CHAPTER

1 Problem Solving

Spotlight on Teaching

Excerpts from NCTM's Standards for School Mathematics Prekindergarten through Grade 12*

Problem solving can and should be used to help students develop fluency with specific skills. For example, consider the following problem, which is adapted from the *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989, p. 24):

> I have pennies, nickels, and dimes in my pocket. If I take three coins out of my pocket, how much money could I have taken?

This problem leads children to adopt a trial-and-error strategy. They can also act out the problem by using real coins. Children verify that their answers meet the problem conditions. Follow-up questions can also be posed: "Is it possible for me to have 4 cents? 11 cents? Can you list all the possible amounts I can have when I pick three coins?" The last question provides a challenge for older or more mathematically sophisticated children and requires them to make an organized list, perhaps like the one shown here.

Pennies	Nickels	Dimes	Total Value
0	0	3	30
0	1	2	25
0	2	1	20
0	3	0	15
1	0	2	21
⋮	⋮	⋮	⋮

Working on this problem offers good practice in addition skills. But the important mathematical goal of this problem—helping students to think systematically about possibilities and to organize and record their thinking—need not wait until students can add fluently.

* *Principles and Standards for School Mathematics* (Reston, VA: National Council of Teachers of Mathematics, 2000), p. 52.

MATH ACTIVITY 1.1

Peg-Jumping Puzzle

Virtual Manipulatives

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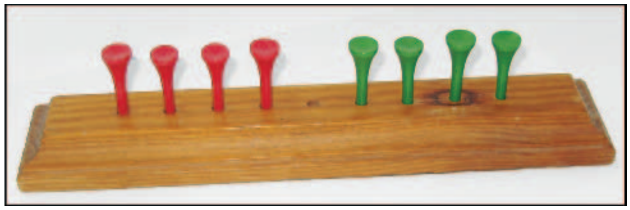
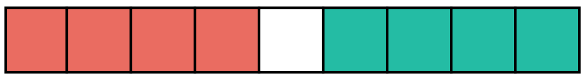
Materials: Colored tiles in the Manipulative Kit or Virtual Manipulatives.

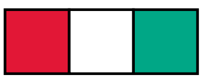


Puzzle: There are four movable red pegs in the holes at one end of a board, four movable green pegs in the holes at the other end, and one empty hole in the center. The challenge is to interchange the pegs so that the red pegs occupy the positions of the green pegs and vice versa, in the fewest moves. Here are the legal moves: Any peg can move to an adjacent empty hole, pegs do not move backward, and a peg of one color can jump over a single peg of another color if there is a hole to jump into.

1. **Using a model:** Sketch nine 1- by 1-inch squares and place four red tiles on the left end and four green tiles on the right. Try solving this problem by moving the tiles according to the rules.



2. **Solving a simpler problem:** Sketch three squares and use one red tile and one green tile to solve this simpler problem. Then sketch five squares and solve the problem with two tiles of each color.



*3. **Making a table:** Sketch the following table and record the minimum number of moves and your strategy when there are three tiles on each side. For example, with one tile on each end you may have moved the red tile first (**R**), then jumped that with the green (**G**), and finally moved the red (**R**). So your strategy could be recorded **RGR.**

Tiles (Pegs) on a Side	Minimum Number of Moves	Strategy
1	3	RGR
2	8	RGGRRGGR
3		

4. **Finding patterns:** You may have noticed one or more patterns in your table. List at least one pattern in your strategies. There is also a pattern in the numbers of moves. Try finding this pattern and predict the number of moves for four tiles on a side. Then test the strategy for solving the Peg Puzzle with four tiles on a side.

*5. **Extending patterns:** Use one of the patterns you discovered to predict the fewest number of moves for solving the puzzle with five or more pegs on each side.

*Answer is given in answer section at back of book.

Section 1.1 INTRODUCTION TO PROBLEM SOLVING

There is no more significant privilege than to release the creative power of a child's mind.
Franz F. Hohn

PROBLEM OPENER

Alice counted 7 cycle riders and 19 cycle wheels going past her house. How many tricycles were there?

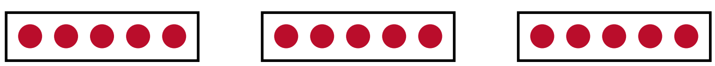
NCTM Standards

Problem solving is the hallmark of mathematical activity and a major means of developing mathematical knowledge. p. 116

"Learning to solve problems is the principal reason for studying mathematics."* This statement by the National Council of Supervisors of Mathematics represents a widespread opinion that problem solving should be the central focus of the mathematics curriculum.

A **problem** exists when there is a situation you want to resolve but no solution is readily apparent. **Problem solving** is the process by which the unfamiliar situation is resolved. A situation that is a problem to one person may not be a problem to someone else. For example, determining the number of people in 3 cars when each car contains 5 people may be a problem to some elementary school students. They might solve this problem by placing chips in boxes or by making a drawing to represent each car and each person (Figure 1.1) and then counting to determine the total number of people.

Figure 1.1



You may be surprised to know that some problems in mathematics are unsolved and have resisted the efforts of some of the best mathematicians to solve them. One such

*National Council of Supervisors of Mathematics, *Essential Mathematics for the 21st Century* (Minneapolis, MN: Essential Mathematics Task Force, 1988).

NCTM Standards

Doing mathematics involves discovery. Conjecture—that is, informed guessing—is a major pathway to discovery. Teachers and researchers agree that students can learn to make, refine, and test conjectures in elementary school. p. 57

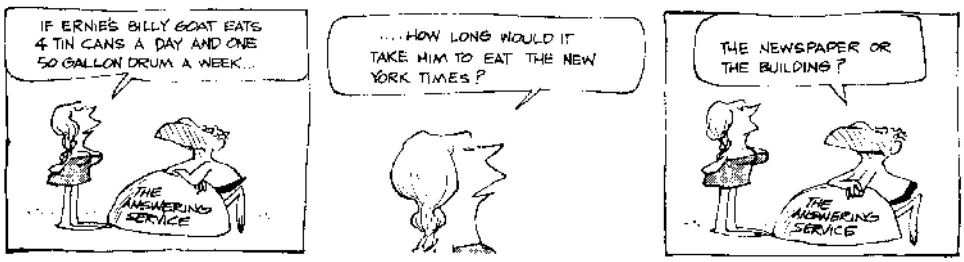
problem was discovered by Arthur Hamann, a seventh-grade student. He noticed that every even number could be written as the difference of two primes.* For example,

$$2 = 5 - 3 \qquad 4 = 11 - 7 \qquad 6 = 11 - 5 \qquad 8 = 13 - 5 \qquad 10 = 13 - 3$$

After showing that this was true for all even numbers less than 250, he predicted that every even number could be written as the difference of two primes. No one has been able to prove or disprove this statement. When a statement is thought to be true but remains unproved, it is called a **conjecture.**

Problem solving is the subject of a major portion of research and publishing in mathematics education. Much of this research is founded on the problem-solving writings of George Polya, one of the foremost twentieth-century mathematicians. Polya devoted much of his teaching to helping students become better problem solvers. His book *How to Solve It* has been translated into 18 languages. In this book, he outlines the following four-step process for solving problems.

Understanding the Problem Polya suggests that a problem solver needs to become better acquainted with a problem and work toward a clearer understanding of it before progressing toward a solution. Increased understanding can come from rereading the statement of the problem, drawing a sketch or diagram to show connections and relationships, restating the problem in your own words, or making a reasonable guess at the solution to help become acquainted with the details.



Sometimes the main difficulty in solving a problem is knowing what question is to be answered.

Devising a Plan The path from understanding a problem to devising a plan may sometimes be long. Most interesting problems do not have obvious solutions. Experience and practice are the best teachers for devising plans. Throughout the text you will be introduced to strategies for devising plans to solve problems.

Carrying Out the Plan The plan gives a general outline of direction. Write down your thinking so your steps can be retraced. Is it clear that each step has been done correctly? Also, it's all right to be stuck, and if this happens, it is sometimes better to put aside the problem and return to it later.

Looking Back When a result has been reached, verify or check it by referring to the original problem. In the process of reaching a solution, other ways of looking at the problem may become apparent. Quite often after you become familiar with a problem, new or perhaps more novel approaches may occur to you. Also, while solving a problem, you may find other interesting questions or variations that are worth exploring.

*M. R. Frame, "Hamann's Conjecture," *Arithmetic Teacher* 23, no. 1 (January 1976): 34–35.

I will learn to draw a diagram to solve problems.

24.3

Problem Solving: Strategy

Draw a Diagram

Read

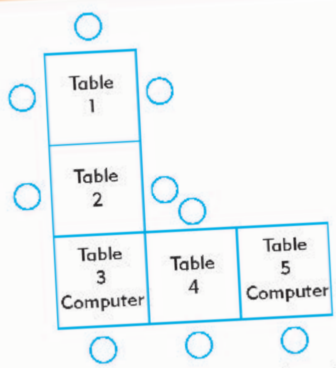
The two new computers for your classroom have finally arrived! You can help design the layout for the new computer center in your room. There are 5 tables available. At least 8 students have to be able to sit in the center at a time. Only 1 person can sit at a table with a computer. All the tables must be touching at least one other table. If one person can sit on each side of the other tables, how would you design the space?

- **What do you know?**
  5 tables available; must seat at least 8 students; 1 person at a table with a computer
- **What do you need to find?**
  How to arrange the tables

Plan

To solve, draw a diagram to represent the tables and chairs. Use squares for tables and circles for chairs.

Solve



Using the arrangement in this diagram, there are 9 seats available.

Look Back

How could you solve this problem in a different way?
Sample answer: If the computer is moved from table 5 to table 2 you would have room for 10 students

1. **Write About It** **Explain** how making a diagram helps you solve the problem.
Sample answer: The diagram helps me see different ways the tables and chairs can be arranged.

Polya's problem-solving steps will be used throughout the text. The purpose of this section is to help you become familiar with the four-step process and to acquaint you with some of the common strategies for solving problems: *making a drawing, guessing and checking, making a table, using a model,* and *working backward.* Additional strategies will be introduced throughout the text.

MAKING A DRAWING

One of the most helpful strategies for understanding a problem and obtaining ideas for a solution is to *draw sketches and diagrams.* Most likely you have heard the expression "A picture is worth a thousand words." In the following problem, the drawings will help you to think through the solution.

Problem

For his wife's birthday, Mr. Jones is planning a dinner party in a large recreation room. There will be 22 people, and in order to seat them he needs to borrow card tables, the size that seats one person on each side. He wants to arrange the tables in a rectangular shape so that they will look like one large table. What is the smallest number of tables that Mr. Jones needs to borrow?

NCTM Standards

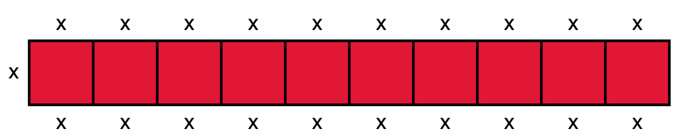
Of the many descriptions of problem-solving strategies, some of the best known can be found in the work of Polya (1957). Frequently cited strategies include using diagrams, looking for patterns, listing all possibilities, trying special values or cases, working backward, guessing and checking, creating an equivalent problem, and creating a simpler problem. p. 53

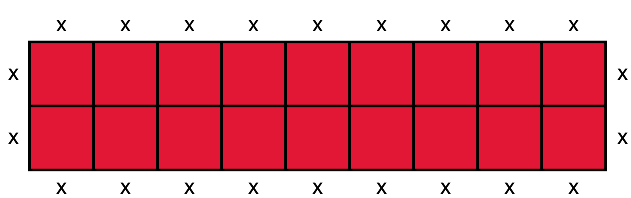
Understanding the Problem The tables must be placed next to each other, edge to edge, so that they form one large rectangular table. **Question 1:** If two tables are placed end to end, how many people can be seated?

One large table

Devising a Plan Drawing pictures of the different arrangements of card tables is a natural approach to solving this problem. There are only a few possibilities. The tables can be placed in one long row; they can be placed side by side with two abreast; etc. **Question 2:** How many people can be seated at five tables if they are placed end to end in a single row?

Carrying Out the Plan The following drawings show two of the five possible arrangements that will seat 22 people. The X's show that 22 people can be seated in each arrangement. The remaining arrangements—3 by 8, 4 by 7, and 5 by 6—require 24, 28, and 30 card tables, respectively. **Question 3:** What is the smallest number of card tables needed?





Looking Back The drawings show that a single row of tables requires the fewest tables because each end table has places for 3 people and each of the remaining tables has places for 2 people. In all the other arrangements, the corner tables seat only 2 people and the remaining tables seat only 1 person. Therefore, regardless of the number of people, a single row is the arrangement that uses the smallest number of card tables, provided the room is long enough. **Question 4:** What is the smallest number of card tables required to seat 38 people?

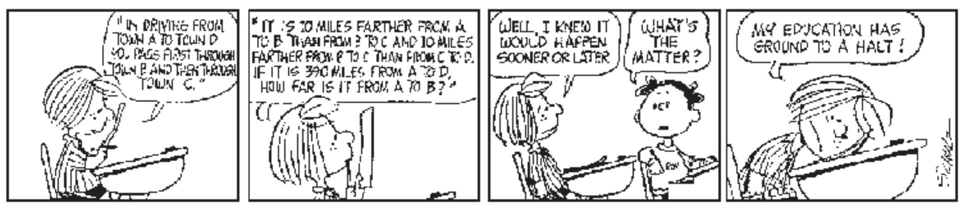
Answers to Questions 1–4 **1.** 6 **2.** 12 **3.** 10 **4.** There will be 3 people at each end table and 32 people in between. Therefore, 2 end tables and 16 tables in between will be needed to seat 38 people.

GUESSING AND CHECKING

Sometimes it doesn't pay to guess, as illustrated by the bus driver in this cartoon. However, many problems can be better understood and even solved by trial-and-error procedures. As Polya said, "Mathematics in the making consists of guesses." If your first guess is off, it may lead to a better guess. Even if guessing doesn't produce the correct answer, you may increase your understanding of the problem and obtain an idea for solving it. The *guess-and-check* approach is especially appropriate for elementary schoolchildren because it puts many problems within their reach.

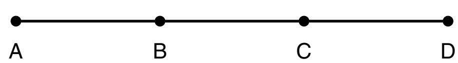
Problem

How far is it from town *A* to town *B* in this cartoon?

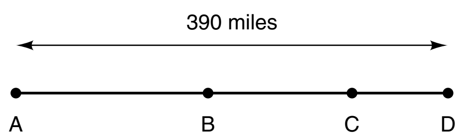


Peanuts: © United Feature Syndicate, Inc.

Understanding the Problem There are several bits of information in this problem. Let's see how Peppermint Patty could have obtained a better understanding of the problem with a diagram. First, let us assume these towns lie in a straight line, so they can be illustrated by points *A*, *B*, *C*, and *D*, as shown in (a). Next, it is 10 miles farther from *A* to *B* than from *B* to *C*, so we can move point *B* closer to point *C*, as in (b). It is also 10 miles farther from *B* to *C* than from *C* to *D*, so point *C* can be moved closer to point *D*. Finally, the distance from *A* to *D* is given as 390 miles. **Question 1:** The problem requires finding what distance?



(a)



(b)

NCTM Standards

Problem solving is not a distinct topic, but a process that should permeate the study of mathematics and provide a context in which concepts and skills are learned. p. 182

Devising a Plan One method of solving this problem is to make a reasonable guess and then use the result to make a better guess. If the 4 towns were equally spaced, as in (a), the distance between each town would be 130 miles ($390 \div 3$). However, the distance from town *A* to town *B* is the greatest. So let's begin with a guess of 150 miles for the distance from *A* to *B*. **Question 2:** In this case, what is the distance from *B* to *C* and *C* to *D*?

Carrying Out the Plan Using a guess of 150 for the distance from *A* to *B* produces a total distance from *A* to *D* that is greater than 390. If the distance from *A* to *B* is 145, then the *B*-to-*C* distance is 135 and the *C*-to-*D* distance is 125. The sum of these distances is 405, which is still too great. **Question 3:** What happens if we use a guess of 140 for the distance from *A* to *B*?

Looking Back One of the reasons for *looking back* at a problem is to consider different solutions or approaches. For example, you might have noticed that the first guess, which produced a distance of 420 miles, was 30 miles too great. **Question 4:** How can this observation be used to lead quickly to a correct solution of the original problem?

Answers to Questions 1–4 **1.** The problem requires finding the distance from *A* to *B*. **2.** The *B*-to-*C* distance is 140, and the *C*-to-*D* distance is 130. **3.** If the *A*-to-*B* distance is 140, then the *B*-to-*C* distance is 130 and the *C*-to-*D* distance is 120. Since the total of these distances is 390, the correct distance from *A* to *B* is 140 miles. **4.** If the distance between each of the 3 towns is decreased by 10 miles, the incorrect distance of 420 will be decreased to the correct distance of 390. Therefore, the distance between town *A* and town *B* is 140 miles.

MAKING A TABLE

A problem can sometimes be solved by listing some of or all the possibilities. A *table* is often convenient for organizing such a list.

Problem

Sue and Ann earned the same amount of money, although one worked 6 days more than the other. If Sue earned $36 per day and Ann earned $60 per day, how many days did each work?

Understanding the Problem Answer a few simple questions to get a feeling for the problem. **Question 1:** How much did Sue earn in 3 days? Did Sue earn as much in 3 days as Ann did in 2 days? Who worked more days?

Devising a Plan One method of solving this problem is to list each day and each person's total earnings through that day. **Question 2:** What is the first amount of total pay that is the same for Sue and Ann, and how many days did it take each to earn this amount?

Carrying Out the Plan The complete table is shown on page 9. There are three amounts in Sue's column that equal amounts in Ann's column. It took Sue 15 days to earn $540. **Question 3:** How many days did it take Ann to earn $540, and what is the difference between the numbers of days they each required?

Technology Connection

Four-Digit Numbers

If any four-digit number is selected and its digits reversed, will the sum of these two numbers be divisible by 11? Use your calculator to explore this and similar questions in this investigation.

Mathematics Investigation
Chapter 1, Section 1
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Number of Days	Sue's Pay	Ann's Pay
1	36	60
2	72	120
3	108	180
4	144	240
5	180	300
6	216	360
7	252	420
8	288	480
9	324	540
10	360	600
11	396	660
12	432	720
13	468	780
14	504	840
15	540	900

Looking Back You may have noticed that every 5 days Sue earns \$180 and every 3 days Ann earns \$180. **Question 4:** How does this observation suggest a different way to answer the original question?

Answers to Questions 1–4 **1.** Sue earned \$108 in 3 days. Sue did not earn as much in 3 days as Ann did in 2 days. Sue must have worked more days than Ann to have earned the same amount. **2.** \$180. It took Sue 5 days to earn \$180, and it took Ann 3 days to earn \$180. **3.** It took Ann 9 days to earn \$540, and the difference between the numbers of days Sue and Ann worked is 6. **4.** When Sue has worked 10 days and Ann has worked 6 days (a difference of 4 days), each has earned \$360; when they have worked 15 days and 9 days (a difference of 6 days), respectively, each has earned \$540.

USING A MODEL

Models are important aids for visualizing a problem and suggesting a solution. The recommendations by the Committee on the Undergraduate Program in Mathematics (CUPM) contain frequent references to the use of models for illustrating number relationships and geometric properties.*

The next problem uses **whole numbers** 0, 1, 2, 3, . . . and is solved by *using a model.* It involves a well-known story about the German mathematician Karl Gauss. When Gauss was 10 years old, his schoolmaster gave him the problem of computing the sum of whole numbers from 1 to 100. Within a few moments the young Gauss wrote the answer on his slate and passed it to the teacher. Before you read the solution to the following problem, try to find a quick method for computing the sum of whole numbers from 1 to 100.

*Committee on the Undergraduate Program in Mathematics, *Recommendations on the Mathematical Preparation of Teachers* (Berkeley, CA: Mathematical Association of America, 1983).

Problem

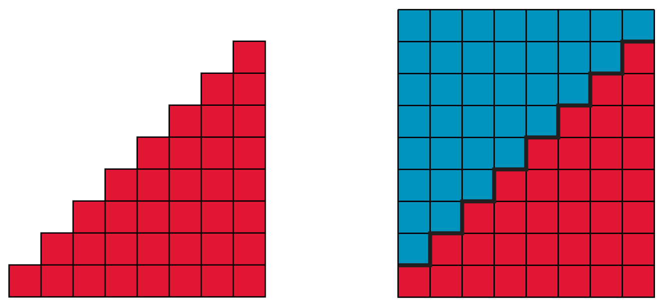
Find an easy method for computing the sum of consecutive whole numbers from 1 to any given number.

Understanding the Problem If the last number in the sum is 8, then the sum is $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8$. If the last number in the sum is 100, then the sum is $1 + 2 + 3 + \cdots + 100$. **Question 1:** What is the sum of whole numbers from 1 to 8?

Devising a Plan One method of solving this problem is to cut staircases out of graph paper. The one shown in (a) is a 1-through-8 staircase: There is 1 square in the first step, there are 2 squares in the second step, and so forth, to the last step, which has a column of 8 squares. The total number of squares is the sum $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8$. By using two copies of a staircase and placing them together, as in (b), we can obtain a rectangle whose total number of squares can easily be found by multiplying length by width. **Question 2:** What are the dimensions of the rectangle in (b), and how many small squares does it contain?

NCTM Standards

Problem solving is not only a goal of learning mathematics but also a major means of doing so. p. 52

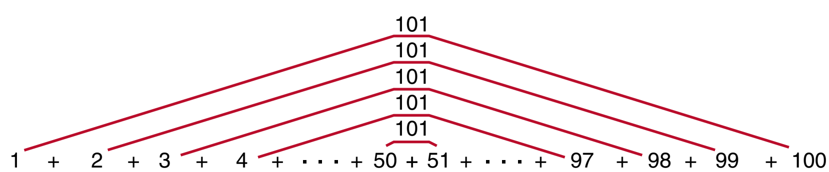


1-through-8 staircase (a)

Two 1-through-8 staircases (b)

Carrying Out the Plan Cut out two copies of the 1-through-8 staircase and place them together to form a rectangle. Since the total number of squares is 8×9, the number of squares in one of these staircases is $(8 \times 9)/2 = 36$. So the sum of whole numbers from 1 to 8 is 36. By placing two staircases together to form a rectangle, we see that the number of squares in one staircase is just half the number of squares in the rectangle. This geometric approach to the problem suggests that the sum of consecutive whole numbers from 1 to any specific number is the product of the last number and the next number, divided by 2. **Question 3:** What is the sum of whole numbers from 1 to 100?

Looking Back Another approach to computing the sum of whole numbers from 1 to 100 is suggested by the following diagram, and it may have been the method used by Gauss. If the numbers from 1 to 100 are paired as shown, the sum of each pair of numbers is 101.



Question 4: How can this sum be used to obtain the sum of whole numbers from 1 to 100?

Answers to Questions 1–4 **1.** 36 **2.** The dimensions are 8 by 9, and there are $8 \times 9 = 72$ small squares. **3.** Think of combining two 1-through-100 staircases to obtain a rectangle with 100×101 squares. The sum of whole numbers from 1 to 100 is $100(101)/2 = 5050$. **4.** Since there are 50 pairs of numbers and the sum for each pair is 101, the sum of numbers from 1 to 100 is $50 \times 101 = 5050$.

Hypatia, 370–415

HISTORICAL HIGHLIGHT

Athenaeus, a Greek writer (ca. 200), in his book *Deipnosophistoe* mentions a number of women who were superior mathematicians. However, Hypatia in the fourth century is the first woman in mathematics of whom we have considerable knowledge. Her father, Theon, was a professor of mathematics at the University of Alexandria and was influential in her intellectual development, which eventually surpassed his own. She became a student of Athens at the school conducted by Plutarch the Younger, and it was there that her fame as a mathematician became established. Upon her return to Alexandria, she accepted an invitation to teach mathematics at the university. Her contemporaries wrote about her great genius. Socrates, the historian, wrote that her home as well as her lecture room was frequented by the most unrelenting scholars of the day. Hypatia was the author of several treatises on mathematics, but only fragments of her work remain. A portion of her original treatise *On the Astronomical Canon of Diophantus* was found during the fifteenth century in the Vatican library. She also wrote *On the Conics of Apollonius.* She invented an astrolabe and a planesphere, both devices for studying astronomy, and apparatuses for distilling water and determining the specific gravity of water.*

*L. M. Osen, *Women in Mathematics* (Cambridge, MA: MIT Press, 1974), pp. 21–32.

WORKING BACKWARD

Problem

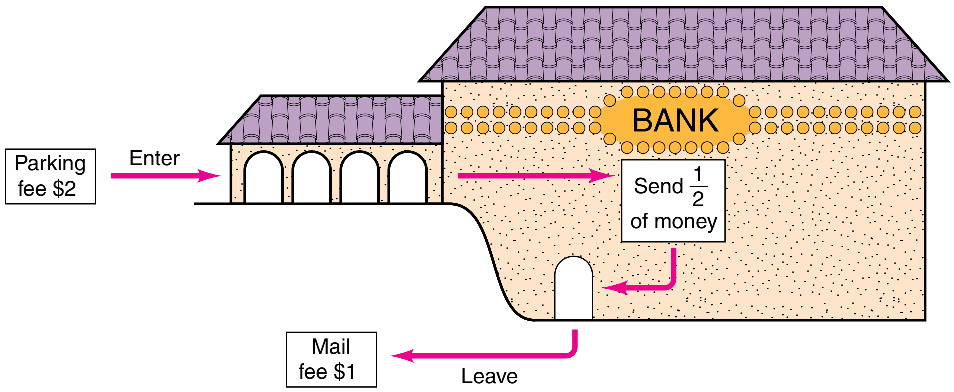
A businesswoman went to the bank and sent half of her money to a stockbroker. Other than a $2 parking fee before she entered the bank and a $1 mail fee after she left the bank, this was all the money she spent. On the second day she returned to the bank and sent half of her remaining money to the stockbroker. Once again, the only other expenses were the $2 parking fee and the $1 mail fee. If she had $182 left, how much money did she have before the trip to the bank on the first day?

Understanding the Problem Let's begin by guessing the original amount of money, say, $800, to get a better feel for the problem. **Question 1:** If the businesswoman begins the day with $800, how much money will she have at the end of the first day, after paying the mail fee?

Devising a Plan Guessing the original amount of money is one possible strategy, but it requires too many computations. Since we know the businesswoman has $182 at the end of the second day, a more appropriate strategy for solving the problem is to retrace her steps back through the bank (see the following diagram). First she receives $1 back from the mail fee. Continue to work back through the second day in the bank. **Question 2:** How much money did the businesswoman have at the beginning of the second day?

NCTM Standards

The goal of school mathematics should be for all students to become increasingly able and willing to engage with and solve problems. p. 182



Carrying Out the Plan The businesswoman had $368 at the beginning of the second day. Continue to work backward through the first day to determine how much money she had at the beginning of that day. **Question 3:** What was this amount?

Looking Back You can now check the solution by beginning with $740, the original amount of money, and going through the expenditures for both days to see if $182 is the remaining amount. The problem can be varied by replacing $182 at the end of the second day by any amount and working backward to the beginning of the first day. **Question 4:** For example, if there was $240 at the end of the second day, what was the original amount of money?

Answers to Questions 1–4 **1.** $398 **2.** The following diagram shows that the businesswoman had $368 at the beginning of the second day.

End of day 2			Beginning of day 2
$182 →	$183 →	$366 →	$368
	Receive $1 mail fee	Receive 1/2 of money sent	Receive $2 parking fee

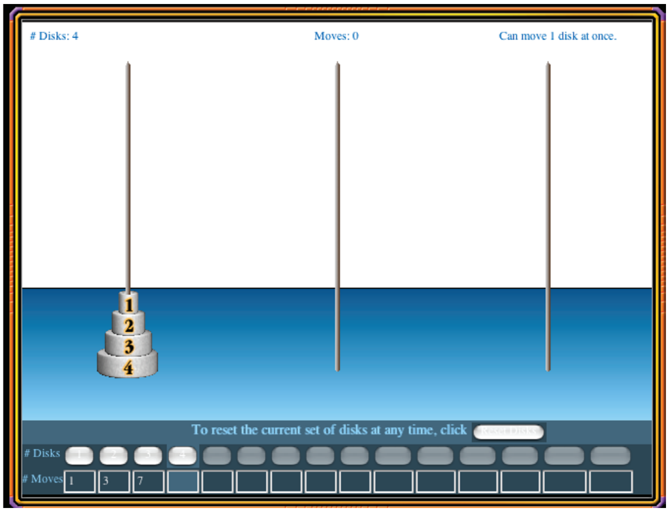
3. The diagram shows that the businesswoman had $740 at the beginning of the first day, so this is the original amount of money.

End of day 1			Beginning of day 1
$368 →	$369 →	$738 →	$740
	Receive $1 mail fee	Receive 1/2 of money sent	Receive $2 parking fee

4. $972

Technology Connection

What is the least number of moves to transfer four disks from one tower to another if only one disk can be moved at a time and a disk cannot be placed on top of a smaller disk? In this applet, you will solve an ancient problem by finding patterns to determine the minimum number of moves for transferring an arbitrary number of disks.



Tower Puzzle Patterns Applet, Chapter 1
www.mhhe.com/bennett-nelson

Exercises and Problems 1.1

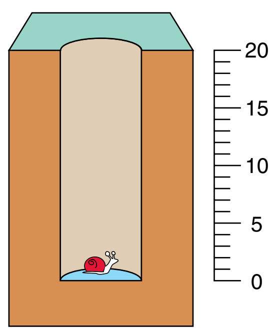
Problems 1 through 20 involve strategies that were presented in this section. Some of these problems are analyzed by Polya's four-step process. See if you can solve these problems before answering parts a, b, c, and d. Other strategies may occur to you, and you are encouraged to use the ones you wish. Often a good problem requires several strategies.

Making a Drawing (1–4)

1. A well is 20 feet deep. A snail at the bottom climbs up 4 feet each day and slips back 2 feet each night. How many days will it take the snail to reach the top of the well?

a. Understanding the Problem What is the greatest height the snail reaches during the first 24 hours? How far up the well will the snail be at the end of the first 24 hours?

b. Devising a Plan One plan that is commonly chosen is to compute $20/2$, since it appears that the snail gains 2 feet each day. However, 10 days is not the correct answer. A second plan is to *make a drawing* and plot the snail's daily progress. What is the snail's greatest height during the second day?



c. Carrying Out the Plan Trace out the snail's daily progress, and mark its position at the end of each day. On which day does the snail get out of the well?

d. Looking Back There is a "surprise ending" at the top of the well because the snail does not slip back on the ninth day. Make up a new snail problem by changing the numbers so that there will be a similar surprise ending at the top of the well.

PS **2.** Five people enter a racquetball tournament in which each person must play every other person exactly once. Determine the total number of games that will be played.

PS **3.** When two pieces of rope are placed end to end, their combined length is 130 feet. When the two pieces are placed side by side, one is 26 feet longer than the other. What are the lengths of the two pieces?

PS **4.** There are 560 third- and fourth-grade students in King Elementary School. If there are 80 more third-graders than fourth-graders, how many third-graders are there in the school?

Making a Table (5–8)

PS **5.** A bank that has been charging a monthly service fee of \$2 for checking accounts plus 15 cents for each check announces that it will change its monthly fee to \$3 and that each check will cost 8 cents. The bank claims the new plan will save the customer money. How many checks must a customer write per month before the new plan is cheaper than the old plan?

a. Understanding the Problem Try some numbers to get a feel for the problem. Compute the cost of 10 checks under the old plan and under the new plan. Which plan is cheaper for a customer who writes 10 checks per month?

b. Devising a Plan One method of solving this problem is to make a table showing the cost of 1 check, 2 checks, etc., such as that shown here. How much more does the new plan cost than the old plan for 6 checks?

Checks	Cost for Old Plan, \$	Cost for New Plan, \$
1	2.15	3.08
2	2.30	3.16
3	2.45	3.24
4	2.60	3.32
5	2.75	3.40
6		
7		
8		

c. Carrying Out the Plan Extend the table until you reach a point at which the new plan is cheaper than the old plan. How many checks must be written per month for the new plan to be cheaper?

d. Looking Back For customers who write 1 check per month, the difference in cost between the old plan and the new plan is 93 cents. What happens to the difference as the number of checks increases? How many checks must a customer write per month before the new plan is 33 cents cheaper?

PS **6.** Sasha and Francisco were selling lemonade for 25 cents per half cup and 50 cents per full cup. At the end of the day they had collected \$15 and had used 37 cups. How many full cups and how many half cups did they sell?

PS **7.** Harold wrote to 15 people, and the cost of postage was \$4.71. If it cost 23 cents to mail a postcard and 37 cents to mail a letter, how many postcards did he write?

PS **8.** I had some pennies, nickels, dimes, and quarters in my pocket. When I reached in and pulled out some change, I had less than 10 coins whose value was 42 cents. What are all the possibilities for the coins I had in my hand?

Guessing and Checking (9–12)

PS **9.** There are two 2-digit numbers that satisfy the following conditions: (1) Each number has the same digits, (2) the sum of the digits in each number is 10, and (3) the difference between the 2 numbers is 54. What are the two numbers?

a. Understanding the Problem The numbers 58 and 85 are 2-digit numbers that have the same digits, and the sum of the digits in each number is 13. Find two 2-digit numbers such that the sum of the digits is 10 and both numbers have the same digits.

b. Devising a Plan Since there are only nine 2-digit numbers whose digits have a sum of 10, the problem can be easily solved by guessing. What is the difference of your two 2-digit numbers from part a? If this difference is not 54, it can provide information about your next guess.

c. Carrying Out the Plan Continue to guess and check. Which pair of numbers has a difference of 54?

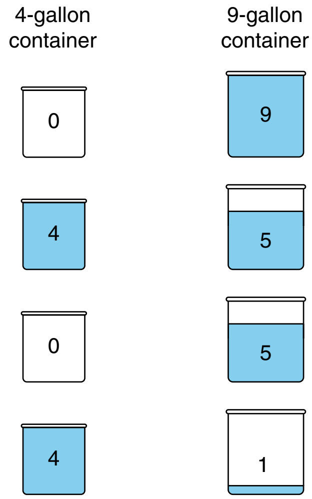
d. Looking Back This problem can be extended by changing the requirement that the sum of the two digits equals 10. Solve the problem for the case in which the digits have a sum of 12.

PS **10.** When two numbers are multiplied, their product is 759; but when one is subtracted from the other, their difference is 10. What are these two numbers?

11. When asked how a person can measure out 1 gallon of water with only a 4-gallon container and a 9-gallon container, a student used this “picture.”

a. Briefly describe what the student could have shown by this sketch.

b. Use a similar sketch to show how 6 gallons can be measured out by using these same containers.

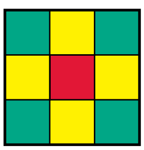


12. Carmela opened her piggy bank and found she had $15.30. If she had only nickels, dimes, quarters, and half-dollars and an equal number of coins of each kind, how many coins in all did she have?

Using a Model (13–16)

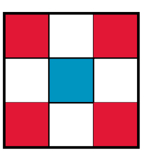
13. Suppose that you have a supply of red, blue, green, and yellow square tiles. What is the fewest number of different colors needed to form a 3×3 square of tiles so that no tile touches another tile of the same color at any point?

a. Understanding the Problem Why is the square arrangement of tiles shown here not a correct solution?



b. Devising a Plan One plan is to choose a tile for the center of the grid and then place others around it so that no two of the same color touch. Why must the center tile be a different color than the other eight tiles?

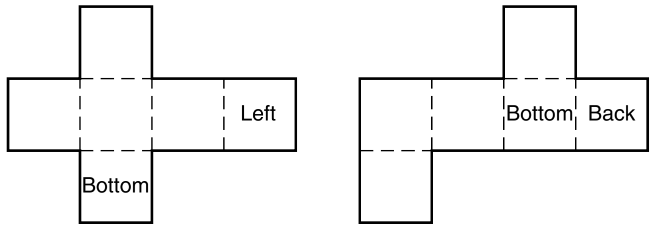
c. Carrying Out the Plan Suppose that you put a blue tile in the center and a red tile in each corner, as shown here. Why will it require two more colors for the remaining openings?



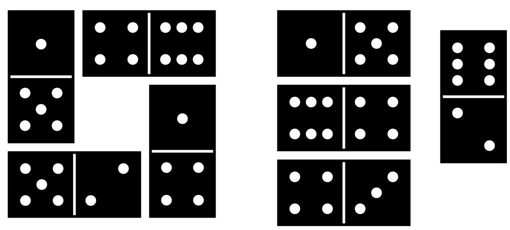
d. Looking Back Suppose the problem had asked for the smallest number of colors to form a square of nine tiles so that no tile touches another tile of the same color along an entire edge. Can it be done in fewer colors; if so, how many?

14. What is the smallest number of different colors of tile needed to form a 4×4 square so that no tile touches another of the same color along an entire edge?

15. The following patterns can be used to form a cube. A cube has six faces: the top and bottom faces, the left and right faces, and the front and back faces. Two faces have been labeled on each of the following patterns. Label the remaining four faces on each pattern so that when the cube is assembled with the labels on the outside, each face will be in the correct place.



16. At the left in the following figure is a domino doughnut with 11 dots on each side. Arrange the four single dominoes on the right into a domino doughnut so that all four sides have 12 dots.



Domino doughnut

Working Backward (17–20)

17. Three girls play three rounds of a game. On each round there are two winners and one loser. The girl who loses on a round has to double the number of chips that each of the other girls has by giving up some of her own chips. Each girl loses one round. At the end of three rounds, each girl has 40 chips. How many chips did each girl have at the beginning of the game?

a. Understanding the Problem Let’s select some numbers to get a feel for this game. Suppose girl *A*, girl *B*, and girl *C* have 70, 30, and 20 chips, respectively, and girl *A* loses the first round. Girl *B* and girl *C* will receive chips from girl *A*, and thus their supply of chips will be doubled. How many chips will each girl have after this round?

b. Devising a Plan Since we know the end result (each girl finished with 40 chips), a natural strategy is to work backward through the three rounds to the beginning. Assume that girl *C* loses the third round. How many chips did each girl have at the end of the second round?

	A	*B*	*C*
Beginning			
End of first round			
End of second round			
End of third round	40	40	40

c. Carrying Out the Plan Assume that girl *B* loses the second round and girl *A* loses the first round. Continue working back through the three rounds to determine the number of chips each of the girls had at the beginning of the game.

d. Looking Back Check your answer by working forward from the beginning. The girl with the most chips at the beginning of this game lost the first round. Could the girl with the fewest chips at the beginning of the game have lost the first round? Try it.

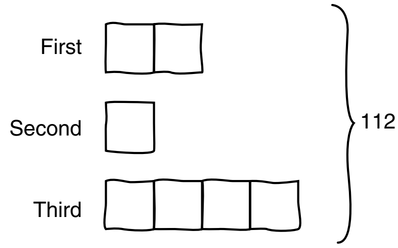
PS **18.** Sue Ellen and Angela both have $510 in their savings accounts now. They opened their accounts on the same day, at which time Sue Ellen started with $70 more than Angela. From then on Sue Ellen added $10 to her account each week, and Angela put in $20 each week. How much money did Sue Ellen open her account with?

PS **19.** Ramon took a collection of colored tiles from a box. Amelia took 13 tiles from his collection, and Keiko took half of those remaining. Ramon had 11 left. How many did he start with?

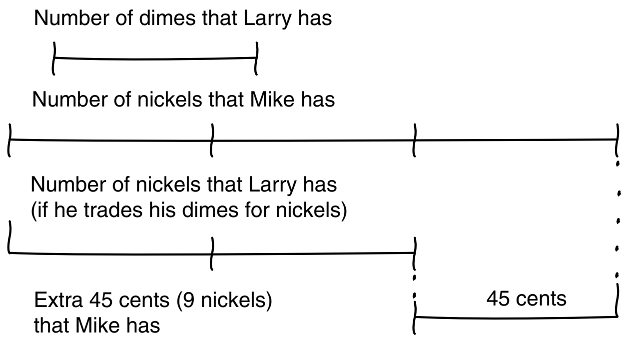
PS **20.** Keiko had 6 more red tiles than yellow tiles. She gave half of her red tiles to Amelia and half of her yellow tiles to Ramon. If Ramon has 7 yellow tiles, how many tiles does Keiko have?

Each of problems 21 to 24 is accompanied by a sketch or diagram that was used by a student to solve it. Describe how you think the student used the diagram, and use this method to solve the problem.

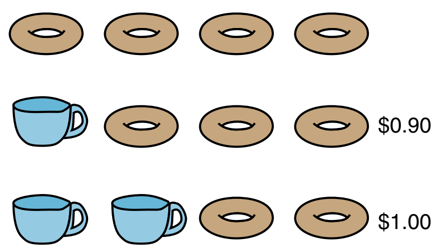
PS **21.** There are three numbers. The first number is twice the second number. The third is twice the first number. Their sum is 112. What are the numbers?



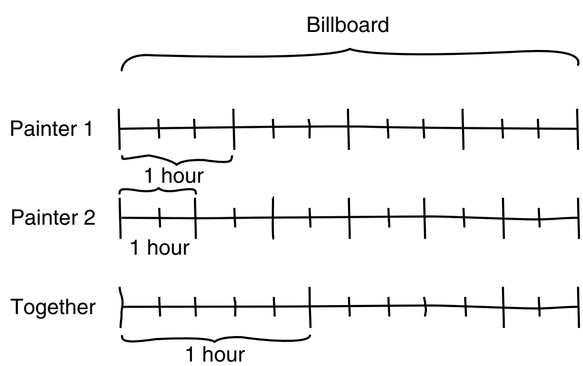
PS **22.** Mike has 3 times as many nickels as Larry has dimes. Mike has 45 cents more than Larry. How much money does Mike have?



PS **23.** At Joe's Cafe 1 cup of coffee and 3 doughnuts cost $0.90, and 2 cups of coffee and 2 doughnuts cost $1.00. What is the cost of 1 cup of coffee? 1 doughnut?



PS **24.** One painter can letter a billboard in 4 hours and another requires 6 hours. How long will it take them together to letter the billboard?



Problems 25 through 34 can be solved by using strategies presented in this section. While you are problem-solving, try to record the strategies you are using. If you are using a strategy different from those of this section, try to identify and record it.

25. There were ships with 3 masts and ships with 4 masts at the Tall Ships Exhibition. Millie counted a total of 30 masts on the 8 ships she saw. How many of these ships had 4 masts?

26. When a teacher counted her students in groups of 4, there were 2 students left over. When she counted them in groups of 5, she had 1 student left over. If 15 of her students were girls and she had more girls than boys, how many students did she have?

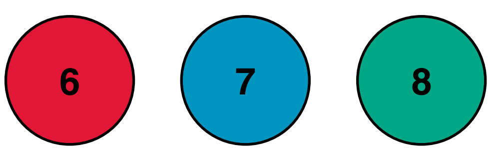
27. The video club to which Lin belongs allows her to receive a free movie video for every three videos she rents. If she pays $3 for each movie video and paid $132 over a 4-month period, how many free movie videos did she obtain?

28. Linda picked a basket of apples. She gave half of the apples to a neighbor, then 8 apples to her mother, then half of the remaining apples to her best friend, and she kept the 3 remaining apples for herself. How many apples did she start with in the basket?

29. Four people want to cross the river. There is only one boat available, and it can carry a maximum of 200 pounds. The weights of the four people are 190, 170, 110, and 90 pounds. How can they all manage to get across the river, and what is the minimum number of crossings required for the boat?

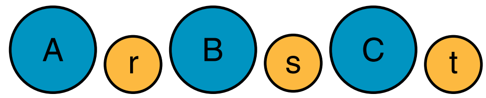
30. A farmer has to get a fox, a goose, and a bag of corn across a river in a boat that is only large enough for her and one of these three items. She does not want to leave the fox alone with the goose nor the goose alone with the corn. How can she get all these items across the river?

31. Three circular cardboard disks have numbers written on the front and back sides. The front sides have the numbers shown here.

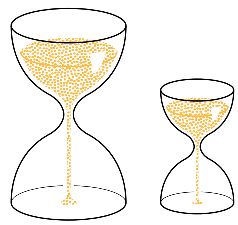


By tossing all three disks and adding the numbers that show face up, we can obtain these totals: 15, 16, 17, 18, 19, 20, 21, and 22. What numbers are written on the back sides of these disks?

32. By moving adjacent disks two at a time, you can change the arrangement of large and small disks shown below to an arrangement in which 3 big disks are side by side followed by the 3 little disks. Describe the steps.



33. How can a chef use an 11-minute hourglass and a 7-minute hourglass to time vegetables that must steam for 15 minutes?



34. The curator of an art exhibit wants to place security guards along the four walls of a large auditorium so that each wall has the same number of guards. Any guard who is placed in a corner can watch the two adjacent walls, but each of the other guards can watch only the wall by which she or he is placed.

a. Draw a sketch to show how this can be done with 6 security guards.

b. Show how this can be done for each of the following numbers of security guards: 7, 8, 9, 10, 11, and 12.

c. List all the numbers less than 100 that are solutions to this problem.

35. Trick questions like the following are fun, and they can help improve problem-solving ability because they require that a person listen and think carefully about the information and the question.

a. Take 2 apples from 3 apples, and what do you have?

b. A farmer had 17 sheep, and all but 9 died. How many sheep did he have left?

c. I have two U.S. coins that total 30 cents. One is not a nickel. What are the two coins?

d. A bottle of cider costs 86 cents. The cider costs 60 cents more than the bottle. How much does the bottle cost?

e. How much dirt is in a hole 3 feet long, 2 feet wide, and 2 feet deep?

f. A hen weighs 3 pounds plus half its weight. How much does it weigh?

g. There are nine brothers in a family and each brother has a sister. How many children are in the family?

h. Which of the following expressions is correct? (1) The whites of the egg are yellow. (2) The whites of the egg is yellow.

Writing and Discussion

1. Suppose one of your elementary school students was having trouble solving the following problem and asked for help: "Tauna gave half of her marbles away. If she gave some to her sister and twice as many to her brother, and had 6 marbles left, how many marbles did she give to her brother?" List a few suggestions you can give to this student to help her solve this problem.

2. When an elementary schoolteacher who had been teaching problem solving introduced the strategy of *making a drawing,* one of her students said that he was not good at drawing. Give examples of three problems you can give this student that would illustrate that artistic ability is not required. Accompany the problems with solution sketches.

3. In years past, it was a common practice for teachers to tell students not to draw pictures or sketches because "you can't prove anything with drawings." Today it is common for teachers to encourage students to form sketches to solve problems. Discuss this change in approach to teaching mathematics. Give examples of advantages and disadvantages of solving problems by *making drawings.*

4. At one time, teachers scolded students for guessing the answers to problems. In recent years, mathematics educators have recommended that *guessing and checking* be taught to school students. Write a few sentences to discuss the advantages of teaching students to "guess and check." Include examples of problems for which this strategy may be helpful.

5. Write a definition of what it means for a question to involve "problem solving." Create a problem that is appropriate for middle school students and explain how it satisfies your definition of problem solving.

Making Connections

1. The **Spotlight on Teaching** at the beginning of Chapter 1 poses the following problem: I have pennies, nickels, and dimes in my pocket. If I take three coins out of my pocket, how much money could I have taken? The solution in this spotlight involves *forming a table.* Explain and illustrate how a different organized list can lead to a solution by noting that the greatest value of the coins is 30 cents and the least value is 3 cents.

2. On page 5, the example from the **Elementary School Text** poses a problem and solves it by the strategy of *making a drawing.* Then it asks the students how the problem can be solved in a different way. (a) Find another solution that has seats for exactly 8 students. (b) Name a strategy from the **Standards** quote on page 6 that is helpful in solving part (a) and explain why the strategy is helpful.

3. The **Standards** quote on page 8 says that problem solving should "provide a context in which concepts and skills are learned." Explain how the staircase model, page 10, provides this context.

4. In the **Process Standard on Problem Solving** (see inside front cover), read the fourth expectation and explain several ways in which Polya's fourth problem-solving step addresses the fourth expectation.

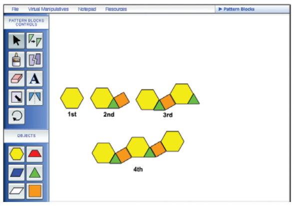
5. The **Historical Highlight** on page 11 has some examples of the accomplishments of Hypatia, one of the first women mathematicians. Learn more about her by researching history of math books or searching the Internet. Record some interesting facts or anecdotes about Hypatia that you could use to enhance your elementary school teaching.

6. The **Problem Opener** for this section, on page 3, says that "Alice counted 7 cycle riders and 19 cycle wheels" and it asks for the number of tricycles. Use one or more of the problem-solving strategies in this section to find all the different answers that are possible if the riders might have been using unicycles, bicycles, or tricycles.

MATH ACTIVITY 1.2

Pattern Block Sequences

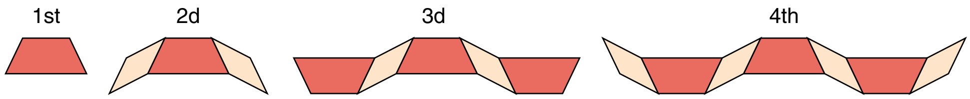
Virtual Manipulatives



www.mhhe.com/bennett-nelson

Materials: Pattern block pieces in the Manipulative Kit or Virtual Manipulatives.

1. Here are the first four pattern block figures of a sequence composed of trapezoids (red) and parallelograms (tan).



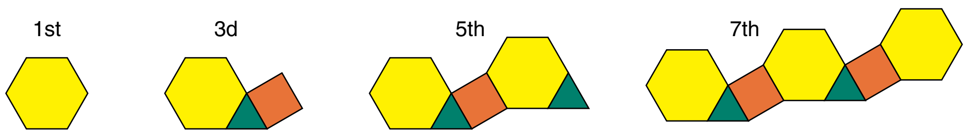
***a.** Find a pattern and use your pattern blocks to build a fifth figure. Sketch this figure.

***b.** If the pattern is continued, how many trapezoids and parallelograms will be in the 10th figure?

c. What pattern blocks are on each end of the 35th figure in the sequence, and how many of each shape are in that figure?

d. Determine the total number of pattern blocks in the 75th figure, and write an explanation describing how you reached your conclusion.

2. Figures 1, 3, 5, and 7 are shown from a sequence using hexagons, squares, and triangles.

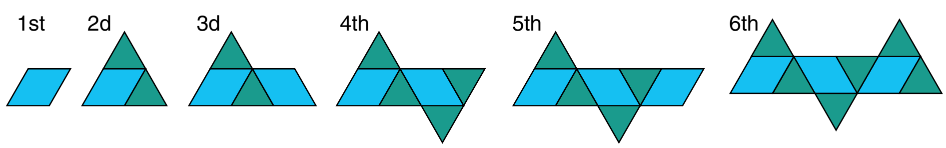


a. Find a pattern and use your pattern blocks to build the eighth and ninth figures.

***b.** Write a description of the 20th figure.

c. Write a description of the 174th, 175th, and 176th figures, and include the number of hexagons, squares, and triangles in each.

3. Use your pattern blocks to build figures 8 and 9 of the following sequence.



***a.** Describe the pattern by which you extend the sequence. Determine the number of triangles and parallelograms in the 20th figure.

b. How many pattern blocks are in the 45th figure?

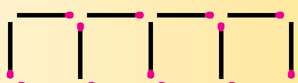
c. The fifth figure in the sequence has 7 pattern blocks. What is the number of the figure that has 87 blocks? Explain your reasoning.

Section 1.2 PATTERNS AND PROBLEM SOLVING

The graceful winding arms of the majestic spiral galaxy M51 look like a winding spiral staircase sweeping through space. This sharpest-ever image of the Whirlpool Galaxy was captured by the *Hubble Space Telescope* in January 2005 and released on April 24, 2005, to mark the 15th anniversary of Hubble's launch.

PROBLEM OPENER

This matchstick track has 4 squares. If the pattern of squares is continued, how many matches will be needed to build a track with 60 squares?



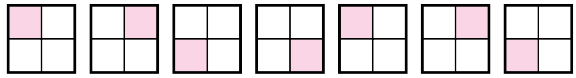
Patterns play a major role in the solution of problems in all areas of life. Psychologists analyze patterns of human behavior; meteorologists study weather patterns; astronomers seek patterns in the movements of stars and galaxies; and detectives look for patterns among clues. Finding a pattern is such a useful problem-solving strategy in mathematics that some have called it the *art of mathematics.*

To find patterns, we need to compare and contrast. We must compare to find features that remain constant and contrast to find those that are changing. Patterns appear in many forms. There are number patterns, geometric patterns, word patterns, and letter patterns, to name a few. Try finding a pattern in each of the following sequences, and write or sketch the next term.

EXAMPLE A 1, 2, 4,

Solution One possibility: Each term is twice the previous term. The next term is 8.

EXAMPLE B



Solution One possibility: In each block of four squares, one square is shaded. The upper left, upper right, lower left, and lower right corners are shaded in order. The next term in this sequence has the shaded block in the lower right corner.

EXAMPLE C

Al, Bev, Carl, Donna

Solution One possibility: The first letters of the names are consecutive letters of the alphabet. The next name begins with E.

NCTM Standards

Historically, much of the mathematics used today was developed to model real-world situations, with the goal of making predictions about those situations. Students in grades 3–5 develop the idea that a mathematical model has both descriptive and predictive power. p. 162

Finding a pattern requires making educated guesses. You are guessing the pattern based on some observation, and a different observation may lead to another pattern. In Example A, the difference between the first and second terms is 1, and the difference between the second and third terms is 2. So using differences between consecutive terms as the basis of the pattern, we would have a difference of 3 between the third and fourth terms, and the fourth term would be 7 rather than 8. In Example C, we might use the pattern of alternating masculine and feminine names or of increasing numbers of letters in the names.

PATTERNS IN NATURE

The spiral is a common pattern in nature. It is found in spiderwebs, seashells, plants, animals, weather patterns, and the shapes of galaxies. The frequent occurrence of spirals in living things can be explained by different growth rates. Living forms curl because the faster-growing (longer) surface lies outside and the slower growing (shorter) surface lies inside. An example of a living spiral is the shell of the mollusk chambered nautilus (Figure 1.2). As it grows, the creature lives in successively larger compartments.

Figure 1.2
Chambered nautilus

A variety of patterns occur in plants and trees. Many of these patterns are related to a famous sequence of numbers called the **Fibonacci numbers.** After the first two numbers of this sequence, which are 1 and 1, each successive number can be obtained by adding the two previous numbers.

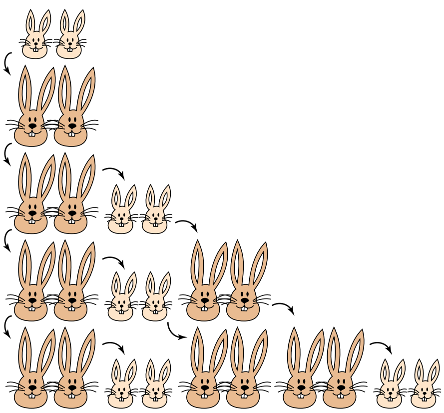
$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

NCTM Standards

Initially, students may describe the regularity in patterns verbally rather than with mathematical symbols (English and Warren 1998). In grades 3–5, they can begin to use variables and algebraic expressions as they describe and extend patterns. p. 38

The seeds in the center of a daisy are arranged in two intersecting sets of spirals, one turning clockwise and the other turning counterclockwise. The number of spirals in each set is a Fibonacci number. Also, the number of petals will often be a Fibonacci number. The daisy in Figure 1.3 has 21 petals.

Figure 1.3



HISTORICAL HIGHLIGHT

Fibonacci numbers were discovered by the Italian mathematician Leonardo Fibonacci (ca. 1175–1250) while studying the birthrates of rabbits. Suppose that a pair of baby rabbits is too young to produce more rabbits the first month, but produces a pair of baby rabbits every month thereafter. Each new pair of rabbits will follow the same rule. The pairs of rabbits for the first 5 months are shown here. The numbers of pairs of rabbits for the first 5 months are the Fibonacci numbers 1, 1, 2, 3, 5. If this birthrate pattern is continued, the numbers of pairs of rabbits in succeeding months will be Fibonacci numbers. The realization that Fibonacci numbers could be applied to the science of plants and trees occurred several hundred years after the discovery of this number sequence.

NUMBER PATTERNS

Number patterns have fascinated people since the beginning of recorded history. One of the earliest patterns to be recognized led to the distinction between **even numbers**

0, 2, 4, 6, 8, 10, 12, 14, . . .

and **odd numbers**

1, 3, 5, 7, 9, 11, 13, 15, . . .

The game Even and Odd has been played for generations. To play this game, one person picks up some stones, and a second person guesses whether the number of stones is odd or even. If the guess is correct, the second person wins.

NCTM Standards

The recognition, comparison, and analysis of patterns are important components of a student's intellectual development. p. 91

Pascal's Triangle The triangular pattern of numbers shown in Figure 1.4 is Pascal's triangle. It has been of interest to mathematicians for hundreds of years, appearing in China as early as 1303. This triangle is named after the French mathematician Blaise Pascal (1623–1662), who wrote a book on some of its uses.

Figure 1.4

```
Row 0            1
Row 1          1   1
Row 2        1   2   1
Row 3      1   3   3   1
Row 4    1   4   6   4   1
```

EXAMPLE D

1. Find a pattern that might explain the numbering of the rows as 0, 1, 2, 3, etc.
2. In the fourth row, each of the numbers 4, 6, and 4 can be obtained by adding the two adjacent numbers from the row above it. What numbers are in the fifth row of Pascal's triangle?

Solution **1.** Except for row 0, the second number in each row is the number of the row. **2.** 1, 5, 10, 10, 5, 1

Arithmetic Sequence Sequences of numbers are often generated by patterns. The sequences 1, 2, 3, 4, 5, . . . and 2, 4, 6, 8, 10, . . . are among the first that children learn. In such sequences, each new number is obtained from the previous number in the sequence by adding a selected number throughout. This selected number is called the **common difference,** and the sequence is called an **arithmetic sequence.**

EXAMPLE E

7, 11, 15, 19, 23, . . .

172, 256, 340, 424, 508, . . .

The first arithmetic sequence has a common difference of 4. What is the common difference for the second sequence? Write the next three terms in each sequence.

Solution The next three terms in the first sequence are 27, 31, and 35. The common difference for the second sequence is 84, and the next three terms are 592, 676, and 760.

Geometric Sequence In a geometric sequence, each new number is obtained by multiplying the previous number by a selected number. This selected number is called the **common ratio,** and the resulting sequence is called a **geometric sequence.**

EXAMPLE F

3, 6, 12, 24, 48, . . .

1, 5, 25, 125, 625, . . .

The common ratio in the first sequence is 2. What is the common ratio in the second sequence? Write the next two terms in each sequence.

Solution The next two terms in the first sequence are 96 and 192. The common ratio for the second sequence is 5, and the next two terms are 3125 and 15,625.

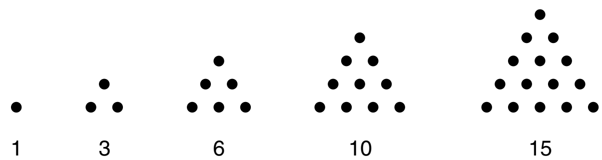
Technology Connection

Triangular Numbers
There is an interesting pattern in the units digits of the triangular numbers 1, 3, 6, 10, 15, ... Look for this pattern and others using the *Mathematics Investigator* software to quickly gather and display data for the triangular numbers.

Mathematics Investigation Chapter 1, Section 2 www.mhhe.com/bennett.nelson

Triangular Numbers The sequence of numbers illustrated in Figure 1.5 is neither arithmetic nor geometric. These numbers are called **triangular numbers** because of the arrangement of dots that is associated with each number. Since each triangular number is the sum of whole numbers beginning with 1, the formula for the sum of consecutive whole numbers can be used to obtain triangular numbers.*

Figure 1.5



EXAMPLE G

The first triangular number is 1, and the fifth triangular number is 15. What is the sixth triangular number?

Solution The sixth triangular number is 21.

HISTORICAL HIGHLIGHT

Karl Friedrich Gauss, 1777–1855

Archimedes, Newton, and the German mathematician Karl Friedrich Gauss are considered to be the three greatest mathematicians of all time. Gauss exhibited a cleverness with numbers at an early age. The story is told that at age 3, as he watched his father making out the weekly payroll for laborers of a small bricklaying business, Gauss pointed out an error in the computation. Gauss enjoyed telling the story later in life and joked that he could figure before he could talk. Gauss kept a mathematical diary, which contained records of many of his discoveries. Some of the results were entered cryptically. For example,

$$\text{Num} = \Delta + \Delta + \Delta$$

is an abbreviated statement that every whole number greater than zero is the sum of three or fewer triangular numbers.**

**H. W. Eves, *In Mathematical Circles* (Boston: Prindle, Weber, and Schmidt, 1969), pp. 111–115.

There are other types of numbers that receive their names from the numbers of dots in geometric figures (see 28–30 in Exercises and Problems 1.2). Such numbers are called **figurate numbers,** and they represent one kind of link between geometry and arithmetic.

*The online 1.2/Investigation, Triangular numbers, *Mathematics* prints sequences of triangular numbers.

I will find a pattern to solve a problem.

10.6 Problem Solving: Strategy

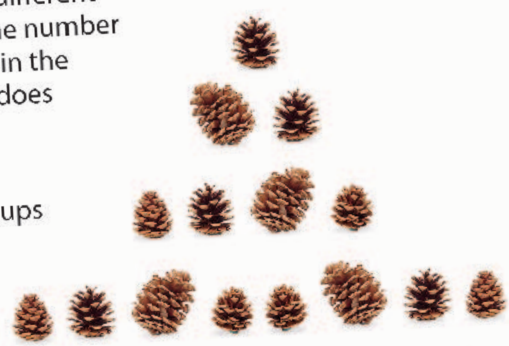
Find a Pattern

VOCABULARY
pattern

Read

Read the problem carefully.

Tamara arranged pine cones into different groups to look like this photo. If the number in each row continues to increase in the same way, how many pine cones does she put in the eighth group?



- **What do you know?**
  The number in the first four groups
- **What are you asked to find?**
  The number to put in the eighth group

Plan

One way to solve the problem is to find a **pattern**. Look for the function that describes how the number of pine cones in each group relates to the number that comes before it.

Solve

The number of pine cones doubles each time.

1 →(x 2)→ 2 →(x 2)→ 4 →(x 2)→

Continue the pattern to find how many pine cones will be in the eighth group.

1 →(x 2)→ 2 →(x 2)→ 4 →(x 2)→ 8 →(x 2)→ 16 →(x 2)→ 32 →(x 2)→ 64 →(x 2)→ 128

Tamara puts 128 pine cones in the eighth group.

Look Back

Is there another way to describe the pattern above?

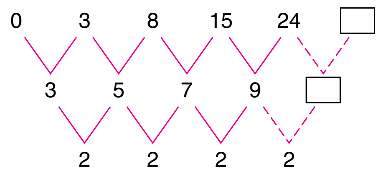
1. **Write About It** **Generalize** What other strategies could you use to solve this problem?

Finite Differences Often sequences of numbers don't appear to have a pattern. However, sometimes number patterns can be found by looking at the differences between consecutive terms. This approach is called the method of **finite differences.**

EXAMPLE H

Consider the sequence 0, 3, 8, 15, 24, ... Find a pattern and determine the next term.

Solution Using the method of finite differences, we can obtain a second sequence of numbers by computing the differences between numbers from the original sequence, as shown below. Then a third sequence is obtained by computing the differences from the second sequence. The process stops when all the numbers in the sequence of differences are equal. In this example, when the sequence becomes all 2s, we stop and work our way back from the bottom row to the original sequence. Assuming the pattern of 2s continues, the next number after 9 is 11, so the next number after 24 is 35.



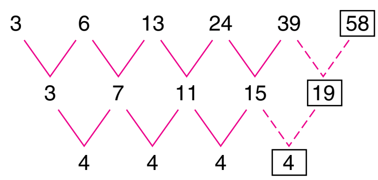
EXAMPLE I

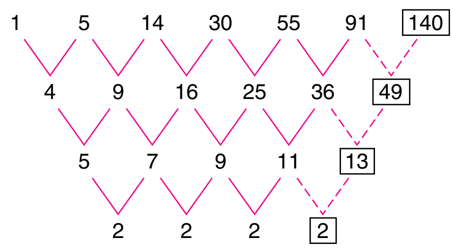
Use the method of finite differences to determine the next term in each sequence.

1. 3, 6, 13, 24, 39

2. 1, 5, 14, 30, 55, 91

Solution **1.** The next number in the sequence is 58. **2.** The next number in the sequence is 140.





INDUCTIVE REASONING

The process of forming conclusions on the basis of patterns, observations, examples, or experiments is called **inductive reasoning.**

NCTM Standards

Identifying patterns is a powerful problem-solving strategy. It is also the essence of inductive reasoning. As students explore problem situations appropriate to their grade level, they can often consider or generate a set of specific instances, organize them, and look for a pattern. These, in turn, can lead to conjectures about the problem.*

* *Curriculum and Evaluation Standards for School Mathematics* (Reston, VA: National Council of Teachers of Mathematics, 1989), p. 82.

EXAMPLE J

Each of these sums of three consecutive whole numbers is divisible by 3.

$$4 + 5 + 6 = 15 \qquad 2 + 3 + 4 = 9 \qquad 7 + 8 + 9 = 24$$

If we conclude, on the basis of these sums, that the sum of any three consecutive whole numbers is divisible by 3, we are using inductive reasoning.

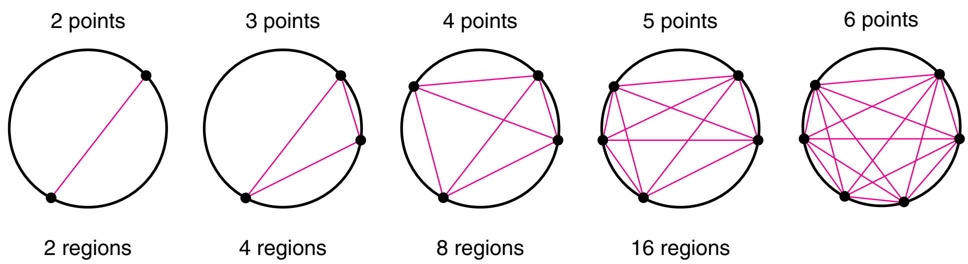
Inductive reasoning can be thought of as making an "informed guess." Although this type of reasoning is important in mathematics, it sometimes leads to incorrect results.

EXAMPLE K

Consider the number of regions that can be obtained in a circle by connecting points on the circumference of the circle. Connecting 2 points produces 2 regions, connecting 3 points produces 4 regions, and so on. Each time a new point on the circle is used, the number of regions appears to double.

NCTM Standards

Because many elementary and middle school tasks rely on inductive reasoning, teachers need to be aware that students might develop an incorrect expectation that patterns always generalize in ways that would be expected on the basis of the regularities found in the first few terms. p. 265



The numbers of regions in the circles shown here are the beginning of the geometric sequence 2, 4, 8, 16, . . . , and it is tempting to conclude that 6 points will produce 32 regions. However, no matter how the 6 points are located on the circle, there will not be more than 31 regions.

Counterexample An example that shows a statement to be false is called a **counterexample.** If you have a general statement, test it to see if it is true for a few special cases. You may be able to find a counterexample to show that the statement is not true, or that a conjecture cannot be proved.

EXAMPLE L

Find two whole numbers for which the following statement is false: The sum of any two whole numbers is divisible by 2.

Solution It is not true for 7 and 4, since $7 + 4 = 11$, and 11 is not divisible by 2. There are pairs of whole numbers for which the statement is true. For example, $3 + 7 = 10$, and 10 is divisible by 2. However, the counterexample of the sum of 7 and 4 shows that the statement is not true for all pairs of whole numbers.

Counterexamples can help us to restate a conjecture. The statement in Example L is false, but if it is changed to read "The sum of two odd numbers is divisible by 2," it becomes a true statement.

EXAMPLE M

For which of the following statements is there a counterexample? If a statement is false, change a condition to produce a true statement.

1. The sum of any four whole numbers is divisible by 2.
2. The sum of any two even numbers is divisible by 2.
3. The sum of any three consecutive whole numbers is divisible by 2.

Solution **1.** The following counterexample shows that statement 1 is false: $4 + 12 + 6 + 3 = 25$, which is not divisible by 2. If the condition "four whole numbers" is replaced by "four even numbers," the statement becomes true. **2.** Statement 2 is true. **3.** The following counterexample shows that statement 3 is false: $8 + 9 + 10 = 27$, which is not divisible by 2. If the condition "three consecutive whole numbers" is replaced by "three consecutive whole numbers beginning with an odd number," the statement becomes true.

Leaning Tower of Pisa
Pisa, Italy.

HISTORICAL HIGHLIGHT

Aristotle (384–322 B.C.), Greek scientist and philosopher, believed that heavy objects fall faster than lighter ones, and this principle was accepted as true for hundreds of years. Then in the sixteenth century, Galileo produced a counterexample by dropping two pieces of metal from the Leaning Tower of Pisa. In spite of the fact that one was twice as heavy as the other, both hit the ground at the same time.

PROBLEM-SOLVING APPLICATION

The strategies of **solving a simpler problem** and **finding a pattern** are introduced in the following problem. Simplifying a problem or solving a related but easier problem can help in understanding the given information and devising a plan for the solution. Sometimes the numbers in a problem are large or inconvenient, and finding a solution for smaller numbers can lead to a plan or reveal a pattern for solving the original problem. Read this problem and try to solve it. Then read the following four-step solution and compare it to your solution.

Problem

There are 15 people in a room, and each person shakes hands exactly once with everyone else. How many handshakes take place?

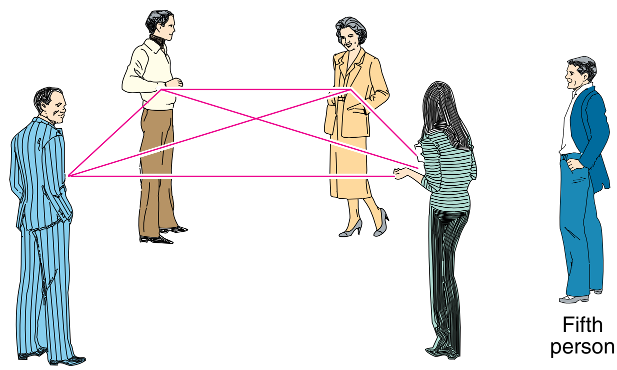
Understanding the Problem For each pair of people, there will be 1 handshake. For example, if Sue and Paul shake hands, this is counted as 1 handshake. Thus, the problem is to determine the total number of different ways that 15 people can be paired. **Question 1:** How many handshakes will occur when 3 people shake hands?

NCTM Standards

During grades 3–5, students should be involved in an important transition in their mathematical reasoning. Many students begin this grade band believing that something is true because it has occurred before, because they have seen examples of it, or because their experience to date seems to confirm it. During these grades, formulating conjectures and assessing them on the basis of evidence should become the norm. p. 188

Devising a Plan Fifteen people are a lot of people to work with at one time. Let's simplify the problem and count the number of handshakes for small groups of people. Solving these special cases may give us an idea for solving the original problem. **Question 2:** What is the number of handshakes in a group of 4 people?

Carrying Out the Plan We have already noted that there is 1 handshake for 2 people, 3 handshakes for 3 people, and 6 handshakes for 4 people. The following figure illustrates how 6 handshakes will occur among 4 people. Suppose a fifth person joins the group. This person will shake hands with each of the first 4 people, accounting for 4 more handshakes.



Similarly, if we bring in a 6th person, this person will shake hands with the first 5 people, and so there will be 5 new handshakes. Suddenly we can see a pattern developing: The 5th person adds 4 new handshakes, the 6th person adds 5 new handshakes, the 7th person adds 6 new handshakes, and so on until the 15th person adds 14 new handshakes. **Question 3:** How many handshakes will there be for 15 people?

Looking Back By looking at special cases with numbers smaller than 15, we obtained a better understanding of the problem and an insight for solving it. The pattern we found suggests a method for determining the number of handshakes for any number of people: Add the whole numbers from 1 to the number that is 1 less than the number of people. You may recall from Section 1.1 that staircases were used to develop a formula for computing such a sum. **Question 4:** How can this formula be used to determine the number of handshakes for 15 people?

Answers to Questions 1–4 **1.** 3 **2.** 6 **3.** $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14 = 105$ **4.** The sum of whole numbers from 1 to 14 is $(14 \times 15)/2 = 105$.

Exercises and Problems 1.2

NCTM's K–4 Standard *Patterns and Relationships* notes that identifying the *core* of a pattern helps children become aware of the structure.* For example, in some patterns there is a core that repeats, as in exercise 1a. In some patterns there is a core that grows, as in exercise 2b. Classify each of the sequences in 1 and 2 as having a core that repeats or that grows, and determine the next few elements in each sequence.

1. a. ▲+○▲+○▲+○▲+○ . . .

b. ××+×××+×××+× . . .

c. ○∗○○○∗○○○○∗○○○○ . . .

2. a. ┌ ┌ ┐ ┘└ ┌ ┌ ┐ ┘└ . . .

b. 1, 2, 1, 1, 2, 3, 2, 1, 1, 2, 3, 4, 3, 2, 1, . . .

c. 2, 3, 5, 7, 2, 3, 5, 7, 2, 3, 5, 7, . . .

Some sequences have a pattern, but they do not have a core. Determine the next three numbers in each of the sequences in exercises 3 and 4.

3. a. 2, 5, 8, 11, 14, 17, 20, 23, . . .

b. 13, 16, 19, 23, 27, 32, 37, 43, . . .

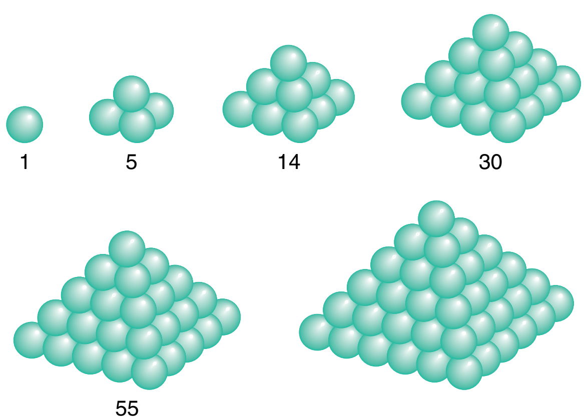
c. 17, 22, 20, 25, 23, 28, 26, 31, . . .

4. a. 31, 28, 25, 22, 19, 16, . . .

b. 46, 48, 50, 54, 58, 64, 70, 78, 86, . . .

c. 43, 46, 49, 45, 41, 44, 47, 43, 39, . . .

One method of stacking cannonballs is to form a pyramid with a square base. The first six such pyramids are shown. Use these figures in exercises 5 and 6.



Curriculum and Evaluation Standards for School Mathematics (Reston, VA: National Council of Teachers of Mathematics, 1989), p. 61.

5. a. How many cannonballs are in the sixth figure?

b. Can the method of finite differences be used to find the number of cannonballs in the sixth figure?

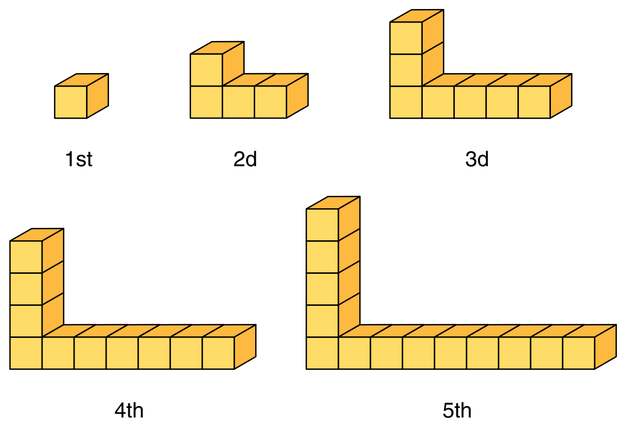
c. Describe the 10th pyramid, and determine the number of cannonballs.

6. a. Describe the seventh pyramid, and determine the number of cannonballs.

b. Do the numbers of cannonballs in successive figures form an arithmetic sequence?

c. Write an expression for the number of cannonballs in the 20th figure. (*Note:* It is not necessary to compute the number.)

Use the following sequence of figures in exercises 7 and 8.



7. a. What type of sequence is formed by the numbers of cubes in successive figures?

b. Describe the 20th figure and determine the number of cubes.

8. a. Can the method of finite differences be used to determine the number of cubes in the 6th figure?

b. Describe the 100th figure and determine the number of cubes.

c. Write an expression for the number of cubes in the nth figure, for any whole number n.

There are many patterns and number relationships that can be easily discovered on a calendar. Some of these patterns are explored in exercises 9 through 11.

NOVEMBER 2002

Sun	Mon	Tue	Wed	Thu	Fri	Sat
					1	2
3	4	5	6	7	8	9
10	11	12	13	14	15	16
17	18	19	20	21	22	23
24	25	26	27	28	29	30

9. The sum of the three circled numbers on the preceding calendar is 45. For any sum of three consecutive numbers (from the rows), there is a quick method for determining the numbers. Explain how this can be done. Try your method to find three consecutive numbers whose sum is 54.

10. If you are told the sum of any three adjacent numbers from a column, it is possible to determine the three numbers. Explain how this can be done, and use your method to find the numbers whose sum is 48.

11. The sum of the 3×3 array of numbers outlined on the preceding calendar is 99. There is a shortcut method for using this sum to find the 3×3 array of numbers. Explain how this can be done. Try using your method to find the 3×3 array with sum 198.

12. Here are the first few Fibonacci numbers: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55. Compute the sums shown below, and compare the answers with the Fibonacci numbers. Find a pattern and explain how this pattern can be used to find the sums of consecutive Fibonacci numbers.

$$1 + 1 + 2 =$$
$$1 + 1 + 2 + 3 =$$
$$1 + 1 + 2 + 3 + 5 =$$
$$1 + 1 + 2 + 3 + 5 + 8 =$$
$$1 + 1 + 2 + 3 + 5 + 8 + 13 =$$
$$1 + 1 + 2 + 3 + 5 + 8 + 13 + 21 =$$

13. The sums of the squares of consecutive Fibonacci numbers form a pattern when written as a product of two numbers.

a. Complete the missing sums and find a pattern.

b. Use your pattern to explain how the sum of the squares of the first few consecutive Fibonacci numbers can be found.

$$1^2 + 1^2 = 1 \times 2$$
$$1^2 + 1^2 + 2^2 = 2 \times 3$$
$$1^2 + 1^2 + 2^2 + 3^2 = 3 \times 5$$
$$1^2 + 1^2 + 2^2 + 3^2 + 5^2 =$$
$$1^2 + 1^2 + 2^2 + 3^2 + 5^2 + 8^2 =$$
$$1^2 + 1^2 + 2^2 + 3^2 + 5^2 + 8^2 + 13^2 =$$

A Fibonacci-type sequence can be started with any two numbers. Then each successive number is formed by adding the two previous numbers. Each number after 3 and 4 in the sequence 3, 4, 7, 11, 18, 29, etc. was obtained by adding the previous two numbers. Find the missing numbers among the first 10 numbers of the Fibonacci-type sequences in exercises 14 and 15.

14. a. 10, ______, 24, ______, ______, 100, ______, ______, ______, 686

b. 2, ______, ______, 16, 25, ______, ______, ______, ______, 280

c. The sum of the first 10 numbers in the sequence in part a is equal to 11 times the seventh number, 162. What is this sum?

d. Can the sum of the first 10 numbers in the sequence in part b be obtained by multiplying the seventh number by 11?

e. Do you think the sum of the first 10 numbers in any Fibonacci-type sequence will always be 11 times the seventh number? Try some other Fibonacci-type sequences to support your conclusion.

15. a. 1, ______, ______, 11, ______, ______, ______, ______, 118, ______

b. 14, ______, 20, 26, ______, ______, 118, ______, ______, 498

c. The sum of the first 10 numbers in part a is equal to 11 times the seventh number. Is this true for the sequence in part b?

d. Is the sum of the first 10 numbers in the Fibonacci sequence equal to 11 times the seventh number in that sequence?

e. Form a conjecture based on your observations in parts c and d.

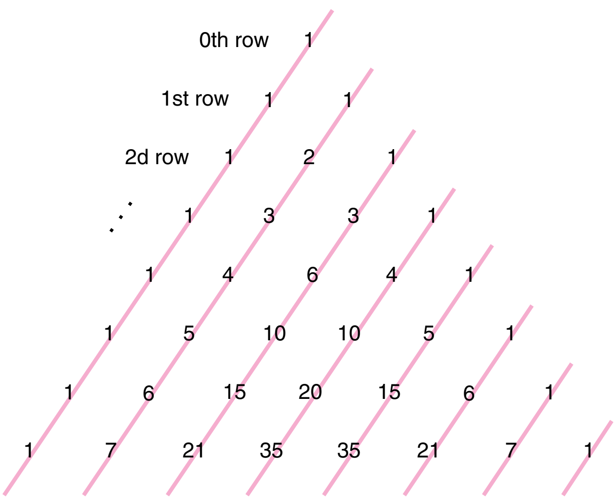
16. The products of 1089 and the first few digits produce some interesting number patterns. Describe one of these patterns. Will this pattern continue if 1089 is multiplied by 5, 6, 7, 8, and 9?

$$1 \times 1089 = 1089$$
$$2 \times 1089 = 2178$$
$$3 \times 1089 = 3267$$
$$4 \times 1089 = 4356$$
$$5 \times 1089 =$$

17. **a.** Find a pattern in the following equations, and use your pattern to write the next equation.
b. If the pattern in the first three equations is continued, what will be the 20th equation?

$$1 + 2 = 3$$
$$4 + 5 + 6 = 7 + 8$$
$$9 + 10 + 11 + 12 = 13 + 14 + 15$$

In Pascal's triangle, which is shown here, there are many patterns. Use this triangle of numbers in exercises 18 through 21.



18. Add the first few numbers in the first diagonal of Pascal's triangle (diagonals are marked by lines), starting from the top. This sum will be another number from the triangle. Will this be true for the sums of the first few numbers in the other diagonals? Support your conclusion with examples.

19. The third diagonal in Pascal's triangle has the numbers 1, 3, 6, ...
a. What is the 10th number in this diagonal?
b. What is the 10th number in the fourth diagonal?

20. Compute the sums of the numbers in the first few rows of Pascal's triangle. What kind of sequence (arithmetic or geometric) do these sums form?

21. What will be the sum of the numbers in the 12th row of Pascal's triangle?

Identify each of the sequences in exercises 22 and 23 as arithmetic or geometric. State a rule for obtaining each number from the preceding number. What is the 12th number in each sequence?

22. **a.** 280, 257, 234, 211, . . .
b. 17, 51, 153, 459, . . .
c. 32, 64, 128, 256, . . .
d. 87, 102, 117, 132, . . .

23. **a.** 4, 9, 14, 19, . . .
b. 15, 30, 60, 120, . . .
c. 24, 20, 16, 12, . . .
d. 4, 12, 36, 108, . . .

The method of finite differences is used in exercises 24 and 25. This method will sometimes enable you to find the next number in a sequence, but not always.

24. **a.** Write the first eight numbers of a geometric sequence, and try using the method of finite differences to find the ninth number. Will this method work?
b. Repeat part a for an arithmetic sequence. Support your conclusions.

25. **a.** Will the method of finite differences produce the next number in the diagonals of Pascal's triangle? Support your conclusions with examples.
b. The sums of the numbers in the first few rows of Pascal's triangle are 1, 2, 4, 8, ... Will the method of finite differences produce the next number in this sequence?

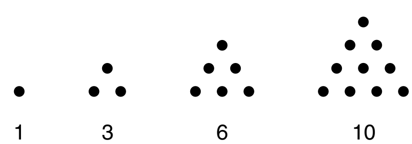
Use the method of finite differences in exercises 26 and 27 to find the next number in each sequence.

26. **a.** 3, 7, 13, 21, 31, 43, . . .
b. 215, 124, 63, 26, 7, . . .

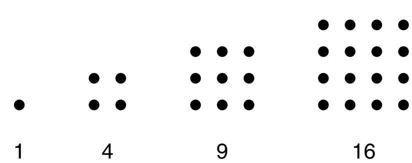
27. **a.** 1, 2, 7, 22, 53, 106, . . .
b. 1, 3, 11, 25, 45, 71, . . .

As early as 500 B.C., the Greeks were interested in numbers associated with patterns of dots in the shape of geometric figures. Write the next three numbers and the 100th number in each sequence in exercises 28 through 30.

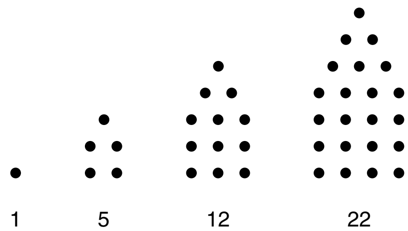
28. Triangular numbers:



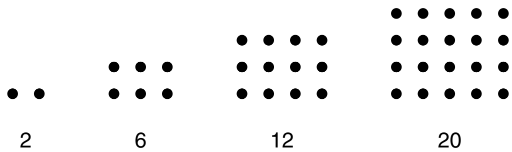
29. Square numbers:



30. Pentagonal numbers. (After the first figure these are five-sided figures composed of figures for triangular numbers and square numbers.)



The Greeks called the numbers represented by the following arrays of dots **oblong numbers.** Use this pattern in exercises 31 and 32.



31. a. What is the next oblong number?
b. What is the 20th oblong number?

32. a. Can the method of finite differences be used to obtain the number of dots in the 5th oblong number?
b. What is the 25th oblong number?

33. The numbers in the following sequence are the first six pentagonal numbers: 1, 5, 12, 22, 35, 51.
a. If the method of finite differences is used, what type of sequence is produced by the first sequence of differences?
b. Can the method of finite differences be used to obtain the next few pentagonal numbers from the first six?

34. Use the method of finite differences to create a new sequence of numbers for the following sequence of square numbers.

1, 4, 9, 16, 25, 36, 49, 64, 81

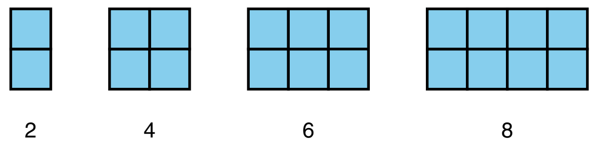
a. What kind of a sequence do you obtain?
b. How can a square arrays of dots (see exercise 29) be used to show that the difference of two consecutive square numbers will be an odd number?

What kind of reasoning is used to arrive at the conclusions in the studies in exercises 35 and 36?

35. In a research study involving 600 people, there was a 30-percent reduction in the severity of colds by using less vitamin C than previously recommended. The researchers concluded that smaller amounts of vitamin C are as effective in reducing colds as large amounts.

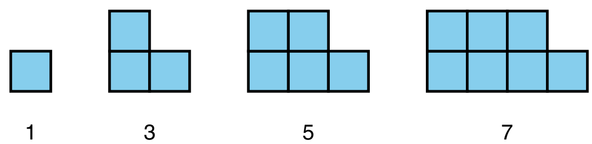
36. A large survey of hospitals found there is an increase in cancer and other disease rates in operating room personnel. The researchers conducting the survey concluded that exposure to anesthetic agents causes health hazards.

37. Continue the pattern of even numbers illustrated here.



a. The fourth even number is 8. Sketch the figure for the ninth even number and determine this number.
b. What is the 45th even number?

38. Continue the pattern of odd numbers illustrated here.



a. The fourth odd number is 7. Sketch the figure for the 12th odd number.
b. What is the 35th odd number?

39. If we begin with the number 6, then double it to get 12, and then place the 12 and 6 side by side, the result is 126. This number is divisible by 7. Try this procedure for some other numbers. Find a counterexample that shows that the result is not always evenly divisible by 7.

Find a counterexample for each of the statements in exercises 40 and 41.

40. a. Every whole number greater than 4 and less than 20 is the sum of two or more consecutive whole numbers.
b. Every whole number between 25 and 50 is the product of two whole numbers greater than 1.

41. a. The product of any two whole numbers is evenly divisible by 2.
b. Every whole number greater than 5 is the sum of either two or three consecutive whole numbers, for example, $11 = 5 + 6$ and $18 = 5 + 6 + 7$.

Determine which statements in exercises 42 and 43 are false, and show a counterexample for each false statement. If a statement is false, change one of the conditions to obtain a true statement.

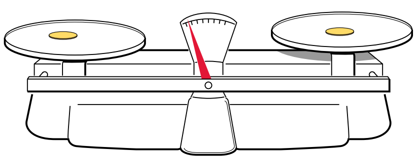
42. **a.** The product of any three consecutive whole numbers is divisible by 2.
b. The sum of any two consecutive whole numbers is divisible by 2.

43. **a.** The sum of any four consecutive whole numbers is divisible by 4.
b. Every whole number greater than 0 and less than 15 is either a triangular number or the sum of two or three triangular numbers.

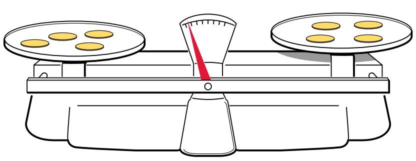
Reasoning and Problem Solving

44. Featured Strategy: Solving a Simpler Problem You are given eight coins and a balance scale. The coins look alike, but one is counterfeit and lighter than the others. Find the counterfeit coin, using just two weighings on the balance scale.

a. Understanding the Problem If there were only two coins and one was counterfeit and lighter, the bad coin could be determined in just one weighing. The balance scale here shows this situation. Is the counterfeit coin on the left or right side of the balance beam?



b. Devising a Plan One method of solving this problem is to guess and check. It is natural to begin with four coins on each side of the balance beam. Explain why this approach will not produce the counterfeit coin in just two weighings. Another method is to simplify the problem and try to solve it for fewer coins.



c. Carrying Out the Plan Explain how the counterfeit coin can be found with one weighing if there are only three coins and with two weighings if there are six coins. By now you may have an idea for solving the original problem. How can the counterfeit coin be found in two weighings?

d. Looking Back Explain how the counterfeit coin can be found in two weighings when there are nine coins.

45. Kay started a computer club, and for a while she was the only member. She planned to have each member find two new members each month. By the end of the first month she had found two new members. If her plan is carried out, how many members will the club have at the end of the following periods?
a. 6 months **b.** 1 year

46. For several years Charlie has had a tree farm where he grows blue spruce. The trees are planted in a square array (square arrays are shown in exercise 29). This year he planted 87 new trees along two adjacent edges of the square to form a larger square. How many trees are in the new square?

47. In the familiar song "The Twelve Days of Christmas," the total number of gifts received each day is a triangular number. On the first day there was 1 gift, on the second day there were 3 gifts, on the third day 6 gifts, etc., until the 12th day of Christmas.
a. How many gifts were received on the 12th day?
b. What is the total number of gifts received during all 12 days?

48. One hundred eighty seedling maple trees are to be set out in a straight line such that the distance between the centers of two adjacent trees is 12 feet. What is the distance from the center of the first tree to the center of the 180th tree?

49. In a long line of railroad cars, an Agco Refrigeration car is the 147th from the beginning of the line, and by counting from the end of the line, the refrigeration car is the 198th car. How many railroad cars are in the line?

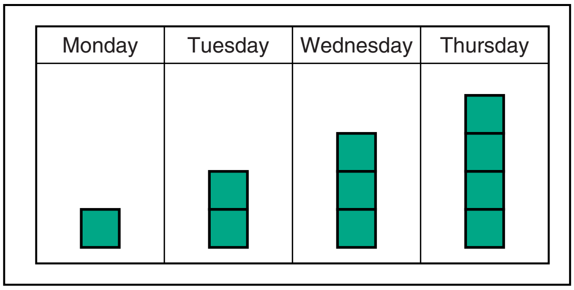
50. If 255 square tiles with colors of blue, red, green, or yellow are placed side by side in a single row so that two tiles of the same color are not next to each other, what is the maximum possible number of red tiles?

51. A card is to be selected at random from 500 cards that are numbered with whole numbers from 1 to 500. How many of these cards have at least one 6 printed on them?

52. A deck of 300 cards is numbered with whole numbers from 1 to 300, with each card having just one number. How many of these cards do not have a 4 printed on them?

Writing and Discussion

1. Suppose you were teaching an elementary school class and for the first four days of a week you put the following tiles on the calendar for Monday through Thursday, as shown here. If the pattern you had in mind for Friday was five tiles in a column, and one student formed a different arrangement, what would you say to this student? Is it possible that this student might be "right"? Explain.



2. An elementary school student discovered a way to get from one square number to the next square number and wanted to know why this was true. For example, if you know that 7^2 is 49, then 8^2 is just $49 + 7 + 8$, or 64. Similarly, 9^2 is $8^2 + 8 + 9$, or 81. Write an explanation with a diagram that illustrates why this relationship holds for all consecutive pairs of square numbers.

3. The beginning of the number pattern, 1 2 4 8, was used by two teachers in separate classes. Teacher *A* asked, "What is the next number in this pattern?" Teacher *B* asked, "What are some possibilities for the next number in this pattern?" List more than one way this number pattern can be continued and explain your reasoning for each way. Discuss the difference between the two questions in terms of the expected student responses.

4. It has been said that mathematics is the study of patterns. How would you explain this point of view to the parents of the children in your classroom? Provide examples to support your position.

5. The NCTM Standards inference at the beginning of exercise set 1.2 refers to the importance of the "core" of a pattern to help children become aware of structure. Write examples of number sequences for the following cases: a pattern with a core that repeats; a pattern with a core that grows; a pattern that is a geometric sequence; and a pattern that is an arithmetic sequence.

Making Connections

1. Compare the **Standards** quotes on pages 4 and 27. (a) Explain why these two statements do not contradict each other. (b) If conjectures and inductive reasoning sometimes lead to false statements, explain why these methods of reasoning are taught in schools.

2. The **Standards** quote on page 27 speaks of students "incorrect expectations" when generalizing patterns. Give some examples of possible incorrect student expectations that may result from their generalizing patterns.

3. On page 25 the example from the **Elementary School Text** shows a triangular pattern of pinecones. (a) The number of cones in each group forms what kind of a sequence? (b) Explain in general how the number of pinecones in any given row is related to the total number of cones in all of the preceding rows.

4. Read the three expectations in the **PreK–2 Standards—Algebra** (see front inside cover) under *Understand patterns, relations . . .* , and explain with examples how the third expectation is satisfied in the Exercises and Problems 1.2.

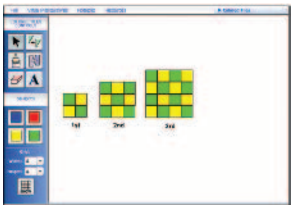
5. The origin of Fibonacci numbers is explained in the **Historical Highlight** on page 22. Use the bibliography and the links for section 1.2 on the companion website and/or browse the Internet to find further applications of Fibonacci numbers. Describe some of these applications.

6. The **Historical Highlight** on page 24 has information on Karl Friedrich Gauss, one of the greatest mathematicians of all time. Gauss kept a mathematical diary and one of his notes claims that every whole number greater than zero can be written as the sum of three or fewer triangular numbers. Verify this statement for numbers less than 20, or find a counterexample.

MATH ACTIVITY 1.3

Extending Tile Patterns

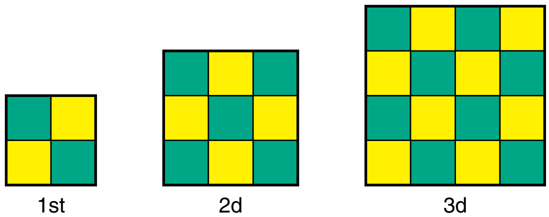
Virtual Manipulatives



www.mhhe.com/bennett-nelson

Materials: Colored tiles in the Manipulative Kit or Virtual Manipulatives.

1. Here are the first three figures in a sequence. Find a pattern and build the fourth figure.



***a.** For each of the first five figures, determine how many tiles there are of each color.

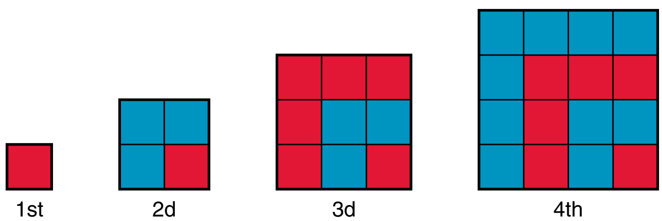
b. Find a pattern and determine the number of tiles of each color for the 10th figure.

c. What is the total number of tiles for the 10th figure?

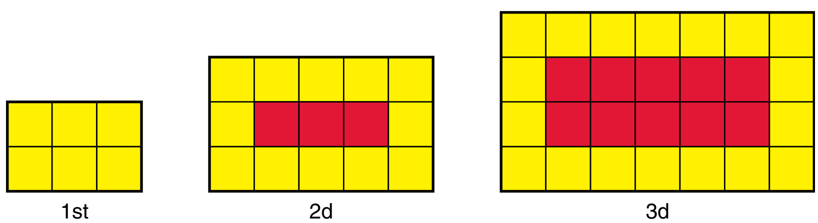
d. Write a description of the 25th figure so that someone reading it could build the figure. Include in your description the number of tiles with each of the different colors and the total number of tiles in the figure.

2. Extend each of the following sequences to the fifth figure, and record the numbers of different-colored tiles in each figure. Find a pattern that enables you to determine the numbers of different-colored tiles in the 10th and 25th figures of each sequence. Describe your reasoning.

***a.**



b.

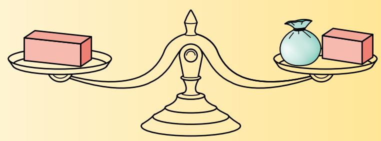


Section 1.3 PROBLEM SOLVING WITH ALGEBRA

"If he could only think in abstract terms"

PROBLEM OPENER

A whole brick is balanced with $\frac{3}{4}$ pound and $\frac{3}{4}$ brick. What is the weight of the whole brick?



NCTM Standards

By viewing algebra as a strand in the curriculum from prekindergarten on, teachers can help students build a solid foundation of understanding and experience as a preparation for more-sophisticated work in algebra in the middle grades and high school. p. 37

Algebra is a powerful tool for representing information and solving problems. It originated in Babylonia and Egypt more than 4000 years ago. At first there were no equations, and words rather than letters were used for variables. The Egyptians used words that have been translated as *heap* and *aha* for unknown quantities in their word problems. Here is a problem from the Rhind Papyrus, written by the Egyptian priest Ahmes about 1650 B.C.:

Heap and one-seventh of heap is 19. What is heap?

Today we would use a letter for the unknown quantity and express the given information in an equation.

$$x + \frac{1}{7}x = 19$$

You may wish to try solving this equation. Its solution is in Example D on the following pages.

Emmy Noether

HISTORICAL HIGHLIGHT

Germany's Amalie Emmy Noether is considered to be the greatest woman mathematician of her time. She studied mathematics at the University of Erlangen, where she was one of only two women among nearly a thousand students. In 1907 she received her doctorate in mathematics from the University of Erlangen. In 1916, the legendary David Hilbert was working on the mathematics of a general relativity theory at the University of Göttingen and invited Emmy Noether to assist him. Although Göttingen had been the first university to grant a doctorate degree to a woman, it was still reluctant to offer a teaching position to a woman, no matter how great her ability and learning. When her appointment failed, Hilbert let her deliver lectures in courses that were announced under his name. Eventually she was appointed to a lectureship at the University of Göttingen. Noether became the center of an active group of algebraists in Europe, and the mathematics that grew out of her papers and lectures at Göttingen made her one of the pioneers of modern algebra. Her famous papers "The Theory of Ideals in Rings" and "Abstract Construction of Ideal Theory in the Domain of Algebraic Number Fields" are the cornerstones of modern algebra courses now presented to mathematics graduate students.*

*D. M. Burton, *The History of Mathematics,* 4th ed. (New York: McGraw-Hill, 1999), pp. 660–662.

NCTM Standards

Research indicates a variety of students have difficulties with the concept of variable (Kuchmann 1978; Kieran 1983; Wafner and Parker 1993) . . . A thorough understanding of variable develops over a long time, and it needs to be grounded in extensive experience. p. 39

VARIABLES AND EQUATIONS

A letter or symbol that is used to denote an unknown number is called a **variable.** One method of introducing variables in elementary schools is with geometric shapes such as $\square$ and $\triangle$. For example, students might be asked to find the number for $\square$ such that $\square + 7 = 12$, or to find some possibilities for $\triangle$ and $\square$ such that $\triangle + \square = 15$. These geometric symbols are less intimidating than letters. Students can replace a variable with a number by writing the numeral inside the geometric shape, as if they were filling in a blank.

To indicate the operations of addition, subtraction, and division with numbers and variables, we use the familiar signs for these operations; for example, $3 + x$, $x - 5$, $x \div 4$, and $x/4$. A product is typically indicated by writing a numeral next to a variable. For example, $6x$ represents 6 times x, or 6 times whatever number is used as a replacement for x. An expression containing algebraic symbols, such as $2x + 3$ or $(4x)(7x) - 5$, is called an **algebraic expression.**

EXAMPLE A

Evaluate the following algebraic expressions for $x = 14$ and $n = 28$.

1. $15 + 3x$
2. $4n - 6$
3. $\frac{n}{7} + 20$
4. $6x \div 12$

Solution **1.** $15 + 3(14) = 15 + 42 = 57$. Notice that when the variable is replaced, parentheses are used; 3(14) means 3 times 14. **2.** $4(28) - 6 = 112 - 6 = 106$. **3.** $28/7 + 20 = 4 + 20 = 24$. **4.** $6(14) \div 12 = 84 \div 12 = 7$.

I will explore how to solve addition equations.

18.1 Explore Addition Equations

Hands On Activity

You can use counters and cups to explore how to solve equations.

You Will Need

- counters
- cups

VOCABULARY

inverse operations

Solve: $d + 9 = 24$

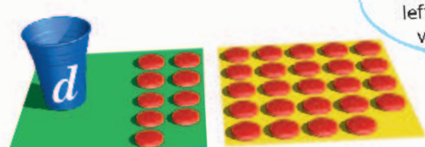
Use Models

STEP 1

Model and write an equation.

- Use a sheet of paper to represent each side of the equation.
- Model $d + 9$ on the left sheet of paper. The cup represents the variable, d. The counters represent numbers.
- Model 24 on the right.

Remember: An equal sign shows that the value on the left side is the same as the value on the right side.

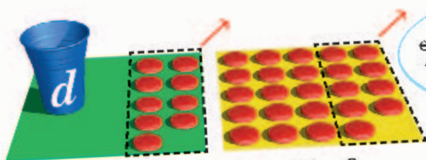


$d + 9 = 24$

STEP 2

Take away 9 counters from the left sheet of paper so only the cup (d) remains. Take away 9 counters from the right side also.

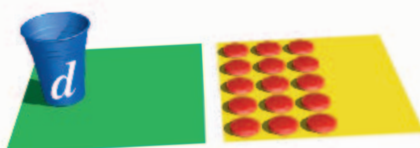
Keep the sides of the equation equal by changing the models on both papers the same way.



$d + 9 - 9 = 24 - 9$

STEP 3

Count the remaining counters. The solution of an equation is the number that makes the equation a true statement when it is substituted for the variable.



$d = 15$

The number 15 makes the equation true. It is the solution. So $d = 15$ since $15 + 9 = 24$.

The elementary ideas of algebra can be presented early in school mathematics. Consider Example B.

EXAMPLE B

Eleanor wins the jackpot in a marble game and doubles her number of marbles. If later she wins 55 more, bringing her total to 127, how many marbles did she have at the beginning?

Solution One possibility is to work backward from the final total of 127 marbles. Subtracting 55 leaves 72, so we need to find the number that yields 72 when doubled. This number is 36. A second approach is to work forward to obtain 127 by guessing. A guess of 20 for the original number of marbles will result in $2(20) + 55 = 95$, which is less than 127. Guesses of increasingly larger numbers eventually will lead to a solution of 36 marbles.

Example B says that if some unknown number of marbles is doubled and 55 more are added, the total is 127. This numerical information is stated in the following *equation* in which the variable x represents the original number of marbles.

$$2x + 55 = 127$$

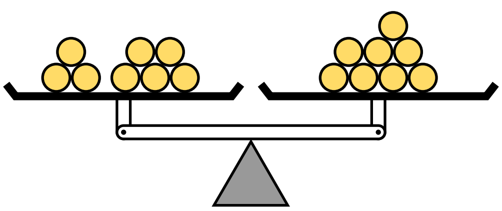


Figure 1.6

An **equation** is a statement of the equality of mathematical expressions; it is a sentence in which the verb is *equals* (=).

A *balance scale* is one model for introducing equations in the elementary school. The idea of *balance* is related to the concept of *equality*. A balance scale with its corresponding equation is shown in Figure 1.6. If each chip on the scale has the same weight, the weight on the left side of the scale *equals* (is the same as) the weight on the right side. Similarly, the sum of numbers on the left side of the equation *equals* the number on the right side.

The balance scale in Figure 1.7 models the *missing-addend* form of subtraction, that is, what number must be added to 5 to obtain 11. The box on the scale may be thought of as *taking the place of,* or *hiding,* the chips needed to balance the scale.

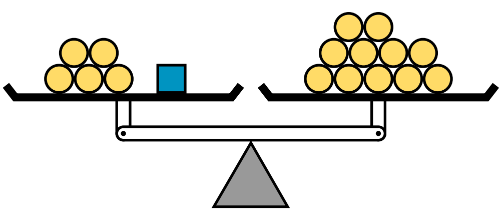


Figure 1.7

One approach to determining the number of chips needed to balance the scale is to *guess and check*. Another approach is to notice that by removing 5 chips from both sides of the scale in Figure 1.7, we obtain the scale shown in Figure 1.8. This scale shows that the box must be replaced by (or is hiding) 6 chips.

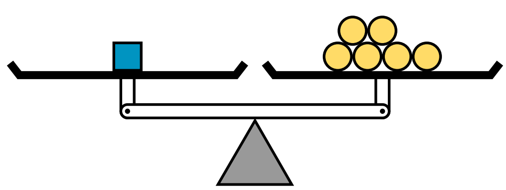


Figure 1.8

Similarly, the equation $5 + x = 11$ can be simplified by subtracting 5 from both sides to obtain $x = 6$. This simpler equation shows that the variable must be replaced by 6.

SOLVING EQUATIONS

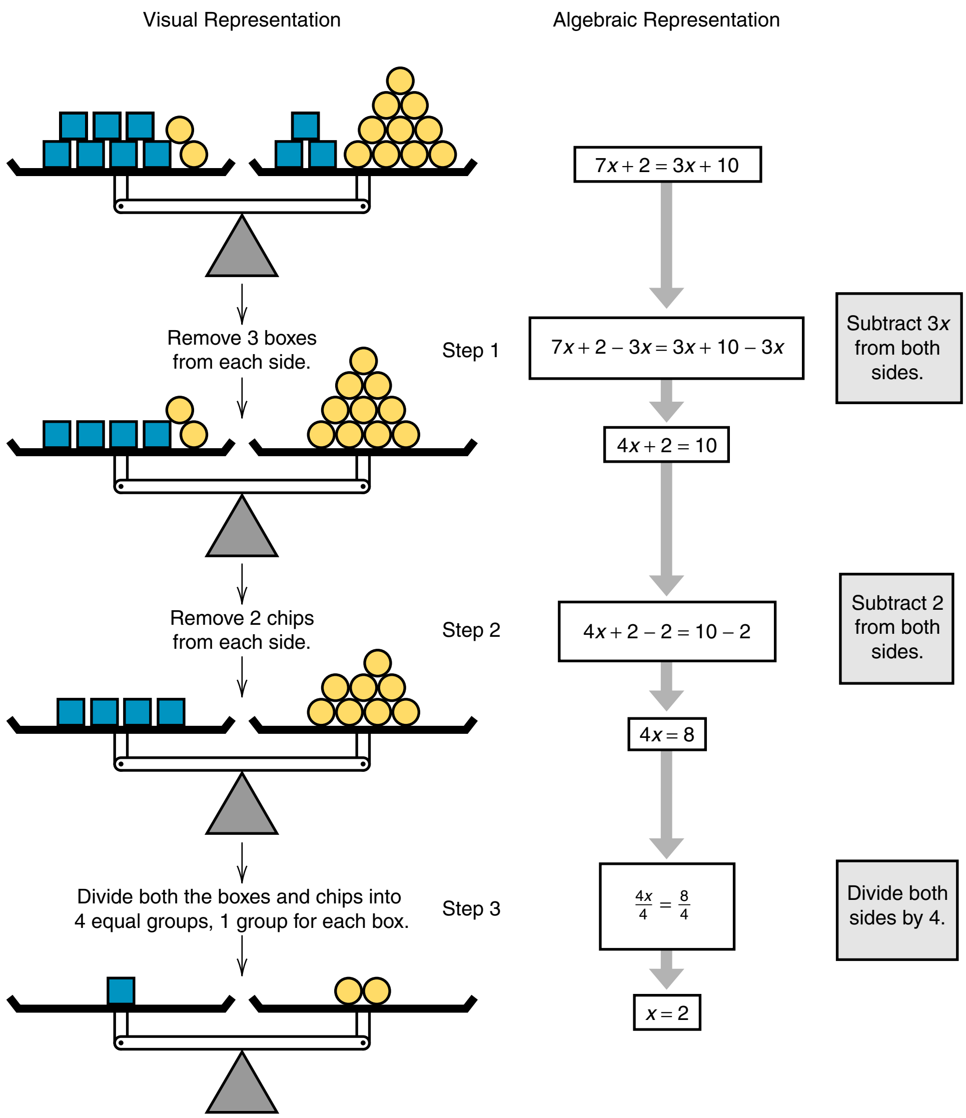
To **solve an equation** or **find the solution(s)** means to find all replacements for the variable that make the equation true. The usual approach to solving an equation is to replace it by a simpler equation whose solutions are the same as those of the original equation. Two equations that have exactly the same solution are called **equivalent equations.**

The *balance-scale model* is used in Example C to illustrate solving an equation. Each step in simplifying the balance scale corresponds to a step in solving the equation.

EXAMPLE C

Solve $7x + 2 = 3x + 10$, using the balance-scale model and equations.

Solution



Check: If each box on the first scale is replaced by 2 chips, the scale will balance with 16 chips on each side. Replacing x by 2 in the equation $7x + 2 = 3x + 10$ makes the equation a true statement and shows that 2 is a solution to this equation.

NCTM Standards

The notion of equality also should be developed throughout the curriculum. They [students] should come to view the equals sign as a symbol of equivalence and balance. p.39

When the balance-scale model is used, the same amount must be *put on* or *removed from* each side to maintain a balance. Similarly, with an equation, the *same operation* must be performed on each side to maintain an equality. In other words, *whatever is done to one side of an equation must be done to the other side.* Specifically, three methods for obtaining equivalent equations are stated next as properties of equality.

Properties of Equality

1. **Addition or Subtraction Property of Equality:** Add the same number or subtract the same number from both sides of an equation.
2. **Multiplication or Division Property of Equality:** Multiply or divide both sides of an equation by the same nonzero number.
3. **Simplification:** Replace an expression in an equation by an equivalent expression.

These methods of obtaining equivalent equations are illustrated in Example D.

EXAMPLE D

Solve these equations.

1. $5x - 9 = 2x + 15$
2. $x + \frac{1}{7}x = 19$ (This is the problem posed by the Egyptian priest Ahmes, described on the opening page of this section.)

Research Statement

Students' difficulties in constructing equations stem in part from their inability to grasp the notion of the equivalence between the two expressions in the left and right sides of the equation.

MacGregor 1998

Solution **1.**

$5x - 9 = 2x + 15$	
$5x - 9 - 2x = 2x + 15 - 2x$	subtraction property of equality; subtract $2x$ from both sides
$3x - 9 = 15$	simplification
$3x - 9 + 9 = 15 + 9$	addition property of equality; add 9 to both sides
$3x = 24$	simplification
$\frac{3x}{3} = \frac{24}{3}$	division property of equality; divide both sides by 3
$x = 8$	simplification

Check: When x is replaced by 8 in the original equation (or in any of the equivalent equations), the equation is true.

$$5(8) - 9 = 2(8) + 15$$
$$31 = 31$$

2.

$x + \frac{1}{7}x = 19$	
$7\left(x + \frac{1}{7}x\right) = 7(19)$	multiplication property of equality; multiply both sides by 7
$8x = 133$	simplification; $7(x + \frac{1}{7}x) = 7x + \frac{7}{7}x = 8x$. This is an example of the **distributive property.***
$\frac{8x}{8} = \frac{133}{8}$	division property of equality; divide both sides by 8
$x = 16\frac{5}{8} = 16.625$	simplification

*For examples of the distributive property, as well as several other number properties, see the subsection Number Properties in Section 3.3.

Check: When x is replaced by 16.625 in the original equation, the equation is true.

$$16.625 + \frac{1}{7}(16.625) = 16.625 + 2.375 = 19$$

SOLVING INEQUALITIES

Not all algebra problems are solved by equations. Consider Example E.

EXAMPLE E

John has $19 to spend at a carnival. After paying the entrance fee of $3, he finds that each ride costs $2. What are the possibilities for the number of rides he can take?

Solution This table shows John's total expenses with different numbers of rides. John can take any number of rides from 0 to 8 and not spend more than $19.

Number of Rides	Expense
0	$ 3
1	5
2	7
3	9
4	11
5	13
6	15
7	17
8	19

Example E says that $3 plus some number of $2 rides must be less than or equal to $19. This numerical information is stated in the following *inequality,* where x represents the unknown number of rides:

$$3 + 2x \leq 19$$

An **inequality** is a statement that uses one of the following phrases: *is less than* ($<$), *is less than or equal to* ($\leq$), *is greater than* ($>$), *is greater than or equal to* ($\geq$), or *is not equal to* ($\neq$).

The *balance-scale model* can also be used for illustrating inequalities. Figure 1.9 illustrates the inequality in Example E. The box can be replaced by any number of chips as long as the beam doesn't tip down on the left side. Some elementary schoolteachers who use the balance-scale model have students tip their arms to imitate the balance scale. Sometimes the teacher places a heavy weight in one of a student's hands and a light weight in the other. This helps students become accustomed to the fact that the amount on the side of the scale that is tipped down is *greater than* the amount on the other side of the scale.

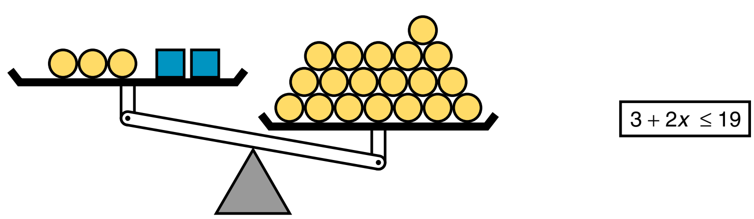


Figure 1.9

One method of finding the number of chips that can be used in place of the box in Figure 1.9 is to think of replacing each box on the scale by the same number of chips, keeping the total number of chips on the left side of the scale less than or equal to 19. Another method is to simplify the scale to determine the possibilities for the number of chips for the box. First, we can remove 3 chips from both sides to obtain the scale setting in Figure 1.10.

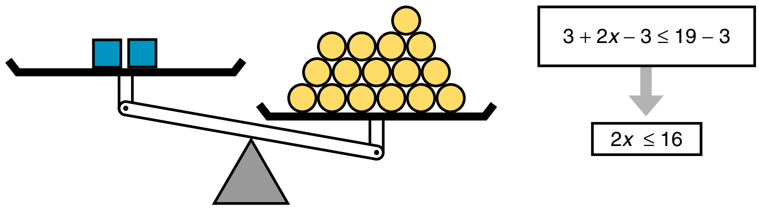


Figure 1.10

Next, we can divide the chips on the right side of the scale into two groups, one group for each box on the left side of the scale. The simplified scale in Figure 1.11 shows that replacing the box by 7 or fewer chips will keep the scale tipped down on the right side and if the box is replaced by 8 chips the scale will be balanced.

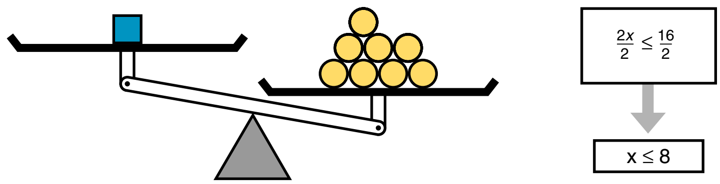


Figure 1.11

To the right of each balance scale above, there is a corresponding inequality. These inequalities are replaced by simpler inequalities to obtain $x \leq 8$. To make this inequality true, we must replace the variable by a number less than or equal to 8.

To **solve an inequality** means to find all the replacements for the variable that make the inequality true. The replacements that make the inequality true are called **solutions.** Like an equation, an inequality is solved by replacing it by simpler inequalities. Two inequalities that have exactly the same solution are called **equivalent inequalities.**

Equivalent inequalities can be obtained using the same steps as those for obtaining *equivalent equations* (performing the same operation on both sides and replacing an expression by an equivalent expression), with one exception: Multiplying or dividing both sides of an inequality by a negative number *reverses the inequality*. For example, $8 > 3$; but if we multiply both sides of the inequality by $^{-}1$, we obtain $^{-}8$ and $^{-}3$, and $^{-}8$ is less than $^{-}3$ ($^{-}8 < {}^{-}3$). These inequalities are illustrated in Figure 1.12.

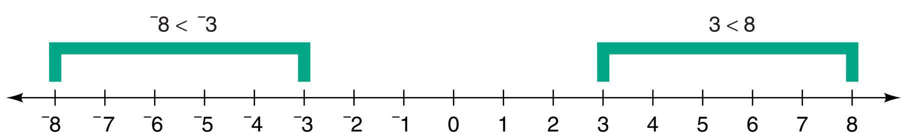


Figure 1.12

Three methods for obtaining equivalent inequalities are stated next as the properties of inequality. (These properties also apply to the inequalities $\leq$, $>$, and $\geq$.)

Properties of Inequality

1. **Addition or Subtraction Property of Inequality:** Add the same number or subtract the same number from both sides of an inequality.
2. **Multiplication or Division Property of Inequality:** Multiply or divide both sides of an inequality by the same nonzero number; and if the number is negative, reverse the inequality sign.
3. **Simplification:** Replace an expression in an inequality by an equivalent expression.

EXAMPLE F

Solve the inequality $4(3x) + 16 < 52$.

NCTM Standards

In the middle grades it is essential that students become comfortable in relating symbolic expressions containing variables to verbal, tabular, and graphical representations or numerical and quantitative relationships. p. 223

Solution

$$4(3x) + 16 < 52$$

$12x + 16 < 52$	simplification
$12x + 16 - 16 < 52 - 16$	subtraction property for inequality; subtract 16 from both sides
$12x < 36$	simplification
$\dfrac{12x}{12} < \dfrac{36}{12}$	division property for inequality; divide both sides by 12
$x < 3$	simplification

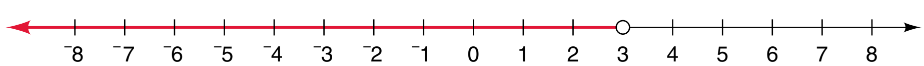
Check: We can get some indication of whether the inequality was solved correctly by trying a number less than 3 to see if it is a solution. When we replace x in the original inequality by 2 we can see that the inequality holds.

$$\begin{aligned} 4[3(2)] + 16 &= 4(6) + 16 \\ &= 24 + 16 \\ &= 40 \end{aligned}$$

and 40 is less than 52.

The solutions for an inequality in one variable can be visualized on a number line. The solutions for the inequality in Example F are shown in Figure 1.13. The circle about the point for 3 indicates that this point is not part of the solution. So the solution includes all the points on the half-line extending to the left of the point for 3.

Figure 1.13

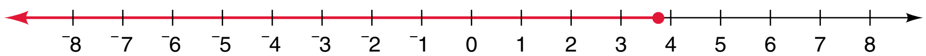


EXAMPLE G Solve the inequality $11x - 7 \leq 3x + 23$, and illustrate its solution by using a number line.

Solution

$11x - 7 \leq 3x + 23$	
$11x - 7 + 7 \leq 3x + 23 + 7$	addition property for inequality; add 7 to both sides
$11x \leq 3x + 30$	simplification
$11x - 3x \leq 3x - 3x + 30$	subtraction property for inequality; subtract $3x$ from both sides
$8x \leq 30$	simplification
$\frac{8x}{8} \leq \frac{30}{8}$	division property for inequality; divide both sides by 8
$x \leq 3\frac{3}{4}$	simplification

Every number less than or equal to $3\frac{3}{4}$ is a solution for the original inequality, and these solutions can be shown on a number line by shading the points to the left of the point for $3\frac{3}{4}$.



USING ALGEBRA FOR PROBLEM SOLVING

One application of algebra is solving problems whose solutions involve equations and inequalities.

EXAMPLE H The manager of a garden center wants to order a total of 138 trees consisting of two types: Japanese red maple and flowering pears. Each maple tree costs \$156 and each pear tree costs \$114. If the manager has a budget of \$18,000, which must all be spent for the trees, how many maple trees will be in the order?

Solution If x equals the number of maple trees, then $138 - x$ will equal the number of pear trees. The following equation shows that the total cost of both types of trees is \$18,000. Notice the use of the distributive property in going from the first to the second equation.

$$156x + 114(138 - x) = 18{,}000$$
$$156x + 15{,}732 - 114x = 18{,}000$$
$$42x = 2268$$
$$x = 54$$

There will be 54 Japanese red maple trees in the order.

Example I is a variation of Example H, but its solution requires an inequality.

EXAMPLE I The manager of a garden center wants to place an order for Hawthorne trees and Service Berry trees so that the number of Service Berry trees is 6 times the number of Hawthorne trees. Each Hawthorne tree costs \$250 and each Service Berry tree costs \$125. If the budget requires that the total cost of the trees be less than \$30,000 and that there be at least 20 Hawthorne trees, what are the different possibilities for the number of Hawthorne trees in the order?

Solution If x equals the number of Hawthorne trees, then $6x$ is the number of Service Berry trees, and the following inequality shows that the total cost of the two types of trees is less than $30,000.

$$\begin{aligned} 250x + 125(6x) &< 30{,}000 \\ 250x + 750x &< 30{,}000 \\ 1000x &< 30{,}000 \\ x &< 30 \end{aligned}$$

Since there is a requirement that the order contain at least 20 Hawthorne trees, the possibilities for the number of Hawthorne trees is 20, 21, 22, . . . 29.

Technology Connection

Palindromic Sums

Notice that 48 + 84 = 132 and 132 + 231 = 363, and 363 is a palindromic number. Will all two-digit numbers eventually go to palindromic numbers using this process of reversing digits and adding? The online 1.3 Mathematics Investigation will help you explore this and similar questions.

Mathematics Investigation
Chapter 1, Section 3
www.mhhe.com/bennett-nelson

Another application of algebra is in analyzing number tricks and so-called magic formulas. Select any number and perform the following operations:

> Add 4 to any number; multiply the result by 6; subtract 9; divide by 3; add 13; divide by 2; and then subtract the number you started with.

If you performed these operations correctly, your final answer will be 9, regardless of the number you started with. This can be proved using the variable x to represent the number selected and performing the following algebraic operations:

1. Select any number $\quad x$
2. Add 4 $\quad x + 4$
3. Multiply by 6 $\quad 6(x + 4) = 6x + 24$
4. Subtract 9 $\quad 6x + 24 - 9 = 6x + 15$
5. Divide by 3 $\quad (6x + 15)/3 = 2x + 5$
6. Add 13 $\quad 2x + 5 + 13 = 2x + 18$
7. Divide by 2 $\quad (2x + 18)/2 = x + 9$
8. Subtract the number x $\quad x + 9 - x = 9$

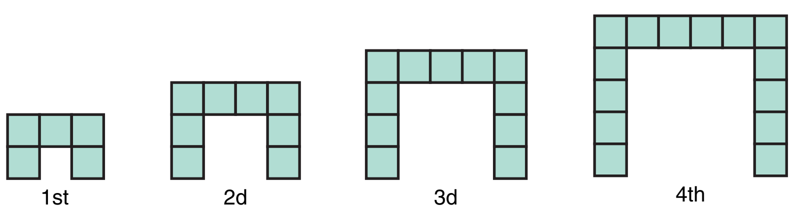
The preceding steps show that it doesn't matter what number is selected for x, for in the final step x is subtracted and the end result is always 9.

PROBLEM-SOLVING APPLICATION

The problem-solving strategy of *using algebra* is illustrated in the solution to the next problem.

Problem

A class of students is shown the following figures formed with tiles and is told that there is a pattern that, if continued, will result in one of the figures having 290 tiles. Which figure will have this many tiles?



NCTM Standards

Two central themes of algebraic thinking are appropriate for young students. The first involves making generalizations and using symbols to represent mathematical ideas, and the second is representing and solving problems (Carpenter and Levi 1999). p. 93

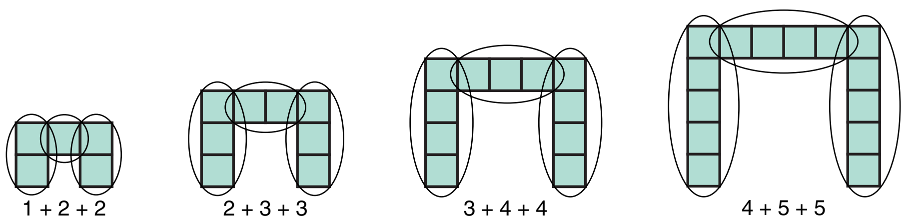
Understanding the Problem The fourth figure has 14 tiles. Find a pattern in the formation of the first few figures, and sketch the fifth and sixth figures. **Question 1:** How many tiles are in the fifth and sixth figures?

Devising a Plan One approach to solving this problem is to use a variable and write an algebraic expression for the nth term. This expression can then be used to determine which figure has 290 tiles.

Notice that the third figure has 3 tiles in each "leg," 3 tiles in the middle of the top row, and 2 corner tiles. The fourth figure has 4 tiles in each leg, 4 in the middle of the top row, and 2 corner tiles. **Question 2:** By extending this reasoning, how many tiles are in the 20th figure? the 100th?

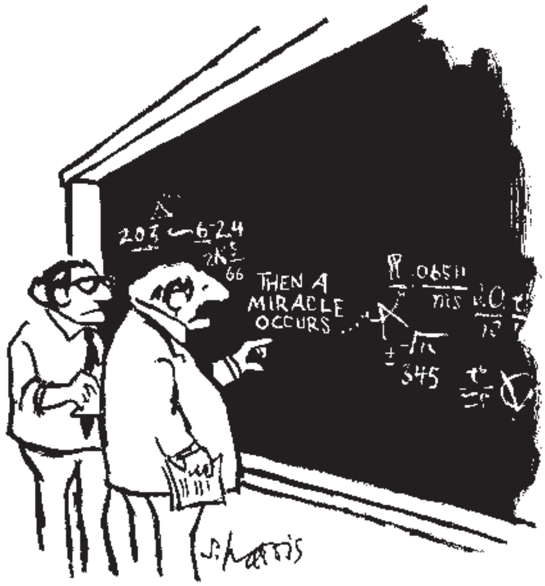
Carrying Out the Plan The nth figure will have n tiles in each leg, n tiles in the middle of the top row, and 2 corner tiles. So the algebraic expression for the number of tiles in the nth figure is $n + n + n + 2$, or $3n + 2$. **Question 3:** What number for n gives the expression $3n + 2$ a value of 290?

Looking Back Perhaps you saw a different way to group the tiles in the first four figures. **Question 4:** If you saw the pattern developing as follows, what would be the algebraic expression for the nth figure?



Answers to Questions 1–4 **1.** The fifth figure has 17 tiles and the sixth has 20. **2.** The 20th figure has $20 + 20 + 20 + 2 = 62$ tiles, and the 100th figure has $100 + 100 + 100 + 2 = 302$ tiles. **3.** $n = 96$. **4.** $n + (n + 1) + (n + 1)$ or $n + 2(n + 1)$.

Exercises and Problems 1.3



"I THINK YOU SHOULD BE MORE EXPLICIT HERE IN STEP TWO."

1. a. At a depth of x feet under water, the pressure in pounds per square inch is $.43x + 14.7$. What is the pressure in pounds per square inch for a depth of 10 feet? 100 feet? 0 feet (surface of the water)?

b. The temperature (Fahrenheit) can be approximated by $x/4 + 40$, where x is the number of cricket chirps in 1 minute. What is the temperature for 20 chirps per minute? 100 chirps per minute?

c. A person's normal blood pressure increases with age and is approximated by $x/2 + 110$, where x is the person's age. The blood pressure for people between 20 and 30 years old should be between what two numbers?

2. a. A woman's shoe size is given by $3x - 22$, where x is the length of her foot in inches. What is a woman's shoe size for a length of 9 inches? 11 inches?

b. The number of words in a child's vocabulary for children between 20 and 50 months is $60x - 900$, where x is the child's age in months. What is the number of vocabulary words for a child whose age is 20 months? 35 months? 4 years?

c. A person's maximum heart rate is $220 - x$, where x is the person's age, and the heart rate for aerobic activity should be between $.7(220 - x)$ and $.8(220 - x)$. A 20-year-old person's heart rate for aerobic activity should be between what two numbers?

3. Tickets for the historical review of ballroom dancing at the Portsmouth Music Hall cost \$28 each for the main-floor seats and \$19 each for the balcony seats. Let m represent the number of tickets sold for main-floor seats and let b represent the number of tickets sold for balcony seats. Write an algebraic expression for the following amounts.

a. The cost in dollars of all the main-floor seats that were sold

b. The total number of seats that were sold for the performance

c. The difference in dollars in the total amount of money paid for all main-floor seats and the total amount paid for all balcony seats, if the total for all main-floor seats was the greater of the two amounts

4. At the Saturday farmers' market, melons cost \$1.20 each and coconuts cost \$1.45 each. Let m represent the number of melons sold during the day, and let c represent the number of coconuts sold. Write an algebraic expression for each of the following.

a. The total number of melons and coconuts sold

b. The cost of all the coconuts sold

c. The total cost of all the melons and coconuts sold

5. In research conducted at the University of Massachusetts, Peter Rosnick found that $37\frac{1}{3}$ percent of a group of 150 engineering students were unable to write the correct equation for the following problem.* Write an equation using variables s and p to represent the following statement: "At this university there are 6 times as many students as professors." Use s for the number of students and p for the number of professors.

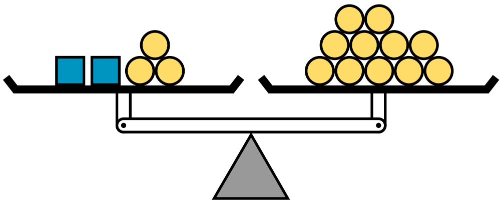
a. What is the correct equation?

b. The most common erroneous answer was $6s = p$. Give a possible explanation for this.

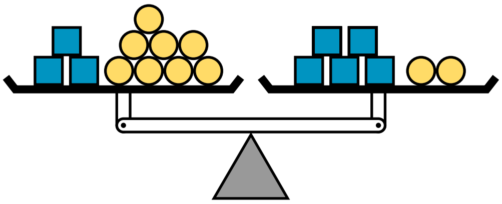
Determine the number of chips needed to replace each box in order for the scales in exercises 6 and 7 to balance. Then using x to represent the number of chips for each box, write the corresponding equation that represents each scale and solve the equation.

*Peter Rosnick, "Some Misconceptions Concerning the Concept of a Variable," *The Mathematics Teacher* 74 (September 1981): 418–420.

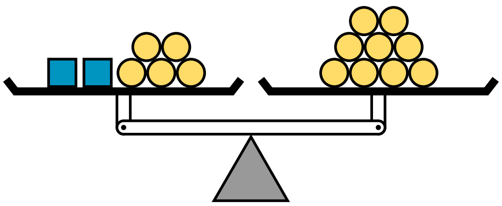
6. a.



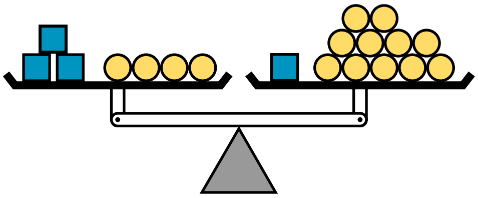
b.



7. a.



b.



Each of the equations in exercises 8 and 9 has been replaced by a similar equivalent equation. Write the property of equality that has been used in each step.

8. a.

$$6x - 14 = 2x$$
$$6x - 14 + 14 = 2x + 14 \quad \text{(step 1)}$$
$$6x = 2x + 14 \quad \text{(step 2)}$$
$$6x - 2x = 2x + 14 - 2x \quad \text{(step 3)}$$
$$4x = 14 \quad \text{(step 4)}$$
$$\frac{4x}{4} = \frac{14}{4} \quad \text{(step 5)}$$
$$x = 3\tfrac{1}{2} \quad \text{(step 6)}$$

b.

$$42x + 102 = 6(3x + 45)$$
$$42x + 102 = 18x + 270 \quad \text{(step 1)}$$
$$42x + 102 - 18x = 18x + 270 - 18x \quad \text{(step 2)}$$
$$24x + 102 = 270 \quad \text{(step 3)}$$
$$24x + 102 - 102 = 270 - 102 \quad \text{(step 4)}$$
$$24x = 168 \quad \text{(step 5)}$$
$$\frac{24x}{24} = \frac{168}{24} \quad \text{(step 6)}$$
$$x = 7 \quad \text{(step 7)}$$

9. a.

$$6(2x - 5) = 7x + 15$$
$$12x - 30 = 7x + 15 \quad \text{(step 1)}$$
$$12x - 30 + 30 = 7x + 15 + 30 \quad \text{(step 2)}$$
$$12x = 7x + 45 \quad \text{(step 3)}$$
$$12x - 7x = 7x + 45 - 7x \quad \text{(step 4)}$$
$$5x = 45 \quad \text{(step 5)}$$
$$\frac{5x}{5} = \frac{45}{5} \quad \text{(step 6)}$$
$$x = 9 \quad \text{(step 7)}$$

b.

$$11(3x) + 2 = 35$$
$$33x + 2 = 35 \quad \text{(step 1)}$$
$$33x + 2 - 2 = 35 - 2 \quad \text{(step 2)}$$
$$33x = 33 \quad \text{(step 3)}$$
$$\frac{33x}{33} = \frac{33}{33} \quad \text{(step 4)}$$
$$x = 1 \quad \text{(step 5)}$$

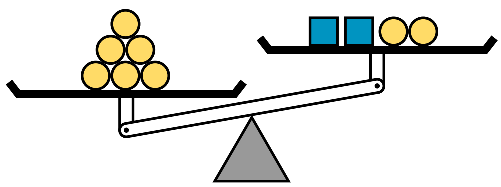
Solve each equation in exercises 10 and 11.

10. a. $2x + 30 = 18 + 5x$
b. $3x - 17 = 22$
c. $13(2x) + 20 = 6(5x + 2)$
d. $8\left(\frac{x}{2} - 5\right) = 2x - 6$

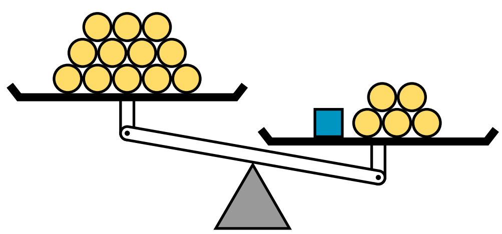
11. a. $43x - 281 = 17x + 8117$
b. $17(3x - 4) = 25x + 218$
c. $56(x + 1) + 7x = 45{,}353$
d. $3x + 5 = 2(2x - 7)$

Determine the number of chips for each box that will keep the scales in exercises 12 and 13 tipped as shown. Then, using x for a variable, write the corresponding inequality for each scale and solve the inequality.

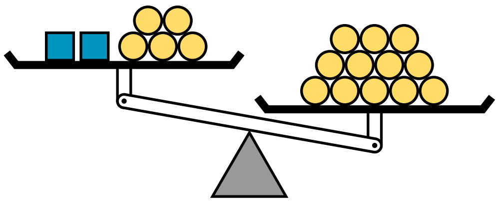
12. a.



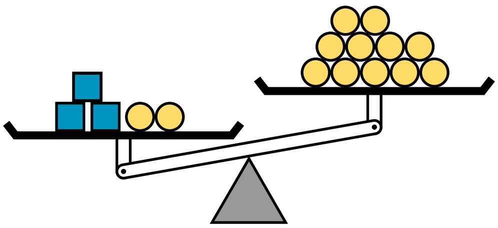
b.



13. a.



b.



Each of the inequalities in exercises 14 and 15 has been replaced by a similar equivalent inequality. Write the property of inequality that has been used in each step.

14. a.
$3x + 14 < 55$
$3x + 14 - 14 < 55 - 14$ (step 1)
$3x < 41$ (step 2)
$\frac{3x}{3} < \frac{41}{3}$ (step 3)
$x < 13\frac{2}{3}$ (step 4)

b. $10x < 55$
$\frac{10x}{10} < \frac{55}{10}$ (step 1)
$x < 5\frac{1}{2}$ (step 2)

15. a.
$6x + 11 > 2x + 19$
$6x + 11 - 2x > 2x + 19 - 2x$ (step 1)
$4x + 11 > 19$ (step 2)
$4x + 11 - 11 > 19 - 11$ (step 3)
$4x > 8$ (step 4)
$\frac{4x}{4} > \frac{8}{4}$ (step 5)
$x > 2$ (step 6)

b.
$2x > 3x - 12$
$2x - 3x > 3x - 12 - 3x$ (step 1)
$^{-}x > {}^{-}12$ (step 2)
$(^{-}1)(^{-}x) < (^{-}1)(^{-}12)$ (step 3)
$x < 12$ (step 4)

Solve each inequality in exercises 16 and 17, and illustrate the solution by using a number line.

16. a. $3x + 5 < x + 17$
b. $3(2x + 7) > 36$

17. a. $6(x + 5) > 11x$
b. $5(x + 8) - 6 > 44$

18. Mr. Dawson purchased some artichokes for 80 cents each and twice as many pineapples for 95 cents each. Altogether he spent \$18.90. Let x represent the number of artichokes, and write an algebraic expression for each item in parts a through c.
a. The total cost in dollars of the artichokes
b. The number of pineapples
c. The total cost in dollars of the pineapples
d. The sum of the costs in parts a and c is \$18.90. Write and solve an equation to determine the number of artichokes Mr. Dawson bought.

19. It cost Marci 24 cents to mail a postcard and 39 cents to mail a letter. She sent either a postcard or a letter to each of 18 people and spent \$5.22. Let x represent the number of postcards she wrote, and write an algebraic expression for each item in parts a through c.
a. The total cost in dollars of the postcards
b. The number of letters
c. The total cost in dollars of the letters
d. The sum of the costs in parts a and c is \$5.22. Write and solve an equation to determine the number of postcards Marci mailed.

20. Teresa purchased pens for 50 cents each and pencils for 25 cents each. She purchased 10 more pencils than pens and gave the clerk a five-dollar bill, which was more than enough to pay the total cost. Let x represent the number of pens, and write an algebraic expression for each item in parts a through c.
a. The total cost in dollars of the pens
b. The number of pencils
c. The total cost in dollars of the pencils
d. The sum of the costs in parts a and c is less than \$5.00. Write and solve an inequality to determine the possibilities for the number of pens and pencils that Teresa purchased.

21. Merle spent \$10.50 for each compact disk and \$8 for each tape. He purchased three more tapes than compact disks, and the total amount of money he spent was less than \$120. Let x represent the number of compact disks he purchased, and write an algebraic expression for each item in parts a through c.
a. The total cost in dollars of the compact disks
b. The number of tapes
c. The total cost in dollars of the tapes
d. The sum of the costs in parts a and c is less than \$120. Write and solve an inequality to determine the possibilities for the number of compact disks Merle purchased.

Reasoning and Problem Solving

Solve word problems 22 to 24 by writing an equation with a variable to represent the given information and then solve the equation.

22. Jeri spends \$60 of her paycheck on clothes and then spends one-half of her remaining money on food. If she has \$80 left, what was the amount of her paycheck?

23. Marcia has 350 feet of fence. After fencing in a square region, she has 110 feet of fence left. What is the length of one side of the square?

24. Rico noticed that if he began with his age, added 24, divided the result by 2, and then subtracted 6, he got his age back. What is his age?

Solve word problems 25 and 26 by writing an inequality with a variable and then solve the inequality.

25. If you add 14 to a certain number, the sum is less than 3 times the number. For what numbers is this true?

26. The length of the first side of a triangle is a whole number greater than 3. The second side is 3 inches longer than the first, and the third side is 3 inches longer than the second. How many such triangles have perimeters less than 36 inches?

27. How can 350 be written as the sum of four consecutive whole numbers?

a. Understanding the Problem If 75 is the first of four consecutive numbers, then the others are 76, 77, and 78. If n is the first number, write an algebraic expression for each of the other three.

b. Devising a Plan One plan is to let n be the first of four consecutive numbers, write an algebraic expression for their sum, and determine which number for n gives that expression a value of 350. If n is the first of four consecutive numbers, what is an algebraic expression for their sum?

c. Carrying Out the Plan What number for n gives a sum of 350?

d. Looking Back Can 350 be written as the sum of other consecutive whole numbers? Use algebraic expressions to determine if 350 can be written as the sum of three consecutive numbers or five consecutive numbers.

28. The teacher asks the class to select a 4×4 array of numbers from a 10×10 number chart and to use only those numbers for the four-step process described here.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	~~36~~	37	38	39	40
41	42	43	44	45	46	~~47~~	~~48~~	~~49~~	50
51	52	53	54	55	~~56~~	57	58	59	60
61	62	63	64	65	~~66~~	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

1. Circle any number and cross out the remaining numbers in its row and column.
2. Circle another number that has not been crossed out and cross out the remaining numbers in its row and column.
3. Repeat step 2 until there are four circled numbers.
4. Add the four circled numbers.

No matter what sum a student comes up with, the teacher will be able to predict the number in the upper-left corner of the student's square by subtracting 66 from the sum and dividing the result by 4. Show why this formula works.

a. Understanding the Problem Let's carry out the steps on the 4×4 array shown above. The first circled number is 46, and the remaining numbers in its row and column have been crossed out. The next circled number is 38. Continue the four-step process. Does the teacher's formula produce the number in the upper-left corner of this 4×4 array?

b. Devising a Plan This problem can be solved by algebra. If we represent the number in the upper-left corner by x, the remaining numbers can be represented in terms of x. Complete the next two rows of the 4×4 array of algebraic expressions shown here.

x	$x + 1$	$x + 2$	$x + 3$
$x + 10$	$x + 11$	$x + 12$	$x + 13$

c. Carrying Out the Plan Carry out the four-step process on the 4×4 array of algebraic expressions you completed in part b. Use the results to show that the teacher's formula works.

d. Looking Back One variation on this number trick is to change the size of the array that the student selects. For example, what formula will the teacher use if a 3×3 array is selected from the 10×10 number chart? (*Hint:* Use a 3×3 algebraic array.) Another variation is to change the number chart. Suppose that a 3×3 array is selected from a calendar. What is the formula in this case?

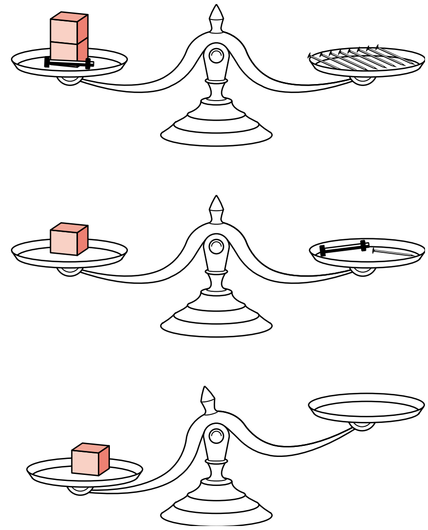
29. Many number tricks can be explained by algebra. Select any number and perform the steps here to see what number you obtain. Add 221 to the number, multiply by 2652, subtract 1326, divide by 663, subtract 870, divide by 4, and subtract the original number. Use algebra to show that the steps always result in the same number.

30. Ask a person to write the number of the month of his or her birth and perform the following operations: Multiply by 5, add 6, multiply by 4, add 9, multiply by 5, and add the number of the day of birth. When 165 is subtracted from this number, the result is a number that represents the person's month and day of birth. Try it.

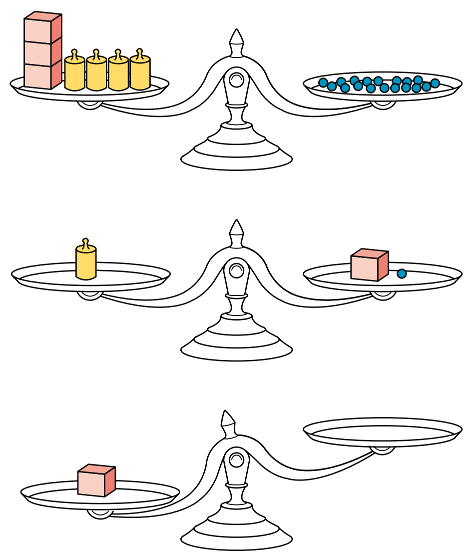
Analysis: Let d and m equal the day and month, respectively. The first three of the preceding steps are represented by the following algebraic expressions. Continue these expressions to show that the final expression is equal to $100m + d$. This shows that the units digit of the final result is the day and the remaining digits are the month.

$$5m$$
$$5m + 6$$
$$4(5m + 6) = 20m + 24$$
$$\vdots$$

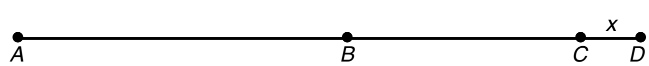
31. Use the information from the first two balance beams shown at the top of the next column to determine the number of nails needed to balance one cube.* (*Note:* There are 8 nails in the pan on the right side.)



32. Use the information from the first two balance beams below to determine the number of marbles needed to balance a cube. (*Note:* There are 18 marbles in the pan on the right.)



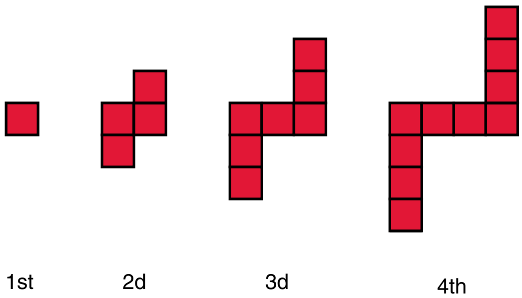
33. In driving from town A to town D, you pass first through town B and then through town C. It is 10 times farther from town A to town B than from towns B to C and 10 times farther from towns B to C than from towns C to D. If it is 1332 miles from towns A to D, how far is it from towns A to B? (Let the variable x represent the distance from towns C to D.)



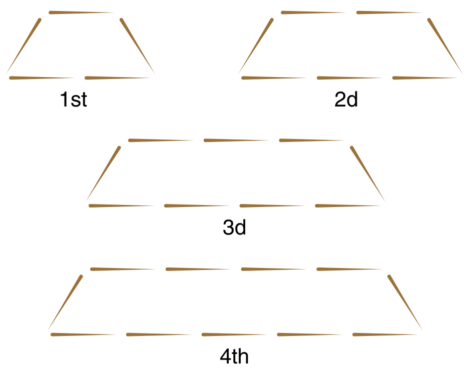
*Balance beam art from *The Arithmetic Teacher* 19 (October 1972): 460–461.

34. The cost of a bottle of cologne, \$28.90, was determined from the cost of the bottle plus the cost of the perfume. If the perfume costs \$14.10 more than the bottle, how much does the bottle cost? (Let the variable x represent the cost of the bottle.)

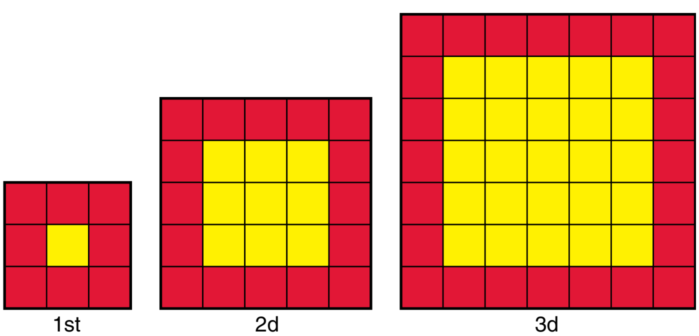
35. If the following tile figures are continued, will there be a figure with 8230 tiles? If so, which figure will it be? If not, what figure has the number of tiles closest to 8230? (*Hint:* Write an algebraic expression for the nth figure and set it equal to 8230.)



36. Find a pattern to extend the toothpick figures. Determine the number of toothpicks in the 50th figure, and write an algebraic expression for the number of toothpicks in the nth figure.



37. Here are the first three figures of a sequence formed by color tiles.

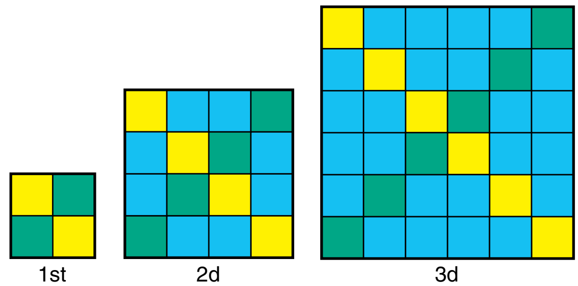


a. Find a pattern and describe the next two figures in the sequence. Then determine how many tiles of each color are used in the fifth figure. Answer this question for each of the first four figures.

b. Describe the 100th figure. Include the number of each color of tile and the total number of tiles in the figure.

c. Write algebraic expressions for the nth figure for (1) the number of yellow tiles, (2) the number of red tiles, and (3) the total number of tiles.

38. These three figures were formed by color tiles.



a. Find a pattern and describe the next two figures in the sequence. How many tiles of each different color are in the fifth figure? Answer this question for each of the first four figures.

b. Describe the 100th figure and determine (1) the number of yellow tiles, (2) the number of green tiles, (3) the number of blue tiles, and (4) the total number of tiles.

c. Write algebraic expressions for the nth figure for (1) the number of yellow tiles, (2) the number of green tiles, (3) the number of blue tiles, and (4) the total number of tiles.

Writing and Discussion

1. Suppose one of your students wrote the following statement in his math journal: "I do not understand what an equation is." Explain how would you help him with his understanding.

2. Suppose that your elementary school math text had a page labeled ALGEBRA with exercises for the students like the following:

$\square + \Delta = 15, \quad \square + 4 = 9, \quad 3 \times \square = 12, \quad 14 - \square = 8$

If your students asked why this was algebra, how would you answer their question?

3. The following type of question has been used to assess algebraic thinking: "On a certain field trip there were 8 times as many students as chaperons. If c represents the number of chaperons and s represents the number of students, what equation represents the given information?" When answering this question, students are sometimes confused over which two equations, $8c = s$ or $8s = c$, to use. Write an activity or series of questions that would help middle school students understand which of these two equations is correct.

Making Connections

1. On page 39 the example from the **Elementary School Text** illustrates the solution of an equation involving a variable. (a) How is this method of solution similar to finding solutions with the balance-scale model from this section? (b) Does either the model on this elementary text page or the balance-scale model in this section help students overcome the difficulties cited in the **Standards** and **Research** statements on page 42? Explain.

2. The **Standards** quote on page 48 discusses two central themes of algebraic thinking. Give an example of each of these themes. Explain.

3. Examine the **Algebra Standards** for the three different levels (see inside covers) to see whether or not any standard suggests using models, like the balance scale, for algebra concepts. If so at what grade levels does this occur?

4. The **Historical Highlight** on page 38 features Amalie Emmy Noether, one of the greatest women mathematicians of her time and a pioneer in modern algebra. Search the Internet for more details on her life and accomplishments and write a paragraph with details that would be interesting to school students.

5. The **Standards** quote on page 38 notes that an understanding of variable needs to be grounded in experience, and the **Standards** quote on page 45 lists several types of experiences in which students should encounter expressions with variables. Give examples of a few of these experiences.

6. Explain how the weight of the whole brick on the balance scale in the **Problem Opener** for this section can be solved visually by using this model. Then solve the problem using algebra. Discuss the connections and similarity between the steps in the two solutions.

7. The **Standards** quote on page 37 notes that algebra is a strand in the curriculum from prekindergarten through high school. Give examples of what algebraic thinking is for children in the early grades. Write an explanation to convince a parent that algebra is appropriate for the early grades.

CHAPTER REVIEW

1. **Problem Solving**
   a. **Problem solving** is the process by which an unfamiliar situation is resolved.
   b. **Polya's Four-Step Process**
   Understanding the problem
   Devising a plan
   Carrying out the plan
   Looking back
   c. **Problem-Solving Strategies**
   Making a drawing
   Guessing and checking
   Making a table
   Using a model
   Working backward
   Finding a pattern
   Solving a simpler problem
   Using algebra

2. **Unsolved Problems**
   a. There are many unsolved problems in mathematics.
   b. A **conjecture** is a statement that has not been proved, yet is thought to be true.

3. **Patterns and Sequences**
   a. There are many kinds of patterns. They are found by comparing and contrasting information.
   b. The numbers in the sequence 1, 1, 2, 3, 5, 8, 13, 21, . . . are called **Fibonacci numbers.** The growth patterns of plants and trees frequently can be described by Fibonacci numbers.
   c. **Pascal's triangle** is a triangle of numbers with many patterns. One pattern enables each row to be obtained from the previous row.
   d. An **arithmetic sequence** is a sequence in which each term is obtained by adding a **common difference** to the previous term.
   e. A **geometric sequence** is a sequence in which each term is obtained by multiplying the previous term by a **common ratio.**
   f. The numbers in the sequence 1, 3, 6, 10, 15, 21, . . . are called **triangular numbers.**
   g. **Finite differences** is a method of finding patterns by computing differences of consecutive terms.

4. **Inductive Reasoning**
   a. **Inductive reasoning** is the process of forming conclusions on the basis of observations, patterns, or experiments.
   b. A **counterexample** is an example that shows that a statement is false.

5. **Variables**
   a. A letter or symbol that is used to denote an unknown number is called a **variable.**
   b. An expression containing algebraic symbols is called an **algebraic expression.**

6. **Equations**
   a. An **equation** is a sentence in which the verb is "equals" ($=$). It is a statement of the equality of mathematical expressions. The following are examples of equations; no variables, $17 + 5 = 22$; and one variable, $15x + 3 = 48$.
   b. To **solve an equation** means to find values for the variable which make the equation true.
   c. Two equations that have exactly the same solutions are called **equivalent equations.**
   d. **Properties of Equality**
   (1) Addition or Subtraction Property of Equality: Add the same number to or subtract the same number from both sides of an equation.
   (2) Multiplication or Division Property of Equality: Multiply or divide both sides of an equation by the same nonzero number.
   (3) Simplification: Replace an expression in an equation by an equivalent expression.

7. **Inequalities**
   a. An **inequality** is a sentence that contains $<$, $\leq$, $>$, $\geq$, or $\neq$. It is a statement of the inequalities of mathematical expressions. The following are examples of inequalities: no variables, $12 < 50$; and one variable, $13x + 5 \geq 28$.
   b. To **solve an inequality** means to find all the values for the variable that make the inequality true.
   c. Two inequalities that have exactly the same solutions are called **equivalent inequalities.**
   d. **Properties of Inequality**
   (1) Addition or Subtraction Property of Inequality: Add the same number to or subtract the same number from both sides of an inequality.
   (2) Multiplication or Division Property of Equality: Multiply or divide both sides of an inequality by the same nonzero number; and if this number is negative, reverse the inequality sign.
   (3) Simplification: Replace an expression in an inequality by an equivalent expression.

CHAPTER TEST

1. List Polya's four steps in problem solving.

2. List the eight problem-solving strategies that were introduced in this chapter.

3. The numbers in the following sums were obtained by using every other Fibonacci number (circled).

(1) 1 (2) 3 (5) 8 (13) 21 (34)

$$1 + 2 =$$
$$1 + 2 + 5 =$$
$$1 + 2 + 5 + 13 =$$
$$1 + 2 + 5 + 13 + 34 =$$

Compute these sums. What is the relationship between these sums and the Fibonacci numbers?

4. What is the sum of numbers in row 9 of Pascal's triangle, if row 2 has the numbers 1, 2, and 1?

5. Find a pattern in the following sequences, and write the next term.

a. 1, 3, 9, 27, 81
b. 3, 6, 9, 12, 15
c. 0, 6, 12, 18, 24
d. 1, 4, 9, 16, 25
e. 3, 5, 11, 21, 35

6. Classify each sequence in problem 5 as arithmetic, geometric, or neither.

7. Use the method of finite differences to find the next three terms in each sequence.

a. 1, 5, 14, 30, 55
b. 2, 9, 20, 35

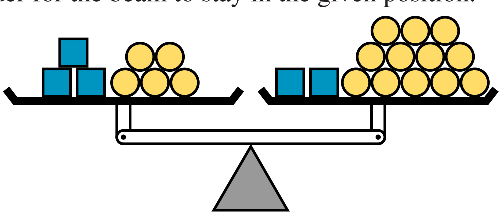
8. What is the fifth number in each of the following sequences of numbers?

a. Triangular numbers
b. Square numbers
c. Pentagonal numbers

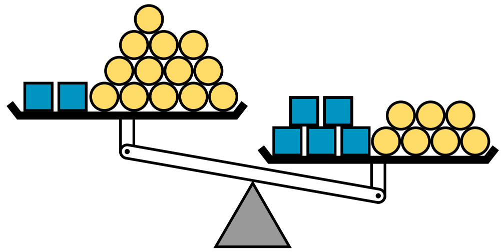
9. Find a counterexample for this conjecture: The sum of any seven consecutive whole numbers is evenly divisible by 4.

10. Determine the number of chips needed for each box in order for the beam to stay in the given position.

a.



b.



11. Solve each equation.

a. $3(x - 40) = x + 16$
b. $4x + 18 = 2(441 - 34x)$

12. Solve each inequality.

a. $7x - 3 < 52 + 2x$
b. $6x - 46 > 79 - 4x$

13. Name the property of equality or inequality used to obtain the new expression in each step.

a.
$$15x - 217 = 2x + 17$$
$$15x - 217 - 2x = 2x + 17 - 2x \quad \text{(step 1)}$$
$$13x - 217 = 17 \quad \text{(step 2)}$$
$$13x - 217 + 217 = 17 + 217 \quad \text{(step 3)}$$
$$13x = 234 \quad \text{(step 4)}$$
$$\frac{13x}{13} = \frac{234}{13} \quad \text{(step 5)}$$
$$x = 18 \quad \text{(step 6)}$$

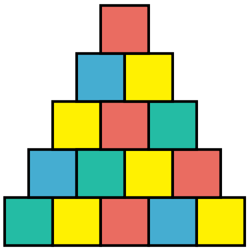
b.
$$38x < 5(23 + 3x)$$
$$38x < 115 + 15x \quad \text{(step 1)}$$
$$38x - 15x < 115 + 15x - 15x \quad \text{(step 2)}$$
$$23x < 115 \quad \text{(step 3)}$$
$$\frac{23x}{23} < \frac{115}{23} \quad \text{(step 4)}$$
$$x < 5 \quad \text{(step 5)}$$

Solve problems 14 through 19, and identify the strategy or strategies you use.

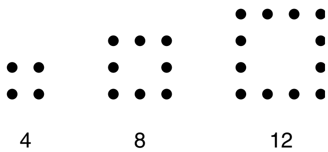
14. A 2000-foot-long straight fence has posts that are set 10 feet on center; that is, the distance between the centers of two adjacent poles is 10 feet. If the fence begins with a post and ends with a post, determine the number of posts in the entire fence.

15. In a game of chips, Pauli lost half her chips in the first round, won 50 chips, then lost half her total, and finally won 80 chips. She finished with 170 chips. How many chips did she have at the beginning of the game?

16. The following tower has 5 tiles along its base and 5 rows of tiles. How many tiles will be required to build a tower like this with 25 tiles along its base and 25 rows of tiles? Write an algebraic expression for the number of tiles in a tower with n tiles along its base and n rows of tiles.



17. Shown below are the first three squares in a pattern. Each square has one more dot on each side than the previous square.



a. How many dots are there in the fourth square?
b. How many dots are there in the 50th square?
c. Write an algebraic expression for the number of dots in the nth square.

18. There are 78 people around a table. Each person shakes hands with the people to his or her immediate right and left. How many handshakes take place? Write an algebraic expression for the number of handshakes, if there are n people around the table.

19. Two men and two boys want to cross a river, using a small canoe. The canoe can carry two boys or one man. What is the least number of times the canoe must cross the river to get everyone to the other side?

20. Find a pattern to extend the following figures of tiles.
a. How many tiles will there be in the 5th figure? The 150th figure?
b. Write an algebraic expression for the number of tiles in the nth figure.