SUPPLEMENT 1 TO CHAPTER 20

Variance-Reducing Techniques

Because considerable computer time usually is required for simulation runs, it is important to obtain as much and as precise information as possible from the amount of simulation that can be done. Unfortunately, there has been a tendency in practice to apply simulation uncritically without giving adequate thought to the efficiency of the experimental design. This tendency has occurred despite the fact that considerable progress has been made in developing special techniques for increasing the precision (i.e., decreasing the variance) of sample estimators.

These variance-reducing techniques often are called **Monte Carlo techniques** (a term sometimes applied to simulation in general). Because they tend to be rather sophisticated, it is not possible to explore them deeply here. However, we shall attempt to impart the flavor of these techniques and the great increase in precision they sometimes provide by presenting two when applied to the following example.

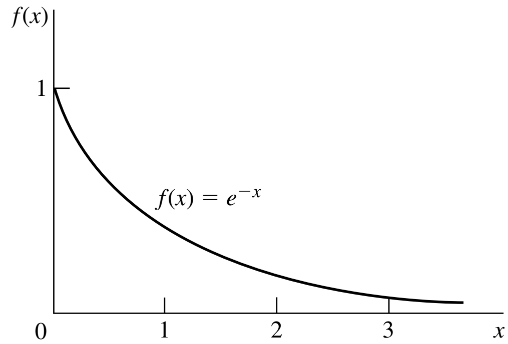
Consider the exponential distribution whose parameter has a value of 1. Thus, its probability density function is $f(x) = e^{-x}$, as shown in Fig. 1, and its cumulative distribution function is $F(x) = 1 - e^{-x}$. It is known that the mean of this distribution is 1. However, suppose that this mean is not known and that we want to estimate this mean by using simulation.

To provide a standard of comparison of the two variance-reducing techniques, we consider first the straightforward simulation approach, sometimes called the **crude Monte Carlo technique.** This approach involves generating some *random observations* from the exponential distribution under consideration and then using the *average* of these observations to estimate the mean. As described in Sec. 20.4, these random observations would be

$$x_i = -\ln(1 - r_i), \qquad \text{for } i = 1, 2, \ldots, n,$$

where $r_1, r_2, \ldots, r_n$ are uniform random numbers between 0 and 1. We use the first three digits in the fifth column of Table 20.3 to obtain 10 such uniform random numbers; the resulting random observations are shown in Table 1. (These same random numbers also are used to illustrate the variance-reducing techniques to sharpen the comparison.)

Notice that the sample average in Table 1 is 0.779, as opposed to the true mean of 1.000. However, because the standard deviation of the sample average happens to be $1/\sqrt{n}$, or $1/\sqrt{10}$ in this case (as could be estimated from the sample), an error of this amount or larger would occur approximately one-half of the time. Furthermore, because the standard deviation of a sample average is always inversely proportional to $\sqrt{n}$, this sample



■ **FIGURE 1**
Probability density function for the example for variance-reducing techniques, where the objective is to estimate the mean of this distribution.

■ **TABLE 1** Application of the crude Monte Carlo technique to the example

i	Random Number* r_i	Random Observation $x_i = -\ln(1 - r_i)$
1	0.495	0.684
2	0.335	0.408
3	0.791	1.568
4	0.469	0.633
5	0.279	0.328
6	0.698	1.199
7	0.013	0.014
8	0.761	1.433
9	0.290	0.343
10	0.693	1.183
		Total = 7.793
		Estimate of mean = 0.779

*Actually, 0.0005 was added to the indicated value for each of the r_i so that the range of their possible values would be from 0.0005 to 0.9995 rather than from 0.000 to 0.999.

size would need to be quadrupled to reduce this standard deviation by one-half. These somewhat disheartening facts suggest the need for other techniques that would obtain such estimates more precisely and more efficiently.

Stratified Sampling

Stratified sampling is a relatively simple Monte Carlo technique for obtaining better estimates. There are two shortcomings of the crude Monte Carlo approach that are rectified by stratified sampling. First, by the very nature of randomness, a random sample may not provide a particularly uniform cross section of the distribution. For example, the random sample given in Table 1 has no observations between 0.014 and 0.328, even though the probability that a random observation will fall inside this interval is greater than $\frac{1}{4}$. Second, certain portions of a distribution may be more critical than others for obtaining a precise estimate, but random sampling gives no special priority to obtaining observations from these portions. For example, the tail of an exponential distribution is especially critical in determining its mean. However, the random sample in Table 1 includes no observations larger than 1.568, even though there is at least a small probability of *much* larger values.

TABLE 2 Formulation of the stratified sampling approach to the example

Stratum	Portion of Distribution	Stratum Random No.	Sample Size	Sampling Weight
1	$0 \le F(x) \le 0.64$	$r_i' = 0 + 0.64r_i$	4	$w_i = \frac{4/10}{0.64} = \frac{5}{8}$
2	$0.64 \le F(x) \le 0.96$	$r_i' = 0.64 + 0.32r_i$	4	$w_i = \frac{4/10}{0.32} = \frac{5}{4}$
3	$0.96 \le F(x) \le 1$	$r_i' = 0.96 + 0.04r_i$	2	$w_i = \frac{2/10}{0.04} = 5$

This explanation is the basic one for why this particular sample average is far below the true mean. Stratified sampling circumvents these difficulties by dividing the distribution into portions called *strata,* where each stratum would be sampled individually with disproportionately heavy sampling of the more critical strata.

To illustrate, suppose that the distribution is divided into three strata in the manner shown in Table 2. These strata were chosen to correspond to observations approximately from 0 to 1, from 1 to 3, and from 3 to ∞, respectively. To ensure that the random observations generated for each stratum actually lie in that portion of the distribution, the uniform random numbers must be converted to the indicated range for $F(x)$, as shown in the third column of Table 2. The number of observations to be generated from each stratum is given in the fourth column.[1] The rightmost column then shows the resulting *sampling weight* for each stratum, i.e., the *ratio* of the *sampling proportion* (the fraction of the total sample to be drawn from the stratum) to the *distribution proportion* (the probability of a random observation falling inside the stratum). These sampling weights roughly reflect the relative importance of the respective strata in determining the mean.

Given the formulation of the stratified sampling approach shown in Table 2, the same uniform random numbers used in Table 1 yield the observations given in the fifth column in Table 3. However, it would not be correct to use the unweighted average of these observations to estimate the mean, because certain portions of the distribution have been sampled more than others. Therefore, before we take the average, we divide the observations from each stratum by the sampling weight for that stratum to give proportionate weightings to the different portions of the distribution, as shown in the rightmost column of Table 3. The resulting *weighted* average of 0.948 provides the desired estimate of the mean.

Method of Complementary Random Numbers

The second variance-reducing technique we shall mention is the method of *complementary random numbers.*[2] The motivation for this method is that the "luck of the draw" on the uniform random numbers generated may cause the average of the resulting random observations to be substantially on one side of the true mean, whereas the *complements* of those uniform random numbers (which are themselves uniform random numbers) would have tended to yield a nearly opposite result. (For example, the uniform random numbers in Table 1 average less than 0.5, and none are as large as 0.8, which led to an estimate substantially below the true mean.) Therefore, using *both* the original uniform random numbers *and* their complements to generate random observations and then calculating the

[1]These sample sizes are roughly based on a recommended guideline that they be proportional to the *product* of the *probability* of a random observation's falling inside the corresponding stratum *times* the *standard deviation* within this stratum.

[2]This method is a special case of the method of *antithetic variates,* which attempts to generate *pairs* of random observations having a high *negative* correlation, so that the combined average will tend to be closer to the mean.

TABLE 3 Application of stratified sampling to the example

Stratum	i	Random Number r_i	Stratum Random No. r_i'	Stratum Random Observation $x_i' = -\ln(1 - r_i')$	Sampling Weight w_i	x_i'/w_i
1	1	0.495	0.317	0.381	$\frac{5}{8}$	0.610
	2	0.335	0.215	0.242	$\frac{5}{8}$	0.387
	3	0.791	0.507	0.707	$\frac{5}{8}$	1.131
	4	0.469	0.300	0.357	$\frac{5}{8}$	0.571
2	5	0.279	0.729	1.306	$\frac{5}{4}$	1.045
	6	0.698	0.864	1.995	$\frac{5}{4}$	1.596
	7	0.013	0.644	1.033	$\frac{5}{4}$	0.826
	8	0.761	0.884	2.154	$\frac{5}{4}$	1.723
3	9	0.290	0.9716	3.561	5	0.712
	10	0.693	0.9877	4.398	5	0.880

Total = 9.481
Estimate of mean = 0.948

TABLE 4 Application of the method of complementary random numbers to the example

i	Random Number r_i	Random Observation $x_i = -\ln(1 - r_i)$	Complementary Random Number $r_i' = 1 - r_i$	Random Observation $x_i' = -\ln(1 - r_i')$
1	0.495	0.684	0.505	0.702
2	0.335	0.408	0.665	1.092
3	0.791	1.568	0.209	0.234
4	0.469	0.633	0.531	0.756
5	0.279	0.328	0.721	1.275
6	0.698	1.199	0.302	0.359
7	0.013	0.014	0.987	4.305
8	0.761	1.433	0.239	0.272
9	0.290	0.343	0.710	1.236
10	0.693	1.183	0.307	0.366
		Total = 7.793		Total = 10.597

Estimate of mean = $\frac{1}{2}(0.779 + 1.060) = 0.920$

combined sample average should provide a more precise estimator of the mean. This approach is illustrated in Table 4,[1] where the first three columns come from Table 1 and the two rightmost columns use the complementary uniform random numbers, which results in a combined sample average of 0.920.

[1]Note that 20 calculations of a logarithm were required in this case, in contrast to the 10 that were required by each of the preceding techniques.

Conclusions

This example has suggested that the variance-reducing techniques provide a much more precise estimator of the mean than does straightforward simulation (the crude Monte Carlo technique). These results definitely were not a coincidence, as a derivation of the variance of the estimators would show. In comparison with straightforward simulation, these techniques (including several more complicated ones not presented here) do indeed provide a much more precise estimator with the same amount of computer time, or they provide an equally precise estimator with much less computer time. Despite the fact that additional analysis may be required to incorporate one or more of these techniques into the simulation study, the rewards should not be forgone readily.

Although this example was particularly simple, it is often possible, though more difficult, to apply these techniques to much more complex problems. For example, suppose that the objective of the simulation study is to estimate the expected waiting time of customers in a queueing system (such as those described in Chap. 17). Because both the probability distribution of interarrival times and the probability distribution of service times are involved, and because consecutive waiting times are not statistically independent, this problem may appear to be beyond the capabilities of the variance-reducing techniques. However, as has been described in detail elsewhere,[1] these techniques and others can indeed be applied to this type of problem very advantageously. For example, the method of *complementary random numbers* can be applied simply by repeating the original simulation run, substituting the complements of the original uniform random numbers to generate the corresponding random observations.

PROBLEMS

20S1-1. Consider the probability distribution whose probability density function is

$$f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

The problem is to perform a simulated experiment, with the help of variance-reducing techniques, for estimating the mean of this distribution. To provide a standard of comparison, also derive the mean analytically.

For each of the following cases, use the same 10 uniform random numbers (obtained as instructed at the beginning of the Problems section for Chap. 20) to generate random observations, and calculate the resulting estimate of the mean.

(a) Use the crude Monte Carlo technique.

(b) Use stratified sampling with three strata—$0 \leq F(x) \leq 0.6$, $0.6 < F(x) \leq 0.9$, and $0.9 < F(x) \leq 1$—with 3, 3, and 4 observations, respectively.

(c) Use the method of complementary random numbers.

20S1-2. Simulation is being used to study a system whose measure of performance X will be partially determined by the outcome of a certain external factor. This factor has three possible outcomes (unfavorable, neutral, and favorable) that will occur with equal probability $\left(\frac{1}{3}\right)$. Because the favorable outcome would greatly increase the spread of possible values of X, this outcome is more critical than the others for estimating the mean and variance of X. Therefore, a stratified sampling approach has been adopted, with six random observations of the value of X generated under the favorable outcome, three generated under the neutral outcome, and one generated under the unfavorable outcome, as follows:

Outcome of External Factor	Simulated Values of *X*
Favorable	8, 5, 1, 6, 3, 7
Neutral	3, 5, 2
Unfavorable	2

[1]S. Ehrenfeld and S. Ben-Tuvia, "The Efficiency of Statistical Simulation Procedures," *Technometrics,* **4**(2): 257–275, 1962. Also see Chap. 11 of Selected Reference 14. For additional information on variance-reducing techniques, see the November 1989 issue of *Management Science* for a special issue on this topic. For a sampling of more recent research in this area, see pp. 69–79 in vol. 44 (1997) of *Naval Research Logistics;* pp. 1295–1312 in vol. 44 (1998) and pp. 1214–1235, 1349–1364 in vol. 46 (2000) of *Management Science;* pp. 900–912 in vol. 49 (2001) of *Operations Research;* and pp. 879–894 in vol. 35 (2003) of *IIE Transactions*.

(a) Develop the resulting estimate of $E(X)$.
(b) Develop the resulting estimate of $E(X^2)$.

20S1-3. A random variable X has $P\{X = 0\} = 0.9$. Given $X \neq 0$, it has a uniform distribution between 5 and 15. Thus, $E(X) = 1$. Obtaining uniform random numbers as instructed at the beginning of the Problems section for Chap. 20, use simulation to estimate $E(X)$.

(a) Estimate $E(X)$ by generating five random observations from the distribution of X and then calculating the sample average. (This is the crude Monte Carlo technique.)
(b) Estimate $E(X)$ by using stratified sampling with two strata—$0 \leq F(x) \leq 0.9$ and $0.9 < F(x) \leq 1$—with 1 and 4 observations, respectively.

20S1-4. Dave's Bicycle Shop repairs bicycles. Forty percent of the bicycles require only a minor repair. The repair time for these bicycles has a uniform distribution between 0 and 1 hour. Sixty percent of the bicycles require a major repair. The repair time for these bicycles has a uniform distribution between 1 hour and 2 hours. You now need to estimate the mean of the overall probability distribution of the repair times for all bicycles by using the following alternative methods.

(a) Use the uniform random numbers—0.7256, 0.0817, and 0.4392—to simulate whether each of three bicycles requires minor repair or major repair. Then use the uniform random numbers—0.2243, 0.9503, and 0.6104—to simulate the repair times of these bicycles. Calculate the average of these repair times to estimate the mean of the overall distribution of repair times.
(b) Draw the cumulative distribution function (CDF) for the overall probability distribution of the repair times for all bicycles.
(c) Use the inverse transformation method with the latter three uniform random numbers given in part (*a*) to generate three random observations from the overall distribution considered in part (*b*). Calculate the average of these observations to estimate the mean of this distribution.
(d) Repeat part (*c*) with the *complements* of the uniform random numbers used there, so the new uniform random numbers are 0.7757, 0.0497, and 0.3896.
(e) Use the method of complementary random numbers to estimate the mean of the overall distribution of repair times by combining the random observations from parts (*c*) and (*d*).
(f) The true mean of the overall probability distribution of repair times is 1.1. Compare the estimates of this mean obtained in parts (*a*), (*c*), (*d*), and (*e*). For the method that provides the closest estimate, give an intuitive explanation for why it performed so well.
(g) Formulate a spreadsheet model to apply the method of complementary random numbers. Use 300 uniform random numbers to generate 600 random observations from the distribution considered in part (*b*) and calculate the average of these random observations. Compare this average with the true mean of the distribution.
(h) The drawbacks of the approach described in part (*a*) are that (1) it does not ensure that the repair times for both minor repairs and major repairs are adequately sampled and (2) it requires two uniform random numbers to generate each random observation of a repair time. To overcome these drawbacks, combine stratified sampling and the method of complementary random numbers by using the first three uniform random numbers given in part (*a*) to generate six random *minor repair* times and the other three uniform random numbers to generate six random *major repair* times. Calculate the resulting estimate of the mean of the overall distribution of repair times.

20S1-5. The employees of General Manufacturing Corp. receive health insurance through a group plan issued by Wellnet. During the past year, 40 percent of the employees did not file any health insurance claims, 40 percent filed only a small claim, and 20 percent filed a large claim. The small claims were spread uniformly between 0 and \$2,000, whereas the large claims were spread uniformly between \$2,000 and \$20,000.

Based on this experience, Wellnet now is negotiating the corporation's premium payment per employee for the upcoming year. You are an OR analyst for the insurance carrier, and you have been assigned the task of estimating the average cost of insurance coverage for the corporation's employees.

Follow the instructions of Prob. 20S1-4, where the size of an employee's health insurance claim (including 0 if no claim was filed) now plays the role that the repair time for a bicycle did in Prob. 20S1-4. [For part (*f*), the true mean of the overall probability distribution of the size of an employee's health insurance claim is \$2,600.]

20S1-6. Consider the probability distribution whose probability density function is

$$f(x) = \begin{cases} 1 - |x| & \text{if } -1 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Use the method of complementary random numbers with two uniform random numbers, 0.096 and 0.569, to estimate the mean of this distribution.

22S1-7. Consider the probability distribution whose probability density function is

$$f(x) = \begin{cases} \dfrac{3}{2}x^2 & \text{if } -1 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Use the method of complementary random numbers with two uniform random numbers, 0.096 and 0.569, to estimate the mean of this distribution.

20S1-8. The probability distribution of the number of heads in 3 flips of a fair coin is the binomial distribution with $n = 3$ and $p = \frac{1}{2}$, so that

$$P\{X = k\} = \binom{3}{k}\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^{3-k} = \frac{3!}{k!(3-k)!}\left(\frac{1}{2}\right)^3 \qquad \text{for } k = 0, 1, 2, 3.$$

The mean is 1.5.

(a) Obtaining uniform random numbers as instructed at the beginning of the Problems section for Chap. 20, use the inverse transformation method to generate three random observations from this distribution, and then calculate the sample average to estimate the mean.

(b) Use the method of complementary random numbers [with the same uniform random numbers as in part (*a*)] to estimate the mean.

(c) Obtaining uniform random numbers as instructed at the beginning of the Problems section for Chap. 20, simulate repeatedly flipping a coin in order to generate three random observations from this distribution, and then calculate the sample average to estimate the mean.

(d) Repeat part (*c*) with the method of complementary random numbers [with the same uniform random numbers as in part (*c*)] to estimate the mean.

20S1-9. Reconsider Prob. 20.6-4. Suppose now that more careful statistical analysis has provided new estimates of the probability distributions of the radii of the shafts and bushings. In actuality, the probability distribution of the radius of a shaft (in inches) has the probability density function

$$f_s(x) = \begin{cases} 400e^{-400(x-1.0000)} & \text{if } x \geq 1.0000 \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, the probability distribution of the radius of a bushing (in inches) has the probability density function

$$f_B(x) = \begin{cases} 100 & \text{if } 1.0000 \leq x \leq 1.0100 \\ 0 & \text{otherwise.} \end{cases}$$

Obtaining uniform random numbers as instructed at the beginning of the Problems section for Chap. 20, perform a simulated experiment for estimating the probability of interference. Notice that almost all cases of interference will occur when the radius of the bushing is much closer to 1.0000 inch than to 1.0100 inches. Therefore, it appears that an efficient experiment would generate most of the simulated bushings from this critical portion of the distribution. Take this observation into account in part *(b)*. For each of the following cases, use the same 10 pairs of uniform random numbers to generate random observations, and calculate the resulting estimate of the probability of interference.

(a) Use the crude Monte Carlo technique.

(b) Develop and apply a stratified sampling approach to this problem.

(c) Use the method of complementary random numbers.