ e^{4x-1} &= e^{2x} && \text{product property} \\ \frac{e^{4x-1}}{e^{2x}} &= 1 && \text{divide} \\ e^{(4x-1)-2x} &= 1 && \text{quotient property} \\ e^{2x-1} &= 1 && \text{result} \end{aligned}$$

NOW TRY EXERCISES 13 THROUGH 18

CAUTION

One of the most common mistakes in solving exponential and logarithmic equations is to apply the inverse function too early—before the equation is simplified. Unless the equation can be written with like bases on both sides, always try to isolate a single logarithmic or exponential term prior to applying the inverse function.

B. Solving Exponential Equations

An exponential equation is one where at least one term has a variable exponent. If an exponential equation can be written with a single term on each side where both have the same base, the equation is most readily solved using the *uniqueness property* as in Section 5.1. If not, we solve a **base-*b* exponential** equation by applying a **base-*b* logarithm** using properties I through IV from Section 5.2. These properties can be applied for any base but are particularly effective when the exponential is 10 or e, since calculators are programmed with these bases. Consider the following illustrations:

base 10
$$\begin{aligned} 10^x &= k && \text{base-10 exponential} \\ \log_{10}10^x &= \log k && \text{apply base-10 logarithms} \\ x &= \log k && \text{property III; find } \log k \text{ using a calculator} \end{aligned}$$

base *e*
$$\begin{aligned} e^x &= k && \text{base } e \text{ exponential} \\ \ln e^x &= \ln k && \text{apply base } e \text{ logarithms} \\ x &= \ln k && \text{property III; find } \ln k \text{ using a calculator} \end{aligned}$$

For exponential bases other than 10 or e, we apply either base and use the power property of logarithms to solve for x: $\log_b k^x = x\log_b k$.

neither 10 nor *e*
$$\begin{aligned} b^x &= k && \text{base-}b\text{ exponential} \\ \log b^x &= \log k && \text{apply either logarithm to both sides (we chose base 10)} \\ x\log b &= \log k && \text{power property} \\ x &= \frac{\log k}{\log b} && \text{solution (divide both sides by } \log b) \end{aligned}$$

The main ideas are summarized here.

SOLVING EXPONENTIAL EQUATIONS

For any real numbers b, x, and k, where $b > 0$ and $b \neq 1$,

1. If $10^x = k$,	2. If $e^x = k$,	3. $b^x = k$
$\log 10^x = \log k$	$\ln e^x = \ln k$	$x \log b = \log k$
$\rightarrow x = \log k$	$\rightarrow x = \ln k$	$\rightarrow x = \dfrac{\log k}{\log b}$

EXAMPLE 3 ▶ Solve each exponential equation and check your answer.

a. $3e^{x+1} - 5 = 7$ **b.** $4^{3x} - 1 = 8$

Solution: ▶ **a.**

$$3e^{x+1} - 5 = 7 \quad \text{given}$$
$$3e^{x+1} = 12 \quad \text{add 5}$$
$$e^{x+1} = 4 \quad \text{divide by 3}$$

Since the left-hand side is a base-e exponential, we apply the base-e logarithm.

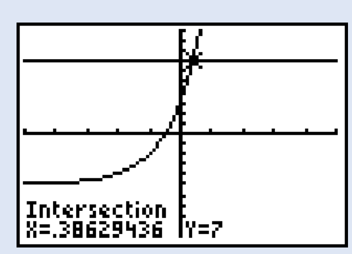
$$\ln e^{x+1} = \ln 4 \quad \text{apply base-}e\text{ logarithms}$$
$$x + 1 = \ln 4 \quad \text{property III}$$
$$x = \ln 4 - 1 \quad \text{solve for } x \text{ (exact form)}$$
$$\approx 0.38629 \quad \text{approximate form (to five decimal places)}$$

Check: ▶

$$3e^{x+1} - 5 = 7 \quad \text{original equation}$$
$$3e^{0.38629+1} - 5 = 7 \quad \text{substitute 0.38629 for } x$$
$$3e^{1.38629} - 5 = 7 \quad \text{simplify exponent}$$
$$3(4) - 5 = 7 \checkmark \quad e^{1.38629} \approx 4$$

GRAPHICAL SUPPORT

We can verify solutions to exponential equations using the same methods as for other equations. For Example 3(a), enter $3e^{x+1} - 5$ as Y_1 and 7 as Y_2 on the Y= screen of your graphing calculator. Using the 2nd TRACE **(CALC) 5: Intersect** option, we find the graphs intersect at $x = 0.38629436$, the solution we found in Example 3(a).



b.

$$4^{3x} - 1 = 8 \quad \text{given}$$
$$4^{3x} = 9 \quad \text{add 1 to both sides}$$

The left-hand side is neither base 10 or base e, so we chose base e to solve:

$$\ln 4^{3x} = \ln 9 \quad \text{apply base-}e\text{ logarithms}$$
$$3x \ln 4 = \ln 9 \quad \text{power property}$$

$$3x = \frac{\ln 9}{\ln 4}$$ divide by ln 4

$$x = \frac{\ln 9}{3 \cdot \ln 4}$$ solve for x (exact form)

$$\approx 0.52832$$ approximate form (to five decimal places)

Check:

$$4^{3x} - 1 = 8$$ original equation

$$4^{3(0.52832)} - 1 = 8$$ substitute 0.52832 for x

$$4^{1.58496} - 1 = 8$$ simplify exponent

$$9 - 1 \approx 8 \checkmark$$ $4^{1.58496} \approx 9$

NOW TRY EXERCISES 19 THROUGH 54

In some cases, there may be exponential terms with unlike bases on *both sides* of the equation. As you apply the solution process to these equations, be sure to distinguish between constant terms like $\ln 5$ and variable terms like $x \ln 5$. As with all equations, the goal is to isolate the variable terms on one side of the equation, with all constant terms on the other.

EXAMPLE 4 Solve the exponential equation: $5^{x+1} = 6^{2x}$.

Solution:

$$5^{x+1} = 6^{2x}$$ given

To begin, we take the natural log (or base-10 log) of both sides:

$$\ln 5^{x+1} = \ln 6^{2x}$$ apply base-e logarithms

$$(x + 1)\ln 5 = 2x \ln 6$$ power property

$$x \ln 5 + \ln 5 = 2x \ln 6$$ distribute

$$\ln 5 = 2x \ln 6 - x \ln 5$$ variable terms to one side

$$\ln 5 = x(2 \ln 6 - \ln 5)$$ factor out x

$$\frac{\ln 5}{2 \ln 6 - \ln 5} = x$$ solve for x (exact form)

$$0.81528 = x$$ approximate form

The check is left to the student.

NOW TRY EXERCISES 55 THROUGH 58

In many important applications of exponential functions, the exponential term appears as part of a quotient. In this case we simply "clear denominators" and attempt to isolate the exponential term as before.

EXAMPLE 5 Solve the exponential equation: $\frac{258}{1 + 20e^{-0.009t}} = 192$

Solution

$$\frac{258}{1 + 20e^{-0.009t}} = 192$$ given

$$258 = 192(1 + 20e^{-0.009t})$$ clear denominators

$$1.34375 = 1 + 20e^{-0.009t}$$ divide by 192

$$0.0171875 = e^{-0.009t} \quad \text{subtract 1, divide by 20}$$
$$\ln 0.0171875 = -0.009t \quad \text{apply base-}e\text{ logarithms}$$
$$\frac{\ln 0.0171875}{-0.009} = t \quad \text{solve for } t \text{ (exact form)}$$
$$451.51 \approx t \quad \text{approximate form}$$

NOW TRY EXERCISES 59 THROUGH 62

C. Solving Logarithmic Equations

As with exponential functions, the fact that logarithmic functions are one-to-one enables us to quickly solve equations that can be rewritten with a single logarithmic term on each side (assuming both have a like base, as in Section 5.1). In particular we have

LOGARITHMIC EQUATIONS WITH LIKE BASES: THE UNIQUENESS PROPERTY

For all real numbers b, m, and n where $b > 0$ and $b \neq 1$,

If $\log_b m = \log_b n$, then $m = n$

If $m = n$, then $\log_b m = \log_b n$

Equal bases imply equal arguments.

EXAMPLE 6 Solve each equation using the uniqueness property of logarithms.

a. $\log(x + 2) = \log 7 + \log x$ **b.** $\log_3 87 - \log_3 x = \log_3 29$

Solution: **a.** $\log(x + 2) = \log 7 + \log x$ **b.** $\log_3 87 - \log_3 x = \log_3 29$

a.		b.
$\log(x + 2) = \log 7x$	properties of logarithms	$\log_3 \frac{87}{x} = \log_3 29$
$x + 2 = 7x$	uniqueness property	$\frac{87}{x} = 29$
$2 = 6x$	solve for x	$87 = 29x$
$\frac{1}{3} = x$	result	$3 = x$

NOW TRY EXERCISES 63 THROUGH 68

If the equation results in a single logarithmic term, the uniqueness property cannot be used and we solve by isolating this term on one side and applying a **base-*b* exponential** (exponentiate both sides) as illustrated here:

$$\log_b x = k \quad \text{exponential equation}$$
$$b^{\log_b x} = b^k \quad \text{exponentiate both sides (using base } b\text{)}$$
$$x = b^k \quad \text{property IV (find } b^k \text{ using a calculator)}$$

Note the end result is simply the exponential form of the equation, and we will actually view the solution process in this way.

SOLVING LOGARITHMIC EQUATIONS

For any algebraic expression X and real numbers b and k, where $b > 0$ and $b \neq 1$,

1. If $\log X = k$, $X = 10^k$
2. If $\ln X = k$, $X = e^k$
3. If $\log_b X = k$, $X = b^k$

As we saw in our study of rational and radical equations, when the domain of a function is something other than all real numbers, **extraneous roots** sometimes arise. Logarithmic equations can also produce such roots, and checking all results is a good practice. See Example 7(b).

EXAMPLE 7 Solve each logarithmic equation and check the solutions.

a. $\ln(x + 7) - \ln 5 = 1.4$ **b.** $\log(x + 12) - \log x = \log(x + 9)$

Solution: **a.** $\ln(x + 7) - \ln 5 = 1.4$ given

Bases are alike → combine terms and write equation in exponential form (uniqueness property cannot be applied).

$$\ln\left(\frac{x + 7}{5}\right) = 1.4 \quad \text{quotient property}$$
$$\left(\frac{x + 7}{5}\right) = e^{1.4} \quad \text{exponential form}$$
$$x + 7 = 5e^{1.4} \quad \text{clear denominator}$$
$$x = 5e^{1.4} - 7 \quad \text{solve for } x \text{ (exact form)}$$
$$\approx 13.27600 \quad \text{approximate form (to five decimal places)}$$

Check $x = 13.276$:

$$\ln(x + 7) - \ln 5 = 1.4 \quad \text{original equation}$$
$$\ln(13.276 + 7) - \ln 5 = 1.4 \quad \text{substitute 13.27600 for } x$$
$$\ln 20.276 - \ln 5 = 1.4 \quad \text{simplify}$$
$$1.4 = 1.4\checkmark \quad \text{result checks}$$

WORTHY OF NOTE

If all digits of the answer given by your calculator are used, the calculator will generally produce "exact" answers when they are checked. Try using the solution $x = 13.27599983$ for Example 7(a).

b. $\log(x + 12) - \log x = \log(x + 9)$ given

Left-hand side can be simplified → write the equation with a *single logarithmic term* on each side and solve using the uniqueness property.

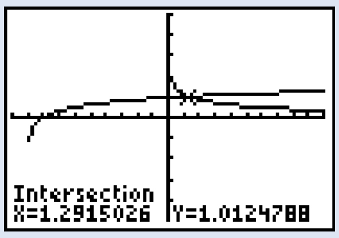
$$\log\left(\frac{x + 12}{x}\right) = \log(x + 9) \quad \text{quotient property}$$
$$\frac{x + 12}{x} = x + 9 \quad \text{uniqueness property}$$
$$x + 12 = x^2 + 9x \quad \text{clear denominator}$$
$$0 = x^2 + 8x - 12 \quad \text{set equal to 0}$$

The quadratic formula gives solutions $x = -4 \pm 2\sqrt{7}$. The solution $x = -4 + 2\sqrt{7}$ ($x \approx 1.29150$) checks, but when $-4 - 2\sqrt{7}$ ($x \approx -9.29150$) is substituted into the original equation, we get $\log 2.7085 - \log(-9.2915) = \log(-0.2915)$ and two of the three terms do not represent real numbers. The "solution" $x = -4 - 2\sqrt{7}$ is an extraneous root.

NOW TRY EXERCISES 69 THROUGH 94

GRAPHICAL SUPPORT

Logarithmic equations can also be checked using the intersection of graphs method. For Example 7(b), we first enter $\log(x + 12) - \log x$ as Y_1 and $\log(x + 9)$ as Y_2 on the Y= screen. Using 2nd TRACE **(CALC) 5:Intersect,** we find the graphs intersect at $x = 1.2915026$, and that *this is the only solution* (knowing the graph's basic shape, we conclude they cannot intersect again).



D. Applications of Exponential and Logarithmic Functions

Applications of exponential and logarithmic functions take many different forms and it would be impossible to illustrate them all. As you work through the exercises, try to adopt a "big picture" approach, applying the general principles illustrated in this section to the various applications. Some may look familiar and may have been introduced in previous sections. The difference here is that we now have the ability to *solve for unknowns* as well as to evaluate the relationships.

Newton's law of cooling relates the temperature of a given object to the constant temperature of a surrounding medium. One form of this relationship is $T = T_1 + (T_0 - T_1)e^{-kh}$, where T_0 is the initial temperature of the object, T_1 is the temperature of the surrounding medium, and T is the temperature after h hours (k is a constant that depends on the materials involved).

EXAMPLE 8 If a can of soft drink is taken from a 50°F cooler and placed in a room where the temperature is 75°F, how long will it take the drink to warm to 70°F? Assume $k = 0.95$ and answer in hours and minutes.

Solution:

$T = T_1 + (T_0 - T_1)e^{-kh}$	given
$70 = 75 + (50 - 75)e^{-0.95h}$	substitute 50 for T_0, 75 for T_1, 70 for T, and 0.95 for k
$-5 = -25e^{-0.95h}$	simplify
$0.2 = e^{-0.95h}$	divide by -25
$\ln 0.2 = \ln e^{-0.95h}$	apply base-e logarithms
$\ln 0.2 = -0.95h$	$\ln e^k = k$
$\dfrac{\ln 0.2}{-0.95} = h$	solve for h
$1.69 \approx h$	result

The can of soda will warm to a temperature of 70°F in approximately 1 hour and 41 min ($0.69 \times 60 \approx 41$).

NOW TRY EXERCISES 97 AND 98

TECHNOLOGY HIGHLIGHT

Using a Graphing Calculator to Explore Exponential/Logarithmic Equations

The keystrokes shown apply to a TI-84 Plus model. Please consult our Internet site or your manual for other models.

Even with the new equation-solving abilities in Section 5.4, there remain a large number of exponential and logarithmic equations that are very difficult to solve using inverse functions and manual methods alone. One example would be the equation $e^{(x-3)} + 1 = 5\ln(x-1) + 2$, in which both exponential and logarithmic functions occur. For equations of this nature, graphing technology remains our best tool. To solve $e^{(x-3)} + 1 = 5\ln(x-1) + 2$, enter the left-hand member as Y_1 and the right-hand member as Y_2 on the Y= screen of your graphing calculator. Based on what we know about the graphs of $y = e^x$ and $y = \ln x$, it is likely that solutions (points of intersection) will occur on the standard screen. Graph both by pressing ZOOM 6 (see Figure 5.14). From the graphs and our knowledge of the basic functions, it is apparent the equation has two solutions (the graphs have two points of intersection). Recall that to find the intersections, we use the 2nd TRACE **(CALC) 5: intersect** option. Press ENTER to identify the first graph, then ENTER once again to identify (select) the second graph. The smaller solution seems to be near $x = 2$, so we enter a "2" when the calculator asks for a guess (Figure 5.15). After a moment, the calculator determines the smaller root is approximately $x = 1.8735744$ (Figure 5.16). Repeating these keystrokes using a guess of "5" reveals the second solution is about $x = 5.0838288$. Recall that the TI-84Plus will temporarily store the last calculated solution as the variable *x*, accessed using the X,T,θ,n key. This will enable a quick check of the solution by simply entering the original expressions on the home screen, as shown in Figure 5.17 Use these ideas to solve the following equations. Note the solution to Exercise 4 can be completed/verified without the aid of a calculator (use a *u*-substitution and the quadratic form).

Figure 5.14

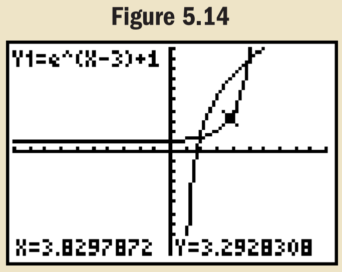


Figure 5.15

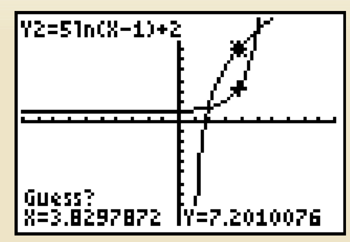


Figure 5.16

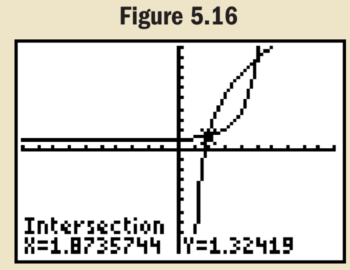
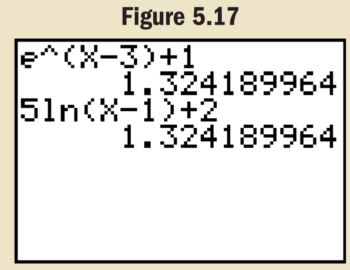


Figure 5.17



Exercise 1: $5\log(2x+3) = e^{2x} - 6$ 1.1311893, −1.467671

Exercise 2: $x^2 + 2 = 3\ln(x+2)$ −0.0506028, 1.2329626

Exercise 3: $-4\log(x-3) + 2 = \ln x$ 4.3556075

Exercise 4: $[\ln(2x)]^2 + 3\ln(2x) - 4 = 0$ $\frac{e}{2} \approx 1.3591409$, $\frac{1}{2e^4} \approx 0.00915782$

5.4 EXERCISES

CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. For $e^{-0.02x+1} = 10$, the solution process is most efficient if we apply a base ___e___ logarithm to both sides.

2. To solve $\ln x - \ln(x+3) = 0$, we can combine terms using the ________ property, or add $\ln(x+3)$ to both sides and use the ___uniqueness___ property.

3. Since logarithmic functions are not defined for all real numbers, we should check all "solutions" for <u>extraneous</u> roots.

4. In the equation $3\ln 5 + x\ln 5 = 2x\ln 6$, $x\ln 5$ is a <u>variable</u> term and should be added to both sides so the variable x can be <u>factored</u> out.

5. Solve the equation here, giving a step-by-step discussion of the solution process: $\ln(4x + 3) + \ln(2) = 3.2$ 2.316566275

6. Describe the difference between *evaluating* the equation below given $x = 9.7$ and *solving* the equation given $y = 9.7$: $y = 3\log_2(x - 1.7) - 2.3$ Answers will vary.

DEVELOPING YOUR SKILLS

Write each equation in the simplified form $\log_b x = k$ (logarithmic term = constant). Do not solve.

7. $3\ln(x + 4) + 7 = 13$ $\ln(x + 4) = 2$

8. $-6 = 2\log_3(x - 5) - 10$ $\log_3(x - 5) = 2$

9. $\log(x + 2) + \log x = 4$ $\log(x^2 + 2x) = 4$

10. $2 = \ln\left(\dfrac{2}{x} + 3\right) + \ln x$ $\ln(2 + 3x) = 2$

11. $2\log_2(x) + \log_2(x - 1) = 2$ $\log_2(x^3 - x^2) = 2$

12. $\ln(x - 2) - \ln x = -\ln(3x)$ $\ln(3x - 6) = 0$

Write each equation in the simplified form $b^x = k$ (exponential term = constant). Do not solve.

13. $4e^{x-2} + 5 = 69$ $e^{x-2} = 16$

14. $2 - 3e^{0.4x} = -7$ $e^{0.4x} = 3$

15. $250e^{0.05x+1} + 175 = 512.5$ $e^{0.05x+1} = 1.35$

16. $-150 = 290.8 - 190e^{-0.75x}$ $e^{-0.75x} = 2.32$

17. $3^x(3^{2x-1}) = 81$ $3^{3x-1} = 81$

18. $(4^{3x+5})4^{-x} = 64$ $4^{2x+5} = 64$

Solve each equation two ways: by equating bases and using the uniqueness properties, and by applying a base-10 or base-e logarithm and using the power property of logarithms.

19. $2^x = 128$ $x = 7$

20. $3^x = 243$ $x = 5$

21. $5^{3x} = 3125$ $x = \frac{5}{3}$

22. $4^{3x} = 1024$ $x = \frac{5}{3}$

23. $5^{x+1} = 625$ $x = 3$

24. $6^{x-1} = 216$ $x = 4$

25. $\left(\dfrac{1}{2}\right)^{n-1} = \dfrac{1}{256}$ $n = 9$

26. $\left(\dfrac{1}{3}\right)^{n-1} = \dfrac{1}{729}$ $n = 7$

27. $\dfrac{1}{625} = \left(\dfrac{1}{5}\right)^{n-1}$ $n = 5$

28. $\dfrac{1}{216} = \left(\dfrac{1}{6}\right)^{n-1}$ $n = 4$

29. $\dfrac{128}{2187} = \left(\dfrac{2}{3}\right)^{n-1}$ $n = 8$

30. $\dfrac{729}{64} = \left(\dfrac{3}{2}\right)^{n-1}$ $n = 7$

31. $\dfrac{16}{625} = \left(\dfrac{2}{5}\right)^{n-1}$ $n = 5$

32. $\dfrac{729}{4096} = \left(\dfrac{3}{4}\right)^{n-1}$ $n = 7$

33. $\dfrac{5}{243} = 5\left(\dfrac{1}{3}\right)^{n-1}$ $n = 6$

34. $\dfrac{5}{32} = 10\left(\dfrac{1}{2}\right)^{n-1}$ $n = 7$

35. $\dfrac{56}{125} = 7\left(\dfrac{2}{5}\right)^{n-1}$ $n = 4$

36. $\dfrac{243}{64} = 16\left(\dfrac{3}{4}\right)^{n-1}$ $n = 6$

Solve using the method of your choice. Answer in exact form and approximate form rounded to four decimal places.

37. $10^x = 97$

38. $10^x = 12{,}089$

39. $879 = 10^{2x}$

40. $10^{3x} = 12{,}089$

41. $879 = 10^{x+3}$

42. $4589 = 10^{x-2}$

43. $e^x = 389$

44. $e^x = 25$

45. $e^{2x} = 1389$

46. $e^{3x} = 2507$

47. $e^{x+1} = 257$

48. $e^{x-3} = 589$

49. $2e^{0.25x} = 5$

50. $3e^{0.08x} = 12$

51. $7^{x+2} = 231$

52. $6^{x+2} = 3589$

53. $5^{3x-2} = 128{,}965$

54. $9^{5x-3} = 78{,}462$

55. $2^{x+1} = 3^x$

56. $7^x = 4^{2x-1}$

57. $5^{2x+1} = 9^{x+1}$

58. $\left(\dfrac{1}{5}\right)^{x-1} = \left(\dfrac{1}{2}\right)^{3-x}$

59. $\dfrac{87}{1 + 3e^{-0.06t}} = 50$

60. $\dfrac{39}{1 + 4e^{-0.04t}} = 24$

61. $160 = \dfrac{200}{1 + 59e^{-0.29t}}$

62. $98 = \dfrac{152}{1 + 20e^{-0.35t}}$

37. $x = \log 97$; $x \approx 1.9868$
38. $x = \log 12{,}089$; $x \approx 4.0824$
39. $x = \dfrac{\log 879}{2}$; $x \approx 1.4720$
40. $x = \dfrac{\log 12{,}089}{3}$; $x \approx 1.3608$
41. $x = \log 879 - 3$; $x \approx -0.0560$
42. $x = \log 4589 + 2$; $x \approx 5.6617$
43. $x = \ln 389$; $x \approx 5.9636$
44. $x = \ln 25$; $x \approx 3.2189$
45. $x = \dfrac{\ln 1389}{2}$; $x \approx 3.6182$
46. $x = \dfrac{\ln 2507}{3}$; $x \approx 2.6089$
47. $x = \ln 257 - 1$; $x \approx 4.5491$
48. $x = \ln 589 + 3$; $x \approx 9.3784$
49. $x = 4\ln\left(\frac{5}{2}\right)$; $x \approx 3.6652$
50. $x = \dfrac{25\ln 4}{2}$; $x \approx 17.3287$
51. $x = \dfrac{\ln 231}{\ln 7} - 2$; $x \approx 0.7968$
52. $x = \dfrac{\ln 3589}{\ln 6} - 2$; $x \approx 2.5685$
53. $x = \dfrac{\ln 128{,}965}{3\ln 5} + \dfrac{2}{3}$; $x \approx 3.1038$
54. $x = \dfrac{\ln 78{,}462}{5\ln 9} + \dfrac{3}{5}$; $x \approx 1.6259$
55. $x = \dfrac{\ln 2}{\ln 3 - \ln 2}$; $x \approx 1.7095$
56. $x = \dfrac{\ln 4}{2\ln 4 - \ln 7}$; $x \approx 1.6769$
57. $x = \dfrac{\ln 9 - \ln 5}{2\ln 5 - \ln 9}$; $x \approx 0.5753$
58. $x = \dfrac{\ln 5 + 3\ln 2}{\ln 2 + \ln 5}$; $x \approx 1.6021$
59. $t = \frac{-50}{3}\ln\left(\frac{37}{150}\right)$; $t \approx 23.3286$
60. $t = -25\ln\left(\frac{5}{32}\right)$; $t \approx 46.4074$
61. $t = \frac{100}{29}\ln(236)$; $t \approx 18.8408$
62. $t = \frac{-20}{7}\ln\left(\frac{27}{980}\right)$; $t \approx 10.2620$

Solve each equation using the uniqueness property of logarithms.

63. $\log(5x + 2) = \log 2$ $x = 0$

64. $\log(2x - 3) = \log 3$ $x = 3$

65. $\log_4(x + 2) - \log_4 3 = \log_4(x - 1)$ $x = \frac{5}{2}$

66. $\log_3(x + 6) - \log_3 x = \log_3 5$ $x = \frac{3}{2}$

67. $\ln(8x - 4) = \ln 2 + \ln x$ $x = \frac{2}{3}$

68. $\ln(x - 1) + \ln 6 = \ln(3x)$ $x = 2$

Solve each equation by converting to exponential form.

69. $\log(3x - 1) = 2$ $x = 33$

70. $\log(2x + 3) = 2$ $x = \frac{97}{2}$

71. $\log_5(x + 7) = 3$ $x = 118$

72. $\log_3(x - 1) = 2$ $x = 10$

73. $\ln(x + 7) = 2$ $x = e^2 - 7$

74. $\ln(x - 2) = 3$ $x = e^3 + 2$

75. $-2 = \log(2x - 1)$ $x = \frac{101}{200}$

76. $-3 = \log(1 + x)$ $x = -0.999$

77. $\log(2x) - 5 = -3$ $x = 50$

78. $\log(3x) + 7 = 8$ $x = \frac{10}{3}$

79. $-2\ln(x + 1) = -6$ $x = e^3 - 1$

80. $-3\ln(x - 3) = -9$ $x = e^3 + 3$

Solve each logarithmic equation using any appropriate method. Clearly identify any extraneous roots. If there are no solutions, so state.

81. $\log(2x - 1) + \log 5 = 1$

82. $\log(x - 7) + \log 3 = 2$

83. $\log_2(9) + \log_2(x + 3) = 3$

84. $\log_3(x - 4) + \log_3(7) = 2$

85. $\ln(x + 7) + \ln 9 = 2$

86. $\ln 5 + \ln(x - 2) = 1$

87. $\log(x + 8) + \log x = \log(x + 18)$

88. $\log(x + 14) - \log x = \log(x + 6)$

89. $\ln(2x + 1) = 3 + \ln 6$

90. $\ln 21 = 1 + \ln(x - 2)$

91. $\log(-x - 1) = \log(5x) - \log x$

92. $\log(1 - x) + \log x = \log(x + 4)$

93. $\ln(2t + 7) = \ln 3 - \ln(t + 1)$

94. $\ln(5 - r) - \ln 6 = \ln(r + 2)$

81. $x = \frac{3}{2}$
82. $x = \frac{121}{3}$
83. $x = \frac{-19}{9}$
84. $x = \frac{37}{7}$
85. $x = \dfrac{e^2 - 63}{9}$
86. $x = \dfrac{e + 10}{5}$
87. $x = 2$; -9 is extraneous
88. $x = 2$; -7 is extraneous
89. $x = 3e^3 - \frac{1}{2}$; $x \approx 59.75661077$
90. $x = 21e^{-1} + 2$; $x \approx 9.725468265$
91. no solution
92. no solution
93. $t = -\frac{1}{2}$; -4 is extraneous
94. $r = -1$

WORKING WITH FORMULAS

95. Half-life of a radioactive substance: $A = A_0\left(\frac{1}{2}\right)^{\frac{t}{h}}$

The **half-life** of radioactive material is the amount of time required for one-half of an initial amount of the substance to vanish due to the decay. The amount of material remaining can be determined using the formula shown, where t represents elapsed time, h is the half-life of the material, A_0 is the initial amount, and A represents the amount remaining. The sodium isotope $^{24}_{11}Na$ has a half-life of 15 hr. If 500 g were initially present, how much is left after 60 hr? (For more on this formula, see page 527 Section 5.5). 31.25 g

96. Forensics—estimating time of death: $h = -3.9\ln\left(\frac{T - t}{98.6 - t}\right)$

Under certain conditions, a forensic expert can determine the approximate time of death for a person found recently expired using the formula shown, where T is the body temperature when it was found, t is the (constant) temperature of the room where the person was found, and h is the number of hours since death. If the body was discovered at 9:00 A.M. in a 73°F air-conditioned room, and had a temperature of 86.2°F, at approximately what time did the person expire? 6:25 A.M.

APPLICATIONS

Newton's law of cooling was discussed in Example 8 of this section: $T = T_1 + (T_0 - T_1)e^{-kh}$, where T_0 is the initial temperature of the object, T_1 is the temperature of the surrounding medium, and T is the temperature after elapsed time h in hours (k is a constant that depends on the materials involved).

97. Cooling time: If a can of soft drink at a room temperature of 75°F is placed in a 40°F refrigerator, how long will it take the drink to cool to 45°F? Assume $k = 0.61$ and answer in hours and minutes. 3 hr 11 min

98. **Cooling time:** Suppose that the temperature in Esconabe, Michigan, was 47°F when a 5°F arctic cold front moved over the state. How long would it take a puddle of water to begin freezing over? (Water freezes at 32°F.) Assume $k = 0.9$ and answer in minutes. 30 min

Use the *barometric equation* $h = (30T + 8000)\ln\left(\frac{P_0}{P}\right)$ for Exercises 99 and 100.

99. **Altitude and pressure:** Determine the atmospheric pressure at the summit of Mount McKinley (in Alaska), a height of 6194 m. Assume the temperature at the summit is -10°C. 34 cmHg

100. **Altitude and pressure:** A plane is flying at an altitude of 10,029 m. If the barometric pre sure is 22 cm of mercury, what is the temperature at this altitude (to the nearest degree)? 3°C

101. **Investment growth:** Use the compound interest formula $A = P\left(1 + \frac{r}{n}\right)^{nt}$ to determine how long it would take \$2500 to grow to \$6000 if the annual rate is 8% and interest was compounded monthly. 11.0 yrs

102. **Radioactive half-lives:** Use the formula discussed in Exercise 95 to find the half-life of polonium, if 1000 g of the substance decayed to 125 g in 420 days. 140 days

103. **Advertising and sales:** An advertising agency determines the number of items sold is related to the amount spent on advertising by the equation $N(A) = 1500 + 315 \ln A$, where A represents the advertising budget and $N(A)$ gives the number of sales. If a company wants to generate 5000 sales, how much money should be set aside for advertising? Round to the nearest dollar. \$66,910

104. **Automobile depreciation:** The amount of time required for a certain new car to depreciate to a given value can be determined using the formula $T(v_c) = 5\ln\left(\frac{v_n}{v_c}\right)$, where v_c represents the current value of the car, v_n represents the value of the car when new, and $T(v_c)$ gives the elapsed time in years. A new car is purchased for \$28,500. Find the current value 3 yr later. \$15,641.13

105. **Spaceship velocity:** In space travel, the change in the velocity of a spaceship V_s (in km/sec) depends on the mass of the ship M_s (in tons), the mass of the fuel that has been burned M_f (in tons), and the escape velocity of the exhaust V_e (in km/sec). Disregarding frictional forces, these are related by the equation $V_s = V_e\ln\left(\frac{M_s}{M_s - M_f}\right)$. Find the mass of the fuel that has been burned when $V_s = 6$ km/sec, if the escape velocity of the exhaust is 8 km/sec and the ship's mass is 100 tons. 52.76 tons

106. **Carbon-14 dating:** After the death of an organism, it no longer absorbs the natural radioactive element known as "carbon-14" (^{14}C) from our atmosphere. Scientists theorize that the age of the organism (now a fossil) can be estimated by measuring the amount of ^{14}C that remains in the fossil, since the half-life of carbon-14 is known. One version of the formula used is $T = -7978 \ln x$, where x is the percentage of ^{14}C that remains in the fossil and T is the time in years since the organism died. If an archeologist claimed the bones of a recently discovered skeleton were 9800 years old, what percent of ^{14}C did she determine remained in the bones? 29.3%

WRITING, RESEARCH, AND DECISION MAKING

107. Virtually all ocean life depends on microscopic plants called *phytoplankton.* These plants can only thrive in what is called the *photic zone* of the ocean, or the top layer of ocean, where there is sufficient light for photosynthesis to take place. The depth of this zone depends on various factors, and is measured using an exponential formula called the Beer-Lambert law. Do some research on this mathematical model and write a report on how it is used. Include several examples and a discussion of the factors that most affect the depth of the photic zone. Answers will vary.

108. In 1798, the English economist Thomas Malthus wrote a paper called "Essay on the Principle of Population," in which he forecast that human populations would grow exponentially, while the supply of food would only grow linearly. This dire predication had a huge impact on the social and economic thinking of the day. Do some research on Thomas Malthus and investigate the mathematical models he used to predict the growth of the food supply versus population growth. Why did his predictions have such an impact? Why were his predictions never realized? Answers will vary.

109. Match each equation with the most appropriate solution strategy, and justify/discuss why.

a.	$e^{x+1} = 25$	d	apply base-10 logarithm to both sides
b.	$\log(2x + 3) = \log 53$	e	rewrite and apply uniqueness property for exponentials
c.	$\log(x^2 - 3x) = 2$	b	apply uniqueness property for logarithms
d.	$10^{2x} = 97$	f	apply either base-10 or base-e logarithm
e.	$2^{5x-3} = 32$	a	apply base-e logarithm
f.	$7^{x+2} = 23$	c	write in exponential form

EXTENDING THE CONCEPT

Solve the following equations.

110. $2e^{2x} - 7e^{x} = 15$ $x = 1.609438$

111. $3e^{2x} - 4e^{x} - 7 = -3$ $x = 0.69319718$

112. $\log_2(x + 5) = \log_4(21x + 1)$ $x = 3$

113. Solve by exponentiating both sides to an appropriate base: $\log_3 3^{2x} = \log_3 5$. $x = \frac{\log_3 5}{2}$

114. Use the algebraic method from Section 3.2 to find the inverse function for $f(x) = 2^{x+1}$. $f^{-1}(x) = \frac{\ln x}{\ln 2} - 1$

Show that $f(x)$ and $g(x)$ are inverse functions by composing the functions and using logarithmic properties.

115. $f(x) = 3^{x-2}$; $g(x) = \log_3 x + 2$

116. $f(x) = e^{x-1}$; $g(x) = \ln(x) + 1$

117. Show $y = 2^x$ is equivalent to $y = e^{x\ln 2}$.

118. Show $y = b^x$ is equivalent to $y = e^{rx}$, where $r = \ln b$.

115. $(f \circ g)(x) = 3^{(\log_3 x+2)-2} = 3^{\log_3 x} = x$; $(g \circ f)(x) = \log_3(3^{x-2}) + 2 = x - 2 + 2 = x$

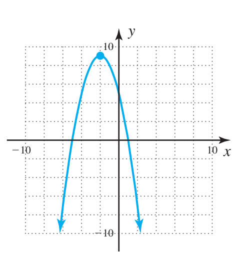
116. $(f \circ g)(x) = e^{[\ln x+1]-1} = e^{\ln x} = x$; $(g \circ f)(x) = \ln e^{x-1} + 1 = x - 1 + 1 = x$

117. $y = e^{x\ln 2} = e^{\ln 2^x} = 2^x$; $y = 2^x = \ln y = x\ln 2$, $e^{\ln y} = e^{x\ln 2} \Rightarrow y = e^{x\ln 2}$

118. $y = bx, \ln y = x\ln b$, $e^{\ln y} = e^{x\ln b}$, $y = e^{xr}$ for $r = \ln b$

MAINTAINING YOUR SKILLS

119. (2.4) Match the graph shown with its correct equation, without actually graphing the function.



a. $y = x^2 + 4x - 5$ **b.** $y = -x^2 - 4x + 5$

c. $y = -x^2 + 4x + 5$ **d.** $y = x^2 - 4x - 5$

120. (R.3) Determine the value of the following expression in exact form (without using a calculator): $\sqrt{(3.2 \times 10^{-23})(2.45 \times 10^{33})}$

121. (3.3) State the domain and range of the functions

a. $y = \sqrt{2x + 3}$ **b.** $y = |x + 2| - 3$

122. (4.6) Graph the function $r(x) = \frac{x^2 - 4}{x - 1}$. Label all intercepts and asymptotes.

123. (4.3) Use synthetic division and the RRT to find all zeroes (real/complex) of $f(x) = x^3 + x + 10$.

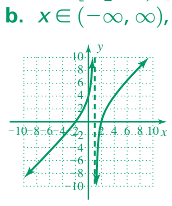
124. (3.6) Suppose the maximum load (in tons) that can be supported by a cylindrical post varies directly with its diameter raised to the fourth power and inversely as the square of its height. A post 8 ft high and 2 ft in diameter can support 6 tons. How many tons can be supported by a post 12 ft high and 3 ft in diameter?

119. b

120. 280,000

121. **a.** $x \in [-\frac{3}{2}, \infty)$, $y \in [0, \infty)$
b. $x \in (-\infty, \infty)$, $y \in [-3, \infty)$

122.



123. $-2, 1 \pm 2i$

124. 13.5 tons

5.5 Applications from Business, Finance, and Physical Science

LEARNING OBJECTIVES

In Section 5.5 you will study applications of:

- **A.** Interest compounded n times per year
- **B.** Interest compounded continuously
- **C.** Exponential growth and decay
- **D.** Annuities and amortization

INTRODUCTION

Would you pay \$950,000 for a home worth only \$250,000? Surprisingly, when a conventional mortgage is repaid over 30 years, this is not at all rare. Over time, the accumulated interest on the mortgage is easily more than two or three times the original value of the house. In this section we explore how interest is paid or charged, and look at other applications of exponential and logarithmic functions from business and finance, and the physical and social sciences.

POINT OF INTEREST

One common application of exponential functions is the calculation of interest. Interest is an amount of money paid *by* you for the use of money that is borrowed, or paid *to* you for money that you invest. The custom of charging or paying interest is very ancient, and there are references to this practice that date back as far as 2000 B.C. in ancient Babylon. In this section, we investigate some of the more common ways interest is charged or paid—applications that require the use of exponential and logarithmic functions.

A. Simple and Compound Interest

Simple interest is an amount of interest that is computed only once during the lifetime of an investment (or loan). In the world of finance, the initial deposit or base amount is referred to as the **principal p,** the **interest rate r** is given as a percentage and is usually stated as an annual rate, with the term of the investment or loan most often given as *time* t in years. Simple interest is merely an application of the basic percent equation, with the additional element of time coming into play. The **simple interest formula** is *interest* = *principal* × *rate* × *time*, or $I = prt$. To find the total amount A that has accumulated (for deposits) or is due (for loans) after t years, we merely add the accumulated interest to the initial principal: $A = p + prt$ or $A = p(1 + rt)$ after factoring.

WORTHY OF NOTE

If a loan is kept for only a certain number of months, weeks, or days, the time t should be stated as a fractional part of a year so the time period for the rate (years) matches the time period over which the loan is repaid.

SIMPLE INTEREST FORMULA

If principal p is deposited or borrowed at interest rate r for a period of t years, the simple interest on this account will be

$$I = prt$$

The total amount A accumulated or due after this period will be:

$$A = p + prt \qquad \text{or} \qquad A = p(1 + rt)$$

EXAMPLE 1 Many finance companies offer what have become known as PayDay Loans—a small \$50 loan to help people get by until payday, usually no longer than 2 weeks. If the cost of this service is \$12.50, determine the annual rate of interest charged by these companies.

Solution: The interest charge is \$12.50, the initial principal is \$50.00 and the time period is 2 weeks or $\frac{2}{52} = \frac{1}{26}$ of a year. The simple interest formula yields

$$I = prt \quad \text{simple interest formula}$$

$$12.50 = 50r\left(\frac{1}{26}\right) \quad \text{substitute \$12.50 for } I\text{, \$50.00 for } p\text{, and } \tfrac{1}{26} \text{ for } t$$

$$6.5 = r \quad \text{result}$$

The annual interest rate on these loans is a whopping 650%!

NOW TRY EXERCISES 7 THROUGH 14

Compound Interest

Many financial institutions pay **compound interest** on deposits they receive, which is interest paid on previously accumulated interest. The most common compounding periods are yearly, semiannually (two times per year), quarterly (four times per year), monthly (12 times per year), and daily (365 times per year). Applications of compound interest typically involve exponential functions. For convenience, consider \$1000 in principal, deposited at 8% for 3 yr. The simple interest calculation shows \$240 in interest is earned and there will be \$1240 in the account: $A = 1000[1 + (0.08)(3)] = \$1240$. If the interest is *compounded each year* $(t = 1)$ instead of once at the start of the three-year period, the interest calculation shows

$$A_1 = 1000(1 + 0.08) = 1080 \text{ in the account at the end of year 1,}$$

$$A_2 = 1080(1 + 0.08) = 1166.40 \text{ in the account at the end of year 2,}$$

$$A_3 = 1166.40(1 + 0.08) \approx 1259.71 \text{ in the account at the end of year 3.}$$

The account has earned an additional \$19.71 interest. More importantly, notice that we're multiplying by $(1 + 0.08)$ each compounding period, meaning results can be computed more efficiently by simply applying the factor $(1 + 0.08)^t$ to the initial principal p. For example:

$$A_3 = 1000(1 + 0.08)^3 \approx \$1259.71.$$

In general, for interest compounded yearly the **accumulated value equation** is $A = p(1 + r)^t$. Notice that solving this equation for p will tell us the amount we need to deposit *now,* in order to accumulate A dollars in t years: $p = \frac{A}{(1 + r)^t}$. This is called the **present value equation.**

INTEREST COMPOUNDED ANNUALLY

If a principal p is deposited at interest rate r and compounded yearly for a period of t years, the **accumulated value** is

$$A = p(1 + r)^t$$

If an accumulated value A is desired after t years, and the money is deposited at interest rate r and compounded yearly, the *present value* is

$$p = \frac{A}{(1 + r)^t}$$

EXAMPLE 2 An initial deposit of \$1000 is made into an account paying 6% compounded yearly. How long will it take for the money to double?

Solution: Using the formula for interest compounded yearly we have

$$\begin{aligned} A &= p(1 + r)^t && \text{given} \\ 2000 &= 1000(1 + 0.06)^t && \text{substitute 2000 for } A\text{, 1000 for } p\text{, and 0.06 for } r \\ 2 &= 1.06^t && \text{isolate variable term} \\ \ln 2 &= t \ln 1.06 && \text{apply base-}e\text{ logarithms} \\ \frac{\ln 2}{\ln 1.06} &= t && \text{solve for } t \\ 11.9 &\approx t && \text{approximate form} \end{aligned}$$

The money will double in just under 12 years.

NOW TRY EXERCISES 15 THROUGH 20

When interest is compounded more than once a year, say monthly, the bank will divide the interest rate by 12 (the number of compoundings) to maintain a constant yearly rate, but then pays you interest 12 times per year (interest is compounded). The net effect is an increased gain in the interest you earn, and the final compound interest formula takes this form:

$$\text{total amount} = \text{principal}\left(1 + \frac{\text{interest rate}}{\text{number of compoundings per year}}\right)^{(\text{number of years} \times \text{number of compoundings per year})}$$

COMPOUNDED INTEREST FORMULA

If principal p is deposited at interest rate r and compounded n times per year for a period of t years, the *accumulated value* will be:

$$A = p\left(1 + \frac{r}{n}\right)^{nt}$$

EXAMPLE 3 Macalyn won \$150,000 in the Missouri lottery and decides to invest the money for retirement in 20 yr. Of all the options available here, which one will produce the most money for retirement?

a. A certificate of deposit paying 5.4% compounded yearly.

b. A money market certificate paying 5.35% compounded semiannually.

c. A bank account paying 5.25% compounded monthly.

d. A bond issue paying 5.2% compounded daily.

Solution:

a. $A = \$150{,}000\left(1 + \frac{0.054}{1}\right)^{(20\times1)}$

$\approx \$429{,}440.97$

b. $A = \$150{,}000\left(1 + \frac{0.0535}{2}\right)^{(20\times2)}$

$\approx \$431{,}200.96$

c. $A = \$150{,}000\left(1 + \frac{0.0525}{4}\right)^{(20\times 4)}$

$\approx \$425{,}729.59$

d. $A = \$150{,}000\left(1 + \frac{0.052}{365}\right)^{(20\times 365)}$

$\approx \$424{,}351.12$

The best choice is (b), semiannual compounding at 5.35% for 20 yr.

NOW TRY EXERCISES 21 THROUGH 24

B. Interest Compounded Continuously

It seems natural to wonder what happens to the interest accumulation as n (the number of compounding periods) becomes very large. It appears the interest rate becomes very small (because we're dividing by n), but the exponent becomes very large (since we're multiplying by n). To see the result of this interplay more clearly, it will help to rewrite the compound interest formula $A = p\left(1 + \frac{r}{n}\right)^{nt}$ using the substitution $n = xr$. This gives $\frac{r}{n} = \frac{1}{x}$, and by direct substitution $\left(xr \text{ for } n \text{ and } \frac{1}{x} \text{ for } \frac{r}{n}\right)$ we obtain the form $A = p\left[\left(1 + \frac{1}{x}\right)^{x}\right]^{rt}$ by regrouping. This allows for a more careful study of the "denominator versus exponent" relationship using $\left(1 + \frac{1}{x}\right)^{x}$, *the same expression we used in Section 5.4 to define the number e.* Once again, note what happens as $x \to \infty$ (meaning the number of compounding periods increase without bound).

x	1	10	100	1000	10,000	100,000	1,000,000
$\left(1 + \frac{1}{x}\right)^{x}$	2	2.56374	2.70481	2.71692	2.71815	2.71827	2.71828

As before, we have as $x \to \infty$, $\left(1 + \frac{1}{x}\right)^{x} \to e$. The net result of this investigation is a formula for **interest compounded continuously,** derived by replacing $\left(1 + \frac{1}{x}\right)^{x}$ with the number e in the formula for compound interest, where $A = p\left[\left(1 + \frac{1}{x}\right)^{x}\right]^{rt}$ becomes $A = p[e]^{rt}$.

INTEREST COMPOUNDED CONTINUOUSLY

If a principal p is deposited at interest rate r and compounded continuously for a period of t years, the *accumulated value* will be

$$A = pe^{rt}$$

EXAMPLE 4 Jaimin has \$10,000 to invest and wants to have at least \$25,000 in the account in 10 yr for his daughter's college education fund. If the account pays interest compounded continuously, what interest rate is required?

Solution: In this case, $P = \$10{,}000$, $A = \$25{,}000$, and $t = 10$.

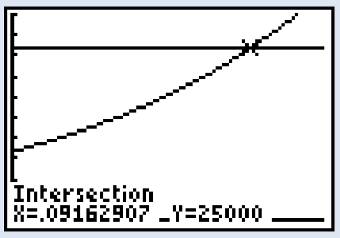
$$
\begin{aligned}
A &= pe^{rt} && \text{given} \\
25{,}000 &= 10{,}000e^{10r} && \text{substitute 25,000 for } A\text{, 10,000 for } p\text{, and 10 for } t \\
2.5 &= e^{10r} && \text{isolate variable term} \\
\ln 2.5 &= 10r \ln e && \text{apply base-}e\text{ logarithms } (\ln e = 1) \\
\frac{\ln 2.5}{10} &= r && \text{solve for } t \\
0.092 &\approx r && \text{approximate form}
\end{aligned}
$$

Jaimin will need an interest rate of about 9.2% to meet his goal.

NOW TRY EXERCISES 25 THROUGH 34

GRAPHICAL SUPPORT

To check the result from Example 4, use $Y_1 = 10{,}000e^{10x}$ and $Y_2 = 25{,}000$, then look for their point of intersection. We need only set an appropriate window size to ensure the answer will appear in the viewing window. Since 25,000 is the goal, $y \in [0, 30{,}000]$ seems reasonable for y. Although 12% interest ($x = 0.12$) is too good to be true, $x \in [0, 0.12]$ leaves a nice frame for the x-values. Verify that the calculator's answer is equal to $\frac{\ln 2.5}{10}$.



There are a number of interesting applications in the exercise set (see Exercises 37 through 46).

WORTHY OF NOTE

Notice the formula for exponential growth is virtually identical to the formula for interest compounded continuously. In fact, both are based on the same principles. If we let $A(t)$ represent the amount in an account after t years and A_0 represent the initial deposit (instead of P), we have: $A(t) = A_0e^{rt}$ versus $Q(t) = Q_0e^{rt}$ and the two can hardly be distinguished.

C. Applications Involving Exponential Growth and Decay

Closely related to the formula for interest compounded continuously are applications of **exponential growth** and **exponential decay.** If Q (quantity) and t (time) are variables, then Q grows exponentially as a function of t if $Q(t) = Q_0e^{rt}$ for the positive constants Q_0 and r. Careful studies have shown that population growth, whether it be humans, bats, or bacteria, can be modeled by these "base-e" exponential growth functions. If $Q(t) = Q_0e^{-rt}$, then we say Q decreases or **decays exponentially** over time. The constant r determines how rapidly a quantity grows or decays and is known as the **growth rate** or **decay rate** constant. Graphs of exponential growth and decay functions are shown here for arbitrary Q_0 and r. Note the graph of $Q(t) = Q_0e^{-rt}$ (Figure 5.19) is simply a reflection across the y-axis of $Q(t) = Q_0e^{rt}$ (Figure 5.18).

Figure 5.18

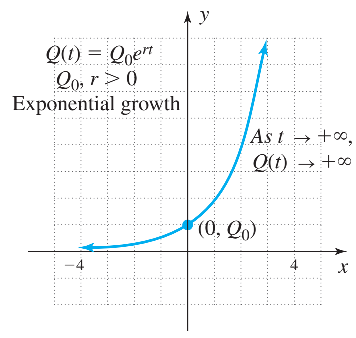
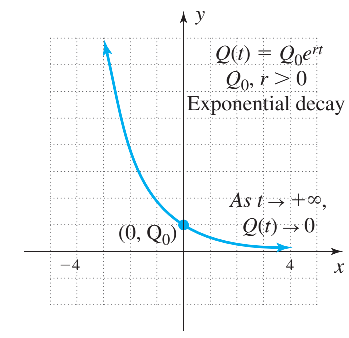


Figure 5.19



EXAMPLE 5 Because fruit flies multiply very quickly, they are often used in a study of genetics. Given the necessary space and food supply, a certain population of fruit flies is known to double every 12 days. If there were 100 flies to begin, find (a) the growth rate r and (b) the number of days until the population reaches 2000 flies.

Solution: **a.** Using the formula for exponential growth with $Q_0 = 100$, $t = 12$, and $Q(t) = 200$, we can solve for the growth rate r.

$$\begin{aligned} Q(t) &= Q_0e^{rt} && \text{exponential growth function} \\ 200 &= 100e^{12r} && \text{substitute 200 for } Q(t)\text{, 100 for } Q_0\text{, and 12 for } t \\ 2 &= e^{12r} && \text{isolate variable term} \\ \ln 2 &= 12r\ln e && \text{apply base-}e\text{ logarithms } (\ln e = 1) \\ \frac{\ln 2}{12} &= r && \text{solve for } r \\ 0.05776 &\approx r && \text{approximate form} \end{aligned}$$

The growth rate is approximately 5.78%.

b. To find the number of days until the fly population reaches 2000 we substitute 0.05776 for r in the exponential growth function.

$$\begin{aligned} Q(t) &= Q_0e^{rt} && \text{exponential growth function} \\ 2000 &= 100e^{0.05776t} && \text{substitute 2000 for } Q(t)\text{, 100 for } Q_0\text{, and 0.05776 for } r \\ 20 &= e^{0.05776t} && \text{isolate variable term} \\ \ln 20 &= 0.05776t\ln e && \text{apply base-}e\text{ logarithms } (\ln e = 1) \\ \frac{\ln 20}{0.05776} &= t && \text{solve for } t \\ 51.87 &\approx t && \text{approximate form} \end{aligned}$$

The fruit fly population will reach 2000 on day 51.

NOW TRY EXERCISES 47 AND 48

WORTHY OF NOTE

Many population growth models assume an unlimited supply of resources, nutrients, and room for growth. When this is not the case, a logistic growth model often results. See Section 5.6.

Perhaps the best known examples of exponential decay involve radioactivity. Ever since the end of World War II, common citizens have been aware of the existence of **radioactive elements** and the power of atomic energy. Today, hundreds of additional applications have been found for radioactive materials, from areas as diverse as biological research, radiology, medicine, and archeology. Radioactive elements decay of their own accord by emitting radiation. The rate of decay is measured using the **half-life** of the substance, which is the time required for a mass of radioactive material to decay until only one-half of its original mass remains. This half-life is used to find the rate of decay r, first mentioned in Section 5.3. In general, we have

$$Q(t) = Q_0 e^{-rt} \quad \text{exponential decay function}$$
$$\frac{1}{2}Q_0 = Q_0 e^{-rt} \quad \text{substitute } \tfrac{1}{2}Q_0 \text{ for } Q(t)$$
$$\frac{1}{2} = \frac{1}{e^{rt}} \quad \text{divide by } Q_0\text{; use negative exponents to rewrite expression}$$
$$2 = e^{rt} \quad \text{property of ratios}$$
$$\ln 2 = rt\ln e \quad \text{apply base-}e\text{ logarithms } (\ln e = 1)$$
$$\frac{\ln 2}{t} = r \quad \text{solve for } r$$

RADIOACTIVE RATE OF DECAY

If t represents the half-life of a radioactive substance per unit time, the nominal rate of decay per a like unit of time is given by

$$r = \frac{\ln 2}{t}$$

The rate of decay for known radioactive elements varies a great deal. For example, the element carbon-14 has a half-life of about 5730 yr, while the element lead-211 has a half-life of only about 3.5 min. Radioactive elements can be detected in extremely small amounts. If a drug is "labeled" (mixed with) a radioactive element and injected into a living organism, its passage through the organism can be traced and information on the health of internal organs can be obtained.

EXAMPLE 6 The radioactive element potassium-42 is often used in biological experiments, since it has a half-life of only about 12.4 hr and desired results can be measured accurately and experiments repeated if necessary. How much of a 2-g sample will remain after 18 hr and 45 min?

Solution: To begin we must find the nominal rate of decay r and use this value in the exponential decay function.

$$r = \frac{\ln 2}{t} \quad \text{radioactive rate of decay}$$
$$r = \frac{\ln 2}{12.4} \quad \text{substitute 12.4 for } t$$
$$r \approx 0.056 \quad \text{result}$$

The rate of decay is approximately 5.6%. To determine how much of the sample remains after 18.75 hr, we use $r = 0.056$ in the decay function and evaluate it at $t = 18.75$.

$$Q(t) = Q_0 e^{-rt} \quad \text{exponential decay function}$$
$$Q(18.75) = 2e^{(-0.056)(18.75)} \quad \text{substitute 2 for } Q_0\text{, 0.056 for } r\text{, and 18.75 for } t$$
$$Q(18.75) \approx 0.7 \quad \text{evaluate}$$

After 18 hr and 45 min, only 0.7 g of potassium-42 will remain.

NOW TRY EXERCISES 49 THROUGH 52

WORTHY OF NOTE

In the case of regularly scheduled loan payments, an amortization schedule is used. The word **amortize** comes from the French word *amortir,* which literally means "to kill off." To amortize a loan means to pay it off, with the schedule of payments often computed using an annuity formula.

WORTHY OF NOTE

It is often assumed that the first payment into an annuity is made *at the end of a compounding period,* and hence earns no interest. This is why the first $100 deposit is not multiplied by the interest factor. These terms are actually the terms of a **geometric sequence,** which we will study later in Section 8.3.

D. Applications Involving Annuities and Amortization

Our previous calculations for simple and compound interest involved a single deposit (the principal) that accumulated interest over time. Many savings and investment plans involve a regular schedule of deposits (monthly, quarterly, or annual deposits) over the life of the investment. Such an investment plan is called an **annuity.**

Similar to our work with compound interest, formulas exist for the *accumulated value* of an annuity and the *periodic payment required* to meet future goals. Suppose that for 4 yr, $100 is deposited annually into an account paying 8% compounded yearly. Using the compound interest formula we can track the total amount A in the account:

$$A = 100 + 100(1.08)^1 + 100(1.08)^2 + 100(1.08)^3$$

To develop an annuity formula, we multiply the annuity equation by 1.08, then subtract the original equation. This leaves only the first and last terms, since the other (interior) terms sum to zero:

$$\begin{aligned}
1.08A &= 100(1.08) + 100(1.08)^2 + 100(1.08)^3 + 100(1.08)^4 && \text{multiply by 1.08}\\
-A &= -[100 + 100(1.08)^1 + 100(1.08)^2 + 100(1.08)^3] && \text{original equation}\\
1.08A - A &= 100(1.08)^4 - 100 && \text{subtract ("interior terms" sum to zero)}\\
0.08A &= 100[(1.08)^4 - 1] && \text{factor out 100}\\
A &= \frac{100[(1.08)^4 - 1]}{0.08} && \text{solve for } A
\end{aligned}$$

This result can be generalized for any periodic payment p, interest rate r, number of compounding periods n, and number of years t. This would give $A = \dfrac{p\left[\left(1 + \frac{r}{n}\right)^{nt} - 1\right]}{\frac{r}{n}}$.

The formula can be made less formidable using $R = \frac{r}{n}$, where R is the interest rate per compounding period.

ACCUMULATED VALUE OF AN ANNUITY

If a periodic payment P is deposited n times per year at an *annual interest rate* r with interest compounded n times per year for t years, the accumulated value is given by

$$A = \frac{P}{R}[(1 + R)^{nt} - 1], \text{ where } R = \frac{r}{n}$$

This is also referred to as the **future value** of the account.

EXAMPLE 7 Since he was a young child, Fitisemanu's parents have been depositing $50 each month into an annuity that pays 6% annually and is compounded monthly. If the account is now worth $9875, how long has it been open?

Solution: In this case $p = 50$, $r = 0.06$, $n = 12$, $R = 0.005$, and $A = 9875$. The formula gives

$$A = \frac{P}{R}[(1 + R)^{nt} - 1] \qquad \text{future value formula}$$

$$9875 = \frac{50}{0.005}[(1.005)^{(12)(t)} - 1] \qquad \text{substitute 9875 for } A\text{, 50 for } p\text{, 0.005 for } R\text{, and 12 for } n$$

$$1.9875 = 1.005^{12t} \qquad \text{simplify and isolate variable term}$$

$$\ln(1.9875) = 12t(\ln 1.005) \qquad \text{apply base-}e\text{ logarithms}$$

$$\frac{\ln(1.9875)}{12\ln(1.005)} = t \qquad \text{solve for } t$$

$$11.5 \approx t \qquad \text{approximate form}$$

The account has been open approximately 11.5 yr.

NOW TRY EXERCISES 53 THROUGH 56

The periodic payment required to meet a future goal or obligation can be computed by solving for P in the previous formula: $P = \frac{AR}{[(1 + R)^{nt} - 1]}$. In this form, P is referred to as a **sinking fund.**

EXAMPLE 8 Sheila is determined to stay out of debt and decides to save \$20,000 to pay cash for a new car in 4 yr. The best investment vehicle she can find pays 9% compounded monthly. If \$300 is the most she can invest each month, can she meet her "4-yr" goal?

Solution: Here we have $P = 300$, $A = 20{,}000$, $r = 0.09$, $n = 12$, and $R = 0.0075$. The sinking fund formula gives

$$P = \frac{AR}{[(1 + R)^{nt} - 1]} \qquad \text{sinking fund}$$

$$300 = \frac{(20{,}000)(0.0075)}{(1.0075)^{12t} - 1} \qquad \text{substitute 300 for } P\text{, 20,000 for } A\text{, 0.0075 for } R\text{, and 12 for } n$$

$$300(1.0075^{12t} - 1) = 150 \qquad \text{multiply in numerator and clear denominators}$$

$$1.0075^{12t} = 1.5 \qquad \text{isolate variable term}$$

$$12t\ln(1.0075) = \ln 1.5 \qquad \text{apply base-}e\text{ logarithms}$$

$$t = \frac{\ln(1.5)}{12\ln(1.0075)} \qquad \text{solve for } t$$

$$\approx 4.5 \qquad \text{approximate form}$$

No. She is close, but misses her original 4-yr goal.

NOW TRY EXERCISES 57 AND 58

For Example 8, we could have substituted 4 for t and left P unknown, to see if a payment of \$300 per month would be sufficient. You can verify the result would be $P \approx \$347.70$, which is what Sheila would need to invest to meet her 4-yr goal exactly.

Using a graphing calculator allows for various other investigations, as demonstrated in the following *Technology Highlight.*

TECHNOLOGY HIGHLIGHT

Using a Graphing Calculator to Explore Compound Interest

The keystrokes shown apply to a TI-84 Plus model. Please consult our Internet site or your manual for other models.

The graphing calculator is an excellent tool for exploring mathematical relationships, particularly when many variables work simultaneously to produce a single result. For example, the formula $A = P\left(1 + \frac{r}{n}\right)^{nt}$ has five different unknowns: the total amount A, initial principal P, interest rate r, compounding periods per year n, and number of years t. In Example 2, we asked how long it would take \$1000 to double if it were compounded yearly at 6% ($n = 1$, $r = 0.06$). What if we deposited \$5000 instead of \$1000? Compounded daily instead of quarterly? Or invested at 12% rather than 10%? There are many ways a graphing calculator can be used to answer such questions. In this exercise, we make use of the calculator's "alpha constants." The TI 84 Plus can use any of the 26 letters of the English alphabet (and even a few other symbols) to store constant values. One advantage is we can use them to write a formula using these constants on the Y= screen, then change any constant from the home screen to see how other values are affected. On the TI-84Plus, these alpha constants are shown in green and are accessed by pressing the ALPHA key followed by the key with the letter desired. Suppose we wanted to study the relationship between an interest rate r and the time t required for a deposit to double. Using Y_1 in place of A as output variable, and x in place of t, enter $A = P\left(1 + \frac{r}{n}\right)^{nt}$ as Y_1 on the Y= screen (Figure 5.20). To assign initial values to the constants P, r, and n we use the STO▸ and ALPHA keys. Let's start with a deposit of \$1000 at 7% interest compounded monthly. The keystrokes are: 1000 STO▸ ALPHA 8 ENTER, 0.07 STO▸ ALPHA X ENTER, and 12 STO▸ ALPHA LOG ENTER (Figure 5.21). After setting an appropriate window size (perhaps Xmax = 15 and Ymax = 3000), we can begin investigating how the interest rate affects this growth. It will help to enter $Y_2 = 2000$ to easily check "doubling time." Graph both functions and use the intersection of graphs method to find the doubling time. This produces the result in Figure 5.22, where we note it will take about 9.9 yr to double under these conditions. Return to the home screen (2nd MODE), change the interest rate of 10%, and graph the functions again. This time the point of intersection is just less than 7 (yr). Experiment with other rates and compounding periods to explore further.

Figure 5.20

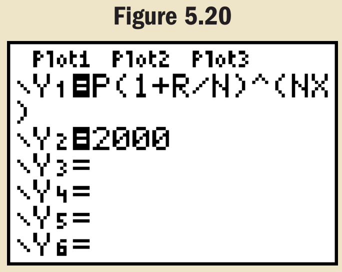


Figure 5.21

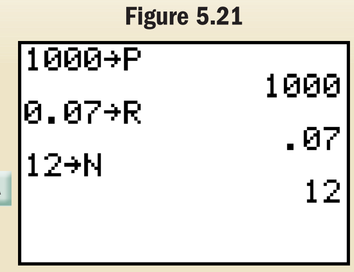
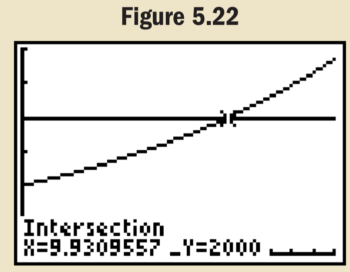


Figure 5.22



Exercise 1: With $P = \$1000$, and $r = 0.08$, investigate the "doubling time" for interest compounded quarterly, monthly, daily, and hourly. 8.75; 8.69; 8.665; 8.664

Exercise 2: With $P = \$1000$, investigate "doubling time" for rates of 6%, 8%, 10%, and 12%, and $n = 4$, $n = 12$, and $n = 365$. Which had a more significant impact, more compounding periods, or a greater interest rate? 6%: 11.64; 11.58; 11.55, 8%: 8.75; 8.69; 8.67, 10%: 7.02; 6.96; 6.93, 12%: 5.86; 5.81; 5.78; a greater interest rate has more impact

Exercise 3: Will a larger principal cause the money to double faster? Investigate and find out. no

5.5 EXERCISES

CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. Compound interest is interest paid to you on previously accumulated interest.

2. The formula for interest compounded continuously is $A = pe^{rt}$, where e is approximately 2.72.

3. Given constants Q_0 and r, and that Q decays exponentially as a function of t, the equation model is $Q(t) = $ Q_0e^{-rt}.

4. Investment plans calling for regularly scheduled deposits are called annuities. The annuity formula gives the future value of the account.

5. Explain/describe the difference between the future value and present value of an annuity. Include an example. Answers will vary.

6. Describe/explain how you would find the rate of growth r, given that a population of ants grew from 250 to 3000 in 6 weeks. Answers will vary.

DEVELOPING YOUR SKILLS

For simple interest accounts, the interest earned or due depends on the principal p, interest rate r, and the time t in years according to the formula $I = prt$.

7. Find p given $I = \$229.50$, $r = 6.25\%$, and $t = 9$ months. \$4896

8. Find r given $I = \$1928.75$, $p = \$8500$, and $t = 3.75$ yr. 6.05%

9. Larry came up a little short one month at bill-paying time and had to take out a title loan on his car at Check Casher's, Inc. He borrowed \$260, and 3 weeks later he paid off the note for \$297.50. What was the annual interest rate on this title loan? (*Hint:* How much *interest* was charged?) 250%

10. Angela has \$750 in a passbook savings account that pays 2.5% simple interest. How long will it take the account balance to hit the \$1000 mark at this rate of interest, if she makes no further deposits? (*Hint:* How much *interest* will be paid?) 13.3 yr

For simple interest accounts, the amount A accumulated or due depends on the principal p, interest rate r, and the time t in years according to the formula $A = p(1 + rt)$.

11. Find p given $A = \$2500$, $r = 6.25\%$, and $t = 31$ months. \$2152.47

12. Find r given $A = \$15,800$, $p = \$10,000$, and $t = 3.75$ yr. 15.47%

13. Olivette Custom Auto Service borrowed \$120,000 at 4.75% simple interest to expand their facility from three service bays to four. If they repaid \$149,925, what was the term of the loan? 5.25 yr

14. Healthy U sells nutritional supplements and borrows \$50,000 to expand their product line. When the note is due 3 yr later, they repay the lender \$62,500. If it was a simple interest note, what was the annual interest rate? 8.33%

For accounts where interest is compounded annually, the amount A accumulated or due depends on the principal p, interest rate r, and the time t in years according to the formula $A = p(1 + r)^t$.

15. Find t given $A = \$48,428$, $p = \$38,000$, and $r = 6.25\%$. 4 yr

16. Find p given $A = \$30,146$, $r = 5.3\%$, and $t = 7$ yr. \$21,000.57

17. How long would it take \$1525 to triple if invested at 7.1%? 16 yr

18. What interest rate will ensure a \$747.26 deposit will be worth \$1000 in 5 yr? 6%

For accounts where interest is compounded annually, the principal P needed to ensure an amount A has been accumulated in the time period t when deposited at interest rate r is given by the formula $P = \frac{A}{(1+r)^t}$.

19. The Stringers need to make a \$10,000 balloon payment in 5 yr. How much should be invested now at 5.75%, so that the money will be available? \$7561.33

20. Morgan is 8 yr old. If her mother wants to have \$25,000 for Morgan's first year of college (in 10 yr), how much should be invested now if the account pays a 6.375% fixed rate? \$13,475.48

For compound interest accounts, the amount A accumulated or due depends on the principal p, interest rate r, number of compoundings per year n, and the time t in years according to the formula $A = p\left(1 + \frac{r}{n}\right)^{nt}$.

21. Find t given $A = \$129{,}500$, $p = \$90{,}000$, and $r = 7.125\%$ compounded weekly. 5 yr

22. Find r given $A = \$95{,}375$, $p = \$65{,}750$, and $t = 15$ yr with interest compounded monthly. 2.48%

23. How long would it take a \$5000 deposit to double, if invested at a 9.25% rate and compounded daily? 7.5 yr

24. What principal should be deposited at 8.375% compounded monthly to ensure the account will be worth \$20,000 in 10 yr? \$8681.04

For accounts where interest is compound continuously, the amount A accumulated or due depends on the principal p, interest rate r, and the time t in years according to the formula $A = pe^{rt}$.

25. Find t given $A = \$2500$, $p = \$1750$, and $r = 4.5\%$. 7.9 yr

26. Find r given $A = \$325{,}000$, $p = \$250{,}000$, and $t = 10$ yr. 2.62%

27. How long would it take \$5000 to double if it is invested at 9.25%. Compare the result to Exercise 23. 7.5 yr

28. What principal should be deposited at 8.375% to ensure the account will be worth \$20,000 in 10 yr? Compare the result to Exercise 24. \$8655.82

Solve for the indicated unknowns.

29. $A = p + prt$
- **a.** solve for t
- **b.** solve for p

30. $A = p(1 + r)^t$
- **a.** solve for t
- **b.** solve for r

31. $A = p\left(1 + \frac{r}{n}\right)^{nt}$
- **a.** solve for r
- **b.** solve for t

32. $A = pe^{rt}$
- **a.** solve for p
- **b.** solve for r

33. $Q(t) = Q_0e^{rt}$
- **a.** solve for Q_0
- **b.** solve for t

34. $p = \frac{AR}{[(1+R)^{nt} - 1]}$
- **a.** solve for A
- **b.** solve for n

WORKING WITH FORMULAS

35. Amount of a mortgage payment: $P = \frac{AR}{1 - (1+R)^{-nt}}$

The mortgage payment required to pay off (or amortize) a loan is given by the formula shown, where P is the payment amount, A is the original amount of the loan, t is the time in years, r is the annual interest rate, n is the number of payments per year, and $R = \frac{r}{n}$. Find the *monthly payment* required to amortize a \$125,000 home, if the interest rate is 5.5%/year and the home is financed over 30 yr. \$709.74

29. a. $t = \frac{A - p}{pr}$
b. $p = \frac{A}{1 + rt}$

30. a. $t = \frac{\ln\left(\frac{A}{p}\right)}{\ln(1 + r)}$
b. $r = \sqrt[t]{\frac{A}{p}} - 1$

31. a. $r = n\left(\sqrt[nt]{\frac{A}{p}} - 1\right)$
b. $t = \frac{\ln\left(\frac{A}{p}\right)}{n\ln\left(1 + \frac{r}{n}\right)}$

32. a. $p = \frac{A}{e^{rt}}$
b. $r = \frac{\ln\left(\frac{A}{p}\right)}{t}$

33. a. $Q_0 = \frac{Q}{e^{rt}}$
b. $t = \frac{\ln\left(\frac{Q}{Q_0}\right)}{r}$

34. a. $A = \frac{p[(1 + R)^{nt} - 1]}{R}$
b. $n = \frac{\ln\left(\frac{AR}{p} + 1\right)}{t\ln(1 + R)}$

36. Total interest paid on a home mortgage: $I = \left[\dfrac{prt}{1 - \left(\dfrac{1}{1 + 0.08\overline{3}r}\right)^{12t}}\right] - p$

The total interest I paid in t years on a home mortgage of p dollars is given by the formula shown, where r is the interest rate on the loan. If the original mortgage was \$198,000 at an interest rate of 6.5%, (a) how much interest has been paid in 10 yr? (b) Use a table of values to determine how many years it will take for the interest paid to exceed the amount of the original mortgage. **a.** \$71,789.99 **b.** 25 yr

APPLICATIONS

37. Simple interest: The owner of Paul's Pawn Shop loans Larry \$200.00 using his Toro riding mower as collateral. Thirteen weeks later Larry comes back to get his mower out of pawn and pays Paul \$240.00. What was the annual simple interest rate on this loan? 80%

38. Simple interest: To open business in a new strip mall, Laurie's Custom Card Shoppe borrows \$50,000 from a group of investors at 4.55% simple interest. Business booms and blossoms, enabling Laurie to repay the loan fairly quickly. If Laurie repays \$62,500, how long did it take? $5\frac{1}{2}$ yr

39. Compound interest: As a curiosity, David decides to invest \$10 in an account paying 10% interest compounded 10 times per year for 10 yr. Is that enough time for the \$10 to triple in value? no

40. Compound interest: As a follow-up experiment (see Exercise 39), David invests \$10 in an account paying 12% interest compounded 10 times per year for 10 yr, and another \$10 in an account paying 10% interest compounded 12 times per year for 10 yr. Which produces the better investment—more compounding periods or a higher interest rate? higher interest rate

41. Compound interest: Due to demand, Donovan's Dairy (Wisconsin, USA) plans to double its size in 4 yr and will need \$250,000 to begin development. If they invest \$175,000 in an account that pays 8.75% compounded semiannually, (a) will there be sufficient funds to break ground in 4 yr? (b) If not, use a table to find the *minimum interest rate* that will allow the dairy to meet its 4-yr goal. no 9.12%

42. Compound interest: To celebrate the birth of a new daughter, Helyn invests 6000 Swiss francs in a college savings plan to pay for her daughter's first year of college in 18 yr. She estimates that 25,000 francs will be needed. If the account pays 7.2% compounded daily, (a) will she meet her investment goal? (b) if not, use a table to find the *minimum rate of interest* that will enable her to meet this 18-yr goal. no 7.93%

43. Interest compounded continuously: Valance wants to build an addition to his home outside Madrid (Spain) so he can watch over and help his parents in their old age. He hopes to have 20,000 euros put aside for this purpose within 5 yr. If he invests 12,500 euros in an account paying 8.6% interest compounded continuously, (a) will he meet his investment goal? (b) If not, find the *minimum rate of interest* that will enable him to meet this 5-yr goal. no 9.4%

44. Interest compounded continuously: Minh-Ho just inherited her father's farm near Mito (Japan), which badly needs a new barn. The estimated cost of the barn is 8,465,000 yen and she would like to begin construction in 4 yr. If she invests 6,250,000 yen in an account paying 6.5% interest compounded continuously, (a) will she meet her investment goal? (b) If not, find the *minimum rate of interest* that will enable her to meet this 4-yr goal. no 7.58%

45. Interest compounded continuously: William and Mary buy a small cottage in Dovershire (England), where they hope to move after retiring in 7 yr. The cottage needs about 20,000 euros worth of improvements to make it the retirement home they desire. If they invest 12,000 euros in an account paying 5.5% interest compounded continuously, (a) will they have enough to make the repairs? (b) If not, find the *minimum amount they need to deposit* that will enable them to meet this goal in 7 yr. no approx 13,609 Euros

46. Interest compounded continuously: After living in Oslo (Norway) for 20 years, Zirkcyt and Shybrt decide to move inland to help operate the family ski resort. They hope to make the move in 6 yr, after they have put aside 140,000 kroner. If they invest 85,000 kroner in an account paying 6.9% interest compounded continuously, (a) will they meet their 140,000 kroner goal? (b) If not, find the *minimum amount they need to deposit* that will allow them to meet this goal in 6 yr. no approx 92,540 kroner

47. Exponential growth: As part of a lab experiment, Luamata needs to grow a culture of 200,000 bacteria, which are known to double in number in 12 hr. If he begins with 1000 bacteria, (a) find the growth rate r and (b) find how many hours it takes for the culture to produce the 200,000 bacteria. 5.78% 91.67 hr

48. Exponential growth: After the wolf population was decimated due to overhunting, the rabbit population in the Boluhti Game Reserve began to double every 6 months. If there were an estimated 120 rabbits to begin, (a) find the growth rate r and (b) find the number of months required for the population to reach 2500. **a.** 11.55% **b.** 26.29 mo

49. Radioactive decay: The radioactive element iodine-131 has a half-life of 8 days and is often used to help diagnose patients with thyroid problems. If a certain thyroid procedure requires 0.5 g and is scheduled to take place in 3 days, what is the minimum amount that must be on hand now (to the nearest hundredth of a gram)? 0.65 g

50. Radioactive decay: The radioactive element sodium-24 has a half-life of 15 hr and is used to help locate obstructions in blood flow. If the procedure requires 0.75 g and is scheduled to take place in 2 days (48 hr), what minimum amount must be on hand *now* (to the nearest hundredth of a gram)? 6.89 g

51. Radioactive decay: The radioactive element americium-241 has a half-life of 432 yr and although extremely small amounts are used (about 0.0002 g), it is the most vital component of standard household smoke detectors. How many years will it take a 10-g mass of americium-241 to decay to 2.7 g? 818 yr

52. Radioactive decay: Carbon-14 is a radioactive compound that occurs naturally in all living organisms, with the amount in the organism constantly renewed. After death, no new carbon-14 is acquired and the amount in the organism begins to decay exponentially. If the half-life of carbon-14 is 5700 yr, how old is a mummy having only 30% of the normal amount of carbon-14? about 9901 yr

Ordinary annuities: If a periodic payment p is deposited n times per year, with annual interest rate r also compounded n times per year for t years, the future value of the account is given by $A = \frac{p[(1 + R)^{nt} - 1]}{R}$, where $R = \frac{r}{n}$ (i.e., if the rate is 9% compounded monthly, $R = \frac{0.09}{12} = 0.0075$).

53. How long would it take Jasmine to save $10,000 if she deposits $90/month at an annual rate of 7.75% compounded monthly? about 7 yr

54. What quarterly investment amount is required to ensure that Larry can save $4700 in 4 yr at an annual rate of 8.5% compounded quarterly? $250.00

55. Saving for college: At the birth of their first child, Latasha and Terrance opened an annuity account and have been depositing $50 per month in the account ever since. If the account is now worth $30,000 and the interest on the account is 6.2% compounded monthly, how old is the child? 23 yr

56. Saving for a bequest: When Cherie (Brandon's first granddaughter) was born, he purchased an annuity account for her and stipulated that she should receive the funds (in trust, if necessary) upon his death. The quarterly annuity payments were $250 and interest on the account was 7.6% compounded quarterly. The account balance of $17,500 was recently given to Cherie. How much longer did Brandon live? 11 yr

57. Saving for a down payment: Tae-Hon is tired of renting and decides that within the next 5 yr he must save $22,500 for the down payment on a home. He finds an investment company that offers

8.5% interest compounded monthly and begins depositing \$250 each month in the account. (a) Is this monthly amount sufficient to help him meet his 5 yr goal? (b) If not, find the *minimum amount he needs to deposit each month* that will allow him to meet his goal in 5 yr. no; \$302.25

58. Saving to open a business: Madeline feels trapped in her current job and decides to save \$75,000 over the next 7 yr to open up a Harley Davidson franchise. To this end, she invests \$145 every week in an account paying $7\frac{1}{2}\%$ interest compounded weekly. (a) Is this weekly amount sufficient to help her meet the seven-year goal? (b) If not, find the *minimum amount she needs to deposit each week* that will allow her to meet this goal in 7 yr? no; \$156.81

WRITING, RESEARCH, AND DECISION MAKING

59. Many claim that inheritance taxes are put in place simply to prevent a massive accumulation of wealth by a select few. Suppose that in 1890, your great-grandfather deposited \$10,000 in an account paying 6.2% compounded continuously. If the account were to pass to you untaxed, what would it be worth in 2005? Do some research on the inheritance tax laws in your state. In particular, what amounts can be inherited untaxed (i.e., before the inheritance tax kicks in)?

59. \$12,488,769.67; answers will vary.

60. One way to compare investment possibilities is to compute what is called the **effective rate of interest.** This is the yearly simple interest rate ($t = 1$) that would generate the same amount of interest as the stated compound interest rate. (a) Would you expect the effective rate to be higher or lower than the stated compound interest rate? (b) Do some research to find a formula that will compute the effective rate of interest, and use it to compare an investment that is compounded monthly at 6.5% with one that is compounded quarterly at 6.75%. Which is the better investment?

60. higher; compounded quarterly at 6.75%

61. Willard Libby, an American chemist, won the 1960 Nobel Prize in Physical Chemistry for his discovery and development of radiocarbon dating. Do some research on how radiocarbon dating is used and write a short report. Include several examples and discuss/illustrate how the concepts in this section are needed for the method to work effectively.
Answers will vary.

EXTENDING THE CONCEPT

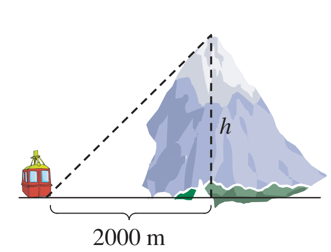
62. If you have not already completed Exercise 42, please do so. For *this* exercise, *solve the compound interest equation for r* to find the exact rate of interest that will allow Helyn to meet her 18-yr goal. 7.93%

63. If you have not already completed Exercise 55, please do so. Suppose the final balance of the account was \$35,100 with interest again being compounded monthly. For *this* exercise, use a graphing calculator to find r, the exact rate of interest the account would have been earning. 7.2%

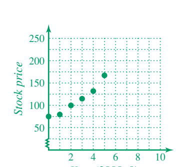
64. Suppose the decay of radioactive elements was measured in terms of a one-fourth life (instead of a half-life). What would the conversion formula be to convert from "fourth-life" to the decay rate r? Polonium-210 has a half-life of 140 days. What is its " fourth-life"? What is its decay rate r? If 20 g of polonium-210 are initially present, how many days until less than 1 g remains? $r = \frac{\ln 4}{t}$; 280 days; ≈ 0.00495; 605 days

MAINTAINING YOUR SKILLS

65. (2.1) In an effort to boost tourism, a trolley car is being built to carry sightseers from a strip mall to the top of Mt. Vernon, 1580-m high. Approximately how long will the trolley cables be? 2548.8 m



66. (2.6) The table shown gives the average price for a share of stock in IBN, a large firm researching sources of alternative energy. Draw a scatter-plot of the data and decide on an appropriate form of regression. Then use a calculator to find a regression equation. If this rate of growth continues, what will a share of stock be worth in 2010? $y = 3.2x^2 + 2.0x + 76.4$; \$416.40

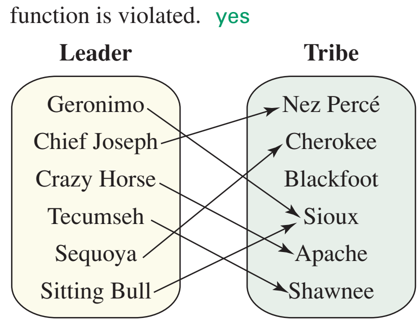
66.


Exercise 66

Year (2000 = 0)	Stock Price
0	76
1	80
2	98
3	112
4	130
5	170

69. a. $f(x) = x^3$, $f(x) = x$, $f(x) = \sqrt{x}$, $f(x) = \sqrt[3]{x}$, $f(x) = \frac{1}{x}$
b. $f(x) = |x|$, $f(x) = x^2$, $f(x) = \frac{1}{x^2}$
c. $f(x) = x$, $f(x) = x^3$, $f(x) = \sqrt{x}$, $f(x) = \sqrt[3]{x}$
d. $f(x) = \frac{1}{x}$, $f(x) = \frac{1}{x^2}$

67. (2.2) Is the following relation a function? If not, state how the definition of a function is violated. yes



68. (4.3) A polynomial is known to have the zeroes $x = 3$, $x = -1$, and $x = 1 + 2i$. Find the equation of the polynomial, given it has degree 4 and a y-intercept of $(0, -15)$.
$P(x) = x^4 - 4x^3 + 6x^2 - 4x - 15$

70. a. $4x^3 + 6x^2 - 8x - 15$ **b.** $x + \frac{9}{2} + \frac{37}{2(2x - 3)}$
c. $8x^2 - 12x + 5$ **d.** $4x^2 + 12x + 7$
e. $\frac{x + 3}{2}$ **f.** $\frac{-3}{2}$

69. (2.4/3.8) Name the toolbox functions that are (a) one-to-one, (b) even, (c) increasing for $x \in \mathbb{R}$, (d) asymptotic.

70. (3.1) Given $f(x) = 2x^2 + 6x + 5$ and $g(x) = 2x - 3$, find (a) $(f \cdot g)(x)$, (b) $\left(\frac{f}{g}\right)(x)$, (c) $(f \circ g)(x)$, (d) $(g \circ f)(x)$, (e) $g^{-1}(x)$, and (f) $(f + g)(-\frac{1}{2})$.

5.6 Exponential, Logarithmic, and Logistic Regression Models

LEARNING OBJECTIVES

In Section 5.6 you will learn how to:

A. Choose an appropriate form of regression using context

B. Use a graphing calculator to obtain exponential, logarithmic, and logistic regression models

C. Use a regression model to answer questions and solve problems

D. Determine when a logistics model is appropriate and apply logistics models to a set of data

INTRODUCTION

The basic concepts involved in calculating a regression equation were presented in Section 2.6. In this section, we apply the regression concept to data sets that are best modeled by power, exponential, logarithmic, or logistic functions. All data sets, while contextual and accurate, *have been carefully chosen* to provide a maximum focus on regression fundamentals and the mathematical concepts that follow. In reality, data sets are often not so "well-behaved" and many require sophisticated statistical tests before any conclusions can be drawn.

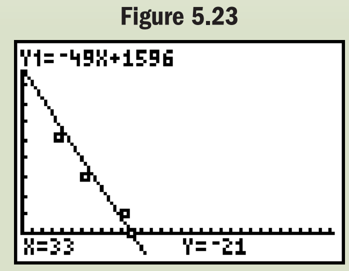
POINT OF INTEREST

Many will likely remember the 2000 presidential election and the debacle regarding the vote count in Florida. The table here shows the number of votes by which Bush led Gore beginning on November 7 ($t = 0$ days) and ending on December 8 ($t = 31$ days), according to the national media. The data are graphed in Figures 5.23, 5.24, and 5.25 with three different forms of regression applied. You are asked to analyze each graph in this context in Exercise 63.

t (days since 11/7)	N (number of votes in Bush lead)
0	1725
11	930
19	537
31	193

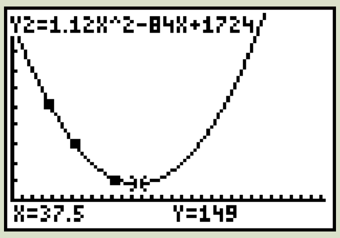
[Source: *USA Today*]

Figure 5.23



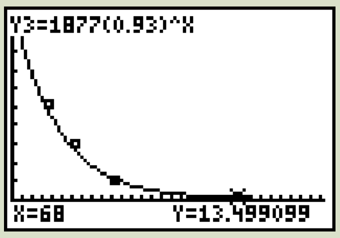
Linear regression

Figure 5.24



Quadratic regression

Figure 5.25



Exponential regression

A. Choosing an Appropriate Form of Regression

WORTHY OF NOTE

Recall that a "best-fit" equation model is one that minimizes the vertical distance between all data points and the graph of the proposed model.

Most graphing calculators have the ability to perform 8 to 10 different forms of regression, and selecting which of these to use is a critical issue. When various forms are applied to a given data set, some are easily discounted due to a poor fit, others may fit very well for only a portion of the data, while still others may compete for being the "best-fit" equation. In a statistical study of regression, an in-depth look at the correlation coefficient (r), the coefficient of determination (r^2 or R^2), and a study of **residuals** are used to help make an appropriate choice. For our purposes, the correct or best choice will generally depend on two things: (1) how well the graph appears to fit the scatter-plot, and (2) the context or situation that generated the data, coupled with a dose of common sense.

WORTHY OF NOTE

For more information on the use of residuals, see the *Calculator Exploration and Discovery* feature from Chapter 3.

As we've noted previously, the final choice of regression can rarely be based on the scatter-plot alone, although relying on the basic characteristics and end behavior of certain graphs can be helpful. With an awareness of the toolbox functions, polynomial graphs and applications of exponential and logarithmic functions, the context of the data can aid a decision.

EXAMPLE 1 Suppose a set of data is generated from each context below. Use common sense, previous experience, or your own knowledge base to state whether a linear, quadratic, logarithmic, exponential or power regression might be most appropriate. Justify your answers.

a. population growth of the United States since 1800

b. the distance covered by a jogger running at a constant speed

c. height of a baseball t seconds after it's thrown

d. the time it takes for a cup of hot coffee to cool to room temperature

Solution: **a.** From examples in Section 5.5 and elsewhere, we've seen that animal and human populations tend to grow exponentially over time. Here, an exponential model is likely most appropriate.

b. Since the jogger is moving at a constant speed, the rate-of-change $\frac{\Delta\text{distance}}{\Delta\text{time}}$ is constant and a linear model would be most appropriate.

c. As seen in numerous places throughout the text, the height of a projectile is modeled by the equation $h(t) = -16t^2 + vt + k$, where $h(t)$ is the height after t seconds. Here, a quadratic model would be most appropriate.

d. Many have had the experience of pouring a cup of hot chocolate, coffee, or tea, only to leave it on the counter as they turn their attention to other things. The hot drink seems to cool quickly at first, then slowly approach room temperature. This experience, perhaps coupled with our awareness of *Newton's Law of Cooling,* shows a logarithmic or exponential model might be appropriate here.

NOW TRY EXERCISES 7 THROUGH 20

B. Exponential and Logarithmic Regression Models

We now focus our attention on regression models that involve exponential and logarithmic functions. Recall the process of developing a regression equation involves these five stages: (1) clearing old data, (2) entering new data; (3) displaying the data; (4) calculating the regression equation; (5) displaying and using the regression graph and equation.

EXAMPLE 2 The number of centenarians (people who are 100 years of age or older) has been climbing steadily over the last half century. The table shows the number of centenarians (per million population) for selected years. Use the data and a graphing calculator to draw the scatter-plot, then use the scatter-plot and context to decide on an appropriate form of regression.

Year "t" (1950 → 0)	Number "N" (per million)
0	16
10	18
20	25
30	74
40	115
50	262

Source: Data from 2004 Statistical Abstract of the United States, Table 14; various other years

Solution: After clearing any existing data in the data lists, enter the input values (years since 1950) in L1 and the output values (number of centenarians per million population) in L2 (Figure 5.26). For the viewing window, scale the x-axis (years since 1950) from –10 to 70 and the y-axis (number per million) from –50 to 500 to comfortably fit the data and allow room for the coordinates to be shown at the bottom of the screen. (Figure 5.27). The scatter-plot rules out a linear model. While a quadratic

Figure 5.26

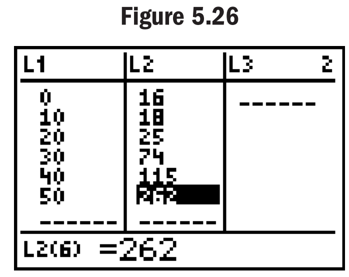
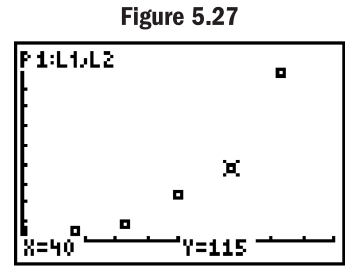


Figure 5.27



WORTHY OF NOTE

The regression equation can be sent directly to the Y= screen, or pasted to the Y= screen from the home screen. To learn how, see the *Technology Highlight* that follows Example 00.

model may fit the data, we expect that the correct model should exhibit asymptotic behavior since extremely few people lived to be 100 years of age prior to dramatic advances in hygiene, diet and medical care. This would lead us toward an exponential equation model. The keystrokes STAT ▶ brings up the **CALC** menu, with **ExpReg** (exponential regression) being option "0." The option can be selected by simply pressing "0" and ENTER, or by using the up arrow ▲ or down arrow ▼ to scroll to **0:ExpReg.** The exponential model seems to fit the data very well and gives a high correlation coefficient (Figures 5.28 and 5.29). To four decimal places the equation model is $y = (11.5090)1.0607^x$.

Figure 5.28

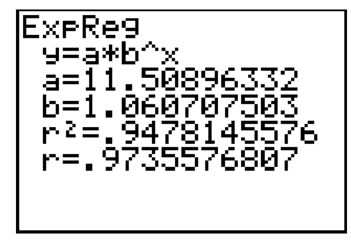
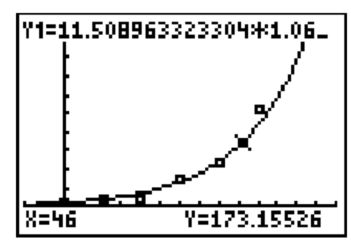


Figure 5.29



NOW TRY EXERCISES 21 AND 22

Given a general exponential function $y = ab^x$, the growth rate constant (discussed in Section 5.3) can be determined by using properties of logarithms to rewrite the relation in the form $y = ae^{kx}$. This is done by setting them equal to each other and solving for k:

$$\begin{aligned} ab^x &= ae^{kx} && \text{set equations equal} \\ b^x &= e^{kx} && \text{divide by } a \\ \ln b^x &= \ln e^{kx} && \text{take natural logs} \\ x \ln b &= kx \ln e && \text{power property} \\ \ln b &= k && \text{solve for } k\ (\ln e = 1) \end{aligned}$$

This shows that $b^x = e^{kx}$ for $k = \ln b$.

EXAMPLE 3 Identify the growth rate constant for the equation model from Example 2, then write the equation as a base e exponential function.

Solution: From Example 2 we have $y = (11.5090)1.0607^x$, with $b = 1.0607$, and $k = \ln 1.0607 \approx 0.0589$. The growth rate is about 5.9%. The corresponding base e function is $y = (11.5090)e^{0.0589x}$.

NOW TRY EXERCISES 23 AND 24

For applications involving exponential and logarithmic functions, it helps to remember that while both basic functions are increasing, a logarithmic function increases at a much slower rate. Consider Example 4.

EXAMPLE 4 One measure used in studies related to infant growth, nutrition and development, is the relation between the circumference of a child's head and their age. The table to the right shows the average circumference of a female child's head for ages 0 to 36 months. Use the data and a graphing calculator to graw the scatter-plot, then use the scatter-plot and context to decide on an appropriate form of regression.

Source: National Center for Health Staistics

Age a (months)	Circumference C (cm)
0	34.8
6	43.0
12	45.2
18	46.5
24	47.5
30	48.2
36	48.6

Solution: After clearing any existing data, enter the child's age (in months) as L1 and the circumference of the head (in cm) as L2. For the viewing window, scale the x-axis from -5 to 50 and the y-axis from 25 to 60 to comfortably fit the data (Figure 5.30). The scatter-plot again rules out a linear model, and the context rules out a polynomial model due to end-behavior. As we expect the circumference of the head to continue increasing slightly for many more months, it appears a logarithmic model may be the best fit. Note that since $\ln(0)$ is undefined, $a = 0.1$ was used to represent the age at birth (rather than $a = 0$), prior to running the regression. The **LnReg** (logarithmic regression) option is option 9, and the keystrokes STAT ▶ **(CALC) 9:LinReg** ENTER gives the equation shown in Figure 5.31, which returns a very high correlation coefficient and fits the data very well (Figure 5.32).

Figure 5.30

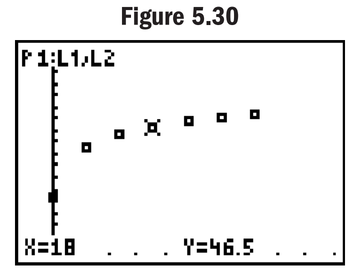


Figure 5.31

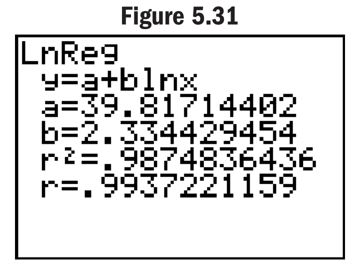
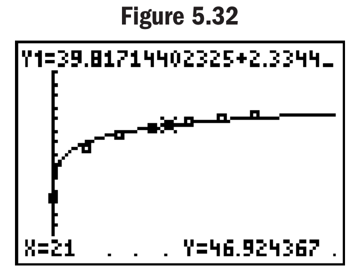


Figure 5.32



NOW TRY EXERCISES 25 AND 26

For more on the correlation coefficient and its use, see Exercise 62.

C. Applications of Regression

Once a model for the data has been obtained, it is generally put to two specific uses. First, it can be used to **extrapolate** or predict future values or occurrences. When using extrapolation, values are projected *beyond* the given set of data. Second, the equation model can be used to **interpolate** or approximate values that occur *between* values given in the data set.

EXAMPLE 5 Use the regression equation from Example 2 to answer the following questions:

a. Approximately how many centenarians (per million) were there in 1995?

b. Approximately how many centenarians will there be (per million) in 2010?

c. In what year will there be approximately 300 centenarians per million?

Solution: **a.** Writing the function using function notation gives $f(x) = 11.509(1.061)^x$. For the year 1995 we have $x = 45$, so we find the value of $f(45)$:

$$\begin{aligned} f(x) &= 11.509(1.061)^x && \text{regression equation} \\ f(45) &= 11.509(1.061)^{45} && \text{substitute } x = 45 \\ &= 11.509(14.3612511) && \text{evaluate exponential first (order of operations)} \\ &= 165.2836389 && \text{result} \end{aligned}$$

In 1995, there were approximately 165 centenarians per million population.

b. For the year 2010 we have $x = 60$, so we find the value of $f(60)$

$$\begin{aligned} f(x) &= 11.509(1.061)^x && \text{regression equation} \\ f(60) &= 11.509(1.061)^{60} && \text{substitute } x = 60 \\ &= 11.509(34.90784461) && \text{evaluate exponential first (order of operations)} \\ &= 401.7543836 && \text{result} \end{aligned}$$

In 2010, there will be approximately 402 centenarians per million population.

c. For Part (c) we are given the *number of centenarians* [the output value $f(x)$] and are asked to find the year x in which this number (300) occurs. So we substitute $f(x) = 300$ and solve for x:

$$\begin{aligned} f(x) &= 11.509(1.061)^x && \text{regression equation} \\ 300 &= 11.509(1.061)^x && \text{substitute } f(x) = 300 \\ \frac{300}{11.509} &= (1.061)^x && \text{divide both sides by 11.509} \\ \ln\left(\frac{300}{11.509}\right) &= \ln(1.061)^x && \text{take the base-}e\text{ (or base-10) logarithm of both sides} \\ \ln\left(\frac{300}{11.509}\right) &= x\ln 1.061 && \text{apply power property of logarithms: } \log_b k^x = x\log_b k \\ \frac{\ln\left(\frac{300}{11.509}\right)}{\ln 1.061} &= x && \text{solve for } x \text{ (exact form): divide both sides by ln(1.601)} \\ 55.06756851 &= x && \text{approximate form} \end{aligned}$$

In the year 2005 (the 55th year after 1950) there will be approximately 300 centenarians per million population.

NOW TRY EXERCISES 54 THROUGH 59

CAUTION

Be very careful when you evaluate expressions like $\frac{\ln\left(\frac{300}{11.509}\right)}{\ln 1.061}$ from Example 5. It is best to compute the result in stages, evaluating the numerator first: $\ln\left(\frac{300}{11.509}\right) =$ 3.260653137, then dividing by the denominator: $3.260653137 \div \ln 1.061 = 55.06756851$.

EXAMPLE 6 Use the regression equation from Example 4 to answer the following questions:

a. What is the average circumference of a female child's head, if the child is 21 months old?

b. According to the equation model, what will the average circumference be when the child turns $3\frac{1}{2}$ years old?

c. If the circumference of the child's head is 46.9 cm, about how old is the child?

Solution:

a. Using function notation we have $C(a) \approx 39.8171 + 2.3344\ln(a)$. Substituting 21 for a gives:

$$\begin{aligned} C(21) &\approx 39.8171 + 2.3344\ln(21) && \text{substitute 21 for } x \\ &\approx 46.9 && \text{result} \end{aligned}$$

The circumference is approximately 46.9 cm.

b. Substituting 3.5 yr $\times$ 12 = 42 months for a gives:

$$\begin{aligned} C(42) &\approx 39.8171 + 2.3344\ln(42) && \text{substitute 42 for } x \\ &\approx 48.5 && \text{result} \end{aligned}$$

The circumference will be approximately 48.5 cm.

c. For Part (c) we're given the circumference C and are asked to find the age "a" in which this circumference (46.9) occurs. Substituting 46.9 for $C(a)$ we obtain:

$$\begin{aligned} 46.9 &= 39.8171 + 2.3344\ln(a) && \text{substitute 1866 for } f(x) \\ \frac{7.0829}{2.3344} &= \ln(a) && \text{subtract 39.8171, then divide by 2.3344} \\ e^{\frac{7.0859}{2.3344}} &= a && \text{write in exponential form} \\ 20.8 &= a && \text{result} \end{aligned}$$

The child must be about 21 months old.

NOW TRY EXERCISES 58 AND 59

D. Logistics Equations and Regression Models

Many population growth models assume an unlimited supply of resources, nutrients, and room for growth, resulting in an exponential growth model. When resources become scarce or room for further expansion is limited, the result is often a **logistic growth model.** At first, growth is very rapid (like an exponential function), but this growth begins to taper off and slow down as nutrients are used up, living space becomes restricted, or due to other factors. Surprisingly, this type of growth can take many forms, including population growth, the spread of a disease, the growth of a tumor, or the spread of a stain in fabric. Specific logistic equations were encountered in Section 5.4. The general equation model for logistic growth is

LOGISTIC GROWTH EQUATION

Given constants a, b, and c, the logistic growth $P(t)$ of a population depends on time t according to the model

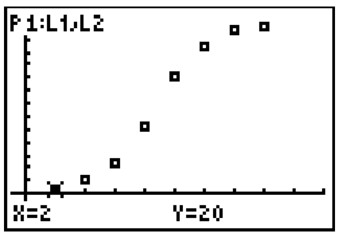
$$P(t) = \frac{c}{1 + ae^{-bt}}$$

The constant c is called the **carrying capacity** of the population, in that as $t \to \infty$, $P(t) \to c$. In words, as the elapsed time becomes very large, growth will approach (but not exceed) c.

EXAMPLE 7 Yeast cultures have a number of applications that are a great benefit to civilization and have been an object of study for centuries. A certain strain of yeast is grown in a lab, with its population checked at 2-hr intervals, and the data gathered are given in the table. Use the data to draw a scatter-plot, and decide on an appropriate form of regression. If a logistic regression is the best model, attempt to estimate the capacity coefficient c prior to using your calculator to find the regression equation. How close were you to the actual value?

Elapsed Time (hours)	Population (100s)
2	20
4	50
6	122
8	260
10	450
12	570
14	630
16	650

Solution: After clearing the data lists, enter the input values (elapsed time) in L1 and the output values (population) in L2. For the viewing window, scale the t-axis from 0 to 20 and the P-axis from 0 to 700 to comfortably fit the data. From the context and scatter-plot, it's apparent the data are



best modeled by a logistic function. Noting that Ymax = 700 and the data seem to level off near the top of the window, a good estimate for c would be about 675. Using logistic regression on the home screen (option **B:Logistic**), we obtain the equation

$$Y_1 = \frac{663}{1 + 123.9e^{-0.553x}} \text{ (rounded).}$$

NOW TRY EXERCISES 60 AND 61

TECHNOLOGY HIGHLIGHT

Pasting the Regression Equation to the Y = Screen

The keystrokes shown apply to a TI-84 Plus model. Please consult our Internet site or your manual for other models.

Once a regression equation has been calculated, the entire equation can be transferred directly from the home screen to the Y= screen of a graphing calculator. The regression curve and scatter-plot can both be viewed by simply pressing GRAPH assuming an appropriate window has been set. On the TI-84 Plus, this feature is accessed using the VARS key. Using the regression from Example 7, we begin by placing the cursor at Y_1 on the Y= screen (CLEAR the old equation and leave the cursor in the empty slot). Pressing the VARS key gives the screen shown in Figure 5.33, where we select option **5:Statistics** and press ENTER. This brings up the screen shown in Figure 5.34, where we select menu option **EQ,** then option **1:EQ** (for equation). Pressing the ENTER key at this point will paste the regression equation directly into the current location of the cursor, which we left at Y_1 on the Y= screen. We can now use the equation to further investigate the population of the yeast culture.

Figure 5.33

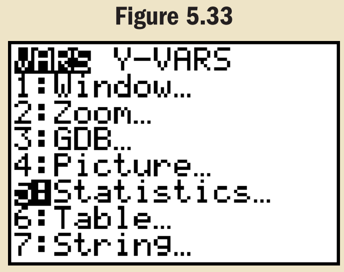
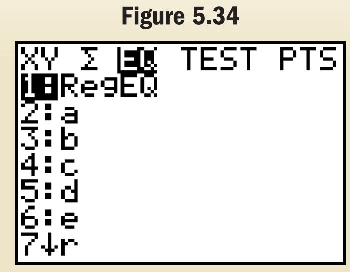


Figure 5.34



As an alternative, you can add the argument "Y_1" to the logistic regression command on the home screen. The sequence **Logistic Y_1** ENTER will calculate the equation and automatically place the result in Y_1.

Exercise 1: Use the ideas from this *Technology Highlight* to paste the equation from Examples 4 and 6 directly into Y_1. Then use the calculator to recompute the answers to questions (a), (b) and (c) of Example 6. Where the answers close?

a. $c \approx 46.92$ cm **b.** $c \approx 48.54$ cm **c.** $a \approx 20.78$ mo yes, very close.

5.6 EXERCISES

CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. The type of regression used often depends on (a) whether a particular graph appears to fit the __data__ and (b) the __context__ or __situation__ that generated the data.

2. The final choice of regression can rarely be based on the __scatter-plot__ alone. Relying on the basic __characteristics__ and __end-behavior__ of certain graphs can be helpful.

3. To extrapolate means to use the data to predict values __beyond__ the given data.

4. To interpolate means to use the data to predict values __between__ the given data.

5. 1. clear old data
2. enter new data
3. display the data
4. calculate the regression equation
5. display and use the regression graph and equation.

6. $f(x) = x$, $f(x) = |x|$, $f(x) = x^2$, $f(x) = x^3$, $f(x) = \sqrt[3]{x}$, $f(x) = \sqrt{x}$

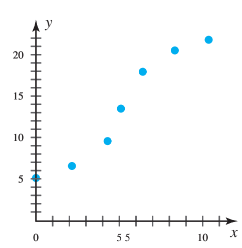
7. e
8. c
9. a
10. b
11. d
12. f

5. List the five steps used to find a regression equation using a calculator. Discuss possible errors that can occur if the first step is skipped. After the new data have been entered, what precautionary step should always be included? Answers will vary.

6. Consider the eight toolbox functions and the exponential and logarithmic functions. How many of these satisfy the condition as $x \to \infty$, $y \to \infty$? For those that satisfy this condition, discuss/explain how you would choose between them judging from the scatter-plot alone. Answers will vary.

DEVELOPING YOUR SKILLS

Match each scatter-plot given with one of the following: (a) likely linear, (b) likely quadratic, (c) likely exponential, (d) likely logarithmic, (e) likely logistic, or (f) none of these.

7.


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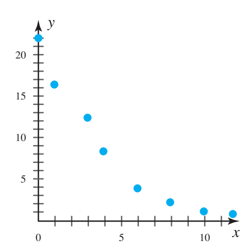

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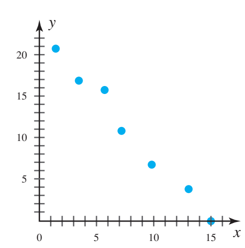

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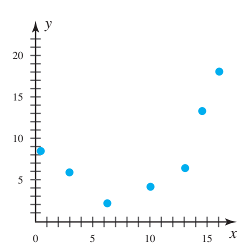

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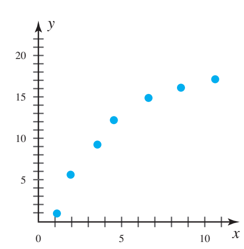

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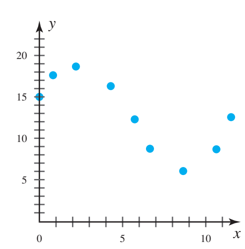


For Exercises 13 to 20, suppose a set of data is generated from the context indicated. Use common sense, previous experience, or your own knowledge base to state whether a linear, quadratic, logarithmic, exponential, power, or logistic regression might be most appropriate. Justify your answers.

13. total revenue and number of units sold linear

14. page count in a book and total number of words linear

15. years on the job and annual salary exponential

16. population growth with unlimited resources exponential

17. population growth with limited resources logistic

18. elapsed time and the height of a projectile quadratic

19. the cost of a gallon of milk over time exponential

20. elapsed time and radioactive decay exponential

Discuss why an exponential model could be an appropriate form of regression for each data set, then find the regression equation.

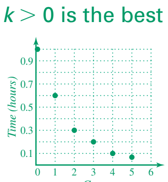
21. Radioactive Studies

Time in Hours	Grams of Material
0.1	1.0
1	0.6
2	0.3
3	0.2
4	0.1
5	0.06

22. Rabbit Population

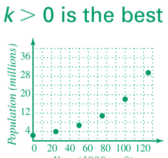
Month	Population (in hundreds)
0	2.5
3	5.0
6	6.1
9	12.3
12	17.8
15	30.2

21. As time increases, the amount of radioactive material decreases but will never truely reach 0 or become negative. Exponential with $b < 1$ and $k > 0$ is the best choice.



$y \approx (1.042)0.5626$

22. Populations usually grow exponentially or logistically. This growth does not appear to tape off. Exponential with $b > 1$ and $k > 0$ is the best choice.



$y \approx (2.6550)1.1754$

23. Identify the growth rate constant for the equation from Exercise 21. $k \approx -0.5752$

24. Identify the growth rate constant for the equation from Exercise 22. $k \approx 0.1616$

Discuss why a logarithmic model could be an appropriate form of regression for each data set, then find the regression equation.

25. Sales will increase rapidly, then level off as the market is saturated with ads and advertising becomes less effective, possibly modeled by a logarithmic function. $y \approx 120.4938 + 217.2705\ln(x)$

26. In a productive mine, we expect that initially, the diamonds may be nearer the surface and more plentiful, becoming more scarce and harder to find as time goes on. A logarithmic model seems to fit this description. $y \approx 454.7845 + 1087.8962\ln(x)$

25. Total number of sales compared to the amount spent on advertising

Advertising costs ($1000s)	Total number of sales
1	125
5	437
10	652
15	710
20	770
25	848
30	858
35	864

26. Cumulative weight of diamonds extracted from a diamond mine

Time (months)	Weight (carats)
1	500
3	1748
6	2263
9	2610
12	3158
15	3501
18	3689
21	3810

The applications that follow require solving equations similar to those below. Solve each equation.

27. $96.35 = (9.4)1.6^x$ 4.95

28. $(3.7)2.9^x = 1253.93$ 5.47

29. $(-10.04)1.046^x = -396.58$ 81.74

30. $12{,}110 = (193.76)0.912^x$ −44.89

31. $4.8x^{2.5} = 468.75$ 6.25

32. $4375 = 1.4x^{-1.25}$ 0.0016

33. $2.103x^{0.6} = 56.781$ 243

34. $52 = 63.9 - 6.8\ln x$ 5.75

35. $498.53 + 2.3\ln x = 2595.9$ $e^{911.9}$

36. $49.05 + 12.7\ln x = 258.6$ 14,650,719.43

37. $9 = 68.76 - 7.2\ln x$ 4023.87

38. $52 = \dfrac{67}{1 + 20e^{-0.62x}}$ 6.84

39. $\dfrac{975}{1 + 82.3e^{-0.423x}} = 890$ 15.98

40. $1080 = \dfrac{1100}{1 + 37.2e^{-0.812x}}$ 9.37

41. $5 = \dfrac{8}{1 + 9.3e^{-0.65x}}$ 4.22

WORKING WITH FORMULAS

42. Learning curve: $P(t) = 5.9 + 12.6\ln(t)$

The number of toy planes a new employee can assemble from its component parts depends on how long the employee has been working on the assembly line. This is modeled by the function shown, where t represents the number of days and $P(t)$ is the number of planes the worker is able to assemble. How many planes is an employee making after 5 days on the job? How long will it take until the employee is able to assemble 35 planes per day? 26 10 days

43. Bicycle sales since 1920: $N(t) = 0.325(1.057)^t$

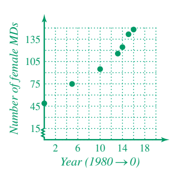
Despite the common use of automobiles and motorcycles, bicycle sales have continued to grow as a means of transportation as well as a form of recreation. The number of bicycles sold each year (in millions) can be approximated by the formula shown, where t is the number of years after 1920 ($1920 \to 0$). According to this model, in what year did bicycle sales exceed 10 million? 1981

Source: 1976/1992 *Statistical Abstract of the United States,* Tables 406/395; various other years

APPLICATIONS

Answer the questions using the given data and the related regression equation. All extrapolations assume the mathematical model will continue to represent future trends.

44.



44. Female physicians: The number of females practicing medicine as MDs is given in the table for selected years. Use the data to draw a scatter-plot, then use the context and scatter-plot to find the best regression equation (see Exercise 50). $y = 50.21(1.07)^x$

Source: 2004 *Statistical Abstract of the United States*, Table 149, various other years

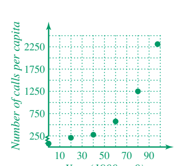
Exercise 44

Year (1980 → 0)	Number (in 1000s)
0	48.7
5	74.8
10	96.1
13	117.2
14	124.9
15	140.1
16	148.3

Exercise 45

Year (1900 → 0)	Number (per capita/ per year)
0	38
20	180
40	260
60	590
80	1250
97	2325

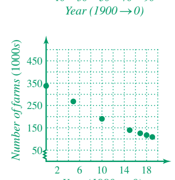
45.



45. Telephone use: The number of telephone calls per capita has been rising dramatically since the invention of the telephone in 1876. The table given shows the number of phone calls per capita per year for selected years. Use the data to draw a scatter-plot, then use the context and scatter-plot to find the best regression equation (also see Exercise 51). $y = 53.24(1.04)^x$

Source: Data from 1997/2000 *Statistical Abstract of the United States*, Tables 922/917

46.



46. Milk production: With milk production becoming a big business, the number of family farms with milk-producing cows has been decreasing as larger corporations take over. The number of farms that keep milk cows for commercial production is given in the table for selected years. Use the data to draw a scatter-plot, then use the context and scatter-plot to find the best regression equation (also see Exercise 52). $y = 346.79(0.94)^x$

Source: 2000 *Statistical Abstract of the United States*, Table 1141

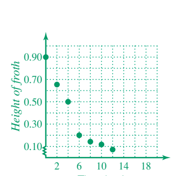
Exercise 46

Year (1980 → 0)	Number (in 1000s)
0	334
5	269
10	193
15	140
17	124
18	117
19	111

Exercise 47

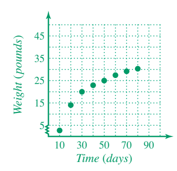
Time (seconds)	Height of Froth (in)
0	0.90
2	0.65
4	0.40
6	0.21
8	0.15
10	0.12
12	0.08

47.



47. Froth height—carbonated beverages: The height of the froth on carbonated drinks and other beverages can be manipulated by the ingredients used in making the beverage and lends itself very well to the modeling process. The data in the table given shows the froth height of a certain beverage as a function of time, after the froth has reached a maximum height. Use the data to draw a scatter-plot, then use the context and scatter-plot to find the best regression equation (see Exercise 53). $y = 0.89\,(0.81)^x$

48.



48. Weight loss: Harold needed to lose weight and started on a new diet and exercise regimen. The number of pounds he's lost since the diet began is given in the table. Draw the scatter-plot, then use a calculator to linearize the data and determine whether an exponential or logarithmic equation model is more appropriate. Finally, find the equation model (see Exercise 54). $y = -27.4 + 13.5 \ln x$

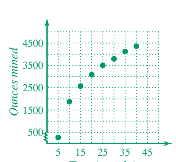
Exercise 48

Time (days)	Pounds Lost
10	2
20	14
30	20
40	23
50	25.5
60	27.6
70	29.2
80	30.7

Exercise 49

Time (months)	Ounces Mined
5	275
10	1890
15	2610
20	3158
25	3501
30	3789
35	4109
40	4309

49.



49. Depletion of resources: The longer an area is mined for gold, the more difficult and expensive it gets to obtain. The cumulative total of the ounces of gold produced by a particular mine is shown in the table. Draw the scatter-plot, and use the graph and the context of the application to determine whether an exponential or logarithmic equation model is more appropriate. Finally, find the equation model (see Exercise 55). $y = -2635.6 + 1904.8 \ln x$

50. Use the regression equation from Exercise 44 to answer the following questions:

- **a.** What was the approximate number of female MDs in 1988? 86,270
- **b.** Approximately how many female MDs will there be in 2005? 272,511
- **c.** In what year did the number of female MDs exceed 100,000? 1990

51. Use the regression equation from Exercise 45 to answer the following questions:

- **a.** What was the approximate number of calls per capita in 1970? 829
- **b.** Approximately how many calls per capita will there be in 2005? 3271
- **c.** In what year did the number of calls per capita exceed 1800? 1900

52. Use the regression equation from Exercise 46 to answer the following questions:

- **a.** What was the approximate number of farms with milk cows in 1993? 155,142
- **b.** Approximately how many farms will have milk cows (for commercial production) in 2004? 78,548
- **c.** In what year did the number of farms with milk cows drop below 150,000? 1994

53. Use the regression equation from Exercise 47 to answer the following questions:

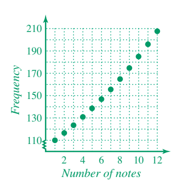
- **a.** What was the approximate height of the froth after 6.5 sec? 0.23
- **b.** How long does it take for the height of the froth to reach one-half of its maximum height? 3.2
- **c.** According to the model, how many seconds until the froth height is 0.02 in.? 18 sec

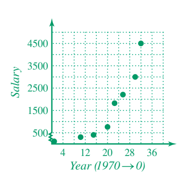
54. Use the regression equation from Exercise 48 to answer the following questions:

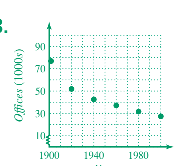
- **a.** What was Harold's total weight loss after 15 days? 9.2 lb
- **b.** Approximately how many days did it take to lose a total of 18 lb? 29 days
- **c.** According to the model, what is the projected weight loss for 100 days? 34.8 lb

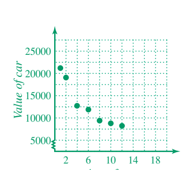
55. Use the regression equation from Exercise 49 to answer the following questions:

- **a.** What was the total number of ounces mined after 18 months? 2870 oz
- **b.** About how many months did it take to mine a total of 4000 oz? 32.6 months
- **c.** According to the model, what is the projected total after 50 months? 4816 oz

56. 

57. 

58. 

59. 

56. Musical notes: The table shown gives the frequency (vibrations per second) for each of the 12 notes in a selected octave from the standard chromatic scale. Use the data to draw a scatter-plot, then use the context and scatter-plot to find the best regression equation. $y = (103.83)1.0595^x$

a. What is the frequency of the "A" note that is an octave higher than the one shown? [*Hint:* The names repeat every 12 notes (one octave) so this would be the 13th note in this sequence.] 220

b. If the frequency is 370.00, what note is being played? The 22nd note, or F#

c. What pattern do you notice for the F#'s in each octave (the 10th, 22nd, 34th, and 46th notes in sequence)? Does the pattern hold for all notes? frequency doubles, yes

Number	Note	Frequency
1	A	110.00
2	A#	116.54
3	B	123.48
4	C	130.82
5	C#	138.60
6	D	146.84
7	D#	155.56
8	E	164.82
9	F	174.62
10	F#	185.00
11	G	196.00
12	G#	207.66

57. Basketball salaries: In 1970, the average player salary for a professional basketball player was about \$43,000. Since that time, player salaries have risen dramatically. The average player salary for a professional player is given in the table for selected years. Use the data to draw a scatter-plot, then use the context and scatter-plot to find the best regression equation. $y = 39.86(1.16)^x$

Source: 1998 *Wall Street Journal Almanac,* p. 985; National Basketball Association Data

a. What was the approximate salary for a player in 1988? \$576,000

b. What is the projection for the average salary in 2005? \$7,187,000

c. In what year did the average salary exceed \$1,000,000? 1992

Year (1970 → 0)	Salary (1000s)
0	43
10	171
15	325
20	750
25	1900
27	2300
30	3500
32	4500

58. Number of U.S. post offices: Due in large part to the ease of travel and increased use of telephones, email, and instant messaging, the number of post offices in the United States has been on the decline since the turn of the century. The data given show the number of post offices (in thousands) for selected years. Use the data to draw a scatter-plot, then use the context and scatter-plot to find the best regression equation. $y = 78.8 - 10.3 \ln x$

Source: 1985/2000 *Statistical Abstract of the United States,* Tables 918/1112

a. Approximately how many post offices were there in 1915? 51,000

b. In what year did the number of post offices drop below 34,000? 1977

c. According to the model, how many post offices will there be in the year 2010? 30,400

Exercise 58

Year 1900 → 0	Offices (1000s)
1	77
20	52
40	43
60	37
80	32
100	28

59. Automobile value: While it is well known that most cars decrease in value over time, what is the best equation model for this decline? Use the data given to draw a scatter-plot, then use the context and scatter-plot to find the best regression equation. Linearize the data if needed. $y = 19{,}943.7 - 5231.4 \ln x$

a. What was the car's value after 7.5 yr? \$9402.94

b. Exactly how old is the car if its current value is \$8150? 9.5 yr

c. Using the model, how old is the car when value $\leq$ \$3000? 25.5 yr

Exercise 59

Age of Car	Value of Car
1	19,500
2	16,950
4	12,420
6	11,350
8	8375
10	7935
12	6900

Exercise 60

Days after Outbreak	Cumulative Total
0	100
14	560
21	870
35	1390
56	1660
70	1710
84	1750

60. **Spread of disease:** Estimates of the cumulative number of SARS (Sudden Acute Respiratory Syndrome) cases reported in Hong Kong during the spring of 2003 are shown in the table, with day 0 corresponding to February 20. Use the data to draw a scatter-plot, then use the context and scatter-plot to find the best regression equation. $y = \dfrac{1719}{1 + 10.2e^{-0.11x}}$

Source: Center for Disease Control @ www.cdc.gov/ncidod/EID/vol9no12

a. What was the approximate number of SARS cases by day 25? 1041

b. About what day did the number of SARS cases exceed 1500? day 38

c. Using this model, what is the projected number of SARS cases on the 84th day? Why does this differ from the value in the table? 1717 The model only approximates the data.

61. **Cable television subscribers:** The percentage of American households having cable television is given in the table to the right for select years from 1976 to 2004. Use the data to draw a scatter-plot, then use the context and scatter-plot to find the best regression equation. $y = \dfrac{69.99}{1 + 4.00e^{-0.22x}}$

Source: Data pooled from the 2001 *New York Times Almanac,* page 393; 2004 *Statistical Abstract of the United States,* Table 1120; various other years

Since 1976 Year	Percentage with Cable TV
0	16
4	22.6
8	43.7
12	53.8
16	61.5
20	66.7
24	68
28	70

a. Approximately what percentage of households had cable TV in 1990? 59.1%

b. In what year did the percentage having cable TV exceed 50%? 1986

c. Using this model, what projected percentage of households will have cable in 2008? 69.7%

WRITING, RESEARCH, AND DECISION MAKING

62. Although correlation coefficients can be very helpful, other factors must also be considered when selecting the most appropriate equation model for a set of data. To see why, use the data given to draw a scatter-plot. Does the scatter-plot clearly suggest particular form of regression? Use the STAT feature of your graphing calculator to (a) find a linear regression equation and note its correlation coefficient, and (b) find an exponential regression equation and note its correlation coefficient. What do you notice? Without knowing the context of the data, would you be able to tell which model might be more suitable?

Input	Output
1	1
2	3.5
3	11
4	9.8
5	15
6	27.5

63. Carefully read the *Point of Interest* paragraph at the beginning of this section, then study the data and the three related graphs. The graphs show the result of applying linear regression, quadratic regression, and exponential regression respectively, to the data.

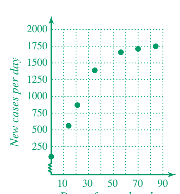
a. If the linear regression is correct, what is the significance of the x-intercept? What would be the outcome of the election?

b. If the quadratic regression is correct, what the significance of the vertex? What would be the outcome of the election?

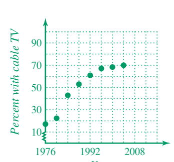
c. If the exponential regression is correct, what is the significance of the asymptote? What would be the outcome of the election?

d. Which of the three regressions would most merit the involvement of the Supreme Court of the United States? Why?

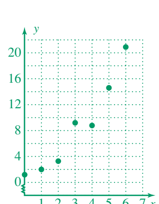
60.



61.



62.



No.

a. $y = 4.737x - 5.280$, $r \approx 0.939$

b. $y = (0.914)1.813^x$, $r \approx 0.939$

The correlation coefficients are equal.

No.

63. **a.** Bush & Gore are tied; Gore wins

b. Bush has his smallest lead; Bush would win

c. Bush would always have a lead. Bush would win

d. exponential; outcome would be too close to call

EXTENDING THE CONCEPT

Planet	Years	Radius
Mercury	0.24	0.39
Venus	0.62	0.72
Earth	1.00	1.00
Mars	1.88	1.52
Jupiter	11.86	5.20

64. The regression fundamentals applied in this section and elsewhere can be extended to other forms as well. The table shown gives the time required for the first six planets to make one complete revolution around the sun (in years), along with the average orbital radius of the planet (in astronomical units, 1 AU = 92.96 million miles). Use a graphing calculator to draw the scatter-plot, then use the scatter-plot, the context and any previous experience to decide whether a polynomial, exponential, logarithmic or power regression is most appropriate. Then find the regression equation and use it to: (a) estimate the average orbital radius of Saturn, given it orbits the Sun every 29.46 years, and (b) estimate how many years it takes Uranus to orbit the Sun, given it has an average orbital radius of 19.2 AU. Power regression; **a.** 9.5 AU; **b.** 84.8 yr

65A. $Y_1 \approx (1.2117)1.2847^x$, $Y_2 \approx -0.7665 + 3.9912\ln(x)$, 1. symmetry about $y = x$; *Z* for (a, b) on Y_1 (b, a) is on Y_2.

65A. Using the points (2, 2) and (7, 7), find the exponential regression equation and the logarithmic regression equation that contains these two points. Enter the functions as Y_1 and Y_2 respectively and graph the points and both equations on the standard screen (ZOOM **6:ZStandard**). Do you notice anything? Enter the function $Y_3 = x$ and re-graph the functions using window size $x \in [0, 10]$ and $y \in [0, 15]$ to obtain an approximately square viewing window. Name two ways we can verify that Y_1 and Y_2 are inverse functions.

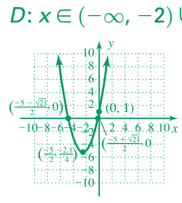
65B. **a.** $y = 1.2117214e^{0.2505524714x}$
b. $a + b\ln(x) = y$
$a + b\ln(y) = x$
$\ln(y) = \dfrac{x - a}{b}$
$y = e^{\frac{x-a}{b}} = e^{\frac{x}{b}}e^{\frac{-a}{b}}$
Substituting $a = -0.7664737786$ and $b = 3.991178001$ at this point gives the result from (a).

65B. Verify that the functions Y_1 and Y_2 from Exercise 65A are inverse functions by: (a) rewriting Y_1 as a base *e* exponential function (see Example 3), and (b) using the algebraic method to find the inverse function for the general form $y = a + b\ln(x)$, then using the corresponding values from Y_2 in the inverse function to reconcile and verify that results match.

MAINTAINING YOUR SKILLS

66. *D*: $x \in (-\infty, -2) \cup (-2, 1) \cup (1, 5) \cup (5, \infty)$

68.



66. (4.3) State the domain of the function, then write it in lowest terms:

$$h(x) = \frac{x^2 - 6x + 5}{x^3 - 4x^2 - 7x + 10} \quad \frac{1}{x + 2}$$

67. (2.3/5.3) Compute the average rate of change for $f(x) = e^{-x}$ and $g(x) = -\ln(x)$ in the interval $x \in [0.9, 1.0]$.
$\frac{\Delta f}{\Delta x} \approx -0.39$, $\frac{\Delta g}{\Delta x} \approx -1.05$

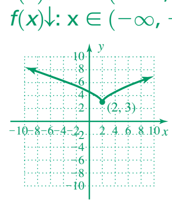
68. (3.4) Graph the function by completing the square. Clearly label the vertex and all intercepts: $p(x) = x^2 + 5x + 1$.

69. (3.7) Find a linear function that will make $p(x)$ continuous. $y = -\frac{3}{2}x + 7;\ 2 \le x < 4$

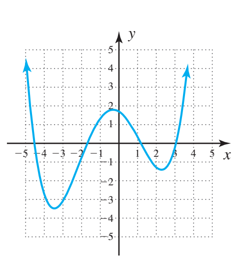
$$p(x) = \begin{cases} x^2 & -2 \le x < 2 \\ ?? & ? \le x < ? \\ \sqrt{x - 4} + 1 & x \ge 4 \end{cases}$$

70. max: $(-0.4, 1.8)$
min: $(-3.5, -3.5)$, $(2.3, -1.4)$
$f(x)\uparrow$: $x \in (-3.5, -0.4) \cup (2.3, \infty)$
$f(x)\downarrow$: $x \in (-\infty, -3.5) \cup (-0.4, 2.3)$

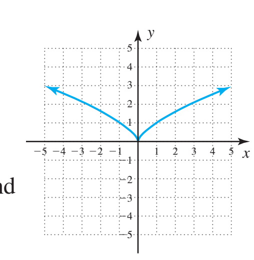
71.



70. (3.8) For the graph of $f(x)$ given, estimate max/min values to the nearest tenth and state intervals where $f(x)\uparrow$, $f(x)\downarrow$.



71. (3.3) The graph of $f(x) = x^{\frac{2}{3}}$ is given. Use it to sketch the graph of $F(x) = (x - 2)^{\frac{2}{3}} + 3$, and use the graph to state the domain and range of *F*.
D: $x \in (-\infty, \infty)$ *R*: $y \in [3, \infty)$



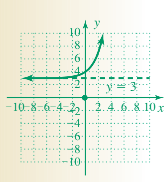
SUMMARY AND CONCEPT REVIEW

SECTION 5.1 Exponential Functions

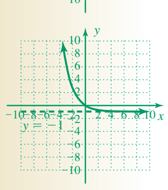
KEY CONCEPTS

- An exponential function is one of the form $f(x) = ab^x$, where $b > 0$, $b \neq 1$, and a, b, and x are real numbers.
- For functions of the form $f(x) = b^x$, $b > 0$ and $b \neq 1$, we have:
  - one-to-one function
  - y-intercept $(0, 1)$
  - domain: $x \in \mathbb{R}$
  - range: $y \in (0, +\infty)$
  - increasing if $b > 1$
  - decreasing if $0 < b < 1$
  - asymptotic to x-axis
- The graph of $y = b^{x \pm h} \pm k$ is a translation of the basic graph of $y = b^x$, horizontally h units opposite the sign and vertically k units in the same direction as the sign.
- To solve exponential equations with like bases, we use the fact that b^x represents a unique number. In other words, if $b^m = b^n$, then $m = n$ (equal bases imply equal exponents). This is referred to as the Uniqueness Property.
- All previous properties of exponents also apply to exponential functions.

1.



2.



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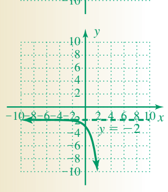


EXERCISES

Graph each function using *transformations of the basic function,* then strategically plotting a few points to check your work and round out the graph. Draw and label the asymptote.

1. $y = 2^x + 3$ **2.** $y = 2^{-x} - 1$ **3.** $y = -3^x - 2$

Solve the exponential equations using the uniqueness property.

4. $3^{2x-1} = 27$ 2 **5.** $4^x = \frac{1}{16}$ −2 **6.** $3^x \cdot 27^{x+1} = 81$ $\frac{1}{4}$

7. A ballast machine is purchased new for \$142,000 by the AT & SF Railroad. The machine loses 15% of its value each year and must be replaced when its value drops below \$20,000. How many years will the machine be in service? 12.1 yr

SECTION 5.2 Logarithms and Logarithmic Functions

KEY CONCEPTS

- A logarithm is an exponent. For $b > 0$, $b \neq 1$, and $x > 0$, $y = \log_b x$ means $b^y = x$ and $b^{\log_b x} = x$.
- A logarithmic *function* is defined as $f(x) = \log_b x$, where $b > 0$, $b \neq 1$, and x is a positive real number. $y = \log_{10} x = \log x$ is called the common logarithmic function.
- For logarithmic functions of the form $f(x) = \log_b x$, $b > 0$ and $b \neq 1$, we have:
  - one-to-one function
  - x-intercept $(1, 0)$
  - domain: $x \in (0, \infty)$
  - range: $y \in \mathbb{R}$
  - increasing if $b > 1$
  - decreasing if $0 < b < 1$
  - asymptotic to y-axis
- The graph of $y = \log_b(x \pm h) \pm k$ is a translation of the basic graph of $y = \log_b x$, horizontally h units opposite the sign and vertically k units in the same direction as the sign.
- To solve logarithmic equations with like bases, we can use the fact that $\log_b x$ is a unique number. In other words, if $\log_b m = \log_b n$, then $m = n$ (equal bases imply equal arguments).

EXERCISES

Write each expression in *exponential* form.

8. $\log_3 9 = 2$ $3^2 = 9$ **9.** $\log_5 \frac{1}{125} = -3$ $5^{-3} = \frac{1}{125}$ **10.** $\log_2 16 = 4$ $2^4 = 16$

Write each expression in *logarithmic* form.

11. $5^2 = 25$ $\log_5 25 = 2$ **12.** $2^{-3} = \frac{1}{8}$ $\log_2(\frac{1}{8}) = -3$ **13.** $3^4 = 81$ $\log_3 81 = 4$

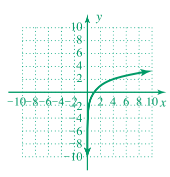
Solve for x.

14. $\log_2 32 = x$ 5 **15.** $\log_x 16 = 2$ 4 **16.** $\log(x - 3) = 1$ 13

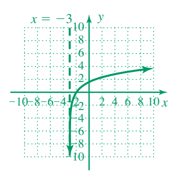
Graph each function using *transformations of the basic function,* then strategically plotting a few points to check your work and round out the graph. Draw and label the asymptote.

17. $f(x) = \log_2 x$ **18.** $f(x) = \log_2(x + 3)$ **19.** $f(x) = 2 + \log_2(x - 1)$

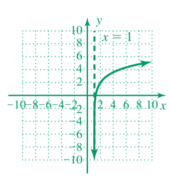
17.



18.



19.



20. The magnitude of an earthquake is given by $M(I) = \log\frac{I}{I_0}$, where I is the intensity and I_0 is the reference intensity. (a) Find $M(I)$ given $I = 62{,}000I_0$ and (b) find the intensity I given $M(I) = 7.3$. **a.** 4.79; **b.** $10^{7.3}\, I_0$

SECTION 5.3 The Exponential Function and Natural Logarithms

KEY CONCEPTS

- The natural exponential function is $y = e^x$, where $e \approx 2.71828$.
- The natural logarithmic function is $y = \log_e x$, most often written in abbreviated form as $y = \ln x$.
- Base-e logarithms can be found using a scientific or graphing calculator and the LN key.
- To evaluate logs with bases other than 10 or e, we use the change-of-base formula: $\log_b x = \frac{\log_d x}{\log_d b}$.
- Since a logarithm is an exponent, it has properties that parallel those of exponents.
  - Product Property: like base and multiplication, add exponents: $b^n b^m = b^{n+m}$
  - Quotient Property: like base and division, subtract exponents: $\log_b(\frac{M}{N}) = \log_b M - \log_b N$
  - Power Property: exponent raised to a power, multiply exponents: $\log_b(k)^x = x\log_b(k)$
- The logarithmic properties can be used to expand an expression, as in $\log(2x) = \log 2 + \log x$, or to contract an expression (combine like terms), as in $\ln(2x) - \ln(x + 3) = \ln\frac{2x}{x + 3}$.

EXERCISES

21. Solve each equation.

a. $\ln(x) = 32$ **b.** $e^x = 9.8$ **c.** $e^x = \sqrt{7}$ **d.** $\ln(x) = 2.38$

22. Evaluate using the change-of-base formula. Answer in exact form and approximate form, rounding to thousandths.

a. $\log_6 45$ **b.** $\log_3 128$ **c.** $\log_2 108$ **d.** $\log_5 200$

23. Use the product or quotient property of logarithms to write each sum or difference as a single term.

a. $\ln 7 + \ln 6$ **b.** $\log_9 2 + \log_9 15$

c. $\ln(x + 3) - \ln(x - 1)$ **d.** $\log x + \log(x + 1)$

24. Use the power property of logarithms to rewrite each term as a product.

a. $\log_5 9^2$ **b.** $\log_7 4^2$ **c.** $\ln 5^{2x-1}$ **d.** $\ln 10^{3x+2}$

25. Evaluate the following logarithmic expressions using only properties of logarithms (without the aid of a calculator or the change-of-base formula), given $\log_5 2 \approx 0.43$ and $\log_5 3 \approx 0.68$.

a. $\log_5 \frac{2}{3}$ **b.** $\log_5 \frac{3}{2}$ **c.** $\log_5 12$ **d.** $\log_5 18$

21. a. e^{32}
b. $\ln 9.8$
c. $\ln\sqrt{7} = \frac{1}{2}\ln 7$
d. $e^{2.38}$

22. a. $\frac{\log 45}{\log 6} \approx 2.125$
b. $\frac{\log 128}{\log 3} \approx 4.417$
c. $\frac{\log 108}{\log 2} \approx 6.755$
d. $\frac{\log 200}{\log 5} \approx 3.292$

23. a. $\ln 42$
b. $\log_9 30$
c. $\ln\left(\frac{x+3}{x-1}\right)$
d. $\log(x^2 + x)$

24. a. $2\log_5 9$
b. $2\log_7 4$
c. $(2x - 1)\ln 5$
d. $(3x + 2)\ln 10$

25. a. ≈ -0.2519
b. ≈ 0.2519
c. ≈ 1.5440
d. ≈ 1.7959

26. a. $4\ln x + \frac{1}{2}\ln y$
b. $\frac{1}{3}\ln p + \ln q$
c. $\frac{5}{3}\log x + \frac{4}{3}\log y - \frac{5}{2}\log x - \frac{3}{2}\log y$
d. $\log 4 + \frac{5}{3}\log p + \frac{4}{3}\log q - \frac{3}{2}\log p - \log q$

26. Use the properties of logarithms to write the following expressions as a sum or difference of simple logarithmic terms.

a. $\ln(x^4\sqrt{y})$ **b.** $\ln(\sqrt[3]{pq})$ **c.** $\log\left(\frac{\sqrt[3]{x^5y^4}}{\sqrt{x^5y^3}}\right)$ **d.** $\log\left(\frac{4\sqrt[3]{p^5q^4}}{\sqrt{p^3q^2}}\right)$

27. The rate of decay for radioactive material is related to its half-life by the formula $R(h) = \frac{\ln 2}{h}$, where h represents the half-life of the material and $R(h)$ is the rate of decay expressed as a decimal. The element radon-222 has a half-life of approximately 3.9 days. (a) Find its rate of decay to the nearest 100th of a percent. (b) Find the half-life of thorium-234 if its rate of decay is 2.89% per day. **a.** 17.77%; **b.** 23.98 days

SECTION 5.4 Exponential/Logarithmic Equations and Applications

KEY CONCEPTS

- To solve an exponential or logarithmic equation, first simplify by combining like terms if possible.
- The way logarithms are defined gives rise to four useful properties: For any base b where $b > 0, b \neq 1$,
  - $\log_b b = 1$ (since $b^1 = b$)
  - $\log_b b^x = x$ (since $b^x = b^x$)
  - $\log_b 1 = 0$ (since $b^0 = 1$)
  - $b^{\log_b x} = x$
- If the equation can be written with like bases on both sides, solve using the uniqueness property.
- If a single logarithmic or exponential term can be isolated on one side, then for any base b:
  - If $b^x = k$, then $x = \frac{\log k}{\log b}$.
  - If $\log_b x = k$, then $x = b^k$.

EXERCISES

Solve each equation.

28. $2^x = 7$ $\frac{\ln 7}{\ln 2}$ **29.** $3^{x+1} = 5$ $\frac{\ln 5}{\ln 3} - 1$ **30.** $4^{x-2} = 3^x$ $\frac{2\ln 4}{\ln 4 - \ln 3}$

31. $\log_5(x + 1) = 2$ 24 **32.** $\log x + \log(x - 3) = 1$ 5; -2 is extraneous **33.** $\log_{25}(x - 2) - \log_{25}(x + 3) = \frac{1}{2}$ no solution

34. The *barometric equation* $H = (30T + 8000)\ln\left(\frac{P_0}{p}\right)$ relates the altitude H to atmospheric pressure P, where $P_0 = 76$ cm of mercury. Find the atmospheric pressure at the summit of Mount Pico de Orizaba (Mexico), whose summit is 5657 m. Assume the temperature at the summit is $T = 12°C$. 38.63 cm Hg

SECTION 5.5 Applications from Investment, Finance, and Physical Science

KEY CONCEPTS

- Simple interest: $I = prt$; p is the initial principal, r is the interest rate per year, and t is the time in years.
- Amount in an account after t years: $A = p + prt$ or $A = p(1 + rt)$.

- Interest compounded n times per year: $A = p\left(1 + \frac{r}{n}\right)^{nt}$; p is the initial principal, r is the interest rate per year, t is the time in years, and n is the number of times per year interest is compounded.
- Interest compounded continuously: $A = pe^{rt}$; p is the initial principal, r is the interest rate per year, and t is the time in years. The base e is the exponential constant $e = 2.71828$.
- Closely related to the formula for interest compounded continuously are the more general formulas for exponential growth and decay, $Q(t) = Q_0e^{rt}$ and $Q(t) = Q_0e^{-rt}$, respectively.
- If a loan or savings plan calls for a regular schedule of deposits, the plan is called an annuity.
- For periodic payment P, deposited or paid n times per year, at annual interest rate r, with interest compounded or calculated n times per year for t years, and $R = \frac{r}{n}$:
  - The accumulated value of the account is $A = \frac{P}{R}\left[(1 + R)^{nt} - 1\right]$.
  - The payment required to meet a future goal is $P = \frac{AR}{\left[(1 + R)^{nt} - 1\right]}$.
  - The payment required to amortize an amount A is $P = \frac{AR}{1 - (1 + R)^{-nt}}$.

EXERCISES

Solve each application.

35. Jeffery borrows \$600.00 from his dad, who decides it's best to charge him interest. Three months later Jeff repays the loan plus interest, a total of \$627.75. What was the annual interest rate on the loan? 18.5%

36. To save money for her first car, Cheryl invests the \$7500 she inherited in an account paying 7.8% interest compounded monthly. She hopes to buy the car in 6 yr and needs \$12,000. Is this possible? almost she needs \$42.15 more.

37. Eighty prairie dogs are released in a wilderness area in an effort to repopulate the species. Five years later a statistical survey reveals the population has reached 1250 dogs. Assuming the growth was exponential, approximate the growth rate to the nearest tenth of a percent. 55.0%

38. To save up for the vacation of a lifetime, Al-Harwi decides to save \$15,000 over the next 4 yr. For this purpose he invests \$260 every month in an account paying $7\frac{1}{2}\%$ interest compounded monthly. (a) Is this monthly amount sufficient to meet the four-year goal? (b) If not, find the *minimum amount he needs to deposit each month* that will allow him to meet this goal in 4 yr. **a.** no **b.** \$268.93

SECTION 5.6 Exponential, Logarithmic, and Logistic Regression Models

KEY CONCEPTS

- The choice of regression models generally depends on: (a) whether the graph appears to fit the data, (b) the context or situation that generated the data, and (c) certain tests applied to the data.
- For a choice between exponential or power regression, linearizing the data can help. In linear form exponential data can be written $(x, \ln y)$; linearized data from power functions has the form $(\ln x, \ln y)$.
- The regression equation can be used to *extrapolate* or predict future values or occurrences. When using extrapolation, values are projected *beyond* the given set of data.

- The regression equation can be used to *interpolate* or approximate intermediate values. When using interpolation, the values occur *between* those given in the data set.

EXERCISES

39. The tremendous surge in cell phone subscriptions that began in the early nineties has continued unabated into the new century. The total number of subscriptions is shown in the table for selected years, with $1990 \to 0$ and the number of subscriptions in millions. Draw a scatter-plot of the data and complete the following.

Source: 2000/2004 *Statistical Abstract of the United States,* Tables 919 and 1144

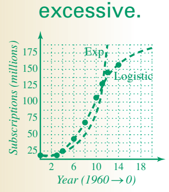
Year $(1990 \to 0)$	Subscriptions (millions)
0	5.3
3	16.0
4	24.1
6	44.0
8	69.2
10	109.5
11	128.4
12	140.0
13	158.7

a. Run both an exponential regression and a logistic regression on the data, and graph them on the same screen with the scatter-plot. Which seems to "fit" the data better?

b. Considering the context of the data and the current population of the United States (approximately 291 million in 2003), which equation model seems more likely to accurately predict the number of cell phone subscriptions in future years? Why?

c. Use both regression equations to approximate the number of subscriptions in 1997. How many subscriptions does each project for 2005? 2010? What do you notice?

39. a. logistic
b. logistic; growth rate exceeds population growth
c. Exp: 44.6 million, Logistic: 56.1 million
exponential: 347.3, 1253 million;
logistic, 179.3, 201.0 million;
projections from exponential equation are excessive.



40. The development of new products, improved health care, greater scientific achievement, and other advances is fueled by huge investments in research and development (R & D). Since 1960, total expenditures in the United States have shown a distinct pattern of growth, and the data is given in the table for selected years from 1960 to 1999. Use the data to draw a scatter-plot and complete the following:

Source: 2004 *Statistical Abstract of the United States,* Table 978

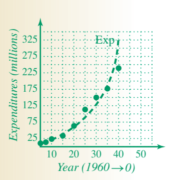
Year $(1960 \to 0)$	Expenditures (billion \$)
0	13.7
5	20.3
10	26.3
15	35.7
20	63.3
25	114.7
30	152.0
35	183.2
39	247.0

a. Decide on an appropriate form of regression and find a regression equation. exponential; $y = 13.29(1.08)^x$

b. Use the equation to estimate R & D expenditures in 1992. 156 billion dollars

c. If current trends continue, how much will be spent on R & D in 2005? 424.2 billion dollars

40.



MIXED REVIEW

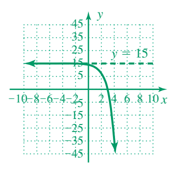
1. Evaluate each expression using the change-of-base formula.
a. $\log_2 30$ $\frac{\log 30}{\log 2} \approx 4.9069$ **b.** $\log_{0.25} 8$ -1.5 **c.** $\log_8 2$ $\frac{1}{3}$

2. Solve each equation using the uniqueness property.
a. $10^{4x-5} = 1000$ 2 **b.** $5^{3x-1} = \sqrt{5}$ $\frac{1}{2}$ **c.** $2^x \cdot 2^{0.5x} = 64$ 4

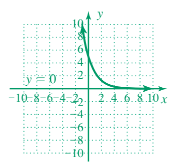
3. Use the power property of logarithms to rewrite each term as a product.
a. $\log_{10} 20^2$ $2\log_{10} 20$ **b.** $\log 10^{0.05x}$ $0.05x$ **c.** $\ln 2^{x-3}$ $(x-3)\ln 2$

Graph each of the following functions by shifting the basic function, then strategically plotting a few points to check your work and round out the graph. Graph and label the asymptote.

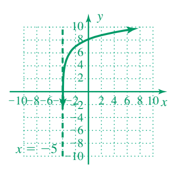
4. $y = -e^x + 15$

4.


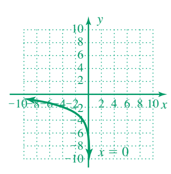
5. $y = 5 \cdot 2^{-x}$

5.


6. $y = \ln(x + 5) + 7$

6.


7. $y = \log_2(-x) - 4$

7.


8. Use the properties of logarithms to write the following expressions as a sum or difference of simple logarithmic terms.

a. $\ln\left(\frac{x^3}{2y}\right)$ $3\ln x - \ln 2 - \ln y$ **b.** $\log\left(10a\sqrt[3]{a^2b}\right)$ $1 + \log a + \frac{2}{3}\log a + \frac{1}{3}\log b$ **c.** $\log_2\left(\frac{8x^4\sqrt{x}}{3\sqrt{y}}\right)$ $3 + \frac{9}{2}\log_2(x) - \log_2 3 - \frac{1}{2}\log_2(y)$

9. Write the following expressions in exponential form.

a. $\log_5 625 = 4$ $5^4 = 625$ **b.** $\ln 0.15x = 0.45$ $e^{0.45} = 0.15x$ **c.** $\log(0.1 \times 10^8) = 7$ $10^7 = 0.1 \times 10^8$

10. Write the following expressions in logarithmic form.

a. $343^{1/3} = 7$ $\log_{343}7 = \frac{1}{3}$ **b.** $256^{3/4} = 64$ $\log_{256}64 = \frac{3}{4}$ **c.** $2^{-3} = \frac{1}{8}$ $\log_2(\frac{1}{8}) = -3$

Solve the following equations. State answers in exact form.

11. $\log_2 128 = x$ 7 **12.** $\log_5(4x + 7) = 0$ $\frac{-3}{2}$ **13.** $10^{x-4} = 200$ $6 + \log 2$

14. $e^{x+1} = 3^x$ $x = \frac{1}{\ln 3 - 1}$ **15.** $\log_2(2x - 5) + \log_2(x - 2) = 4$ 5.09 **16.** $\log(3x - 4) - \log(x - 2) = 1$ $\frac{16}{7}$

Solve each application.

17. The magnitude of an earthquake is given by $M(I) = \log\left(\frac{I}{I_0}\right)$, where I is the intensity of the quake and I_0 is the reference intensity 2×10^{11} (energy released from the smallest detectable quake). On October 23, 2004, the Niigata region of Japan was hit by an earthquake that registered 6.5 on the Richter scale. Find the intensity of this earthquake by solving the following equation for I: $6.5 = \log\left(\frac{I}{2 \times 10^{11}}\right)$. $I = 6.3 \times 10^{17}$

18. Serene is planning to buy a house. She has \$6500 to invest in a certificate of deposit that compounds interest quarterly at an annual rate of 4.4%. (a) Find how long it will take for this account to grow to the \$12,500 she will need for a 10% down payment for a \$125,000 house. Round to the nearest tenth of a year. (b) Suppose instead of investing an initial \$6500, Serene deposits \$500 a quarter in an account paying 4% each quarter. Find how long it will take for this account to grow to \$12,500. Round to the nearest tenth of a year. 14.9 yr 5.6 yr

Exercise 19

19. British artist Simon Thomas designs sculptures he calls hypercones. These sculptures involve rings of exponentially decreasing radii rotated through space. For one sculpture, the radii follow the model $r(n) = 2(0.8)^n$, where n counts the rings (outer most first) and $r(n)$ is radii in meters. Find the radii of the six largest rings in the sculpture. Round to the nearest hundredth of a meter. 1.6 m, 1.28 m, 1.02 m, 0.82 m, 0.66 m, 0.52 m

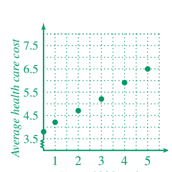
Source: http://www.plus.maths.org/issue8/features/art/

20. The following data report the increase in health care cost per employee for an employer from 1999 to 2004. Use the data to make a scatter-plot, then determine which type of regression should be applied and find a regression equation. Then answer the following questions. $y = 0.55x + 3.69$

a. Use the model to find the projected cost in 2005. 6.99

b. If this pattern of data continued, when would the projected cost exceed \$10,000? 2010

Year (1999 → 0)	Employer Average Health Care Cost per Employee (1000s)
0	3.8
1	4.2
2	4.7
3	5.2
4	5.9
5	6.5

20.


PRACTICE TEST

1. Write the expression $\log_3 81 = 4$ in exponential form. $3^4 = 81$

2. Write the expression $25^{1/2} = 5$ in logarithmic form. $\log_{25} 5 = \frac{1}{2}$

3. Write the expression $\log_b\left(\frac{\sqrt{x^5}\,y^3}{z}\right)$ as a sum or difference of logarithmic terms. $\frac{5}{2}\log_b x + 3\log_b y - \log_b z$

4. Write the expression $\log_b m + \left(\frac{3}{2}\right)\log_b n - \frac{1}{2}\log_b p$ as a single logarithm. $\log_b \frac{m\sqrt{n^3}}{\sqrt{p}}$

Solve for x by writing each expression using the same base.

5. $5^{x-7} = 125$ $x = 10$

6. $2 \cdot 4^{3x} = \frac{8^x}{16}$ $x = \frac{-5}{3}$

Given $\log_a 3 \approx 0.48$ and $\log_a 5 \approx 1.72$, evaluate the following without the use of a calculator:

7. $\log_a 45$ 2.68

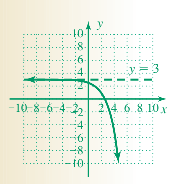
8. $\log_a 0.6$ -1.24

Graph using transformations of the parent function. Verify answers using a graphing calculator.

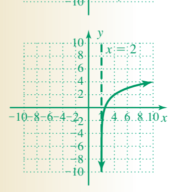
9. $g(x) = -2^{x-1} + 3$

10. $h(x) = \log_2(x - 2) + 1$

9.



10.



Use the change-of-base formula to evaluate. Verify results using a calculator.

11. $\log_3 100$ 4.19

12. $\log_6 0.235$ -0.81

Solve each equation.

13. $3^{x-1} = 89$ $x = 1 + \frac{\ln 89}{\ln 3}$

14. $\log_5 x + \log_5(x + 4) = 1$ $x = 1$; $x = -5$ is extraneous

15. A copier is purchased new for $8000. The machine loses 18% of its value each year and must be replaced when its value drops below $3000. How many years will the machine be in service? ≈ 5 yr

16. How long would it take $1000 to double if invested at 8% annual interest compounded daily? ≈ 8.7 yr

17. The number of ounces of unrefined platinum drawn from a mine is modeled by $Q(t) = -2600 + 1900\ln(t)$, where $Q(t)$ represents the number of ounces mined in t months. How many months did it take for the number of ounces mined to exceed 3000? 19.1 months

18. Septashi can invest his savings in an account paying 7% compounded semi-annually, or in an account paying 6.8% compounded daily. Which is the better investment? 7% compounded semi-annually

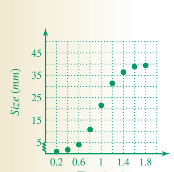
19. Jacob decides to save $4000 over the next 5 yr so that he can present his wife with a new diamond ring for their 20th anniversary. He invests $50 every month in an account paying $8\frac{1}{4}$% interest compounded monthly. (a) Is this amount sufficient to meet the 5-yr goal? no (b) If not, find the *minimum amount he needs to save monthly* that will enable him to meet this goal. $54.09

20. Using time-lapse photography, the growth of a stain is tracked in 0.2 second intervals, as a small amount of liquid is dropped on various fabrics. Use the data given to draw a scatter-plot and decide on an appropriate regression model. Precisely how long, to the nearest hundredth of a second, did it take the stain to reach a size of 15 mm? logistic; $y = \frac{39.1156}{1 + 314.6617e^{-5.9483x}}$; 0.89 sec

Exercise 20

Time (sec)	Size (mm)
0.2	0.39
0.4	1.27
0.6	3.90
0.8	10.60
1.0	21.50
1.2	31.30
1.4	36.30
1.6	38.10
1.8	39.00

20.



Calculator Exploration and Discovery

Investigating Logistic Equations

The keystrokes shown apply to a TI-84Plus model. Please consult our Internet site or your manual for other models.

As we saw in Section 5.6, logistics models have the form $P(t) = \frac{c}{1 + ae^{-bt}}$, where a, b, and c are constants and $P(t)$ represents the population at time t. For populations modeled by a logistics curve (sometimes called an "S" curve) growth is very rapid at first (like an exponential function), but this growth begins to slow down and level off due to various factors. This *Calculator Exploration and Discovery* is designed to investigate the effects that a, b, and c have on the resulting graph.

I. From our earlier observation, as t becomes larger and larger, the term ae^{-bt} becomes smaller and smaller (approaching 0) because it is a decreasing function: as $t \to \infty$, $ae^{-bt} \to 0$. If we allow that the term eventually becomes so small it can be disregarded, what remains is $P(t) = \frac{c}{1}$ or c. This is why c is called the capacity constant and the population can get no larger than c. In Figure 5.35, the graph of $P(t) = \frac{1000}{1 + 50e^{-1x}}$ ($a = 50$, $b = 1$, and $c = 1000$) is shown using a lighter line, while Figure 5.36 shows the graph of $P(t) = \frac{750}{1 + 50e^{-1x}}$ ($a = 50$, $b = 1$, and $c = 750$), is given in bold. The window size is indicated in Figure 5.37.

Figure 5.35

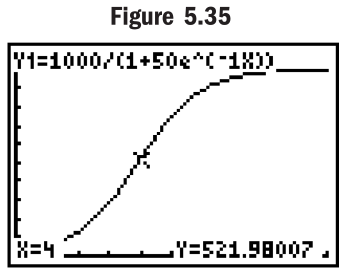


Figure 5.36

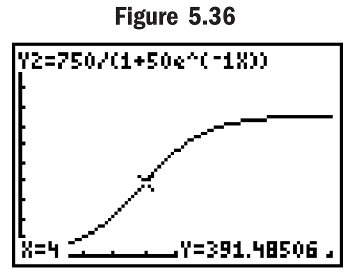
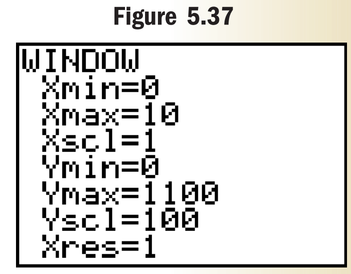


Figure 5.37



Also note that if a is held constant, smaller values of c cause the "interior" of the S curve to grow at a slower rate than larger values, a concept studied in some detail in a Calculus I class.

II. On the other hand, if $t = 0$, $ae^{-bt} = ae^0 = a$, and we note the ratio $P(0) = \frac{c}{1 + a}$ represents the *initial population.* This also means for constant values of c, larger values of a make the ratio $\frac{c}{1 + a}$ smaller; while smaller values of a make the ratio $\frac{c}{1 + a}$ larger. From this we conclude that a primarily affects the initial population. For the screens shown next, $P(t) = \frac{1000}{1 + 50e^{-1x}}$ (from (I) above) is graphed using a lighter line. The graph of $P(t) = \frac{1000}{1 + 5e^{-1x}}$ ($a = 5$, $b = 1$, and $c = 1000$) is shown in bold in Figure 5.38. The graph of $P(t) = \frac{1000}{1 + 500e^{-1x}}$ ($a = 500$, $b = 1$, and $c = 1000$) is graphed in bold in Figure 5.39.

Figure 5.38

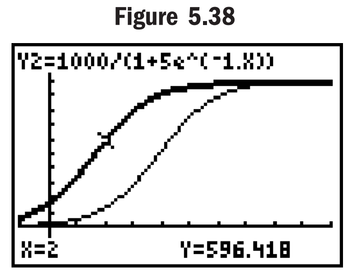
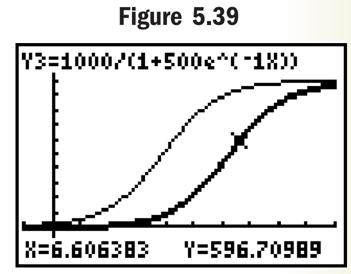


Figure 5.39



Note that changes in a appear to have no effect on the rate of growth in the interior of the S curve.

III. As for the value of b, we might expect that it affects the rate of growth in much the same way as the growth rate r does for exponential functions $Q(t) = Q_0e^{-rt}$. Sure enough, we note from the graphs below that b has no effect on the initial value or the eventual capacity, but causes the population to approach this capacity more quickly for larger values of b, and more slowly for smaller values of b. For the screens shown, $P(t) = \frac{1000}{1 + 50e^{-1x}}$ ($a = 50$, $b = 1$, and $c = 1000$) is graphed using a lighter line. The graph of $P(t) = \frac{1000}{1 + 50e^{-1.2x}}$ ($a = 50$, $b = 1.2$, and $c = 1000$) is graphed in bold on the same screen in Figure 5.40. The graph of $P(t) = \frac{1000}{1 + 50e^{-0.8x}}$ ($a = 50$, $b = 0.8$, and $c = 1000$) is graphed in bold in Figure 5.41.

Figure 5.40

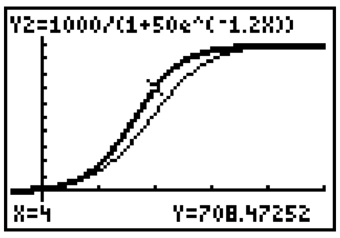
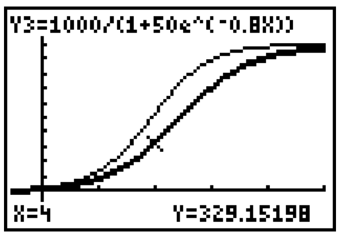


Figure 5.41



The following exercises are based on the population of an ant colony, modeled by the logistic function $P(t) = \frac{2500}{1 + 25e^{-0.5x}}$. Try to respond to each exercise without the use of a calculator.

1. Identify the values of a, b, and c for this logistics curve. $a = 25$ $b = 0.5$ $c = 2500$

2. What was the approximate initial population of the colony? 96 ants

3. Which gives a larger initial population: (a) $c = 2500$ and $a = 25$ or (b) $c = 3000$ and $a = 15$? b

4. What is the maximum population capacity for this colony? 2500

5. Would the population of the colony surpass 2000 more quickly if $b = 0.6$ or if $b = 0.4$? $b = 0.6$

6. Which causes a slower population growth: (a) $c = 2000$ and $a = 25$ or (b) $c = 3000$ and $a = 25$? a

7. Verify your responses to Exercises 2 through 6 using a graphing calculator.

STRENGTHENING CORE SKILLS

More on Solving Exponential and Log Equations

In order to more effectively solve exponential and logarithmic equations, it might help to see how the general process of equation solving applies to a wide range of equation types. Consider the following sequence of equations:

$$2x + 3 = 11 \qquad 2|x| + 3 = 11$$
$$2\sqrt{x} + 3 = 11 \qquad 2e^x + 3 = 11$$
$$2x^2 + 3 = 11 \qquad 2\log x + 3 = 11$$

Each of these equations can be represented generally as $2f(x) + 3 = 11$, where we note that the process of equation solving applies identically to all: first subtract 3, then divide by 2, and finally apply the appropriate inverse function. If we do this symbolically, we have

$$\begin{aligned} 2f(x) + 3 &= 11 && \text{general equation} \\ 2f(x) &= 8 && \text{subtract 3} \\ f(x) &= 4 && \text{divide by 2} \\ f^{-1}[f(x)] &= f^{-1}(4) && \text{apply inverse function} \\ x &= f^{-1}(4) && \text{solution} \end{aligned}$$

The importance of this result cannot be overstated, as it tells us that every solution in the previous sequence is represented by $x = f^{-1}(4)$, where f^{-1} is the inverse function appropriate to the equation. This serves as a dramatic reminder that we must *isolate the term or factor that contains the unknown* before attempting to apply the inverse function or solution process. If we isolate an exponential term with either base 10 or base e, the inverse function is a logarithm of base 10 or base e, respectively. This can easily be seen using the power property of logarithms and the fact that $\ln e = 1$ and $\log 10 = 1$:

ILLUSTRATION 1 Solve for x: **a.** $2e^x + 3 = 11$ **b.** $(2)10^x + 3 = 11$

Solution:

a.
$$\begin{aligned} 2e^x + 3 &= 11 && \text{original equation} \\ 2e^x &= 8 && \text{subtract 3} \\ e^x &= 4 && \text{divide by 2} \\ \ln e^x &= \ln 4 && \text{apply } f^{-1}(x) \\ x\ln e &= \ln 4 && \text{Power Property} \\ x &= \ln 4 && \text{solution} \end{aligned}$$

b.
$$\begin{aligned} (2)10^x + 3 &= 11 \\ (2)10^x &= 8 \\ 10^x &= 4 \\ \log 10^x &= \log 4 \\ x\log 10 &= \log 4 \\ x &= \log 4 \end{aligned}$$

If we isolate an exponential term with base other than 10 or e, the same process is applied using *either* base 10 or base e, with the additional step of dividing by $\log k$ or $\ln k$ to solve for x:

ILLUSTRATION 2 Solve for x: **a.** $(2)5^x + 3 = 11$ **b.** $(2)1.23^x + 3 = 11$

Solution:

a.
$$\begin{aligned} (2)5^x + 3 &= 11 && \text{original equation} \\ (2)5^x &= 8 && \text{subtract 3} \\ 5^x &= 4 && \text{divide by 2} \\ \ln 5^x &= \ln 4 && \text{apply } f^{-1}(x) \\ x\ln 5 &= \ln 4 && \text{power property} \\ x &= \frac{\ln 4}{\ln 5} && \text{solution (exact form)} \end{aligned}$$

b.
$$\begin{aligned} (2)1.23^x + 3 &= 11 \\ (2)1.23^x &= 8 \\ 1.23^x &= 4 \\ \log 1.23^x &= \log 4 \\ x\log 1.23 &= \log 4 \\ x &= \frac{\log 4}{\log 1.23} \end{aligned}$$

In many applications, the coefficients and bases of an exponential equation are not "nice looking" values. In other words, they are often noninteger or even irrational. Regardless, the solution process remains the same and we should try to see that all such equations belong to the same family.

ILLUSTRATION 3 For the St. Louis Cardinals, player salaries on the opening day roster for the 2000–2001 season were closely modeled by the exponential equation $y = (12.268)0.866^x$, where x represents a player's ordinal rank (1st, 2nd, 3rd, etc.) on the salary schedule and y approximates the player's actual salary in millions. According to this model, how many players make more than $5,000,000?

Source: Sports Illustrated at www.sportsillustrated.com/baseball/mlb/news/2001/04/09

Solution:

$$y = (12.268)0.866^x \quad \text{original model}$$
$$5 = (12.268)0.866^x \quad \text{substitute } y = 5$$
$$0.4075644 \approx 0.866^x \quad \text{divide by 12.268}$$
$$\ln(0.4075644) \approx \ln 0.866^x \quad \text{apply natural log to both sides}$$
$$\ln(0.4075644) \approx x \ln 0.866 \quad \text{power property}$$
$$\frac{\ln 0.4075644}{\ln 0.866} \approx x \quad \text{solve for } x \text{ (divide by ln 0.866)}$$
$$6.24 \approx x \quad \text{simplify}$$

According to this model, six or seven players make more than \$5,000,000.

Exercise 1: For the New York Yankees, player salaries on the opening day roster for the 2000–2001 season were closely modeled by the exponential equation $y = (21.303)0.842^x$, where x represents a player's ordinal rank (1st, 2nd, 3rd, etc.) on the salary schedule and y approximates the player's actual salary in millions. According to this model, how many players make more than \$5 million?

Source: (same as Illustration 3) between eight and nine players

CUMULATIVE REVIEW CHAPTERS 1–5

Use the quadratic formula to solve for x.

1. $x^2 - 4x + 53 = 0$ $\quad x = 2 \pm 7i$

2. $6x^2 + 19x = 36$ $\quad x = \frac{4}{3}, x = \frac{-9}{2}$

3. Use substitution to show that $4 + 5i$ is is a zero of $f(x) = x^2 - 8x + 41$.

4. Graph using transformations of a basic function: $y = 2\sqrt{x+2} - 3$.

5. Find $(f \circ g)(x)$ and $(g \circ f)(x)$ and comment on what you notice: $f(x) = x^3 - 2$; $g(x) = \sqrt[3]{x+2}$.

6. State the domain of $h(x)$ in interval notation: $h(x) = \frac{\sqrt{x+3}}{x^2 + 6x + 8}$. $\quad x \in [-3, -2) \cup (-2, \infty)$

7. According to the 2002 *National Vital Statistics Report* (Vol. 50, No. 5, page 19) there were 3100 sets of triplets born in the United States in 1991, and 6740 sets of triplets born in 1999. Assuming the relationship (year, sets of triplets) is linear: (a) find the equation of the line, (b) explain the meaning of the slope in this context, and (c) use the equation to estimate the number of sets born in 1996, and to project the number of sets that will be born in 2007 if this trend continues.

8. State the following geometric formulas:

a. area of a circle $\quad A = \pi r^2$

b. Pythagorean theorem $\quad a^2 + b^2 = c^2$

c. perimeter of a rectangle $\quad P = 2L + 2W$

d. area of a trapezoid $\quad A = \frac{h}{2}(B + b)$

9. Graph the following piecewise-defined function and state its domain, range, and intervals where it is increasing and decreasing.

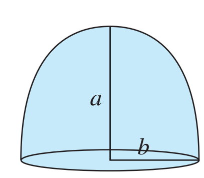
$$h(x) = \begin{cases} -4 & -5 \le x < -2 \\ -x^2 & -2 \le x < 3 \\ 3x - 18 & x \ge 3 \end{cases}$$

10. Solve the inequality and write the solution in interval notation: $\frac{2x+1}{x-3} \ge 0$. $\quad x \in (-\infty, -1] \cup (3, +\infty)$

11. Use the rational roots theorem to find all zeroes of $f(x) = x^4 - 3x^3 - 12x^2 + 52x - 48$. $\quad x = 3, x = 2$ (multiplicity 2); $x = -4$

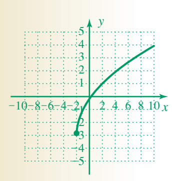
12. Given $f(c) = \frac{9}{5}c + 32$, find k, where $k = f(25)$. Then find the inverse function using the algebraic method, and verify that $f^{-1}(k) = 25$.

13. Solve the formula $V = \frac{1}{2}\pi b^2 a$ (the volume of a paraboloid) for the variable b. $\quad \sqrt{\frac{2V}{\pi a}} = b$



3. $(4 + 5i)^2 - 8(4 + 5i) + 41 = 0$
$-9 + 40i - 32 - 40i + 41 = 0$
$0 = 0$ ✓

4.



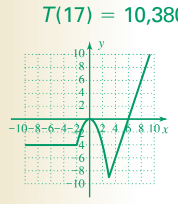
5. $f(g(x)) = x$
$g(f(x)) = x$
Since $(f \circ g)(x) = (g \circ f)(x)$, they are inverse functions.

7. a. $T(t) = 455t + 2645$ (1991 → year 1)

b. $\frac{\Delta T}{\Delta t} = \frac{455}{1}$, triple births increases by 455 each year.

c. $T(6) = 5375$ sets of triplets, $T(17) = 10{,}380$ sets of triplets

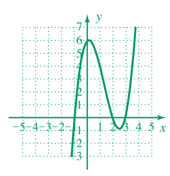
9.



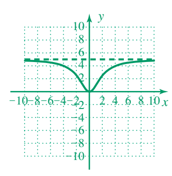
D: $x \in [5, \infty)$, R: $y \in [-9, \infty)$
$h(x)\uparrow$: $x \in (-2, 0) \cup (3, \infty)$
$h(x)\downarrow$: $x \in (0, 3)$

12. $k = 77$
$f^{-1}(c) = \frac{5}{9}(f - 32)$
$f^{-1}(77) = \frac{5}{9}(77 - 32)$
$= 25$ ✓

14.



15.



16. $x \approx -40.298$

17. $x = 5$, $x = -6$ is an extraneous root

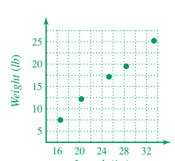
18. **a.** $P(183) \approx 34.7\ W$

b. $P(t) = \dfrac{50}{4}$

$t \approx 693$ days

Approx. 1 yr 11 months

19.

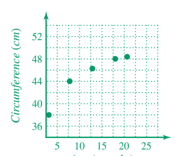


a. linear
$W = 1.24L - 15.83$

b. 32.5 lb

c. 35.3 in

20.



a. $C(a) \approx 37.9694 + 3.4229\ \ln(a)$

b. 49.5 cm

c. 30.9 mo

14. Use the *Guidelines for Graphing* to graph the polynomial $p(x) = x^3 - 4x^2 + x + 6$.

15. Use the *Guidelines for Graphing* to graph the rational function $r(x) = \dfrac{5x^2}{x^2 + 4}$.

16. Solve for x: $10 = -2e^{-0.05x} + 25$.

17. Solve for x: $\ln(x + 3) + \ln(x - 2) = \ln(24)$.

18. Once in orbit, satellites are often powered by radioactive isotopes. From the natural process of radioactive decay, the power output declines over a period of time. For an initial amount of 50 g, suppose the power output is modeled by the function $p(t) = 50e^{-0.002t}$, where $p(t)$ is the power output in watts, t days after the satellite has been put into service. (a) Approximately how much power remains 6 *months* later? (b) How many *years* until only one-fourth of the original power remains?

After reading a report from The National Center for Health Statistics regarding the growth of children from age 0 to 36 months, Maryann decides to track the relationships (length in inches, weight in pounds) and (age in months, circumference of head in centimeters) for her newborn child, a beautiful baby girl—Morgan.

19. For the (length, weight) data given: (a) draw a scatter-plot, decide on an appropriate form of regression, and find a regression equation; (b) use the equation to find Morgan's weight when she reaches a height (length) of 39 in.; and (c) determine her length when she attains a weight of 28 lb.

Exercise 19

Length (inches)	Weight (pounds)
17.5	5.50
21	10.75
25.5	16.25
28.5	19.00
33	25.25

Exercise 20

Age (months)	Circumference (centimeters)
1	38.0
6	44.0
12	46.5
18	48.0
21	48.3

20. For the (age, circumference) data given (a) draw a scatter-plot, decide on an appropriate form of regression, and find a regression equation; (b) use the equation to find the circumference of Morgan's head when she reaches an age of 27 months; and (c) determine the age when the circumference of her head reaches 50 cm.