CHAPTER 3

The Karnaugh Map

The algebraic methods developed in Chapter 2 allow us, in theory, to simplify any function. However, there are a number of problems with that approach. There is no formal method, such as first apply Property 10, then P14, etc. The approach is totally heuristic, depending heavily on experience. After manipulating a function, we often cannot be sure whether it is a minimum. We may not always find the minimum, even though it appears that there is nothing else to do. Furthermore, it gets rather difficult to do algebraic simplification with more than four or five variables. Finally, it is easy to make copying mistakes as we rewrite the equations.

In this chapter, we will examine an approach that is easier to implement, the *Karnaugh map** (sometimes referred to as a K-map). This is a graphical approach to finding suitable product terms for use in sum of product expressions. (The product terms that are "suitable" for use in minimum SOP expressions are referred to as *prime implicants*. We will define that term shortly.) The map is useful for problems of up to six variables and is particularly straightforward for most problems of three or four variables. Although there is no guarantee of finding a minimum solution, the methods we will develop nearly always produce a minimum. We will adapt the approach (with no difficulty) to finding minimum POS expressions, to problems with don't cares, and to multiple output problems.

In Chapter 4, we will introduce two other techniques that can be computerized and could be used for more than six variables (although the amount of work required to use them for hand computation is very large).

3.1 INTRODUCTION TO THE KARNAUGH MAP

In this section, we will look at the layout of two-, three-, and four-variable maps. The Karnaugh map consists of one square for each possible minterm in a function. Thus, a two-variable map has 4 squares, a three-variable map has 8 squares, and a four-variable map has 16 squares.

Three views of the two-variable map are shown in Map 3.1. In each, the upper right square, for example, corresponds to $A = 1$ and $B = 0$, minterm 2.

*This tool was introduced in 1953 by Maurice Karnaugh.

Map 3.1 Two-variable Karnaugh maps.

$A'B'$	AB'
$A'B$	AB

		A
	m_0	m_2
B	m_1	m_3

$B \backslash A$	0	1
0	0	2
1	1	3

When we plot a function, we put a 1 in each square corresponding to a minterm that is included in the function, and put a 0 in or leave blank those squares not included in the function. For functions with don't cares, an X goes in the square for which the minterm is a don't care. Map 3.2 shows examples of these.

Map 3.2 Plotting functions.

$b \backslash a$	0	1
0	1	
1		1

$f(a, b) = \Sigma m\,(0, 3)$

$B \backslash A$	0	1
0	1	X
1		1

$g(A, B) = \Sigma m\,(0, 3) + \Sigma d\,(2)$

Three-variable maps have eight squares, arranged in a rectangle as shown in Map 3.3.*

Map 3.3 Three-variable maps.

$C \backslash AB$	$A'B'$ 00	$A'B$ 01	AB 11	AB' 10
C' 0	$A'B'C'$	$A'BC'$	ABC'	$AB'C'$
C 1	$A'B'C$	$A'BC$	ABC	$AB'C$

$C \backslash AB$	00	01	11	10
0	0	2	6	4
1	1	3	7	5

*Some people label the row(s) of the map with the first variable(s) and the columns with the others. The three-variable map then looks like

$A \backslash BC$	00	01	11	10
0	0	1	3	2
1	4	5	7	6

This version of the map produces the same results as the other.

Notice that the last two columns are not in numeric order. That is the key idea that makes the map work. By organizing the map that way, the minterms in adjacent squares can always be combined using the adjacency property,

P9a. $ab + ab' = a$

EXAMPLE 3.1

$m_0 + m_1$: $A'B'C' + A'B'C = A'B'$

$m_4 + m_6$: $AB'C' + ABC' = AC'$

$m_7 + m_5$: $ABC + AB'C = AC$

Also, the outside columns (and the outside rows when there are four rows) are adjacent. Thus,

$m_0 + m_4$: $A'B'C' + AB'C' = B'C'$

$m_1 + m_5$: $A'B'C + AB'C = B'C$

If we had ordered the columns in numeric order, as shown in Map 3.4 (where the algebraic version of the minterms is shown only for m_2 and m_4), we would not be able to combine adjacent squares:

Map 3.4 **Incorrect** arrangement of the map.

C \ AB	00	01	10	11
0	0	2 $A'BC'$	4 $AB'C'$	6
1	1	3	5	7

$$m_2 + m_4 = A'BC' + AB'C' = C'(A'B + AB')$$

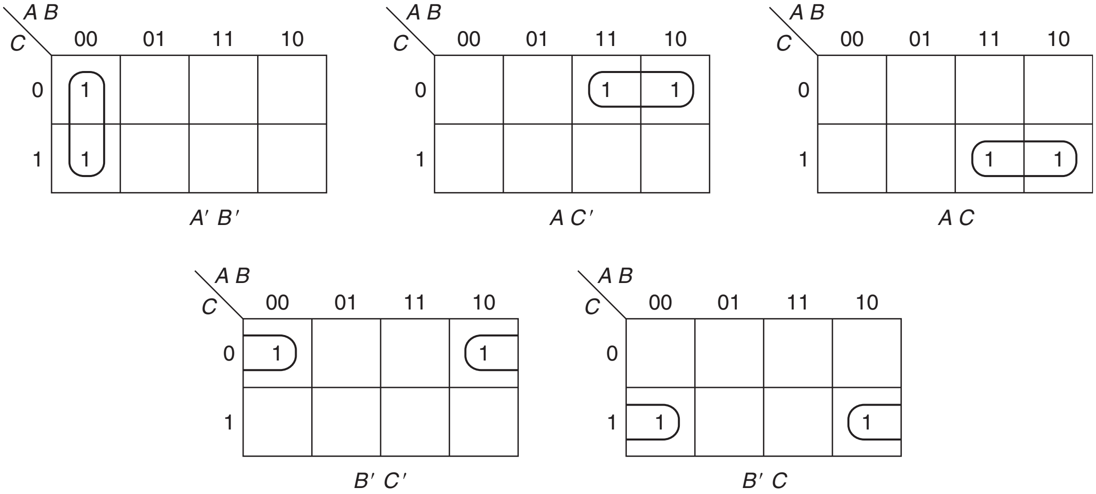
However, we cannot manipulate that into a single term.

Product terms that correspond to the sum of two minterms appear as two adjacent 1's on the map. The terms of Example 3.1 are shown in Map 3.5.

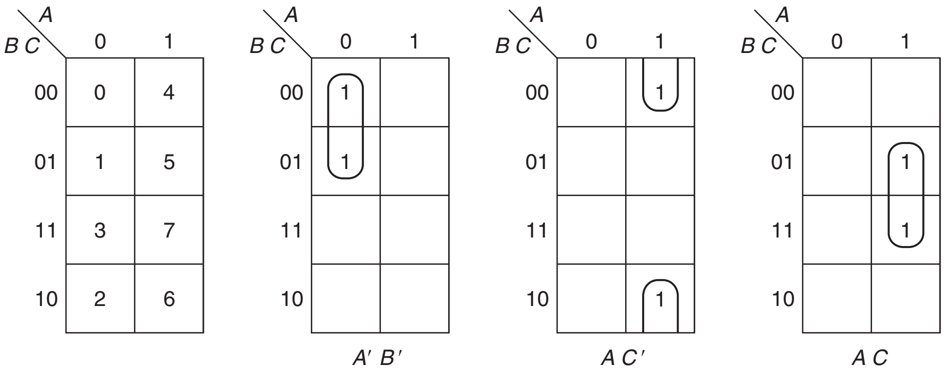
It is sometimes more convenient to draw the map in a vertical orientation (that is, two columns and four rows) as shown in Map 3.6. Both versions of the map produce the same results.

In reading the map, it is useful to label the pairs of columns (in those arrangements where there are four columns) as shown in Map 3.7. Thus, 1's in squares 4 and 6 are in the A columns and the C' row (that is, not in the C row), producing the AC' term as shown earlier.

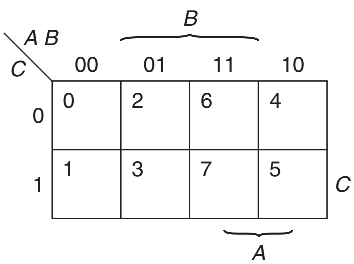
Map 3.5 Product terms corresponding to groups of two.



Map 3.6 Vertical orientation of three-variable map.



Map 3.7 Map with columns labeled.



The four-variable map consists of 16 squares in the 4 by 4 arrangement shown in Map 3.8.

As with the three-variable map, 1's in two adjacent squares (where the top and bottom rows as well as the left and right columns are considered to be adjacent) correspond to a single product term (combined using P9a). Example 3.2 shows three such terms.

Map 3.8 The four-variable map.

CD \ AB	00	01	11	10
00	0	4	12	8
01	1	5	13	9
11	3	7	15	11
10	2	6	14	10

CD \ AB	00	01	11	10
00	$A'B'C'D'$	$A'BC'D'$	$ABC'D'$	$AB'C'D'$
01	$A'B'C'D$	$A'BC'D$	$ABC'D$	$AB'C'D$
11	$A'B'CD$	$A'BCD$	$ABCD$	$AB'CD$
10	$A'B'CD'$	$A'BCD'$	$ABCD'$	$AB'CD'$

EXAMPLE 3.2

$m_{13} + m_9$: $ABC'D + AB'C'D = AC'D$

$m_3 + m_{11}$: $A'B'CD + AB'CD = B'CD$

$m_0 + m_2$: $A'B'C'D' + A'B'CD' = A'B'D'$

CD \ AB	00	01	11	10
00				
01			1	1
11				
10				

$AC'D$

CD \ AB	00	01	11	10
00				
01				
11	1			1
10				

$B'CD$

CD \ AB	00	01	11	10
00	1			
01				
11				
10	1			

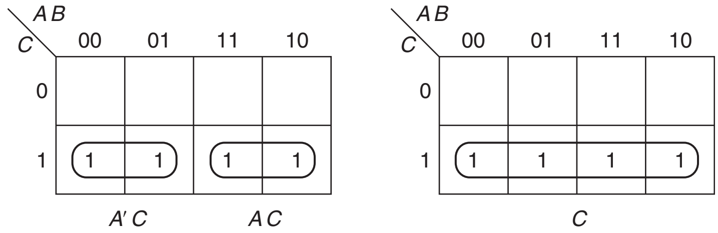
$A'B'D'$

Up to this point, all of the product terms that we have shown correspond to two minterms combined using P9a. These correspond to a product term with one literal missing, that is, with only two literals in a three-variable function and three literals in a four-variable function. Let us next look at the maps of Map 3.9 with a group of four 1's.

On the map to the left, we have circled two groups of two, one forming the term $A'C$ and the other forming the term AC. Obviously, P9a can be applied again to these two terms, producing

$$A'C + AC = C$$

Map 3.9 A group of four 1's.



That is shown on the map to the right as a rectangle of four 1's. In general, rectangles of four 1's will correspond to a product term with two of the variables missing (that is, a single literal for three-variable problems and a two-literal term for four-variable problems).

We could have factored C from all of the terms producing

$$A'B'C + A'BC + ABC + AB'C = C(A'B' + A'B + AB + AB')$$

However, the sum inside the parentheses is just a sum of all of the minterms of A and B; that must be 1. Thus, we can get the result in just that one step. Indeed, we could have added a secondary property to P9, namely,

P9aa. $a'b' + a'b + ab + ab' = 1$

P9bb. $(a' + b')\,(a' + b)\,(a + b)\,(a + b') = 0$

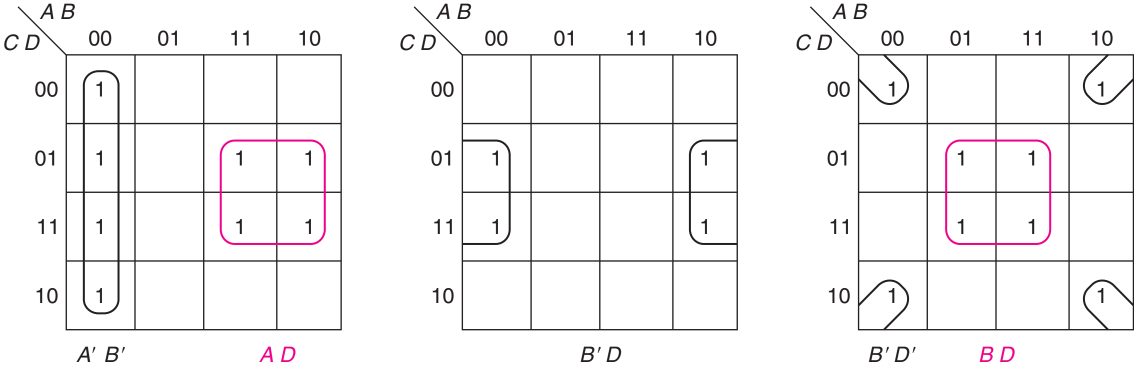
These can be proved by repeated application of P9, first to the first two terms, then to the last two terms, and finally to the resulting terms as shown

$$(a'b' + a'b) + (ab + ab') = (a') + (a) = 1$$
$$[(a' + b')\,(a' + b)][(a + b)\,(a + b')] = [a'][a] = 0$$

Some examples of such groups for four-variable problems are shown in Map 3.10.

The easiest way to identify the term from the map is by determining in which row(s) and column(s) all of the 1's are located. Thus, on the first map, the 1's in the group on the left are all in the 00 ($A'B'$) column and thus the term is $A'B'$. The other group has its 1's in the 11 and 10 columns; the common feature is the 1 in the A position (which corresponds to A). Furthermore, the 1's are in the 01 and 11 rows; there is a common 1 in the D position. Thus, the term is AD. In the middle map, the

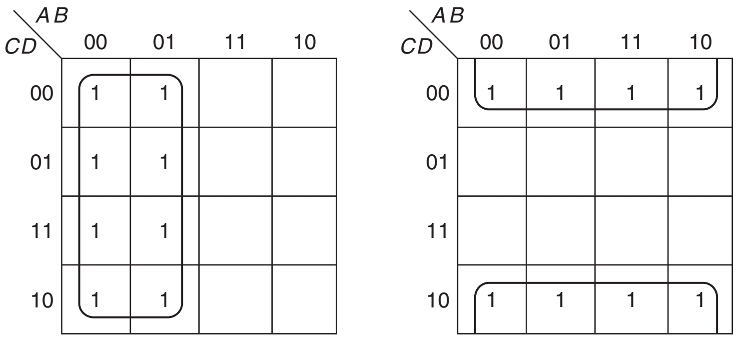
Map 3.10 Examples of groups of four.



1's are in the 00 and 10 columns, producing B' and the 01 and 11 rows, resulting in D; the term is thus $B'D$. (Notice, by the way, that that term also appears on the first map, even though it was not circled.) On the last map, the four corners produce the term $B'D'$ (since all the 1's are in the 00 or 10 columns and the 00 or 10 rows). The middle group is BD. Any of these terms could also be obtained algebraically by first writing the minterms, then applying P10a to pairs of terms, and then applying it again to the two terms that resulted (as we did for the three-variable example). However, the whole idea of the map is to eliminate the need to do algebra.

Two adjacent groups of four can be combined in a similar way to form a group of eight squares (with three of the literals missing). Two such groups are shown in Map 3.11. The terms are A' for the map on the left and D' for the map on the right.

Map 3.11 Groups of eight.



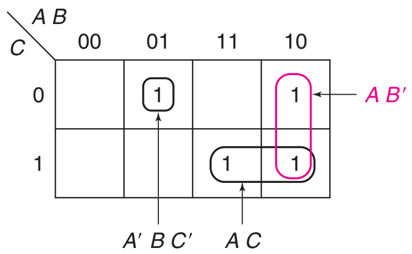
We can plot any function on the map. Either, we know the minterms, and use that form of the map, or we put the function in SOP form and plot each of the product terms.

EXAMPLE 3.3

Map

$$F = AB' + AC + A'BC'$$

The map for F follows, with each of the product terms circled. Each of the two-literal terms corresponds to two squares on the map (since one of the variables is missing). The AB' term is in the 10 column. The AC term is in the $C = 1$ row and in the 11 and 10 columns (with a common 1 in the A position). Finally, the minterm $A'BC'$ corresponds to one square, in the 01 ($A'B$) column and in the $C = 0$ row.



We could have obtained the same map by first expanding F to minterm form algebraically, that is,

$$\begin{aligned} F &= AB'(C' + C) + AC(B' + B) + A'BC' \\ &= AB'C' + AB'C + AB'C + ABC + A'BC' \\ &= m_4 + m_5 + m_5 + m_7 + m_2 \\ &= m_2 + m_4 + m_5 + m_7 \end{aligned}$$

(removing duplicates and reordering)

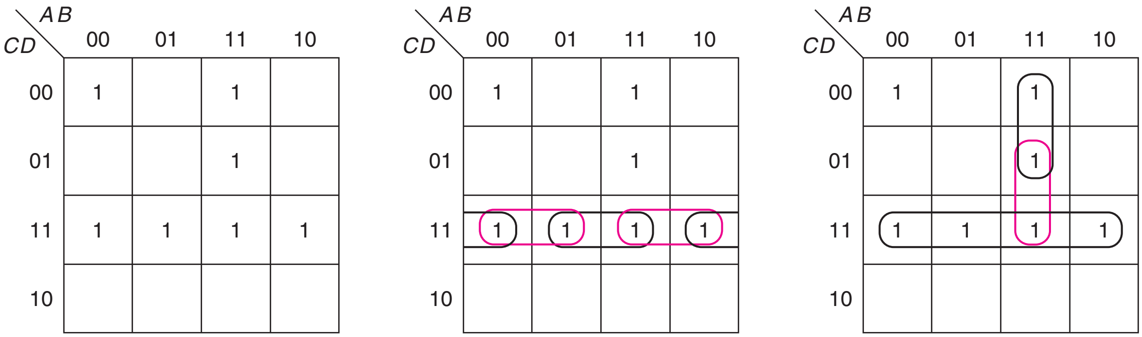
We can then use the numeric map and produce the same result.

C \ AB	00	01	11	10
0	0	2: 1	6	4: 1
1	1	3	7: 1	5: 1

We are now ready to define some terminology related to the Karnaugh map. An *implicant* of a function is a product term that can be used in an SOP expression for that function, that is, the function is 1 whenever the implicant is 1 (and maybe other times, as well). From the point of view of the map, an implicant is a rectangle of 1, 2, 4, 8, . . . (any power of 2) 1's.* That rectangle may not include any 0's. All minterms are implicants.

Consider the function, *F,* of Map 3.12. The second map shows the first four groups of 2; the third map shows the other groups of 2 and the group of 4.

Map 3.12 A function to illustrate definitions.



The implicants of *F* are

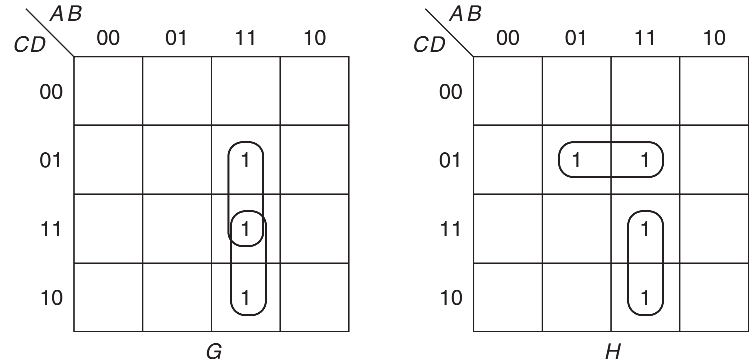
Minterms	*Groups of 2*	*Groups of 4*
$A'B'C'D'$	$A'CD$	CD
$A'B'CD$	BCD	
$A'BCD$	ACD	
$ABC'D'$	$B'CD$	
$ABC'D$	ABC'	
$ABCD$	ABD	
$AB'CD$		

Any SOP expression for *F* must be a sum of implicants. Indeed, we must choose enough implicants such that each of the 1's of *F* are included in at least one of these implicants. Such an SOP expression is sometimes referred to as a *cover* of *F*, and we sometimes say that an implicant *covers* certain minterms (for example, ACD covers m_{11} and m_{15}).

Implicants must be rectangular in shape and the number of 1's in the rectangle must be a power of 2. Thus, neither of the functions whose maps are shown in Example 3.4 are covered by a single implicant, but rather by the sum of two implicants each (in their simplest form).

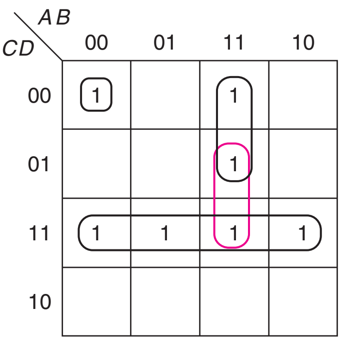
*We will expand the definition of an implicant to include maps with don't cares in Section 3.3.

EXAMPLE 3.4



G consists of three minterms, $ABC'D$, $ABCD$, and $ABCD'$, in the shape of a rectangle. It can be reduced no further than is shown on the map, namely, to $ABC + ABD$, since it is a group of three 1's, not two or four. Similarly, H has the same three minterms plus $A'BC'D$; it is a group of four, but not in the shape of a rectangle. The minimum expression is, as shown on the map, $BC'D + ABC$. (Note that ABD is also an implicant of G, but it includes 1's that are already included in the other terms.)

Map 3.13 Prime implicants.



A *prime implicant* is an implicant that (from the point of view of the map) is not fully contained in any one other implicant. For example, it is a rectangle of two 1's that is not part of a single rectangle of four 1's. On Map 3.13, all of the prime implicants of F are circled. They are $A'B'C'D'$, ABC', ABD, and CD. Note that the only minterm that is not part of a larger group is m_0 and that the other four implicants that are groups of two 1's are all part of the group of four.

From an algebraic point of view, a prime implicant is an implicant such that if any literal is removed from that term, it is no longer an implicant. From that viewpoint, $A'B'C'D'$ is a prime implicant because $B'C'D'$, $A'C'D'$, $A'B'D'$, and $A'B'C'$ are not implicants (that is, if we remove any literal from that term, we get a term that is 1 for some input combinations for which the function is to be 0). However, ACD is not a prime implicant since when we remove A, leaving CD, we still have an implicant. (Surely, the graphical approach of determining which implicants are prime implicants is easier than the algebraic method of attempting to delete literals.)

The purpose of the map is to help us find minimum SOP expressions, where we defined minimum as being minimum number of product terms (implicants), and among those with the same number of implicants, the ones with the fewest number of literals. However, the only product terms that we need consider are prime implicants. Why? Say we found an implicant that was not a prime implicant. Then, it must be contained in some larger implicant, a prime implicant. But that larger implicant (say four 1's rather than two) has fewer literals. That alone makes a solution using the term that is not a prime implicant not a minimum. (For example, CD has two literals, whereas, ACD has three.) Furthermore, that larger implicant covers more 1's, which often will mean that we need fewer terms.

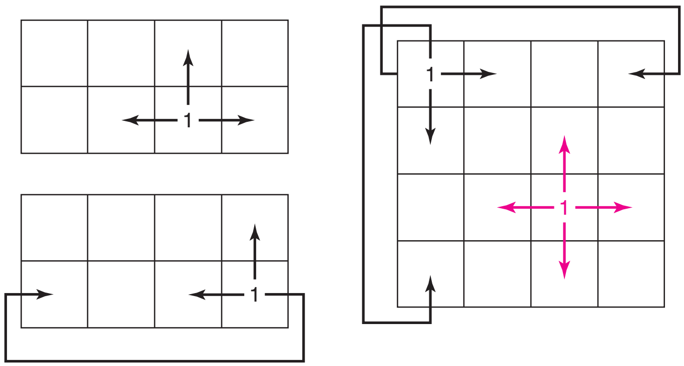
An *essential prime implicant* is a prime implicant that includes at least one 1 that is not included in any other prime implicant. (If we were to circle all of the prime implicants of a function, the essential prime implicants are those that circle at least one 1 that no other prime implicant circles.) In the example of Map 3.13, $A'B'C'D'$, ABC', and CD are essential prime implicants; ABD is not. The term *essential* is derived from the idea that we must use that prime implicant in any minimum SOP expression. A word of caution is in order. There will often be a prime implicant that is used in a minimum solution (even in all minimum solutions when more than one equally good solution exists) that is not "essential." That happens when each of the 1's covered by this prime implicant could be covered in other ways. We will see examples of that in Section 3.2. *[SP 1; EX 1]*

3.2 MINIMUM SUM OF PRODUCT EXPRESSIONS USING THE KARNAUGH MAP

In this section, we will describe two methods for finding minimum SOP expressions using the Karnaugh map. Although these methods involve some heuristics, we can all but guarantee that they will lead to a minimum SOP expression (or more than one when multiple solutions exist) for three- and four-variable problems. (It also works for five- and six-variable maps, but our visualization in three dimensions is more limited. We will discuss this in detail in Section 3.6.)

In the process of finding prime implicants, we will be considering each of the 1's on the map starting with the most *isolated* 1's. By isolated, we mean that there are few (or no) adjacent squares with a 1 in it. In an ***n***-variable map, each square has ***n*** adjacent squares. Examples for three- and four-variable maps are shown in Map 3.14.

Map 3.14 Adjacencies on three- and four-variable maps.



Map Method 1

1. Find all essential prime implicants. Circle them on the map and mark the minterm(s) that make them essential with a star (★). Do this by examining each 1 on the map that has not already been circled. It is usually quickest to start with the most isolated 1's, that is, those that have the fewest adjacent squares with 1's in them.

2. Find *enough* other prime implicants to cover the function. Do this using two criteria:

a. Choose a prime implicant that covers as many new 1's (that is, those not already covered by a chosen prime implicant).

b. Avoid leaving isolated uncovered 1's.

It is often obvious what "enough" is. For example, if there are five uncovered 1's and no prime implicants cover more than two of them, then we need at least three more terms. Sometimes, three may not be sufficient, but it usually is.

We will now look at a number of examples to demonstrate this method. First, we will look at the example used to illustrate the definitions.

EXAMPLE 3.5

As noted, m_0 has no adjacent 1's; therefore, it ($A'B'C'D'$) is a prime implicant. Indeed, it is an essential prime implicant, since no other prime implicant covers this 1. (That is always the case when minterms are prime implicants.) The next place that we look is m_{12}, since it has only one adjacent 1. Those 1's are covered by prime implicant ABC'. Indeed, no other prime implicant covers m_{12}, and thus ABC' is essential. (Whenever we have a 1 with only one adjacent 1, that group of two is an essential prime implicant.) At this point, the map has become

CD \ AB	00	01	11	10
00	1★		1★	
01			1	
11	1	1	1	1
10				

and

$$F = A'B'C'D' + ABC' + \cdots$$

Each of the 1's that have not yet been covered are part of the group of four, *CD*. Each has two adjacent squares with 1's that are part of that group. That will always be the case for a group of four. (Some squares, such as m_{15} may have more than two adjacent 1's.) *CD* is essential because no other prime implicant covers m_3, m_7, or m_{11}. However, once that group is circled, we have covered the function:

CD \ AB	00	01	11	10
00	1*		1*	
01			1	
11	1*	1*	1	1*
10				

resulting in

$$F = A'B'C'D + ABC' + CD$$

In this example, once we have found the essential prime implicants, we are done; all of the 1's have been covered by one (or more) of the essential prime implicants. We do not need step 2. There may be other prime implicants that were not used (such as *ABD* in this example).

EXAMPLE 3.6

We start looking at the most isolated 1, m_{11}. It is covered only by the group of two shown, *wyz*. The other essential prime implicant is $y'z'$ because of m_0, m_8, or m_{12}. None of these are covered by any other prime implicant; each makes that prime implicant essential. The second map shows these two terms circled.

yz \ wx	00	01	11	10
00	1	1	1	1
01		1		
11		1	1	1
10				

yz \ wx	00	01	11	10
00	1*	1	1*	1*
01		1		
11		1	1	1*
10				

That leaves two 1's uncovered. Each of these can be covered by two different prime implicants, but the only way to cover them both with one term is shown on the first of the maps below.

Thus, the minimum sum of product solution is

$$f = y'z' + wyz + w'xz$$

yz \ wx	00	01	11	10
00	1*	1	1*	1*
01		1		
11		1	1	1*
10				

yz \ wx	00	01	11	10
00	1*	1	1*	1*
01		1		
11		1	1	1*
10				

The other two prime implicants are $w'xy'$ and xyz, circled in brown on the last map. They are redundant, however, since they cover no new 1's. Even though $w'xz$ must be used in a minimum solution, it does not meet the definition of an essential prime implicant; each of the 1's covered by it can be covered by other prime implicants.

We will next look at the "dead end" example from Chapter 2 (Example 2.2).

EXAMPLE 3.7

$$f = a'b'c' + a'bc' + a'bc + ab'c'$$

In the first attempt at algebraic manipulation, we grouped the first two minterms. But, as can be seen on the left-hand map below, the two 1's that are left could not be combined and resulted in a three-term solution. Furthermore, $a'c'$ is not an essential prime implicant. If, on the other hand, we used the map, we could see that choosing the two essential prime implicants on the right-hand map includes all of the minterms and produces the solution

$$f = a'b + b'c'$$

c \ ab	00	01	11	10
0	1	1		1
1		1		

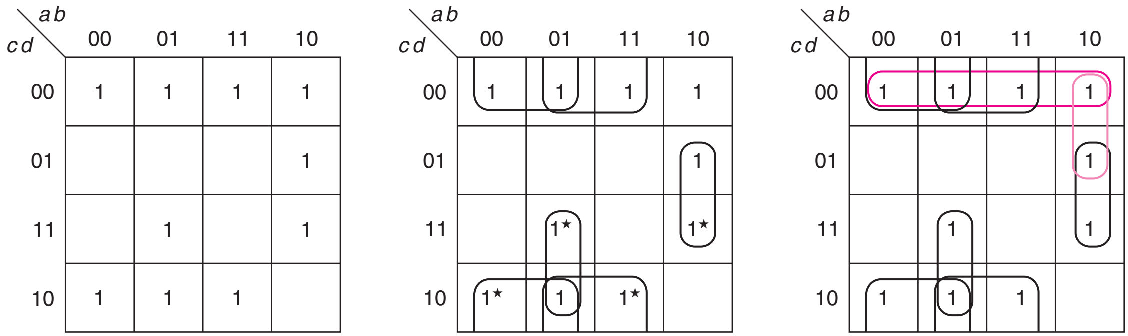
c \ ab	00	01	11	10
0	1	1		1*
1		1*		

Sometimes, after selecting all of the essential prime implicants, there are two choices for covering the remaining 1's, but only one of these produces a minimum solution, as in Example 3.8.

EXAMPLE 3.8

$f(a, b, c, d) = \Sigma m(0, 2, 4, 6, 7, 8, 9, 11, 12, 14)$

The first map shows the function and the second shows all essential prime implicants circled. In each case, one of the 1's (as indicated with a star, *) can be covered by only that prime implicant. (That is obvious from the last map, where the remaining two prime implicants are circled.)



Only one 1 (m_8) is not covered by an essential prime implicant. It can be covered in two ways, by a group of four (in brown) and a group of two (tan). Clearly, the group of four provides a solution with one less literal, namely,

$$f = a'd' + bd' + a'bc + ab'd + c'd'$$

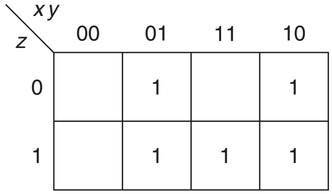
When asking whether a 1 makes a group of four an essential prime implicant on a four-variable map, we need find only two adjacent 0's. If there are fewer than two adjacent 0's, this 1 must be either in a group of eight or part of two or more smaller groups. Note that in Example 3.8, m_2 and m_{14} have two adjacent 0's, and thus each makes a prime implicant essential. In contrast, m_0, m_4, m_8, and m_{12} each have only one adjacent 0 and are each covered by two or three prime implicants.

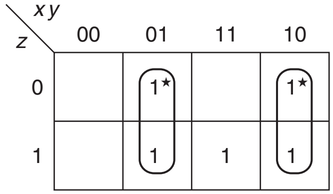
We will now consider some examples with multiple minimum solutions, starting with the three-variable function used to illustrate the definition of terminology in Section 2.2.3.

EXAMPLE 3.9

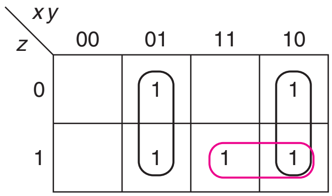
$x'yz' + x'yz + xy'z' + xy'z + xyz$

A map of that function is shown on the left. The two essential prime implicants are shown on the map on the right.

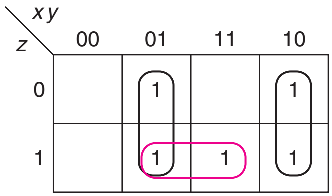




After finding the two essential prime implicants, m_7 is still uncovered. The following maps show the two solutions.



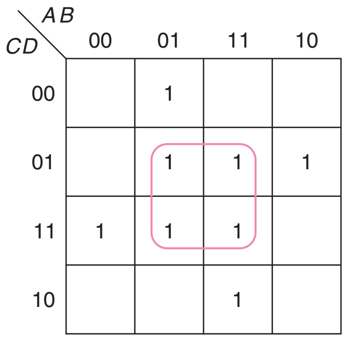
$x'y + xy' + xz$

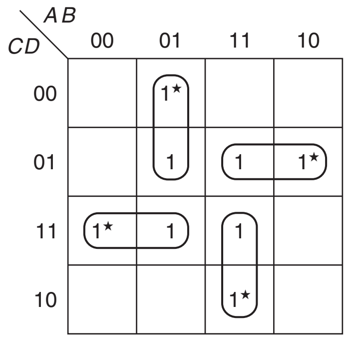


$x'y + xy' + yz$

EXAMPLE 3.10

This example is one we call "don't be greedy."





At first glance, one might want to take the only group of four (circled in tan). However, that term is not an essential prime implicant, as is obvious once we circle all of the essential prime implicants and find that the four 1's in the center are covered. Thus, the minimum solution is

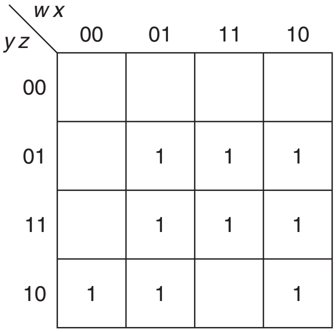
$$G = A'BC' + A'CD + ABC + AC'D$$

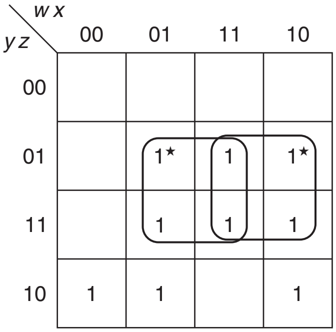
EXAMPLE 3.11

$g(w, x, y, z) = \Sigma m(2, 5, 6, 7, 9, 10, 11, 13, 15)$

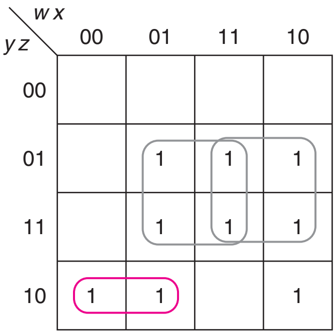
The function is mapped first, and the two essential prime implicants are shown on the second map, giving

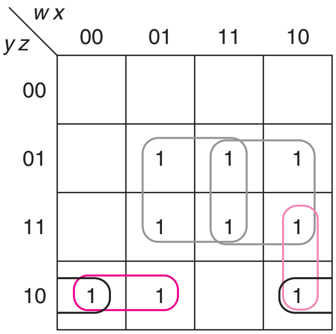
$g = xz + wz + \cdots$

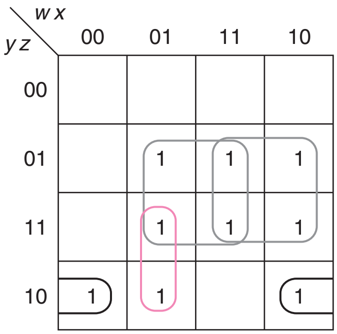




Although m_2 looks rather isolated, it can indeed be covered by $w'yz'$ (with m_6) or by $x'yz'$ (with m_{10}). After choosing the essential prime implicants, the remaining three 1's can each be covered by two different prime implicants. Since three 1's still need to be covered (after choosing the essential prime implicants), and since all the remaining prime implicants are groups of two and thus have three literals, we need at least two more of these prime implicants. Indeed, there are three ways to cover the remaining 1's with two more prime implicants. Using the first criteria, we choose one of the prime implicants that covers two new 1's, $w'yz'$, as shown on the left-hand map.







Then, only m_{10} remains, and it can be covered either by $wx'y$ or by $x'yz'$, as shown on the center map. Similarly, we could have started with $x'yz'$, in which case we could use $w'xy$ to complete the cover, as on the right-hand map. (We could also have chosen $w'yz'$, but that repeats one of the answers from before.) Thus, the three solutions are

$g = xz + wz + w'yz' + wx'y$

$g = xz + wz + w'yz' + x'yz'$

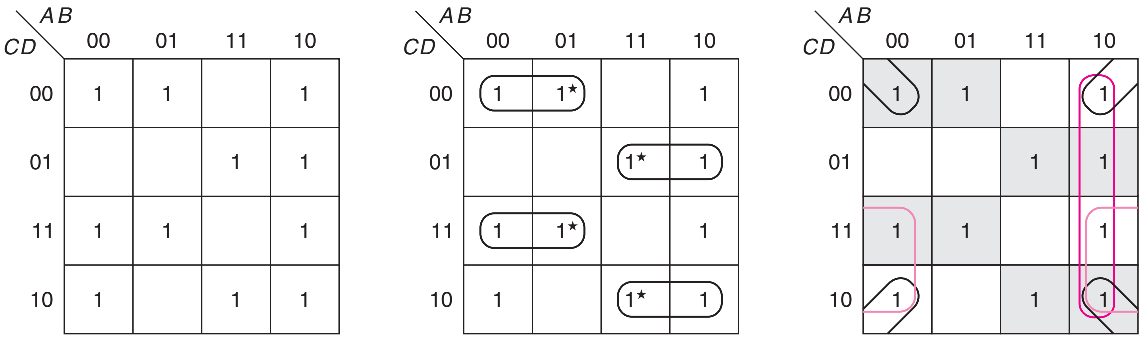
$g = xz + wz + x'yz' + w'xy$

All three minimum solutions require four terms and 10 literals.

At this point, it is worth stating the obvious.

> **COMMON MISTAKE:** If there are multiple solutions, all minimum solutions must have the same number of terms and literals. If, for example, you find a minimum solution with three terms and seven literals, no solution with four terms is minimum, and no solution with three terms and eight literals is minimum.

EXAMPLE 3.12



The four essential prime implicants are shown on the second map, leaving three 1's to be covered:

$$F = A'C'D' + AC'D + A'CD + ACD' + \cdots$$

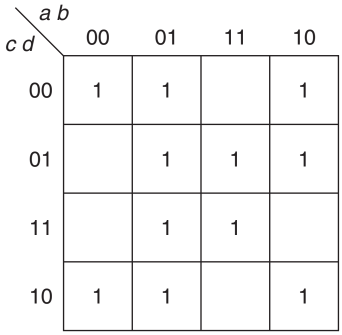
These squares are shaded on the right-hand map. The three other prime implicants, all groups of four, are also shown on the right-hand map. Each of these covers two of the remaining three 1's (no two the same). Thus, any two of $B'D'$, AB', and $B'C$ can be used to complete the minimum SOP expression. The resulting three equally good answers are

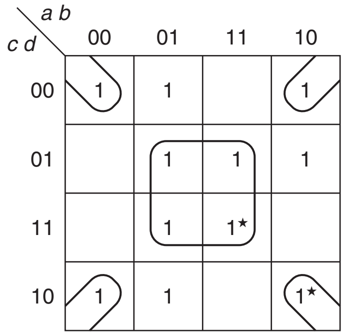
$$F = A'C'D' + AC'D + A'CD + ACD' + B'D' + AB'$$

$$F = A'C'D' + AC'D + A'CD + ACD' + B'D' + B'C$$

$$F = A'C'D' + AC'D + A'CD + ACD' + AB' + B'C$$

EXAMPLE 3.13

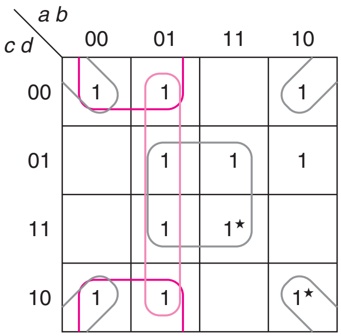


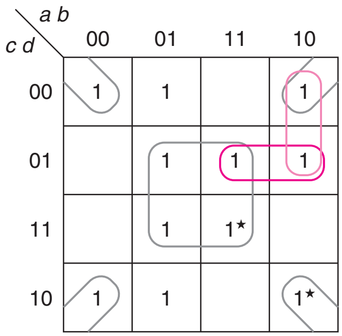


Once again there are two essential prime implicants, as shown on the right-hand map. The most isolated 1's are m_{10} and m_{15}. Each has only two adjacent 1's. But all of the 1's in groups of four have at least two adjacent 1's; if there are only two, then that minterm will make the prime implicant essential. (Each of the other 1's in those groups of four has at least three adjacent 1's.) The essential prime implicants give us

$$f = b'd' + bd + \cdots$$

There are three 1's not covered by the essential prime implicants. There is no single term that will cover all of them. However, the two in the 01 column can be covered by either of two groups of four, as shown on the map on the left ($a'd'$ circled in brown, $a'b$ in tan). And, there are two groups of two that cover m_9 ($ac'd$ circled in brown, $ab'c'$ in tan), shown on the map to the right.





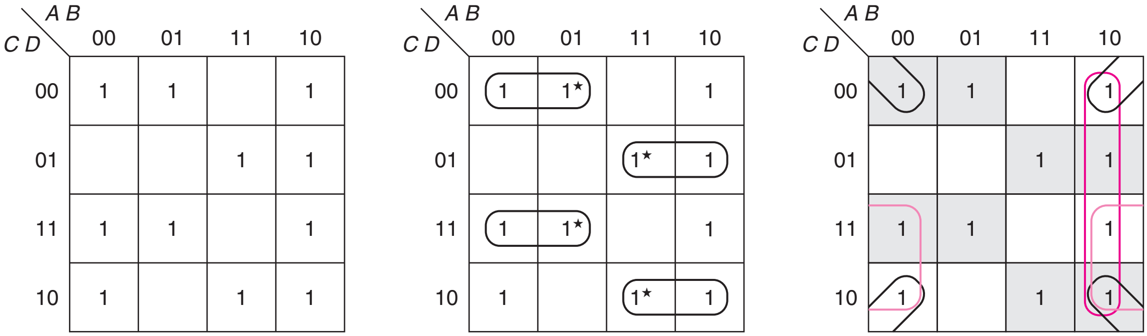
We can choose one term from the first pair and (independently) one from the second pair. Thus, there are four solutions. We can write the solution as shown, where we take one term from within each bracket

$$f = b'd' + bd + \begin{Bmatrix} a'd' \\ a'b \end{Bmatrix} + \begin{Bmatrix} ac'd \\ ab'c' \end{Bmatrix}$$

or we can write out all four expressions

$$
\begin{aligned}
f &= b'd' + bd + a'd' + ac'd \\
&= b'd' + bd + a'd' + ab'c' \\
&= b'd' + bd + a'b + ac'd \\
&= b'd' + bd + a'b + ab'c'
\end{aligned}
$$

EXAMPLE 3.14



The four essential prime implicants are shown on the second map, leaving three 1's to be covered:

$$F = A'C'D' + AC'D + A'CD + ACD' + \cdots$$

These squares are shaded on the third map. The three other prime implicants, all groups of four, are also shown on the third map. Each of these covers two of the remaining three 1's (no two the same). Thus any two of $B'D'$, AB', and $B'C$ can be used to complete the minimum sum of products expression. The resulting three equally good answers are

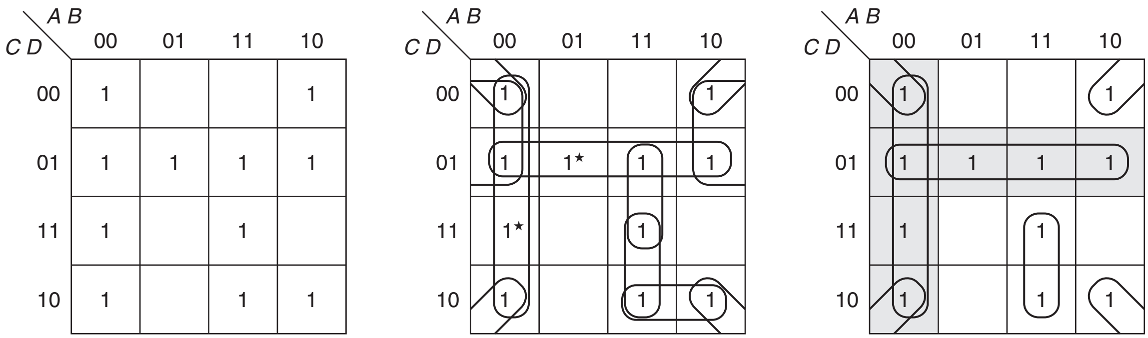
$$
\begin{aligned}
F &= A'C'D' + AC'D + A'CD + ACD' + B'D' + AB' \\
F &= A'C'D' + AC'D + A'CD + ACD' + B'D' + B'C \\
F &= A'C'D' + AC'D + A'CD + ACD' + AB' + B'C
\end{aligned}
$$

Before doing additional (more complex) examples, we will introduce a somewhat different method for finding minimum sum of products expressions.

Map Method 2

1. Circle all of the prime implicants.
2. Select all essential prime implicants; they are easily identified by finding 1's that have only been circled once.
3. Then choose enough of the other prime implicants (as in Method 1). Of course, these prime implicants have already been identified in step 1.

EXAMPLE 3.15



All of the prime implicants have been circled on the center map. Note that m_0 has been circled three times and that several minterms have been circled twice. However, m_3 and m_5 have only been circled once. Thus, the prime implicants that cover them, $A'B'$ and $C'D$ are essential. On the third map, we have shaded the part of the map covered by essential prime implicants to highlight what remains to be covered. There are four 1's, each of which can be covered in two different ways, and five prime implicants not used yet. No prime implicant covers more than two new 1's; thus, we need at least two more terms. Of the groups of four, only $B'D'$ covers two new 1's; $B'C'$ covers only one. Having chosen the first group, we must use ABC to cover the rest of the function, producing

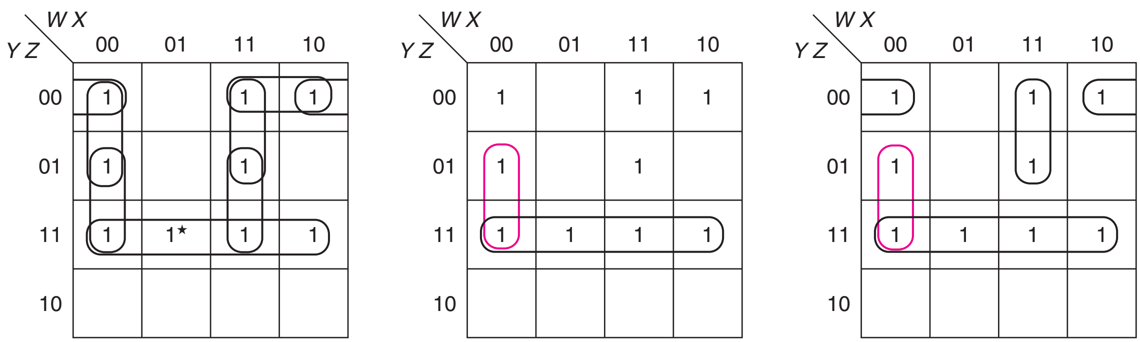
$$F = A'B' + C'D + B'D' + ABC$$

Notice that this is the only set of four prime implicants (regardless of size) that covers the function.

EXAMPLE 3.16

$$G(A, B, C, D) = \Sigma m(0, 1, 3, 7, 8, 11, 12, 13, 15)$$

This is a case with more 1's left uncovered after finding the essential prime implicant. The first map shows all the prime implicants circled. The only essential prime implicant is YZ; there are five 1's remaining to be covered. Since all of the other prime implicants are groups of two, we need three more prime implicants. These 1's are organized in a chain, with each prime implicant linked to one on either side. If we are looking for just one solution, we should follow the guidelines from Method 1, choosing two terms that



each cover new 1's and then select a term to cover the remaining 1. One such example is shown on the third map, starting with WXY' and $X'Y'Z'$. If we wish to find all of the minimum solutions, one approach is to start at one end of the chain (as shown in the second map). (We could have started at the other end, with m_{13}, and achieved the same results.) To cover m_1, we must either use $W'X'Z$, as shown in brown above, or $W'X'Y'$ (as shown on the maps below). Once we have chosen $W'X'Z$, we have no more freedom, since the terms shown on the third map above are the only way to cover the remaining 1's in two additional terms. Thus, one solution is

$$F = YZ + W'X'Z + X'Y'Z' + WXY'$$

The next three maps show the solutions using $W'X'Y'$ to cover m_0.

YZ \ WX	00	01	11	10
00	1		1	1
01	1		1	
11	1	1	1	1
10				

YZ \ WX	00	01	11	10
00	1		1	1
01	1		1	
11	1	1	1	1
10				

YZ \ WX	00	01	11	10
00	1		1	1
01	1		1	
11	1	1	1	1
10				

After choosing $W'X'Y'$, there are now three 1's to be covered. We can use the same last two terms as before (left) or use $WY'Z'$ to cover m_8 (right two maps). The other three solutions are thus

$$F = YZ + W'X'Y' + X'Y'Z' + WXY'$$

$$F = YZ + W'X'Y' + WY'Z' + WXY'$$

$$F = YZ + W'X'Y' + WY'Z' + WXZ$$

We will now look at some examples with no essential prime implicants. A classic example of such a function is shown in Example 3.17.

EXAMPLE 3.17

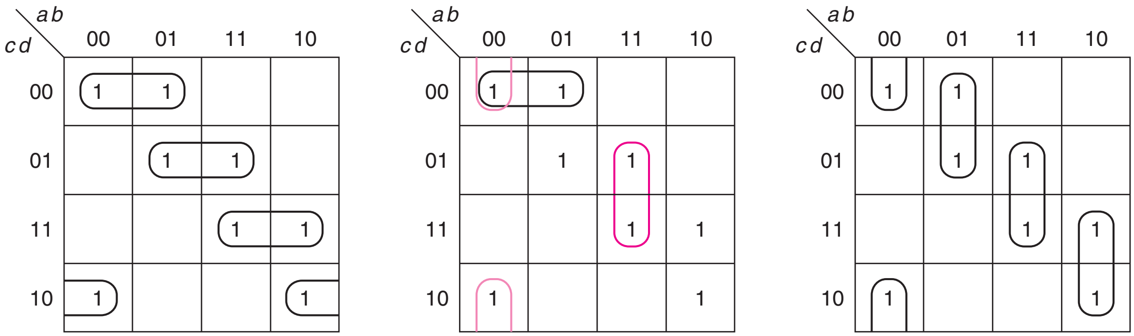
cd \ ab	00	01	11	10
00	1	1		
01		1	1	
11			1	1
10	1			1

cd \ ab	00	01	11	10
00	1	1		
01		1	1	
11			1	1
10	1			1

cd \ ab	00	01	11	10
00	1	1		
01		1	1	
11			1	1
10	1			1

There are eight 1's; all prime implicants are groups of two. Thus, we need at least four terms in a minimum solution. There is no obvious place to start; thus, in the second map, we arbitrarily chose one of the terms, $a'c'd'$. Following the guidelines of step 2, we should then choose a second term that covers two new 1's, in such a way as not to leave an isolated uncovered 1. One such term is $bc'd$, as shown on the third map. Another possibility would be $b'cd'$ (the group in the last row). As we will see, that group will also be used. Repeating that procedure, we get the cover on the left-hand map below,

$$f = a'c'd' + bc'd + acd + b'cd'$$



Notice that if, after starting with $a'c'd'$, we chose one of the prime implicants not included in this solution above, such as abd, shown on the middle map, we leave an isolated uncovered 1 (which would require a third term) plus three more 1's (which would require two more terms). A solution using those two terms would require five terms (obviously not minimum since we found one with four). Another choice would be a term such as $a'b'd'$, which covers only one new 1, leaving five 1's uncovered. That, too, would require at least five terms.

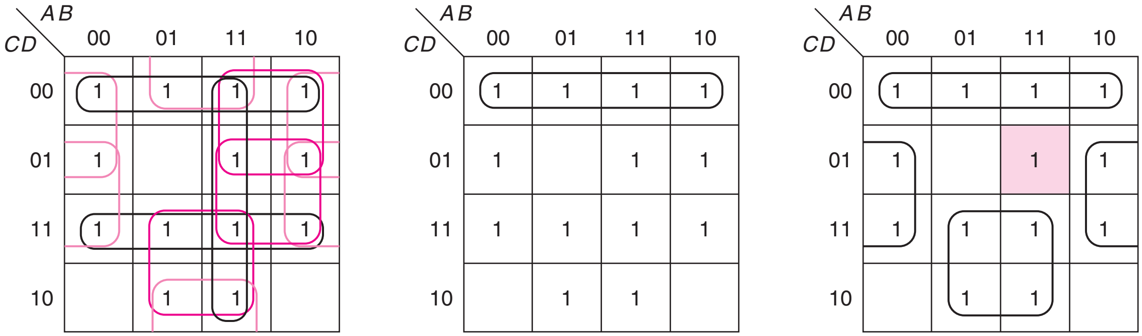
The other solution to this problem starts with $a'b'd'$, the only other prime implicant to cover m_0. Using the same process, we obtain the map on the right and the expression

$$f = a'b'd' + a'bc' + abd + ab'c$$

EXAMPLE 3.18

$$G(A, B, C, D) = \Sigma m(0, 1, 3, 4, 6, 7, 8, 9, 11, 12, 13, 14, 15)$$

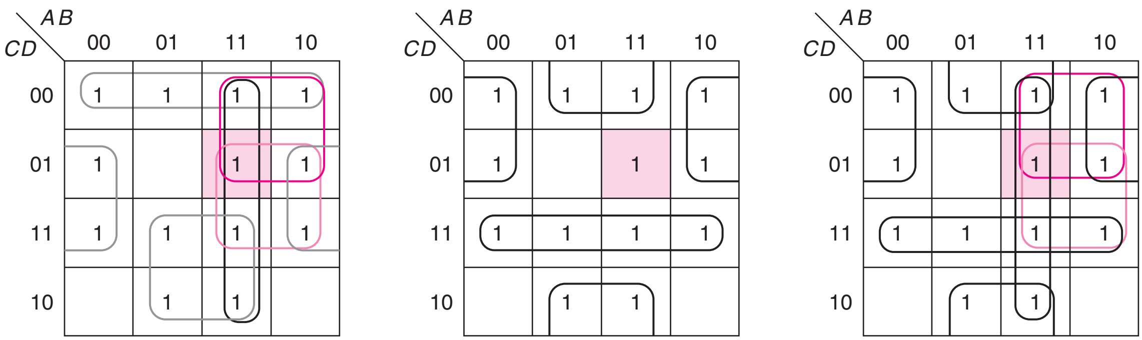
All of the prime implicants are groups of four. Since there are 13 1's, we need at least four terms. The first map shows all of the prime implicants circled; there are nine. There are no 1's circled only once, and thus, there are no essential prime implicants.



As a starting point, we choose one of the minterms covered by only two prime implicants, say m_0. On the second map, we used $C'D'$ to cover it. Next, we found two additional prime implicants that cover four new 1's each, as shown on the third map. That leaves just m_{13} to be covered. As can be seen on the fourth map (shown below), there are three different prime implicants that can be used. Now, we have three of the minimum solutions.

$$F = C'D' + B'D + BC + \{AB \quad \text{or} \quad AC' \quad \text{or} \quad AD\}$$

If, instead of using $C'D'$ to cover m_0, we use $B'C'$ (the only other prime implicant that covers m_0), as shown on the next map, we can find two other groups of four that each cover four new 1's and leave just m_{13} to be covered. Once again, we have three different ways to complete the cover (the same three terms as before).



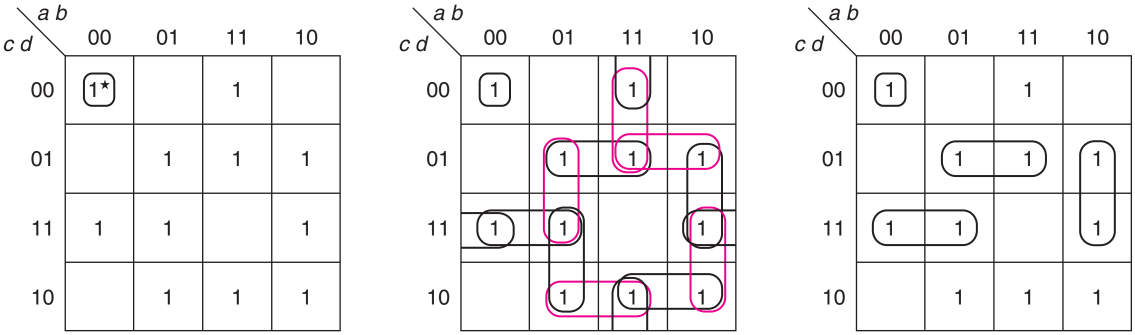
Thus, there are six equally good solutions

$$F = \begin{Bmatrix} C'D' + B'D + BC \\ B'C' + BD' + CD \end{Bmatrix} + \begin{Bmatrix} AB \\ AC' \\ AD \end{Bmatrix}$$

where one group of terms is chosen from the first bracket and an additional term from the second. We are sure that there are no better solutions, since each uses the minimum number of prime implicants, four. Although it may not be obvious without trying other combinations, there are no additional minimum solutions.

A number of other examples are included in Solved Problems 1 and 2. Example 3.19 is one of the most complex four-variable problems, requiring more terms than we might estimate at first.

EXAMPLE 3.19



This function has one essential prime implicant (a minterm) and ten other 1's. All of the other prime implicants are groups of two. The second map shows all 13 prime implicants. The prime implicants of this function are

$a'b'c'd'$	$a'cd$	$b'cd$	$ac'd$	$bc'd$	bcd'	acd'
$a'cd$	$a'bd$	abc'	abd'	$ab'c$	$ab'd$	

Note that every 1 (other than m_0) can be covered by two or three different terms.

Since there are ten 1's to be covered by groups of two, we know that we need at least five terms, in addition to $a'b'c'd'$. The third map shows the beginnings of an attempt to cover the function. Each term covers two new 1's without leaving any isolated uncovered 1. (The 1 at the top could be combined with m_{14}.) The four 1's that are left require three additional terms. After trying several other groupings, we can see that it is not possible to cover this function with less than seven terms. There are 32 different minimum solutions to this problem. A few of the solutions are listed below. The remainder are left as an exercise (Exercise 1p).

$$
\begin{aligned}
f &= a'b'c'd' + a'cd + bc'd + ab'd + abc' + a'bc + acd' \\
&= a'b'c'd' + a'cd + bc'd + ab'd + abd' + bcd' + ab'c \\
&= a'b'c'd' + b'cd + a'bd + ac'd + abd' + acd' + bcd' \\
&= a'b'c'd' + b'cd + abc' + bcd' + a'bd + ab'c + ab'd
\end{aligned}
$$

[SP 2, 3, 4; EX 2, 3, 4]

3.3 DON'T CARES

Finding minimum solutions for functions with don't cares does not significantly change the methods we developed in the last section. We need to modify slightly the definitions of an implicant and a prime implicant, and clarify the definition of an essential prime implicant.

An *implicant* is a rectangle of 1, 2, 4, 8, . . . 1's or X's (containing no 0's).

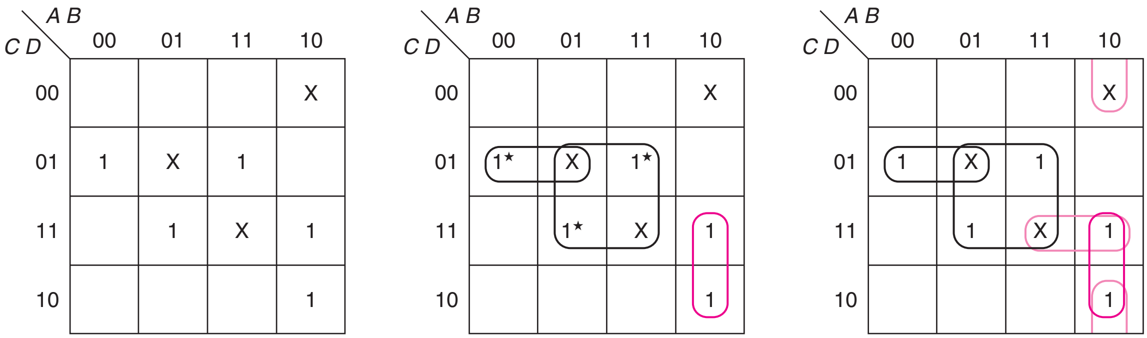
A *prime implicant* is an implicant not included in any one larger rectangle. Thus, from the point of view of finding prime implicants, X's (don't cares) are treated as 1's.

An *essential prime implicant* is a prime implicant that covers at least one 1 not covered by any other prime implicant (as always). Don't cares (X's) do not make a prime implicant essential.

Now, we just apply either of the methods of the last section. When we are done, some of the X's may be included and some may not. But we *don't care* whether or not they are included in the function.

EXAMPLE 3.20

$F(A, B, C, D) = \Sigma m(1, 7, 10, 11, 13) + \Sigma d(5, 8, 15)$



We first mapped the function, entering a 1 for those minterms included in the function and an X for the don't cares. We found two essential prime implicants, as shown on the center map. In each case, the 1's with a star cannot be covered by any other prime implicant. That left the two 1's circled in brown to cover the rest of the function. That is not an essential prime implicant, since each of the 1's could be covered by another prime implicant (as shown in tan on the third map). However, if we did not use $AB'C$, we would need two additional terms, instead of one. Thus, the only minimum solution is

$$F = BD + A'C'D + AB'C$$

and terms $AB'D'$ and ACD are prime implicants not used in the minimum solution. Note that if all of the don't cares were made 1's, we would need a fourth term to cover m_8, making

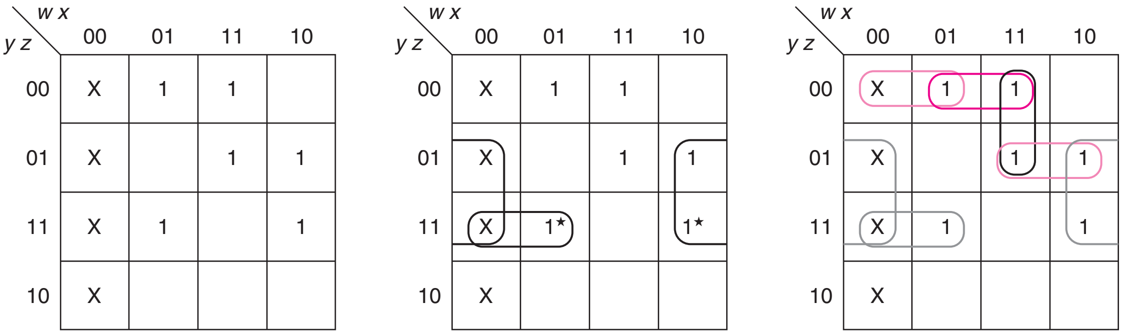
$$F = BD + A'C'D + AB'C + AB'D' \quad \text{or}$$
$$F = BD + A'C'D + ACD + AB'D'$$

and that if all of the don't cares were 0's, the function would become

$$F = A'B'C'D + A'BCD + ABC'D + AB'C$$

In either case, the solution is much more complex then when we treated those terms as don't cares (and made two of them 1's and the other a 0).

EXAMPLE 3.21

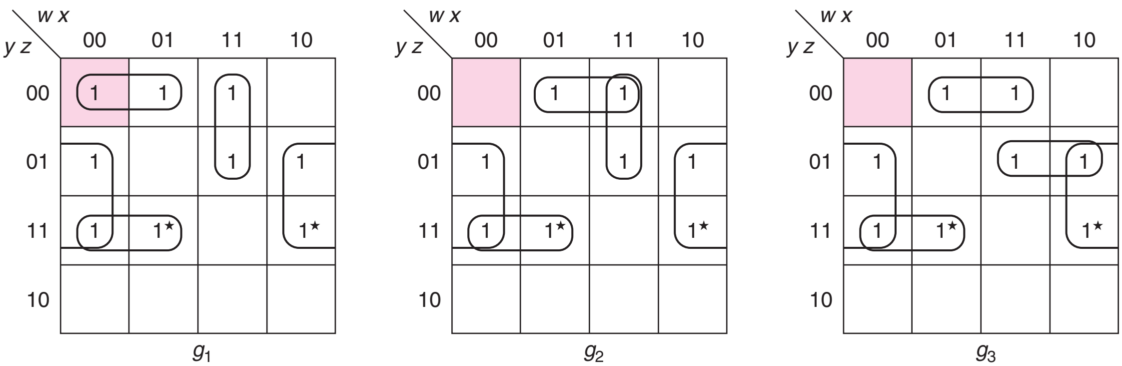


There are two essential prime implicants, as shown on the center map, $x'z$ and $w'yz$. The group of four don't cares, $w'x'$, is a prime implicant (since it is a rectangle of four 1's or X's) but it is not essential (since it does not cover any 1's not covered by some other prime implicant). Surely, a prime implicant made up of all don't cares would never be used, since that would add a term to the sum without covering any additional 1's. The three remaining 1's require two groups of two and thus there are three equally good solutions, each using four terms and 11 literals:

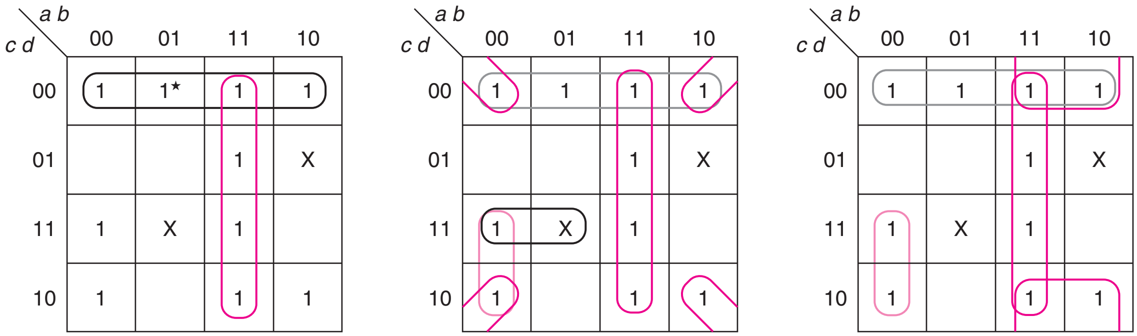
$$g_1 = x'z + w'yz + w'y'z' + wxy'$$
$$g_2 = x'z + w'yz + xy'z' + wxy'$$
$$g_3 = x'z + w'yz + xy'z' + wy'z$$

An important thing to note about Example 3.21 is that the three algebraic expressions are not all equal. The first treats the don't care for m_0 as a 1, whereas the other two (which are equal to each other) treat it as a 0. This will often happen with don't cares. They must treat the specified part of the function (the 1's and the 0's) the same, but the don't cares may take on different values in the various solutions. The maps of Map 3.15 show the three functions.

Map 3.15 The different solutions for Example 3.21.



EXAMPLE 3.22



On the first map, we have shown the only essential prime implicant, $c'd'$, and the other group of four that is used in all three solutions, ab. (This must be used since the only other prime implicant that would cover m_{15} is bcd, which requires one more literal and does not cover any 1's that are not covered by ab.) The three remaining 1's require two terms, one of which must be a group of two (to cover m_3) and the other must be one of the groups of four that cover m_{10}. On the second map, we have shown two of the solutions, those that utilize $b'd'$ as the group of four. On the third map, we have shown the third solution, utilizing ad'. Thus, we have

$$g_1 = c'd' + ab + b'd' + a'cd$$
$$g_2 = c'd' + ab + b'd' + a'b'c$$
$$g_3 = c'd' + ab + ad' + a'b'c$$

We can now ask if these solutions are equal to each other. We can either map all three solutions as we did for Example 3.21, or we can make a table of the behavior of the don't cares—one column for each don't care and one row for each solution.

	m_7	m_9
g_1	1	0
g_2	0	0
g_3	0	0

From the table, it is clear that $g_2 = g_3$, but neither is equal to g_1. A more complex example is found in the solved problems.

Don't cares give us another option for solving map problems for functions with or without don't cares. At any point in the process of using either Map Method 1 or 2, we can replace all 1's covered by the terms already chosen by don't cares. That highlights the 1's remaining to be covered. We then need to choose enough terms to cover the remaining

1's. This works because the 1's already covered can be used again (as part of a term covering some new 1's), but need not be.

EXAMPLE 3.23

$F(A, B, C, D) = \Sigma m(0, 3, 4, 5, 6, 7, 8, 10, 11, 14, 15)$

CD \ AB	00	01	11	10
00	1	1		1
01		1*		
11	1*	1	1	1
10		1	1	1

CD \ AB	00	01	11	10
00	1	X		1
01		X		
11	X	X	X	X
10		X	1	1

CD \ AB	00	01	11	10
00	1	X		1
01		X		
11	X	X	X	X
10		X	1	1

We first found the two essential prime implicants, $A'B$ and CD. On the second map, we converted all of the 1's covered to don't cares. Finally, we can cover the remaining 1's with AC and $B'C'D'$, producing

$$F = A'B + CD + AC + B'C'D'$$

Replacing covered minterms by don't cares accomplishes the same thing as the shading that we did in Examples 3.14 and 3.15; it highlights the 1's that remain to be covered.

EXAMPLE 3.24

The essential prime implicants, xy' and $x'y$, are circled on the first map. The 1's covered by them are changed to don't cares on the second map. It is now clear that the two 1's in the 10 column can be covered by either $w'y$ or $w'x$ and that the other 1 can be covered by $wx'z'$ or $wy'z'$.

yz \ wx	00	01	11	10
00		1	1	1
01		1	1	
11	1	1		1
10	1	1		1

yz \ wx	00	01	11	10
00		X	X	1
01		X	X	
11	X	1		X
10	X	1		X

Thus, the four minimum solutions are

$$xy' + x'y + \begin{Bmatrix} w'y \\ x'y \end{Bmatrix} + \begin{Bmatrix} wx'z' \\ wy'z' \end{Bmatrix}$$

[SP 5, 6; EX 5, 6]

3.4 PRODUCT OF SUMS

Finding a minimum product of sums expression requires no new theory. The following approach is the simplest:

1. Map the complement of the function. (If there is already a map for the function, replace all 0's by 1's, all 1's by 0's, and leave X's unchanged.)
2. Find the minimum sum of products expression for the complement of the function (using the techniques of the last two sections).
3. Use DeMorgan's theorem (P11) to complement that expression, producing a product of sums expression.

Another approach, which we will not pursue here, is to define the dual of prime implicants (referred to as prime implicates) and develop a new method.

EXAMPLE 3.25

$$f(a, b, c, d) = \Sigma m(0, 1, 4, 5, 10, 11, 14)$$

Since all minterms must be either minterms of f or of f', then f' must be the sum of all of the other minterms, that is

$$f'(a, b, c, d) = \Sigma m(2, 3, 6, 7, 8, 9, 12, 13, 15)$$

Maps of both f and f' are shown below:

cd \ ab	00	01	11	10
00	1	1		
01	1	1		
11				1
10			1	1

f

cd \ ab	00	01	11	10
00			1	1
01			1	1
11	1	1	1	
10	1	1		

f'

We did not need to map f, unless we wanted both the sum of products expression and the product of sums expression. Once we mapped f, we did not need to write out all the minterms of f'; we could have just replaced the 1's by 0's and 0's by 1's. Also, instead of mapping f', we could look for rectangles of 0's on the map of f. This function is rather straightforward. The maps for the minimum sum of product expressions for both f and f' are shown next:

$cd \backslash ab$	00	01	11	10
00	1	1		
01	1	1		
11				1
10			1	1

f

$cd \backslash ab$	00	01	11	10
00			1	1
01			1	1
11	1	1	1	
10	1	1		

f'

There is one minimum solution for f and there are two equally good solutions for the sum of products for f':

$$f = a'c' + ab'c + acd' \qquad f' = ac' + a'c + abd$$

$$f' = ac' + a'c + bcd$$

We can then complement the solutions for f' to get the two minimum product of sums solutions for f:

$$f = (a' + c)(a + c')(a' + b' + d')$$

$$f = (a' + c)(a + c')(b' + c' + d')$$

The minimum sum of products solution has three terms and eight literals; the minimum product of sums solutions have three terms and seven literals. (There is no set pattern; sometimes the sum of products solution has fewer terms or literals, sometimes the product of sums does, and sometimes they have the same number of terms and literals.)

EXAMPLE 3.26

Find all of the minimum sum of products and all minimum product of sums solutions for

$$g(w, x, y, z) = \Sigma m(1, 3, 4, 6, 11) + \Sigma d(0, 8, 10, 12, 13)$$

We first find the minimum sum of products expression by mapping g. However, before complicating the map by circling prime implicants, we also map g' (top of next page). Note that the X's are the same on both maps.

$yz \backslash wx$	00	01	11	10
00	X	1	X	X
01	1		X	
11	1			1
10		1		X

g

$yz \backslash wx$	00	01	11	10
00	X	1	X	X
01	1		X	
11	1			1
10		1*		X

$yz \backslash wx$	00	01	11	10
00	X	X	X	X
01	1		X	
11	1			1
10		X		X

g'

yz \ wx	00	01	11	10
00	X		X	X
01		1	X	1
11		1	1	
10	1		1	X

yz \ wx	00	01	11	10
00	X		X	X
01		1	X	1*
11		1*	1	
10	1*		1	X

yz \ wx	00	01	11	10
00	X		X	X
01		X	X	X
11		X	X	
10	X		1	X

For g, the only essential prime implicant, $w'xz'$ is shown on the center map. The 1's covered by it are made don't cares on the right-hand map, and the remaining useful prime implicants are circled. We have seen similar examples before, where we have three 1's to be covered in groups of two. There are three equally good solutions:

$$g = w'xz' + \begin{Bmatrix} w'x'y' + x'yz \\ w'x'z + x'yz \\ w'x'z + wx'y \end{Bmatrix}$$

For g', there are three essential prime implicants, as shown on the center map. Once all of the 1's covered by them have been made don't cares, there is only one 1 left; it can be covered in two ways as shown on the right map:

$$g' = x'z' + xz + wy' + \begin{Bmatrix} wx \\ wz' \end{Bmatrix}$$

$$g = (x + z)(x' + z')(w' + y)\begin{Bmatrix} (w' + x') \\ (w' + z) \end{Bmatrix}$$

Note that in this example, the sum of products solutions each require only three terms (with nine literals), whereas the product of sums solutions each require four terms (with eight literals).

Finally, we want to determine which, if any, of the five solutions are equal. The complication (compared to this same question in the last section) is that when we treat a don't care as a 1 for g', that means that we are treating it as a 0 of g. Labeling the three sum of product solutions as g_1, g_2, and g_3, and the two product of sums solutions as g_4 and g_5, we produce the following table:

	0	**8**	**10**	**12**	**13**
g_1	1	0	0	0	0
g_2	0	0	0	0	0
g_3	0	0	1	0	0
g_4'	1	1	1	1	1
g_4	0	0	0	0	0
g_5'	1	1	1	1	1
g_5	0	0	0	0	0

The product of sum solutions treat all of the don't cares as 1's of g' since each is circled by the essential prime implicants of g'. (Thus, they are 0's of g.) We then note that the three solutions that are equal are

$$g_2 = w'xz' + w'x'z + x'yz$$

$$g_4 = (x + z)(x' + z')(w' + y)(w' + x')$$

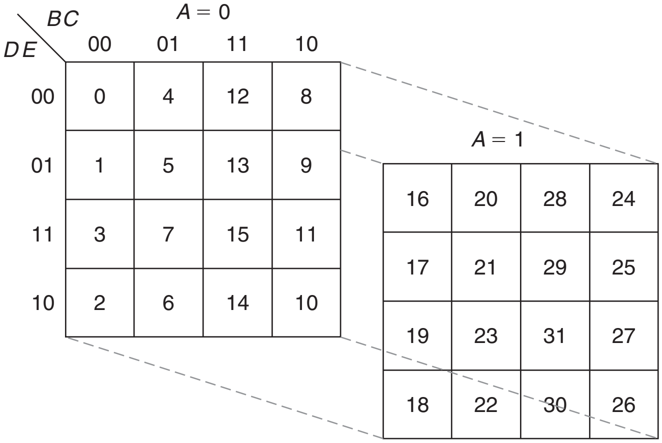
$$g_5 = (x + z)(x' + z')(w' + y)(w' + z)$$

[SP 7, 8; EX 7, 8]

3.5 FIVE- AND SIX-VARIABLE MAPS

A five-variable map consists of $2^5 = 32$ squares. Although there are several arrangements that have been used, we prefer to look at it as two layers of 16 squares each. The top layer (on the left below) contains the squares for the first 16 minterms (for which the first variable, A, is 0) and the bottom layer contains the remaining 16 squares, as pictured in Map 3.16:

Map 3.16 A five-variable map.



Each square in the bottom layer corresponds to the minterm numbered 16 more than the square above it. Product terms appear as rectangular solids of 1, 2, 4, 8, 16, . . . 1's or X's. Squares directly above and below each other are adjacent.

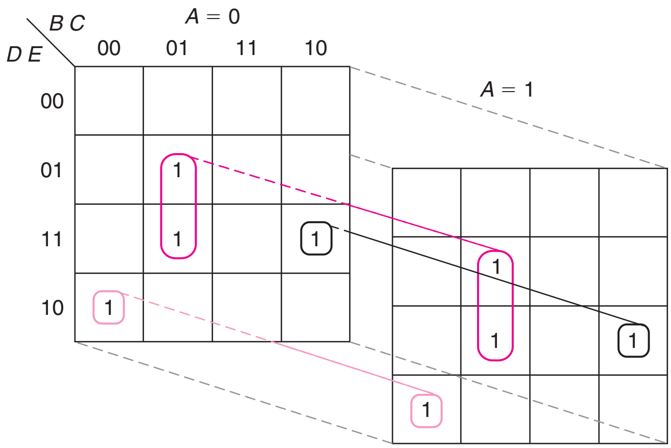
EXAMPLE 3.27

$$m_2 + m_5 = A'B'C'DE' + AB'C'DE' = B'C'DE'$$

$$m_{11} + m_{27} = A'BC'DE + ABC'DE = BC'DE$$

$$m_5 + m_7 + m_{21} + m_{23} = B'CE$$

These terms are circled on the following map.

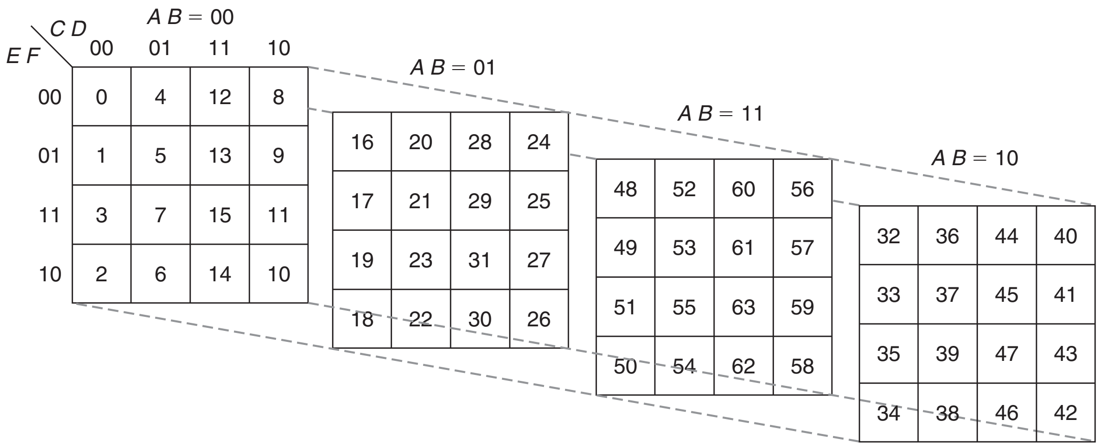


In a similar manner, six-variable maps are drawn as four layers of 16-square maps, where the first two variables determine the layer and the other variables specify the square within the layer. The layout, with minterm numbers shown, is given in Map 3.17. Note that the layers are ordered in the same way as the rows and the columns, that is 00, 01, 11, 10.

In this section, we will concentrate on five-variable maps, although we will also do an example of six-variable maps at the end. The techniques are the same as for four-variable maps; the only thing new is the need to visualize the rectangular solids. Rather than drawing the maps to look like three dimensions, we will draw them side by side. The function, F, is mapped in Map 3.18.

$$F(A, B, C, D, E) = \Sigma m(4, 5, 6, 7, 9, 11, 13, 15, 16, 18, 27, 28, 31)$$

Map 3.17 A six-variable map.



Map 3.18 A five-variable problem.

A = 0

DE \ BC	00	01	11	10
00		1		
01		1	1	1
11		1	1	1
10		1		

A = 1

DE \ BC	00	01	11	10
00	1		1	
01				
11			1	1
10	1			

As always, we first look for the essential prime implicants. A good starting point is to find 1's on one layer for which there is a 0 in the corresponding square on an adjoining layer. Prime implicants that cover that 1 are contained completely on that layer (and thus, we really only have a four-variable map problem). In this example, m_4 meets this criteria (since there is a 0 in square 20 below it). Thus, the only prime implicants covering m_4 must be on the first layer. Indeed, $A'B'C$ is an essential prime implicant. (Note that the A' comes from the fact that this group is contained completely on the $A = 0$ layer of the map and the $B'C$ from the fact that this group is in the second column.) Actually, all four 1's in this term have no counterpart on the other layer and m_6 would also make this prime implicant essential. (The other two 1's in that term are part of another prime implicant, as well.) We also note that m_9, m_{16}, m_{18}, and m_{28} have 0's in the corresponding square on the other layer and make a prime implicant essential. Although m_{14} has a 0 beneath it (m_{30}), it does not make a prime implicant on the A' layer essential. Thus, Map 3.19 shows each of these circled, highlighting the essential prime implicants that are contained on one layer.

Map 3.19 Essential prime implicants on one layer.

A = 0

DE \ BC	00	01	11	10
00		1*		
01		1	1	1*
11		1	1	1
10		1*		

A = 1

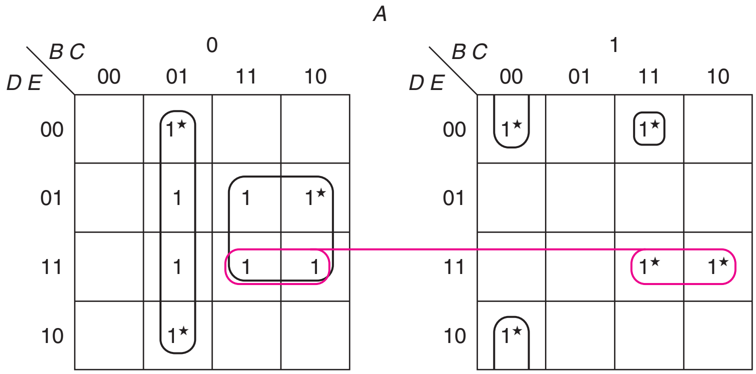
DE \ BC	00	01	11	10
00	1*		1*	
01				
11			1	1
10	1*			

So far, we have

$$F = A'B'C + A'BE + AB'C'E' + ABCD'E' + \cdots$$

The two 1's remaining uncovered do have counterparts on the other layer. However, the only prime implicant that covers them is *BDE*, as shown on Map 3.20 in brown. It, too, is an essential prime implicant. (Note that prime implicants that include 1's from both layers do not have the variable *A* in them. Such prime implicants must, of course, have the same number of 1's on each layer; otherwise, they would not be rectangular.)

Map 3.20 A prime implicant covering 1's on both layers.



The complete solution is thus

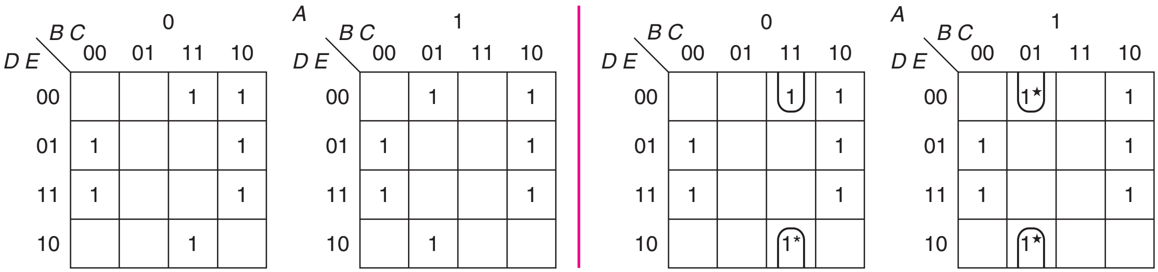
$$F = A'B'C + A'BE + AB'C'E' + ABCD'E' + BDE$$

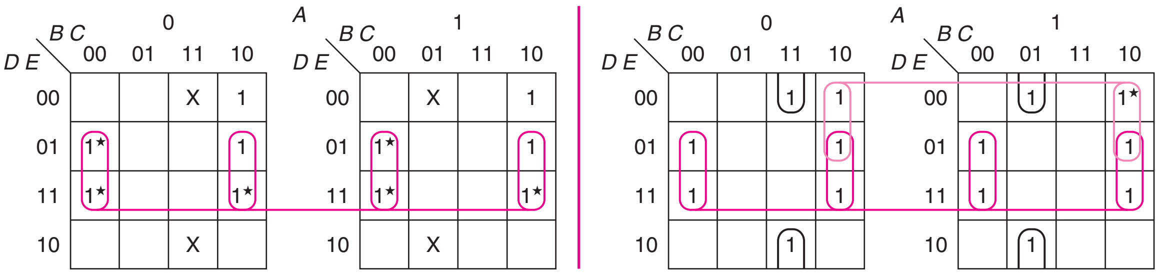
Groups of eight 1's are not uncommon in five-variable problems, as illustrated in Example 3.28.

EXAMPLE 3.28

$$G(A, B, C, D, E) = \Sigma m(1, 3, 8, 9, 11, 12, 14, 17, 19, 20, 22, 24, 25, 27)$$

The first map shows a plot of that function. On the second map, to the right, we have circled the two essential prime implicants that we found by considering 1's on one layer with 0's in the corresponding square on the other layer, $A'BCE'$ and $AB'CE'$. The group of eight 1's, $C'E$ (also an essential prime implicant), is shown in brown on the third map (where the essential prime implicants found on the second map are shown as don't cares). Groups of eight have three literals missing (leaving only two). At this point, only two 1's are left uncovered; that requires the essential prime implicant, $BC'D'$, shown on the fourth map in tan.





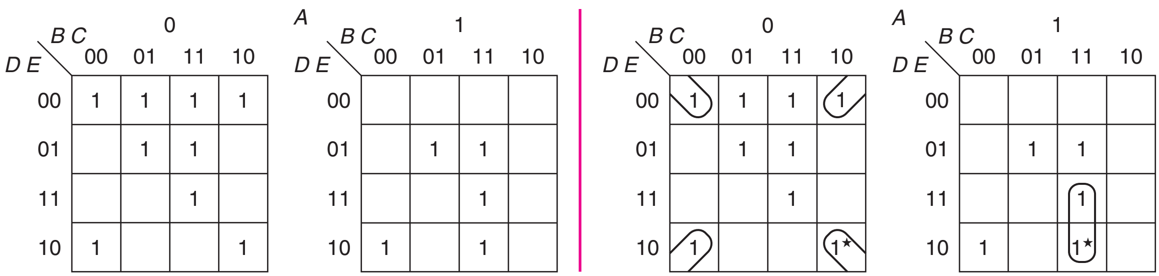
The solution is thus

$$G = A'BCE' + AB'CE' + C'E + BC'D'$$

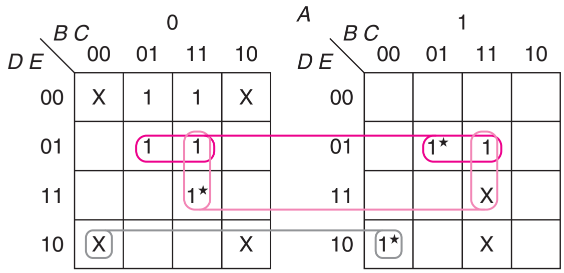
Note that there is only one other prime implicant in this function, $A'BD'E'$; it covers no 1's not already covered.

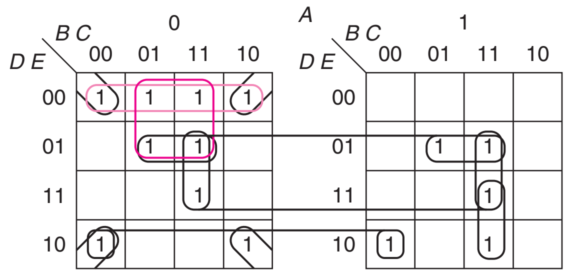
EXAMPLE 3.29

The next problem is shown on the maps below. Once again, we start by looking for 1's that are on one layer, with a corresponding 0 on the other layer. Although there are several such 1's on the $A = 0$ layer, only m_{10} makes a prime implicant essential. Similarly, on the $A = 1$ layer, m_{30} is covered by an essential prime implicant. These terms, $A'C'E'$ and $ABCD$, are shown on the second map. The 1's covered are shown as don't cares on the next map.



Three other essential prime implicants include 1's from both layers of the map; they are $CD'E$, BCE and $B'C'DE'$, as shown on the left-hand map below. These were found by looking for isolated 1's, such as m_{21}, m_{15}, and m_{18}.





Finally, the remaining two 1's (m_4 and m_{12}) can be covered in two ways, as shown on the right-hand map above, $A'CD'$ and $A'D'E'$. Thus, the two solutions are

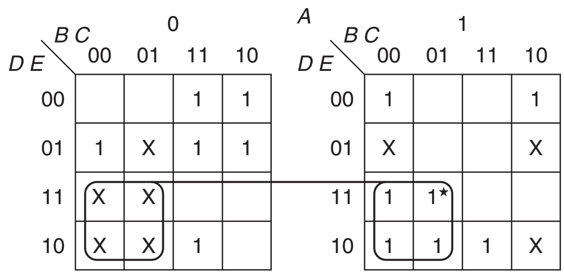
$$F = A'C'E' + ABCD + CD'E + BCE + B'C'DE' + A'CD'$$

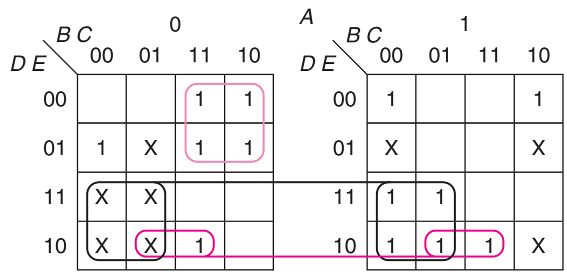
$$F = A'C'E' + ABCD + CD'E + BCE + B'C'DE' + A'D'E'$$

EXAMPLE 3.30

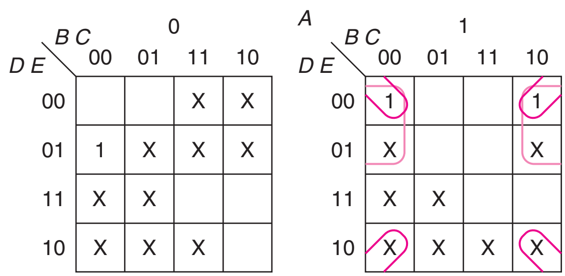
$$H(A, B, C, D, E) = \Sigma m(1, 8, 9, 12, 13, 14, 16, 18, 19, 22, 23, 24, 30) + \Sigma d(2, 3, 5, 6, 7, 17, 25, 26)$$

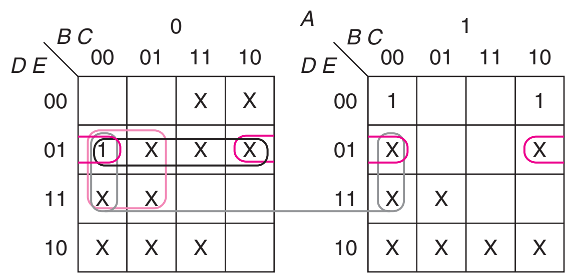
A map of H is shown below on the left with the only essential prime implicant, $B'D$ (a group of eight, including four 1's and four don't cares), circled.





Next, we choose CDE', since otherwise separate terms would be needed to cover m_{14} and m_{30}. We also chose $A'BD'$ since it covers four new 1's. Furthermore, if that were not used, a group of two ($A'BCE'$) would be needed to cover m_{12}. That leaves us with three 1's (m_1, m_{16}, and m_{24}) to be covered. On the maps below, we have replaced all covered 1's by don't cares (X's) to highlight the remaining 1's. No term that covers m_1 also covers either of the other terms. However, m_{16} and m_{24} can be covered with one term in either of two ways ($AC'E'$ or $AC'D'$) as shown on the first map below, and m_1 can





be covered by four different groups of four, as shown on the second map ($A'D'E$, $A'B'E$, $B'C'E$, or $C'D'E$), yielding the eight solutions shown.

$$H = B'D + CDE' + A'BD' + \begin{Bmatrix} AC'E' \\ AC'D' \end{Bmatrix} + \begin{Bmatrix} A'D'E \\ A'B'E \\ B'C'E \\ C'D'E \end{Bmatrix}$$

Finally, we will look at one example of a six-variable function.

EXAMPLE 3.31

$$G(A, B, C, D, E, F) = \Sigma m(1, 3, 6, 8, 9, 13, 14, 17, 19, 24, 25, 29, 32, 33, 34, 35, 38, 40, 46, 49, 51, 53, 55, 56, 61, 63)$$

The map is drawn horizontally, with the first two variables determining the 16-square layer (numbered, of course 00, 01, 11, 10).

AB	00				01				11				10			
EF \ CD	00	01	11	10	00	01	11	10	00	01	11	10	00	01	11	10
00				1				1				1*	1			1
01	1		1	1	1		1	1	1	1	1		1			
11	1*				1*				1	1	1*		1			
10		1	1										1	1	1	

The first map shows three of the essential prime implicants. The only one that is confined to one layer is on the third layer, $ABDF$. The 1's in the upper right-hand corner of each layer form another group of four (without the first two variables), $CD'E'F'$. The brown squares form a group of eight, $C'D'F$. The next map shows 1's covered by the first three prime implicants as don't cares.

AB	00				01				11				10			
EF \ CD	00	01	11	10	00	01	11	10	00	01	11	10	00	01	11	10
00				X				X				X	1			X
01	X		1*	1	X		1	1	X	X	X		X			
11	X				X				X	X	X		X			
10		1*	1*										1	1	1*	

The other two essential prime implicants are $A'CE'F$ and $B'DEF'$. (Remember that the top and bottom layers are adjacent.) Finally, m_{32} and m_{34} (on the fourth layer) remain uncovered; they are covered by the term, $AB'C'D'$. (Each of them could have been covered by a group of two, but that would take two terms.) Thus, the minimum expression is

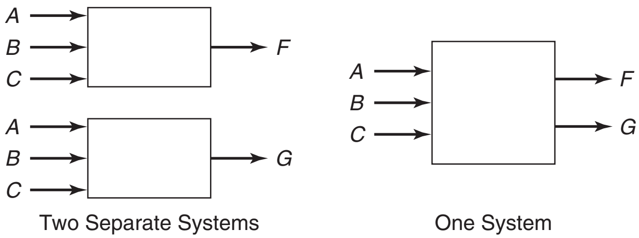
$$G = ABDF + CD'E'F' + C'D'F + A'CE'F + B'DEF' + AB'C'D'$$

[SP 9, 10; EX 9, 10]

3.6 MULTIPLE OUTPUT PROBLEMS

Many real problems involve designing a system with more than one output. If, for example, we had a problem with three inputs, A, B, and C and two outputs, F and G, we could treat this as two separate problems (as shown on the left in Figure 3.1). We would then map each of the functions, and find minimum solutions. However, if we treated this as a single system with three inputs and two outputs (as shown on the right), we may be able to economize by sharing gates.

Figure 3.1 Implementation of two functions.



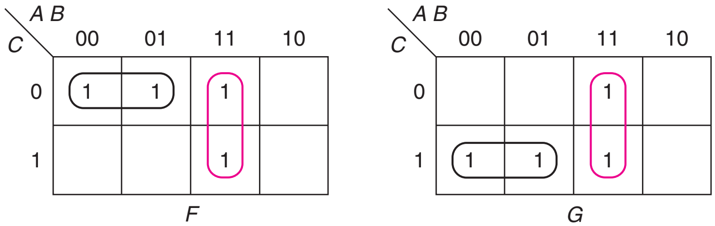
In this section, we will illustrate the process of obtaining minimum two-level solutions using AND and OR gates (sum of products solutions), assuming all variables are available both uncomplemented and complemented.* We could convert each of these solutions into NAND gate circuits (using the same number of gates and gate inputs). We could also find product of sums solutions (by minimizing the complement of each of the functions and then using DeMorgan's theorem).

We will illustrate this by first considering three very simple examples.

EXAMPLE 3.32

$$F(A, B, C) = \Sigma m(0, 2, 6, 7) \qquad G(A, B, C) = \Sigma m(1, 3, 6, 7)$$

If we map each of these and solve them separately,

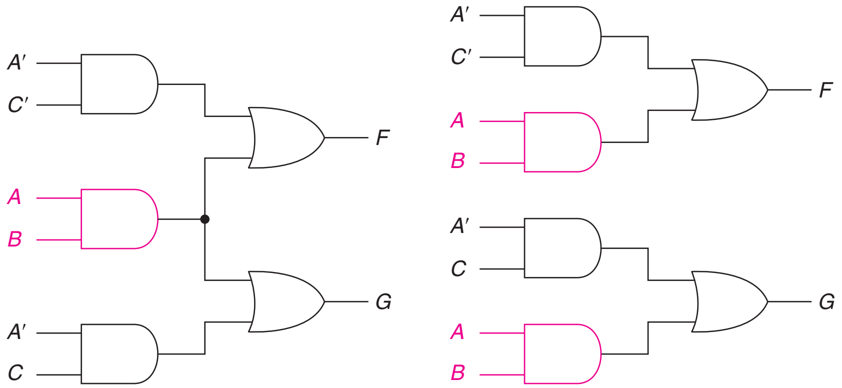


we obtain

$$F = A'C' + AB \qquad G = A'C + AB$$

Looking at the maps, we see that the same term (AB) is circled on both. Thus, we can build the circuit on the left, rather than the two circuits on the right.

*We will use as the definition for *minimum* a circuit containing the minimum number of gates, and among those with the same number of gates, the minimum number of gate inputs.



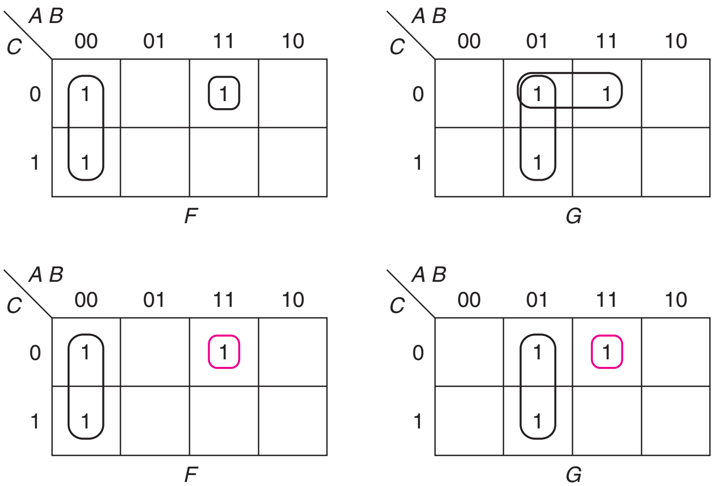
Obviously, the version on the left requires only five gates, whereas the one on the right uses six.

This example is the simplest. Each of the minimum sum of products expressions contains the same term. It would take no special techniques to recognize this and achieve the savings.

Even when the two solutions do not have a common prime implicant, we can share as illustrated in the following example:

EXAMPLE 3.33

$F(A, B, C) = \Sigma m(0, 1, 6) \qquad G(A, B, C) = \Sigma m(2, 3, 6)$



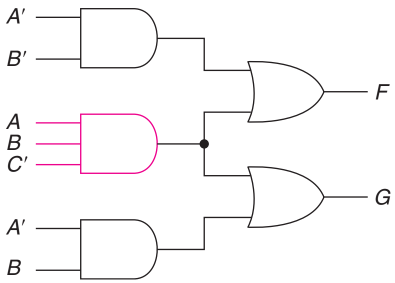
In the top maps, we considered each function separately and obtained

$F = A'B' + ABC' \qquad G = A'B + BC'$

This solution requires six gates (four ANDs and two ORs) with 13 inputs. However, as can be seen from the second pair of maps, we can share the term ABC' and obtain

$$F = A'B' + ABC' \qquad G = A'B + ABC'$$

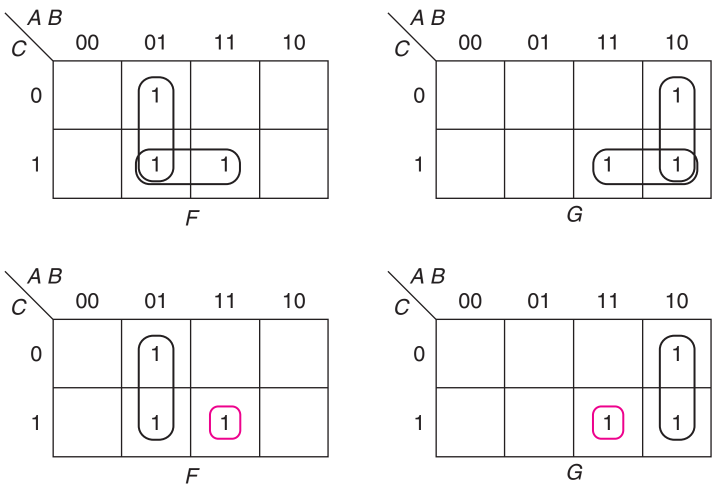
(To emphasize the sharing, we have shown the shared term in brown, and will do that in other examples that follow.) As can be seen from the circuit below, this only requires five gates with 11 inputs.



This example illustrates that a shared term in a minimum solution need not be a prime implicant. (In Example 3.33, ABC' is a prime implicant of F but not of G; in Example 3.34, we will use a term that is not a prime implicant of either function.)

EXAMPLE 3.34

$$F(A, B, C) = \Sigma m(2, 3, 7) \qquad G(A, B, C) = \Sigma m(4, 5, 7)$$



In the first pair of maps, we solved this as two problems. Using essential prime implicants of each function, we obtained

$$F = A'B + BC \qquad G = AB' + AC$$

However, as can be seen in the second set of maps, we can share the term ABC, even though it is not a prime implicant of either function, and once again get a solution that requires only five gates:

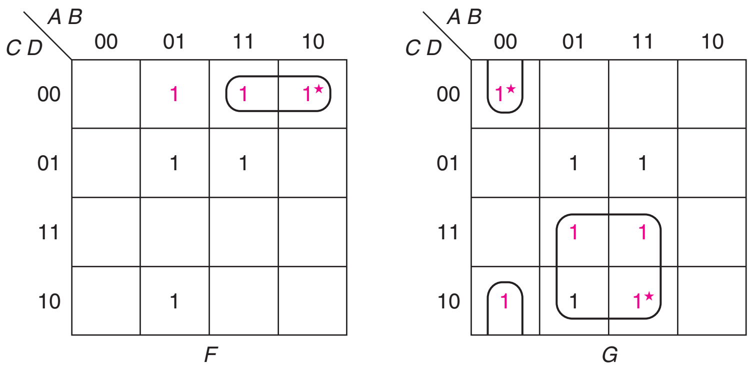
$$F = A'B + ABC \qquad G = AB' + ABC$$

The method for solving this type of problem is to begin by looking at the 1's of each function that are 0's of the other function. They must be covered by prime implicants of that function. Only the shared terms need not be prime implicants. In this last example, we chose $A'B$ for F since m_2 makes that an essential prime implicant of F and we chose AB' for G since m_4 makes that an essential prime implicant of G. That left just one 1 uncovered in each function—the same 1—which we covered with ABC. We will now look at some more complex examples.

EXAMPLE 3.35

$$F(A, B, C, D) = \Sigma m(4, 5, 6, 8, 12, 13)$$
$$G(A, B, C, D) = \Sigma m(0, 2, 5, 6, 7, 13, 14, 15)$$

The maps of these functions are shown below. In them, we have shown in brown the 1's that are included in one function and not the other.



We then circled each of those prime implicants that was made essential by a brown 1. The only brown 1 that was not circled in F is m_4 because that can be covered by two prime implicants. Even though one of the terms would have fewer literals, we must wait. Next, we will use $A'BD'$ for F. Since m_6 was covered by an essential prime implicant of G, we are no longer looking for a term to share. Thus, m_6 will be covered in F by the prime implicant, $A'BD'$. As shown on the maps below, that leaves m_4 and m_{12} to be covered in both functions, allowing us to share the term $BC'D$, as shown on the following maps circled in brown.

CD \ AB	00	01	11	10
00		1	1	1*
01		1	1	
11				
10		1		

F

CD \ AB	00	01	11	10
00	1*			
01		1	1	
11		1	1	
10	1	1	1*	

G

leaving

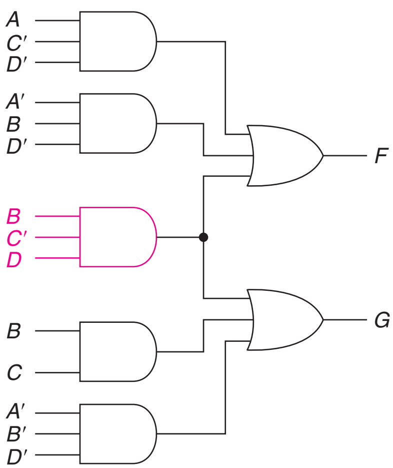
$$F = AC'D' + A'BD' + BC'D$$
$$G = A'B'D' + BC + BC'D$$

for a total of seven gates with 20 gate inputs. Notice that if we had minimized the functions individually, we would have used two separate terms for the third term in each expression, resulting in

$$F = AC'D' + A'BD' + BC'$$
$$G = A'B'D' + BC + BD$$

for a total of eight gates with 21 gate inputs. Clearly, the shared circuit costs less.

The shared version of the circuit is shown below.*



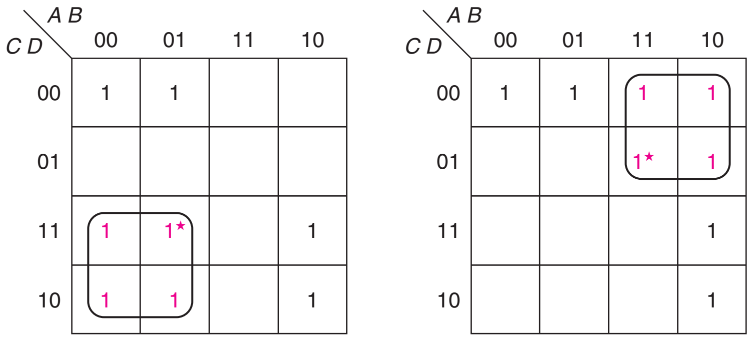
*All of these gates can be changed to NAND gates, even though the output of $BC'D$ goes to two places. There are still two bubbles (NOTs) in each path.

EXAMPLE 3.36

$$F(A, B, C, D) = \Sigma m(0, 2, 3, 4, 6, 7, 10, 11)$$

$$G(A, B, C, D) = \Sigma m(0, 4, 8, 9, 10, 11, 12, 13)$$

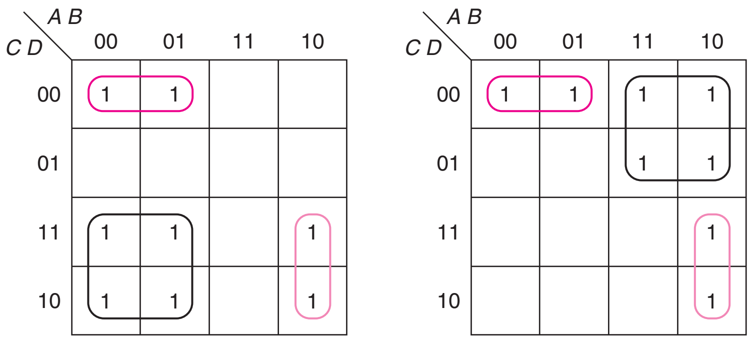
Once again the maps are shown with the unshared 1's in brown and the prime implicants made essential by one of those 1's circled.



Each of the functions can be solved individually with two more groups of four, producing

$$F = A'C + A'D' + B'C \qquad G = AC' + C'D' + AB'$$

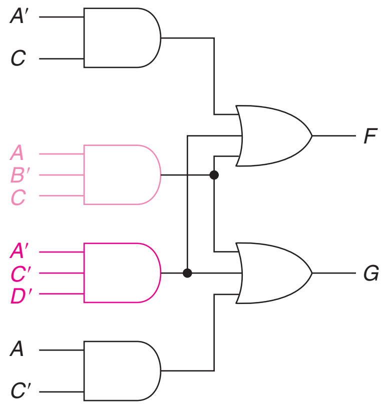
That would require eight gates with 18 gate inputs. However, sharing the groups of two as shown on the next set of maps reduces the number of gates to six and the number of gate inputs to 16. If these functions were implemented with NAND gates, the individual solutions would require a total of three packages, whereas the shared solution would require only two.



F *G*

leaving the equations and the resulting AND/OR circuit.

$$F = A'C + A'C'D' + AB'C \qquad G = AC' + A'C'D' + AB'C$$



EXAMPLE 3.37

$$F(W, X, Y, Z) = \Sigma m(2, 3, 7, 9, 10, 11, 13)$$

$$G(W, X, Y, Z) = \Sigma m(1, 5, 7, 9, 13, 14, 15)$$

On the maps below, the 1's that are not shared are shown in brown and the essential prime implicants that cover these 1's are circled.

YZ \ WX	00	01	11	10
00				
01			1	1
11	1	1		1
10	1*			1*

F

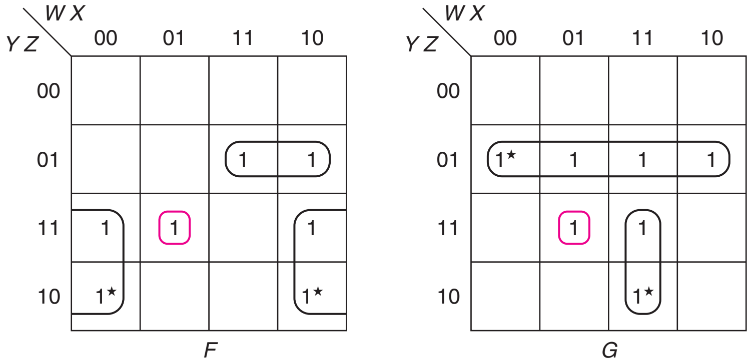
YZ \ WX	00	01	11	10
00				
01	1*	1	1	1
11		1	1	
10			1*	

G

$$F = X'Y + \cdots$$

$$G = Y'Z + WXY + \cdots$$

Now, there are three 1's left in F. Since m_9 and m_{13} have been covered in G by an essential prime implicant, no sharing is possible for these terms in F. Thus, $WY'Z$, a prime implicant of F, is used in the minimum cover. Finally, there is one uncovered 1 in each function, m_7; it can be covered by a shared term, producing the solution



$F = X'Y + WY'Z + W'XYZ$

$G = Y'Z + WXY + W'XYZ$

This requires seven gates and 20 inputs, compared to the solution we obtain by considering these as separate problems

$F = X'Y + WY'Z + W'YZ$

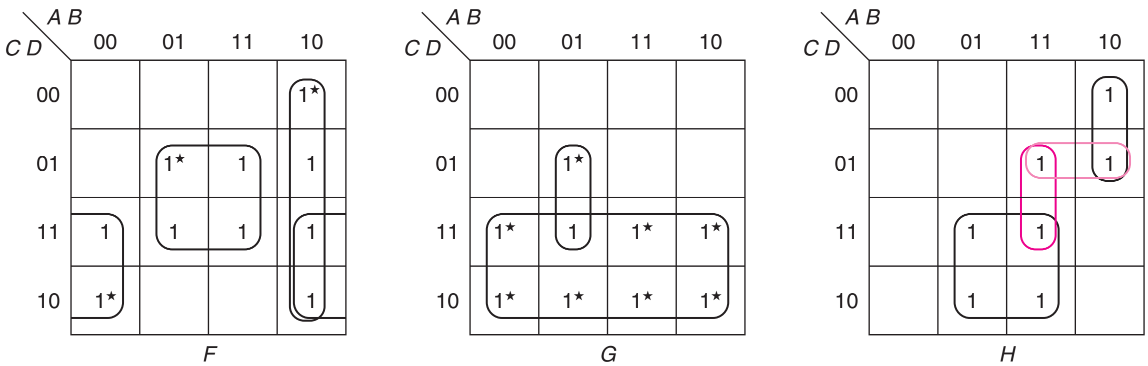
$G = Y'Z + WXY + XZ$

which requires eight gates with 21 inputs.

The same techniques can be applied to problems with three or more outputs.

EXAMPLE 3.38

First, we show the solution obtained if we considered them as three separate problems.



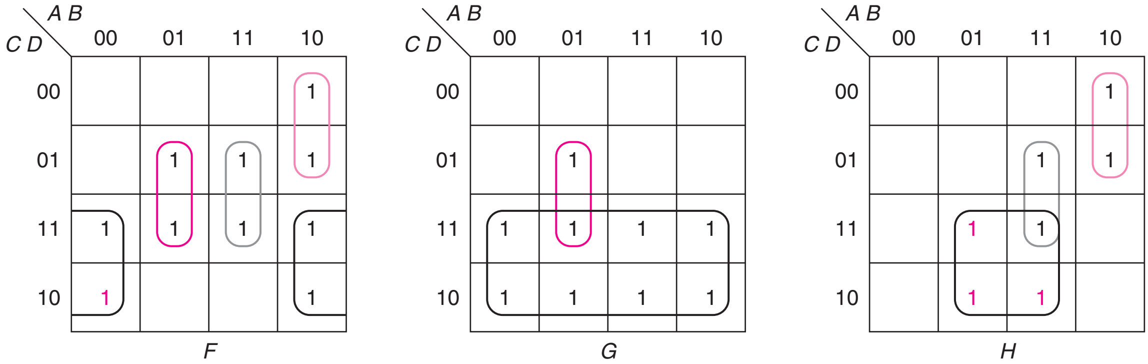
$F = AB' + BD + B'C$

$G = C + A'BD$

$H = BC + AB'C' + (ABD$ or $AC'D)$

This solution requires 10 gates and 25 gate inputs. (Note that the term C in function G does not require an AND gate.)

The technique of first finding 1's that are only minterms of one of the functions does not get us started for this example, since each of the 1's is a minterm of at least two of the functions. The starting point, instead, is to choose C for function G. The product term with only one literal does not require an AND gate and uses only one input to the OR gate. Any other solution, say sharing $B'C$ with F and BC with H, requires at least two inputs to the OR gate. Once we have made that choice, however, we must then choose $B'C$ for F and BC for H because of the 1's shown in brown on the following maps. There is no longer any sharing possible for those 1's and they make those prime implicants essential in F and H.



The term $AB'C'$ (circled in tan) was chosen next for H since it is an essential prime implicant of H and it can be shared (that is, all of the 1's in that term are also 1's of F, the only place where sharing is possible). $AB'C'$ is also used for F, since it covers two 1's and we would otherwise require an additional term, AB', to cover m_8. In a similar fashion, the term $A'BD$ is used for G (it is the only way to cover m_5) and can then be shared with F. Finally, we can finish covering F and H with ABD (a prime implicant of H, one of the choices for covering H when we treated that as a separate problem). It would be used also for F, rather than using another AND gate to create the prime implicant BD. The solution then becomes

$$F = B'C + AB'C' + A'BD + ABD$$
$$G = C + A'BD$$
$$H = BC + AB'C' + ABD$$

which requires only eight gates and 22 gate inputs (a savings of two gates and three-gate inputs).

EXAMPLE 3.39

$$F(A, B, C, D) = \Sigma m(0, 2, 6, 10, 11, 14, 15)$$
$$G(A, B, C, D) = \Sigma m(0, 3, 6, 7, 8, 9, 12, 13, 14, 15)$$
$$H(A, B, C, D) = \Sigma m(0, 3, 4, 5, 7, 10, 11, 12, 13, 14, 15)$$

The map on the next page shows these functions; the only 1 that is not shared and makes a prime implicant essential is m_9 in G. That prime implicant, AC', is shown circled.

CD \ AB	00	01	11	10
00	1			
01				
11			1	1
10	1	1	1	1

F

CD \ AB	00	01	11	10
00	1		1	1
01			1	1*
11	1	1	1	
10		1	1	

G

CD \ AB	00	01	11	10
00	1	1	1	
01		1	1	
11	1	1	1	1
10			1	1

H

Next, we note that AC is an essential prime implicant of F (because of m_{11} and m_{15}) and of H (because of m_{10}). Furthermore, neither m_{10} nor m_{11} are 1's of G. Thus, that term is used for both F and H. Next, we chose BC' for H and BC for G; each covers four new 1's, some of which can no longer be shared (since the 1's that correspond to other functions have already been covered).

CD \ AB	00	01	11	10
00	1			
01				
11			1	1
10	1	1	1	1

F

CD \ AB	00	01	11	10
00	1		1	1
01			1	1
11	1	1	1	
10		1	1	

G

CD \ AB	00	01	11	10
00	1	1	1	
01		1	1	
11	1	1	1	1
10			1	1

H

At this point, we can see that $A'B'C'D'$ can be used to cover m_0 in all three functions; otherwise, we would need three different three-literal terms. $A'CD$ can be used for G and H, and, finally, CD' is used for F, producing the following map and algebraic functions.

CD \ AB	00	01	11	10
00	1			
01				
11			1	1
10	1	1	1	1

F

CD \ AB	00	01	11	10
00	1		1	1
01			1	1
11	1	1	1	
10		1	1	

G

CD \ AB	00	01	11	10
00	1	1	1	
01		1	1	
11	1	1	1	1
10			1	1

H

$F = AC + A'B'C'D' + CD'$
$G = AC' + BC + A'B'C'D' + A'CD$
$H = AC + BC' + A'B'C'D' + A'CD$

This solution requires 10 gates with 28 inputs, compared to 13 gates and 35 inputs if these were implemented separately.

EXAMPLE 3.40

Finally, we will consider an example of a system with don't cares:

$F(A, B, C, D) = \Sigma m(2, 3, 4, 6, 9, 11, 12) + \Sigma d(0, 1, 14, 15)$
$G(A, B, C, D) = \Sigma m(2, 6, 10, 11, 12) + \Sigma d(0, 1, 14, 15)$

A map of the functions, with the only prime implicant made essential by a 1 that is not shared circled, $B'D$, is shown below.

CD \ AB	00	01	11	10
00	X	1	1	
01	X			1*
11	1		X	1
10	1	1	X	

F

CD \ AB	00	01	11	10
00	X		1	
01	X			
11			X	1
10	1	1	X	1

G

Since m_{11} has now been covered in F, we must use the essential prime implicant of G, AC, to cover m_{11} there. Also, as shown on the next maps, ABD' is used for G, since that is an essential prime implicant of G, and the whole term can be shared. (We will share it in the best solution.)

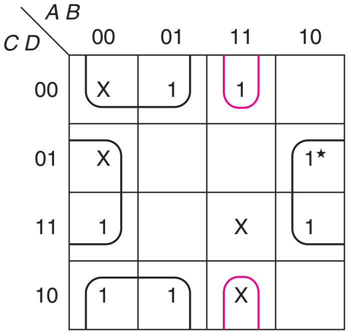
CD \ AB	00	01	11	10
00	X	1	1	
01	X			1*
11	1		X	1
10	1	1	X	

F

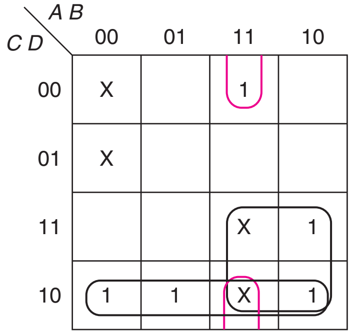
CD \ AB	00	01	11	10
00	X		1	
01	X			
11			X	1
10	1	1	X	1

G

Since we need the term *ABD′* for *G*, one approach is to use it for *F* also. (That only costs a gate input to the OR gate.) If we do that, we could cover the rest of *F* with *A′D′* and the rest of *G* with *CD′*, yielding the map and equations that follow.



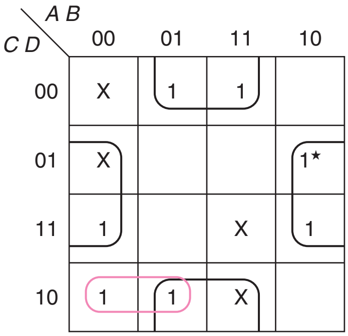
F



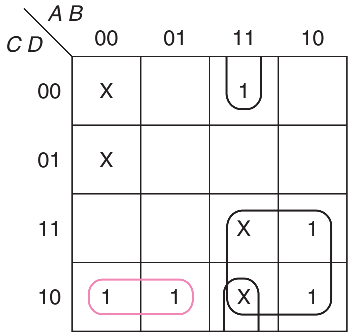
G

$$F = B'D + ABD' + A'D'$$
$$G = AC + ABD' + CD'$$

That solution uses seven gates and 17 inputs. Another solution using the same number of gates but one more input shares *A′CD′*. That completes *G,* and then the cover of *F* is completed with *BD′*. The maps and equations are thus:



F



G

$$F = B'D + A'CD' + BD'$$
$$G = AC + ABD' + A'CD'$$

That, too, requires seven gates, but using a three-input AND gate instead of a two-input one, bringing the total number of inputs to 18. Thus, this solution is not minimum.

[SP 11; EX 11, 12]

3.7 SOLVED PROBLEMS

1. Plot the following functions on a Karnaugh map:

a. $f(a, b, c) = \Sigma m(0, 1, 3, 6)$

b. $g(w, x, y, z) = \Sigma m(3, 4, 7, 10, 11, 14) + \Sigma d(2, 13, 15)$

c. $F = BD' + ABC + AD + A'B'C$

c \ ab	00	01	11	10
0	1		1	
1	1	1		

a.

yz \ wx	00	01	11	10
00		1		
01			X	
11	1	1	X	1
10	X		1	1

b.

CD \ AB	00	01	11	10
00		1	1	
01			1	1
11	1		1	1
10	1	1	1	

c.

2. For each of the following, find all minimum sum of products expressions. (If there is more than one solution, the number of solutions is given in parentheses.)

a. $G(X, Y, Z) = \Sigma m(1, 2, 3, 4, 6, 7)$

b. $f(w, x, y, z) = \Sigma m(2, 5, 7, 8, 10, 12, 13, 15)$

c. $g(a, b, c, d) = \Sigma m(0, 6, 8, 9, 10, 11, 13, 14, 15)$ (2 solutions)

d. $f(a, b, c, d) = \Sigma m(0, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15)$ (2 solutions)

e. $f(a, b, c, d) = \Sigma m(0, 1, 2, 4, 6, 7, 8, 9, 10, 11, 12, 15)$

f. $g(a, b, c, d) = \Sigma m(0, 2, 3, 5, 7, 8, 10, 11, 12, 13, 14, 15)$ (4 solutions)

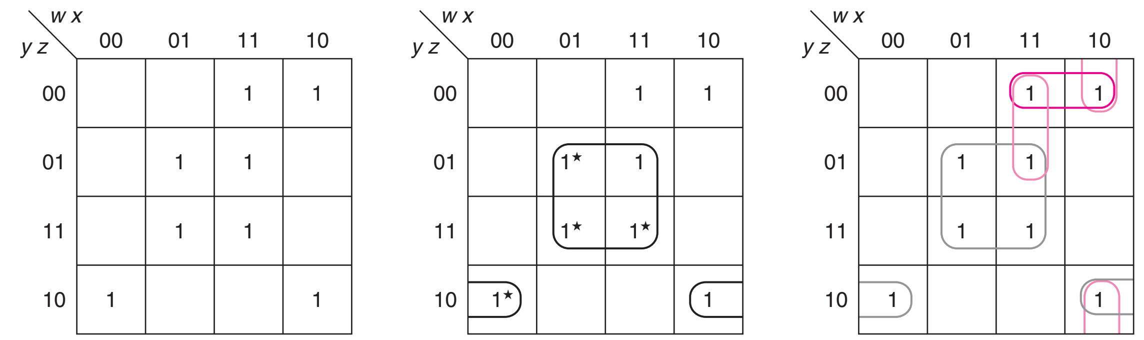
a. All of the prime implicants are essential, as shown on the map to the right.

Z \ XY	00	01	11	10
0		1	1	1
1	1	1	1	

Z \ XY	00	01	11	10
0		1*	1	1*
1	1*	1	1*	

$G = Y + XZ' + X'Z$

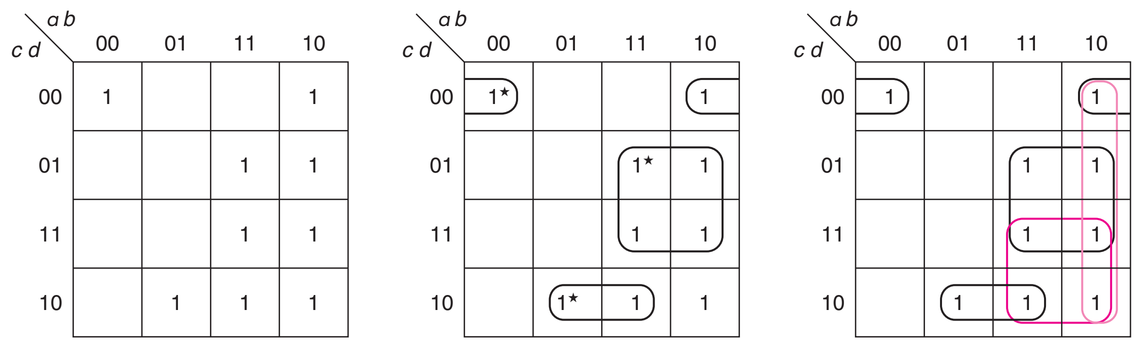
b.



The essential prime implicants are shown on the second map, leaving two 1's to be covered. The third map shows that each can be covered by two different prime implicants, but the brown group shown is the only one that covers both with one term. We would require both tan terms. The minimum is

$$f = xz + x'yz' + wy'z'$$

c.



The three essential prime implicants are shown on the center map. The only 1 left to be covered can be covered by either of two groups of four, as shown circled in brown on the third map, producing

$$g = b'c'd' + bcd' + ad + ab'$$
$$g = b'c'd' + bcd' + ad + ac$$

cd \ ab	00	01	11	10
00	1	1		1
01		1	1	1
11		1	1	1
10		1	1	1

d.

cd \ ab	00	01	11	10
00	1	1		1
01		1	1	1
11		1	1	1
10		1	1	1

cd \ ab	00	01	11	10
00	1	1		1
01		1	1	1
11		1	1	1
10		1	1	1

There are no essential prime implicants. We need one group of two to cover m_0; all other 1's can be covered by groups of four. Once we have chosen $a'c'd'$ to cover m_0 (center map), we would choose ab' to cover m_8. (Otherwise, we must use $b'c'd'$, a group of two, to cover that 1. Not only is that more literals, but it covers nothing else new; ab' covered three additional uncovered 1's.) Once that has been done, the other two prime implicants become obvious, giving

$$f = a'c'd' + ab' + bc + bd$$

In a similar fashion (on the next map), once we choose $b'c'd'$ (the other prime implicant that covers m_0), $a'b$ is the appropriate choice to cover m_4:

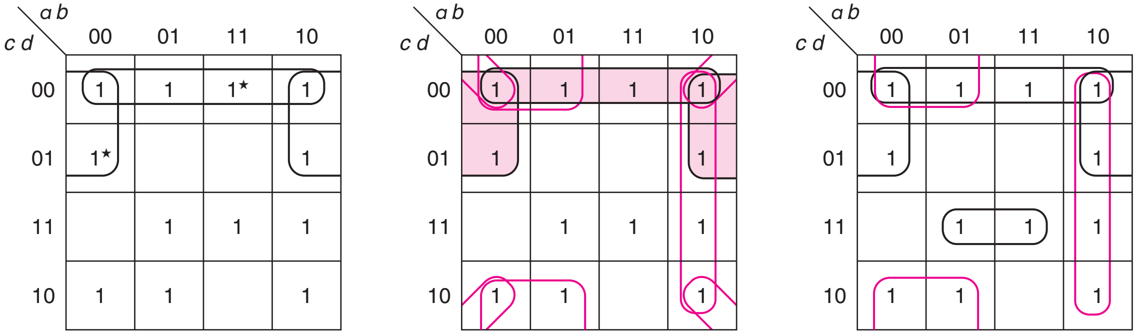
cd \ ab	00	01	11	10
00	1	1		1
01		1	1	1
11		1	1	1
10		1	1	1

cd \ ab	00	01	11	10
00	1	1		1
01		1	1	1
11		1	1	1
10		1	1	1

The only way to cover the remaining 1's in two terms is with ac and ad, as shown on the second map, leaving

$$f = b'c'd' + a'b + ac + ad$$

e. There are two essential prime implicants, as indicated on the first map, leaving six 1's to be covered. The essential prime implicants are shaded on the second map.



No prime implicant covers more than two of the remaining 1's; thus, three more terms are needed. The three groups of four (two literal terms) are circled in brown on the second map. We can cover four new 1's only using $a'd'$ and ab'. Note that m_7 and m_{15} are uncovered; they require a group of two, bcd. The only minimum solution, requiring five terms and 11 literals,

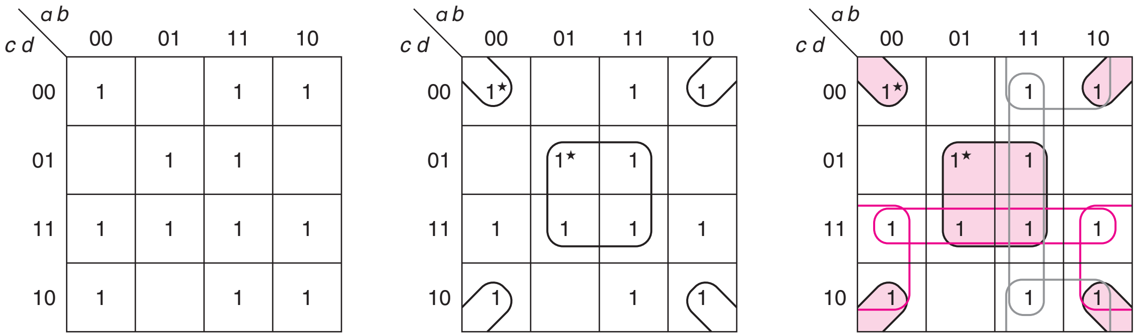
$$f = c'd' + b'c' + a'd' + ab' + bcd$$

is shown on the third map. There is another solution that uses five terms, but it requires 12 literals, namely,

$$f = c'd' + b'c' + b'd' + a'bc + acd$$

Obviously, it is not minimum (since it has an extra literal); it only used one of the groups of four instead of two.

f. On the second map, the two essential prime implicants have been highlighted $(b'd' + bd)$, leaving four 1's uncovered. On the third map, we have shown the 1's covered by these prime implicants shaded.



We can cover m_3 and m_{11} by either cd or $b'c$ (shown with brown lines), and we can cover m_{12} and m_{14} by either ab or ad' (shown in gray lines). Thus, there are four solutions:

$$f = b'd' + bd + cd + ab$$
$$f = b'd' + bd + cd + ad'$$
$$f = b'd' + bd + b'c + ab$$
$$f = b'd' + bd + b'c + ad'$$

The term ac is also a prime implicant. However, it is not useful in a minimum solution since it leaves two isolated 1's to be covered, resulting in a five-term solution.

3. For each of the following functions, find the minimum SOP expression(s). There are two solutions for z.

cd \ ab	00	01	11	10
00			1	1
01		1	1	
11			1	1
10		1	1	

w

cd \ ab	00	01	11	10
00	1		1	1
01		1	1	
11			1	1
10		1	1	

x

cd \ ab	00	01	11	10
00	1		1	1
01	1	1	1	
11			1	1
10		1	1	

y

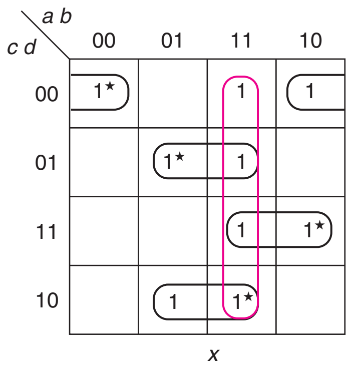
cd \ ab	00	01	11	10
00	1		1	1
01	1	1	1	
11	1		1	1
10	1	1	1	

z

All of the 1's of w are also 1's of the other functions. For x, we added one 1; for y, we added a second 1; and for z, we added two more. Only essential prime implicants are used for w (and the group of four is not needed).

$$w = ac'd' + bc'd + acd + bcd'$$

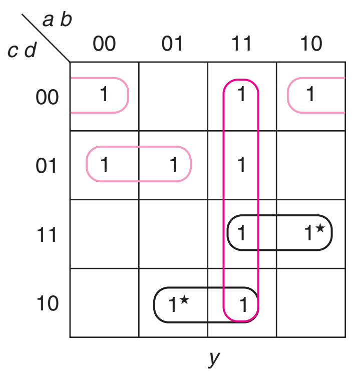
For x, the last three terms of w are still essential prime implicants, as is $b'c'd'$. As can be seen on the next map, that leaves only m_{12} uncovered.



That leaves a choice between ab and $ac'd'$. Obviously, the former has one less literal, leaving the minimum solution:

$$x = bc'd + acd + bcd' + b'c'd' + ab$$

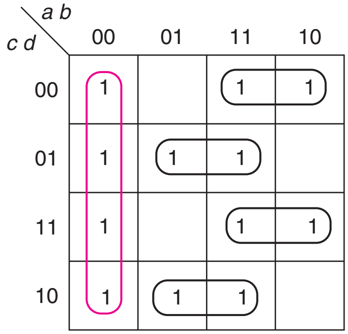
For y, there are only two essential prime implicants, leaving six 1's to cover.



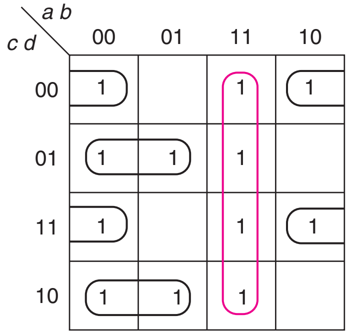
No term covers more than two of them. We must then use the group of four, giving

$$y = acd + bcd' + ab + b'c'd' + a'c'd$$

Finally, for z, we need 4 three-literal terms to cover the 1's in columns 01 and 10.



z



z

We can either use the solution for w and add $a'b'$ (as shown on the left) or use the other 4 three-literal terms along with ab (as shown on the right).

$$f = ac'd' + bc'd + acd + bcd' + a'b'$$
$$= ab + b'c'd' + a'c'd + b'cd + a'cd'$$

4. For the following functions,

i. List all prime implicants, indicating which are essential.

ii. Show the minimum sum of products expression(s).

a. $G(A, B, C, D) = \Sigma m(0, 1, 4, 5, 7, 8, 10, 13, 14, 15)$ (3 solutions)

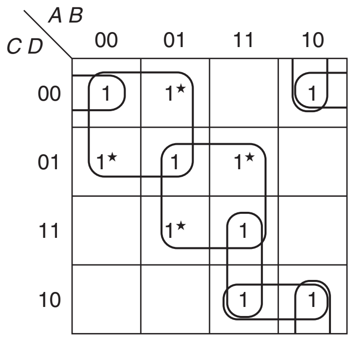
b. $f(w, x, y, z) = \Sigma m(2, 3, 4, 5, 6, 7, 9, 10, 11, 13)$

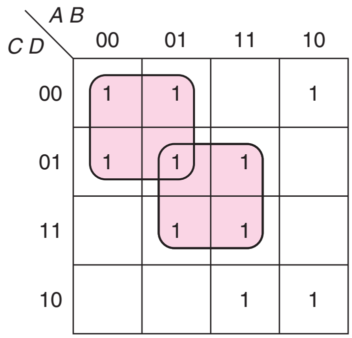
c. $h(a, b, c, d) = \Sigma m(1, 2, 3, 4, 8, 9, 10, 12, 13, 14, 15)$ (2 solutions)

a. The first map shows all of the prime implicants circled; the 1's that have been covered only once are indicated with a star.

Essential prime implicants: $A'C'$, BD

Other prime implicants: $B'C'D'$, $AB'D'$, ACD', ABC

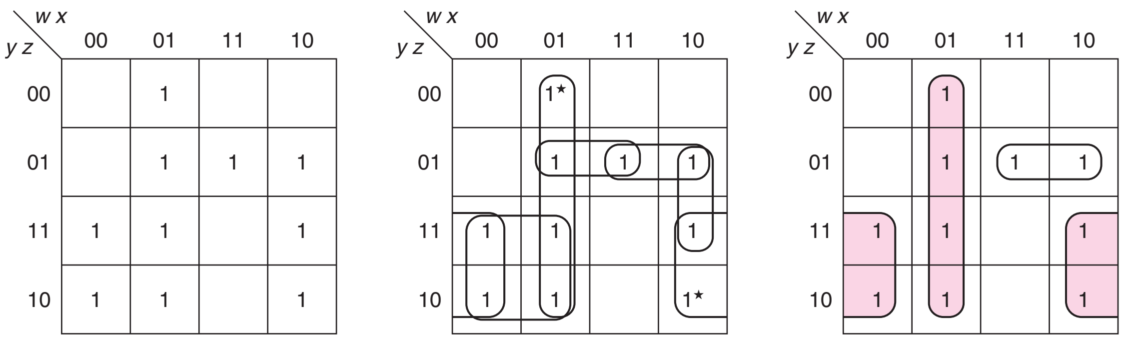




On the second map, the essential prime implicants have been shaded, highlighting the three 1's remaining to be covered. We need two terms to cover them, at least one of which must cover two of these remaining 1's. The three solutions are thus

$$F = A'C' + BD + ACD' + B'C'D'$$
$$F = A'C' + BD + AB'D' + ACD'$$
$$F = A'C' + BD + AB'D' + ABC$$

b.



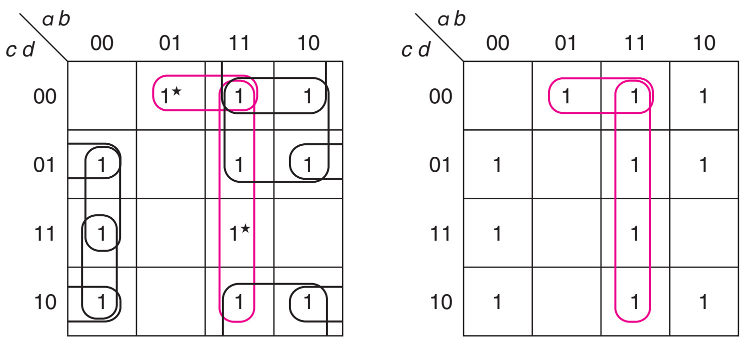
The second map shows all of the prime implicants circled and the 1's that have been covered only once are indicated with a star:

Essential prime implicants: $w'x$, $x'y$
Other prime implicants: $w'y$, $xy'z$, $wy'z$, $wx'z$

With the essential prime implicants shaded on the third map, it is clear that the only minimum solution is

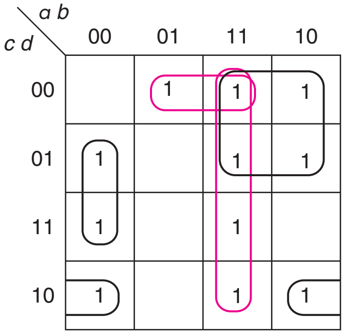
$$f = w'x + x'y + wy'z$$

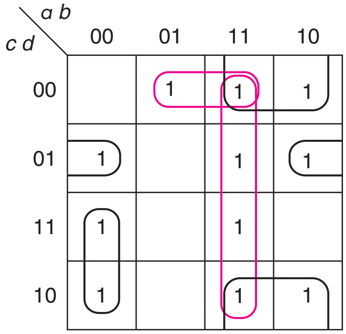
c. All of the prime implicants are circled on the first map, with the essential prime implicants shown in brown.



Essential prime implicants: ab, $bc'd'$
Other prime implicants: ac', ad', $b'c'd$, $b'cd'$, $a'b'c$, $a'b'd$

Once we chose the essential prime implicants, there are six 1's left to be covered. We can only cover two at a time. There are two groups of four 1's, either of which can be used. (We cannot use both, since that would only cover three 1's.) The two solutions are shown on the maps below.





$$h = ab + bc'd' + ac' + a'b'd + b'cd'$$
$$h = ab + bc'd' + ad' + b'c'd + a'b'c$$

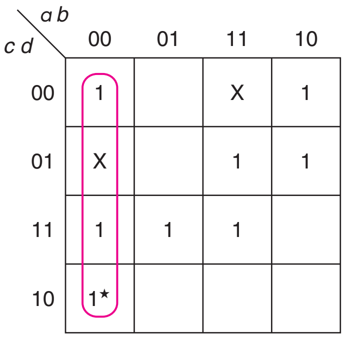
5. For each of the following, find all minimum sum of products expressions. (If there is more than one solution, the number of solutions is given in parentheses.)

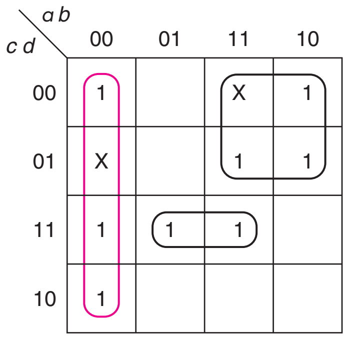
a. $f(a, b, c, d) = \Sigma m(0, 2, 3, 7, 8, 9, 13, 15) + \Sigma d(1, 12)$

b. $F(W, X, Y, Z) = \Sigma m(1, 3, 5, 6, 7, 13, 14) + \Sigma d(8, 10, 12)$ (2 solutions)

c. $f(a, b, c, d) = \Sigma m(3, 8, 10, 13, 15) + \Sigma d(0, 2, 5, 7, 11, 12, 14)$ (8 solutions)

a.



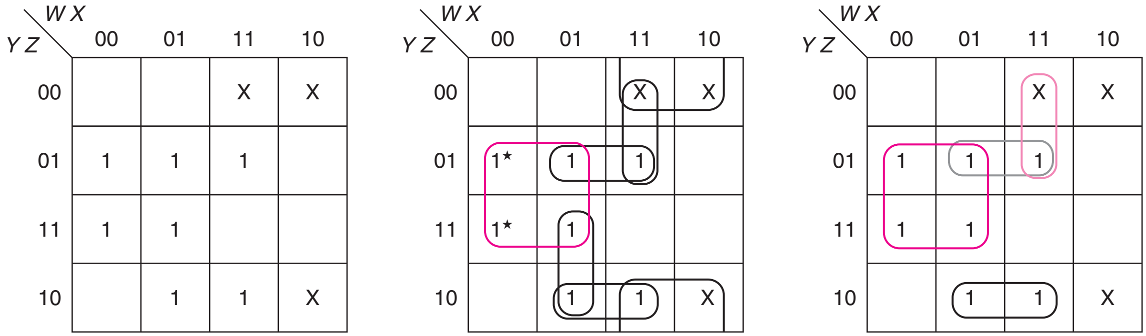


The first map shows the one essential prime implicant, $a'b'$. The remaining 1's can be covered by two additional terms, as shown on the second map. In this example, all don't cares are treated as 1's. The resulting solution is

$$f = a'b' + ac' + bcd$$

Although there are other prime implicants, such as $b'c'$, abd, and $a'cd$, three prime implicants would be needed in addition to $a'b'$ if any of them were chosen.

b.



The second map shows all of the prime implicants circled. It is clear that only $W'Z$ is essential, after which three 1's remain uncovered. The prime implicant XYZ' is the only one that can cover two of these and thus appears in both minimum solutions. That leaves a choice of two terms to cover the remaining one—either WXY' (tan) or $XY'Z$ (gray). Note that they treat the don't care at m_{12} differently, and, thus, although the two solutions shown below both satisfy the requirements of the problem, they are not equal:

$$F = W'Z + XYZ' + WXY'$$
$$F = W'Z + XYZ' + XY'Z$$

Also, the group of four (WZ') is not used; that would require a four-term solution.

c. There are no essential prime implicants in this problem. The left map shows the only two prime implicants that cover m_8; they also cover m_{10}. We must choose one of these. The next map shows the only prime implicants that cover m_{13}; both also cover m_{15}. We must choose one of these also. Finally, the last map shows the only two prime implicants that cover m_3.

cd \ ab	00	01	11	10
00	X		X	1
01		X	1	
11	1	X	1	X
10	X		X	1

cd \ ab	00	01	11	10
00	X		X	1
01		X	1	
11	1	X	1	X
10	X		X	1

cd \ ab	00	01	11	10
00	X		X	1
01		X	1	
11	1	X	1	X
10	X		X	1

So, our final solution takes one from each group, giving us a total of eight solutions:

$$f = \left\{\begin{matrix} ad' \\ b'd' \end{matrix}\right\} + \left\{\begin{matrix} ab \\ bd \end{matrix}\right\} + \left\{\begin{matrix} cd \\ b'c \end{matrix}\right\}$$

or, written out

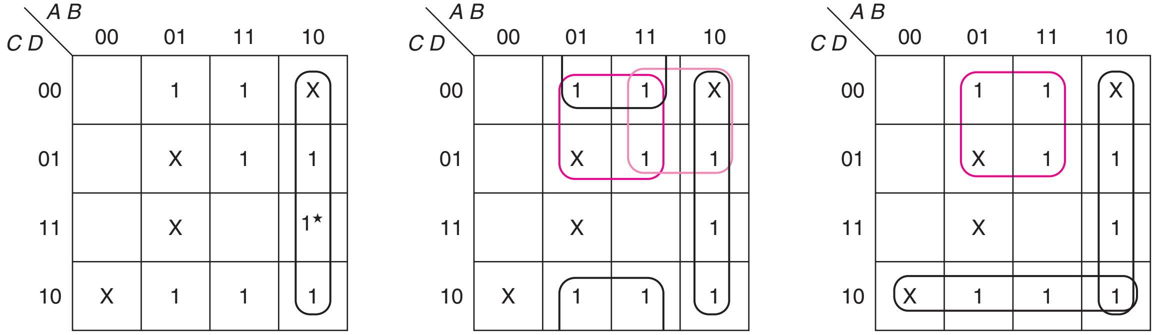
$$f = ad' + ab + cd$$
$$f = ad' + ab + b'c$$
$$f = ad' + bd + cd$$
$$f = ad' + bd + b'c$$
$$f = b'd' + ab + cd$$
$$f = b'd' + ab + b'c$$
$$f = b'd' + bd + cd$$
$$f = b'd' + bd + b'c$$

6. For each of the following, find all minimum sum of products expressions. Label the solutions $f_1, f_2, \ldots$ and indicate which solutions are equal.

a. $F(A, B, C, D) = \Sigma m(4, 6, 9, 10, 11, 12, 13, 14) + \Sigma d(2, 5, 7, 8)$ (3 solutions)

b. $f(a, b, c, d) = \Sigma m(0, 1, 4, 6, 10, 14) + \Sigma d(5, 7, 8, 9, 11, 12, 15)$ (13 solutions)

a.



On the first map, we have shown the one essential prime implicant, AB'. Neither $A'B$ nor CD' are essential, since the 1's covered by them can each be covered by some other prime implicant. (That there is a don't care that can only be covered by one of these terms does not make that term essential.) With five 1's left to be covered, we need two additional terms. The first that stands out is BD', circled on the middle map, since it covers four of the remaining 1's. If that is chosen, it leaves only m_{13}, which can be covered by BC' or AC'. However, the third map shows still another cover, utilizing BC' and CD'. Thus, the three solutions are

$$F_1 = AB' + BD' + BC'$$
$$F_2 = AB' + BD' + AC'$$
$$F_3 = AB' + BC' + CD'$$

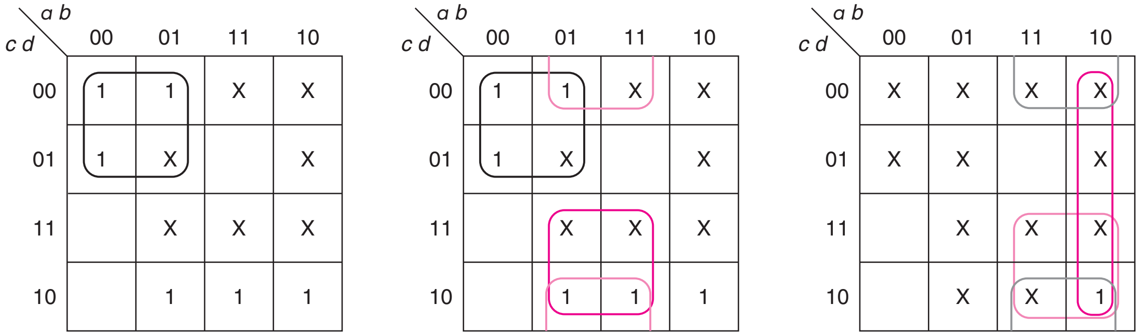
Notice that none of the solutions utilize the remaining prime implicant, $A'B$.

Next is the question of whether or not these three solutions are equal. The answer can be determined by examining how the don't cares are treated by each of the functions. The following table shows that:

	2	**5**	**7**	**8**
F_1	0	1	0	1
F_2	0	0	0	1
F_3	1	1	0	1

In all functions, m_7 is treated as 0 (that is, it is not included in any prime implicant used) and m_8 as 1 (since it is included in the essential prime implicant, AB'); but the first two columns show that no two functions treat m_2 and m_5 the same. Thus, none of these is equal to any other.

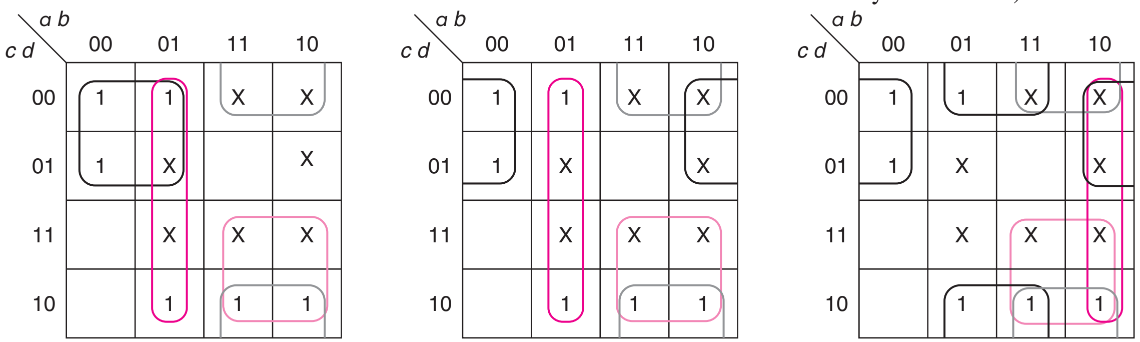
b. There are no essential prime implicants. The best place to start is with a 1 that can only be covered in two ways; in this problem there is only one, m_1. Any solution must contain either the term $a'c'$ (as shown on the first four maps) or the term $b'c'$ (as shown on the remaining two maps). There is no reason to use both, since $b'c'$ does not cover any 1's that are not already covered by $a'c'$. The first map shows $a'c'$. Note that there are three 1's left, requiring two more terms. At least one of these terms must cover two of the remaining 1's.



The second map shows two ways of covering m_6 and m_{14}, bc and bd'. In either case, only one 1 is left to be covered. The third map shows the previously covered 1's as don't cares and three ways of covering the last 1, m_{10}. Thus, we have as the first six solutions

$$f_1 = a'c' + bc + ab'$$
$$f_2 = a'c' + bc + ac$$
$$f_3 = a'c' + bc + ad'$$
$$f_4 = a'c' + bd' + ab'$$
$$f_5 = a'c' + bd' + ac$$
$$f_6 = a'c' + bd' + ad'$$

Next, we consider how we may cover both m_{10} and m_{14} with one term (in addition to those already found). That provides two more solutions shown on the left map below. (Other solutions that use these terms have already been listed.)



$$f_7 = a'c' + a'b + ad'$$
$$f_8 = a'c' + a'b + ac$$

We next consider the solutions that use $b'c'$. The middle map shows two of these, utilizing $a'b$. The last map shows the final three, utilizing bd', instead; it has the same three last terms as in the first series. Thus, we have

$$f_9 = b'c' + a'b + ad'$$
$$f_{10} = b'c' + a'b + ac$$
$$f_{11} = b'c' + bd' + ab'$$
$$f_{12} = b'c' + bd' + ac$$
$$f_{13} = b'c' + bd' + ad'$$

Finally, the table below shows how each of the functions treats the don't cares:

	5	**7**	**8**	**9**	**11**	**12**	**15**
f_1	1	1	1	1	1	0	1
f_2	1	1	0	0	1	0	1
f_3	1	1	1	0	0	1	1
f_4	1	0	1	1	1	1	0
f_5	1	0	0	0	1	1	1
f_6	1	0	1	0	0	1	0
f_7	1	1	1	0	0	1	0
f_8	1	1	0	0	1	0	1
f_9	1	1	1	1	0	1	0
f_{10}	1	1	1	1	1	0	1
f_{11}	0	0	1	1	1	1	0
f_{12}	0	0	1	1	1	1	1
f_{13}	0	0	1	1	0	1	0

Comparing the rows, the only two pairs that are equal are

$$f_1 = f_{10} \quad \text{and} \quad f_2 = f_8.$$

7. For each of the following functions, find all of the minimum sum of products expressions and all of the minimum product of sums expressions:

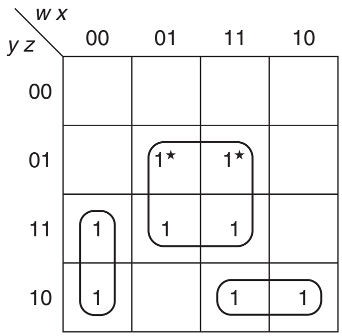
a. $f(w, x, y, z) = \Sigma m(2, 3, 5, 7, 10, 13, 14, 15)$
(1 SOP, 1 POS solution)

b. $f(a, b, c, d) = \Sigma m(3, 4, 9, 13, 14, 15) + \Sigma d(2, 5, 10, 12)$
(1 SOP, 2 POS solutions)

c. $f(a, b, c, d) = \Sigma m(4, 6, 11, 12, 13) + \Sigma d(3, 5, 7, 9, 10, 15)$
(2 SOP and 8 POS solutions)

a. The map of f is shown below.

$yz \backslash wx$	00	01	11	10
00				
01		1	1	
11	1	1	1	
10	1		1	1

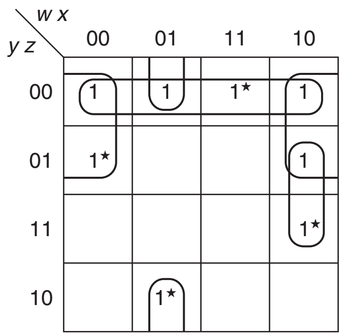


Although there is only one essential prime implicant, there is only one way to complete the cover with two more terms, namely,

$$f = xz + w'x'y + wyz'$$

By replacing all the 1's with 0's and 0's with 1's, or by plotting all the minterms not in f, we get the map for f'

$yz \backslash wx$	00	01	11	10
00	1	1	1	1
01	1			1
11				1
10		1		



There are four essential prime implicants, covering all of f', giving

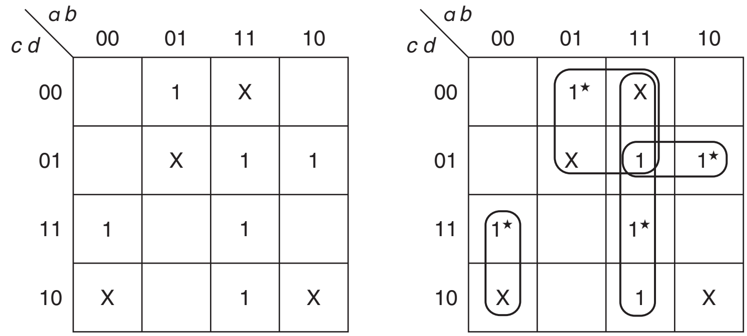
$$f' = x'y' + y'z' + w'xz' + wx'z$$

Using DeMorgan's theorem, we get

$$f = (x + y)(y + z)(w + x' + z)(w' + x + z')$$

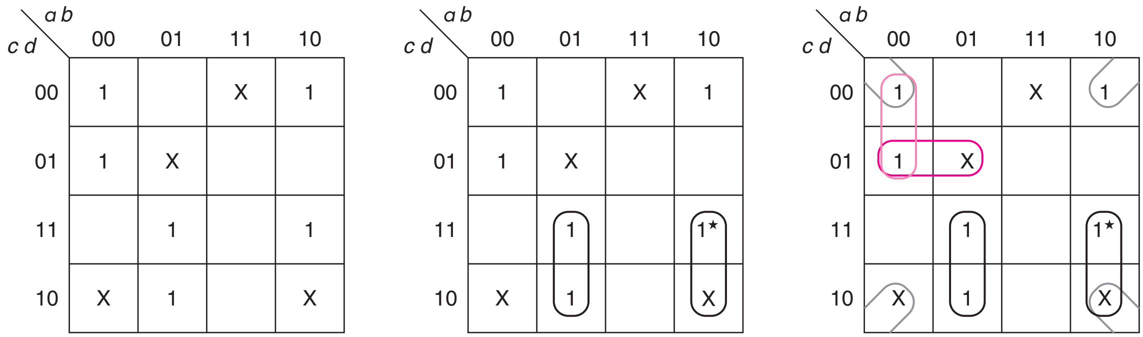
In this case, the sum of products solution requires fewer terms.

b. As indicated on the map below, all of the 1's are covered by essential prime implicants, producing the minimum sum of products expression.



$$f_1 = bc' + ab + a'b'c + ac'd$$

Now, replacing all of the 1's by 0's and 0's by 1's and leaving the X's unchanged, we get the map for f'.



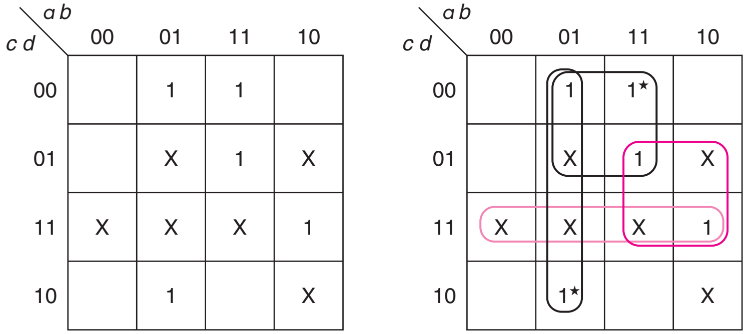
There is one essential prime implicant, $ab'c$. Although m_6 and m_7 can each be covered in two ways, only $a'bc$ covers them both (and neither of the other terms cover additional 1's). The middle map shows each of these terms circled, leaving three 1's to be covered. There is a group of four, covering two of the 1's (as shown on the third map), $b'd'$. That leaves just m_1, which can be covered in two ways, as shown on the third map in brown and tan lines. Thus, the two minimum sum of product expressions for f' are

$$f_2' = ab'c + a'bc + b'd' + a'c'd$$
$$f_3' = ab'c + a'bc + b'd' + a'b'c'$$

producing the two minimum product of sums solutions

$$f_2 = (a' + b + c')(a + b' + c')(b + d)(a + c + d')$$
$$f_3 = (a' + b + c')(a + b' + c')(b + d)(a + b + c)$$

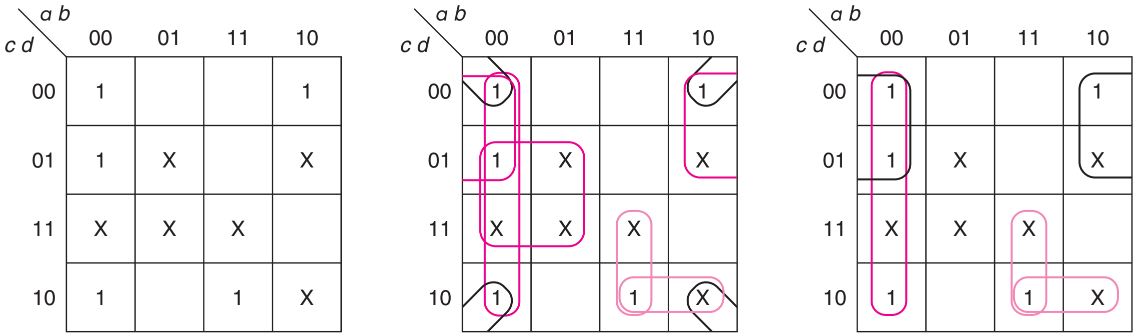
c. The map for f is shown next (on the left). There are two essential prime implicants, leaving only m_{11} to be covered. There are two groups of four that can be used, as indicated on the right-hand map.



Thus the two sum of products solutions are

$$f_1 = a'b + bc' + ad$$
$$f_2 = a'b + bc' + cd$$

We then mapped f' and found no essential prime implicants.



We chose as a starting point m_8. It can be covered either by the four corners, $b'd'$ (as shown on the second map) or by $b'c'$, as shown on the third map. Whichever solution we choose, we need a group of two to cover m_{14} (as shown in tan); neither covers any other 1. After choosing one of these (and $b'd'$), all that remains to be covered is m_1. The three brown lines show the covers. (Notice that one of those is $b'c'$.) If we don't choose $b'd'$, then we must choose $b'c'$ to cover m_0 and $a'b'$ to cover m_2 (since the only other prime implicant that covers m_2 is $b'd'$, and we have already found all of the solutions using that term). Thus, the eight solutions for f' are

$$f_3' = b'd' + abc + a'b'$$
$$f_4' = b'd' + abc + a'd$$
$$f_5' = b'd' + abc + b'c'$$

$$f'_6 = b'd' + acd' + a'b'$$
$$f'_7 = b'd' + acd' + a'd$$
$$f'_8 = b'd' + acd' + b'c'$$
$$f'_9 = b'c' + abc + a'b'$$
$$f'_{10} = b'c' + acd' + a'b'$$

The product of sums solutions for f are thus

$$f_3 = (b + d)(a' + b' + c')(a + b)$$
$$f_4 = (b + d)(a' + b' + c')(a + d')$$
$$f_5 = (b + d)(a' + b' + c')(b + c)$$
$$f_6 = (b + d)(a' + c' + d)(a + b)$$
$$f_7 = (b + d)(a' + c' + d)(a + d')$$
$$f_8 = (b + d)(a' + c' + d)(b + c)$$
$$f_9 = (b + c)(a' + b' + c')(a + b)$$
$$f_{10} = (b + c)(a' + c' + d)(a + b)$$

8. Label the solutions of each part of problem 7 as $f_1, f_2, \ldots$, and indicate which solutions are equal.

a. Since this problem does not involve don't cares, all solutions are equal.

b.

	2	5	10	12
f_1	1	1	0	1
f'_2	1	1	1	0
f_2	0	0	0	1
f'_3	1	0	1	0
f_3	0	1	0	1

All of the solutions are unique. The sum of products solution treats m_2 as a 1; the product of sums treats it as a 0. The two product of sums solutions treat m_5 differently.

c.

	3	5	7	9	10	15
f_1	0	1	1	1	0	1
f_2	1	1	1	0	0	1
f'_3	1	0	0	0	1	1
f'_4	1	1	1	0	1	1
f'_5	0	0	0	1	1	1
f'_6	1	0	0	0	1	0
f'_7	1	1	1	0	1	0
f'_8	0	0	0	1	1	0
f'_9	1	0	0	1	0	1
f'_{10}	1	0	0	1	1	0

For one of the sum of products expressions to be equal to one of the product of sums expressions, the pattern must be opposite (since we are showing the values of the don't cares for f' for the POS forms). Thus, $f_1 = f_6$, and $f_2 = f_8$, that is

$$a'b + bc' + ad = (b + d)(a' + c' + d)(a + b)$$
$$a'b + bc' + cd = (b + d)(a' + c' + d)(b + c)$$

9. Find the minimum sum of products solution(s) for each of the following:

a. $F(A, B, C, D, E) = \Sigma m(0, 5, 7, 9, 11, 13, 15, 18, 19, 22, 23, 25, 27, 28, 29, 31)$

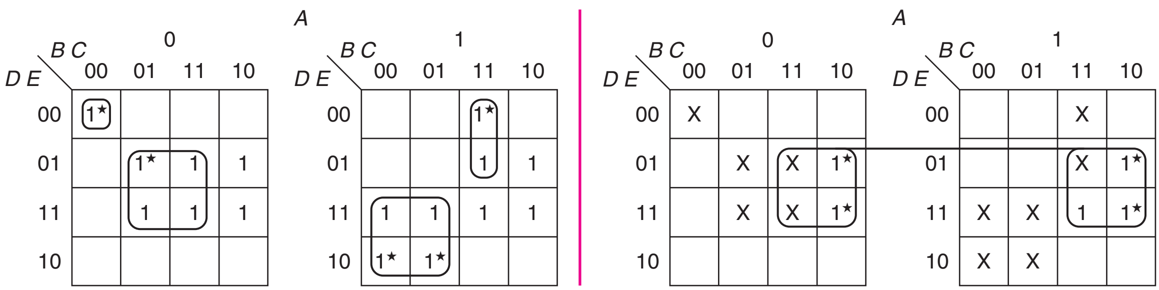
b. $F(A, B, C, D, E) = \Sigma m(0, 2, 4, 7, 8, 10, 15, 17, 20, 21, 23, 25, 26, 27, 29, 31)$

c. $G(V, W, X, Y, Z) = \Sigma m(0, 1, 4, 5, 6, 7, 10, 11, 14, 15, 21, 24, 25, 26, 27)$ (3 solutions)

d. $G(V, W, X, Y, Z) = \Sigma m(0, 1, 5, 6, 7, 8, 9, 14, 17, 20, 21, 22, 23, 25, 28, 29, 30)$ (3 solutions)

e. $H(A, B, C, D, E) = \Sigma m(1, 3, 10, 14, 21, 26, 28, 30) + \Sigma d(5, 12, 17, 29)$

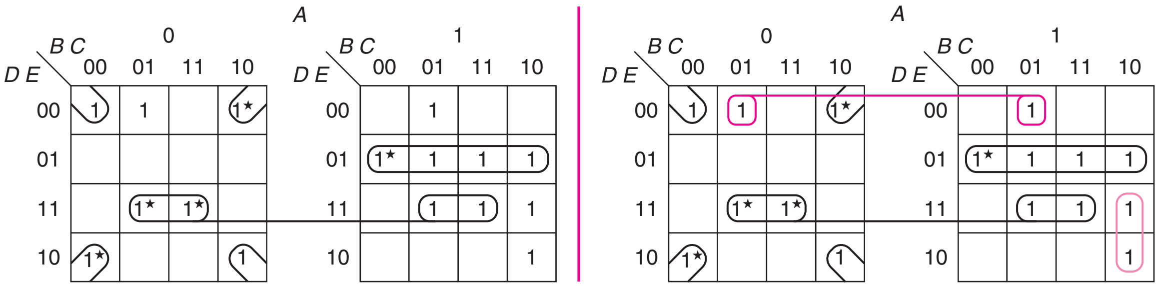
a. We begin by looking at 1's for which the corresponding position on the other layer is 0. On the first map, all of the essential prime implicants that are totally contained on one layer of the map, $A'B'C'D'E'$, $A'CE$, $AB'D$, and $ABCD'$, are circled.



The 1's covered by these essential prime implicants are shown as don't cares on the second map. The remaining 1's are all part of the group of eight, BE, shown on the second map. Thus, the minimum solution is

$$F = A'B'C'D'E' + A'CE + AB'D + ABCD' + BE$$

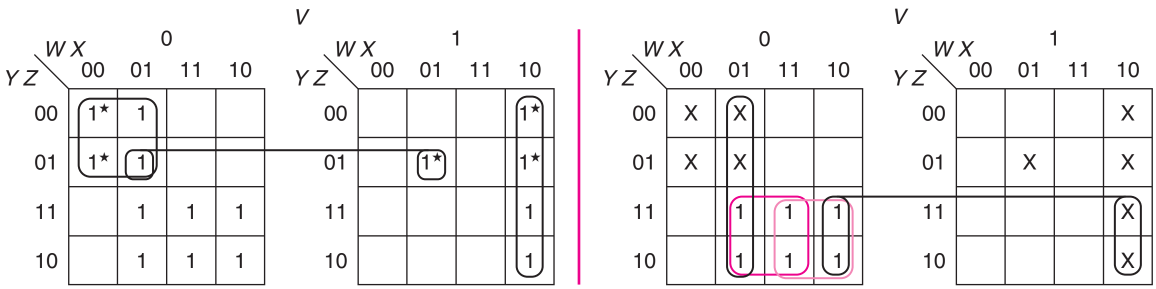
b. On the left-hand map below, the essential prime implicants are circled. Note that $A'C'E'$ is on the top layer, $AD'E$ is on the lower layer, and CDE is split between the layers.



That leaves four 1's to be covered, using two groups of two as shown on the right map. The minimum is thus

$$F = A'C'E' + AD'E + CDE + B'CD'E' + ABC'D$$

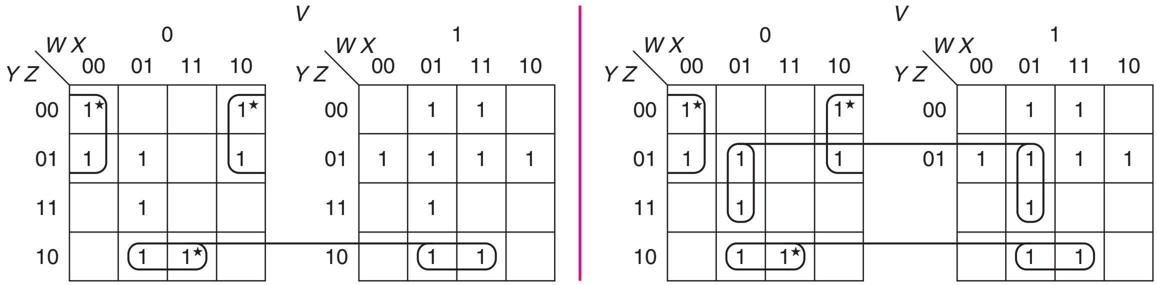
c. The map, with essential prime implicants circled, is shown on the left. After choosing $V'W'Y' + VWX' + W'XY'Z$, there are still six 1's uncovered. On the right-hand map, the minterms covered by essential prime implicants are shown as don't cares. Each of the 1's can be covered by two different groups of four, which are shown on the map on the right.



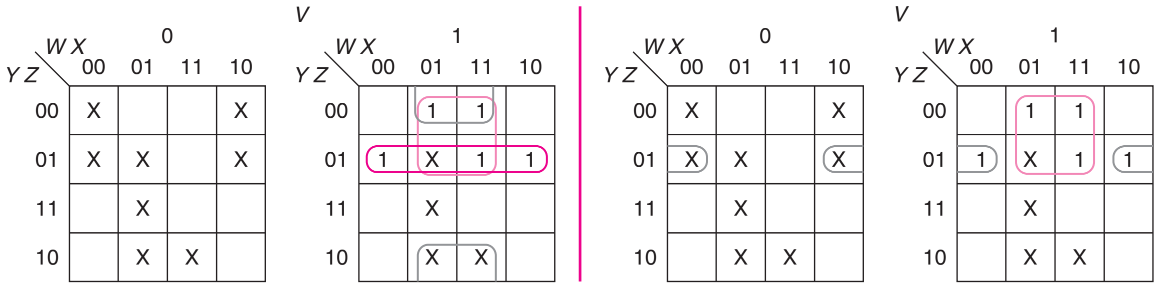
One group that covers four new 1's must be used (or both of them may be used), giving the following solutions:

$$G = V'W'Y' + VWX' + W'XY'Z + V'XY + V'WY$$
$$G = V'W'Y' + VWX' + W'XY'Z + V'XY + WX'Y$$
$$G = V'W'Y' + VWX' + W'XY'Z + V'WY + V'W'X$$

d. On the first map, the two essential prime implicants, $V'X'Y'$ and XYZ', are circled. The term $W'XZ$ is circled on the second map; if it is not used, $W'XY$ would be needed to cover m_7 and m_{23}. But then, three more terms would be needed to cover the function.



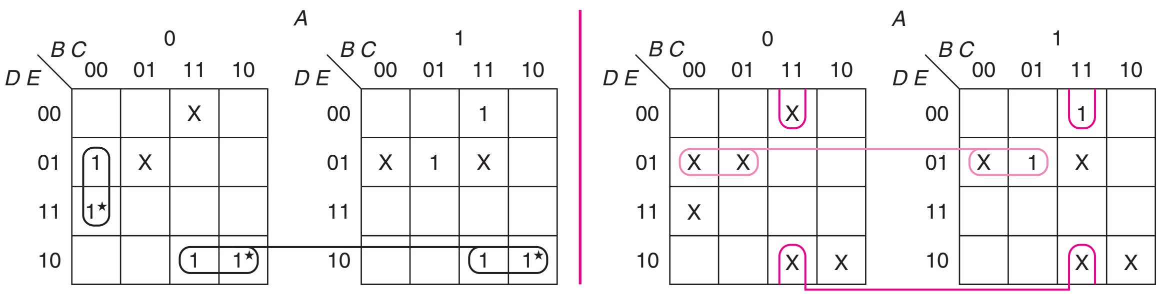
The following maps show the covered terms as don't cares and three ways of covering the remaining 1's. On the left map, the brown term, $VY'Z$, is used with either of the other terms, VXY' or VXZ'. On the right map, VXY' and $X'Y'Z$ are used.



The three minimum solutions are thus

$$G = V'X'Y' + XYZ' + W'XZ + VY'Z + VXY'$$
$$G = V'X'Y' + XYZ' + W'XZ + VY'Z + VXZ'$$
$$G = V'X'Y' + XYZ' + W'XZ + VXY' + X'Y'Z$$

e. The two essential prime implicants, $A'B'C'E$ and BDE', are circled on the first map. Each of the remaining 1's can be covered in two ways, by a group of two contained completely on one layer or by the group of four shown.



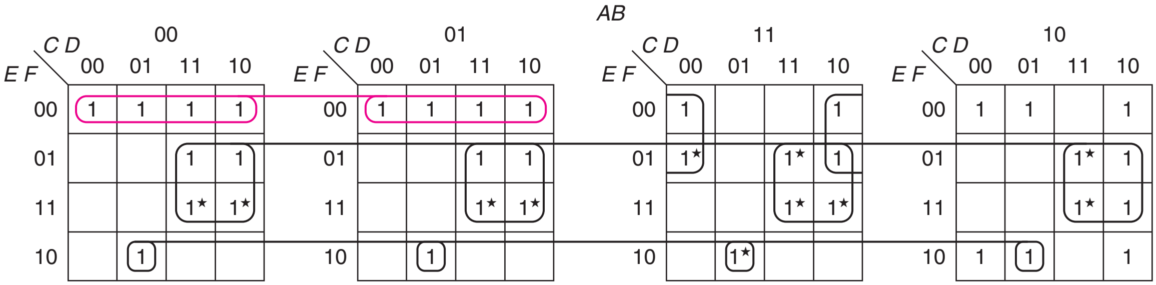
Thus, the minimum solution is

$$H = A'B'C'E + BDE' + BCE' + B'D'E$$

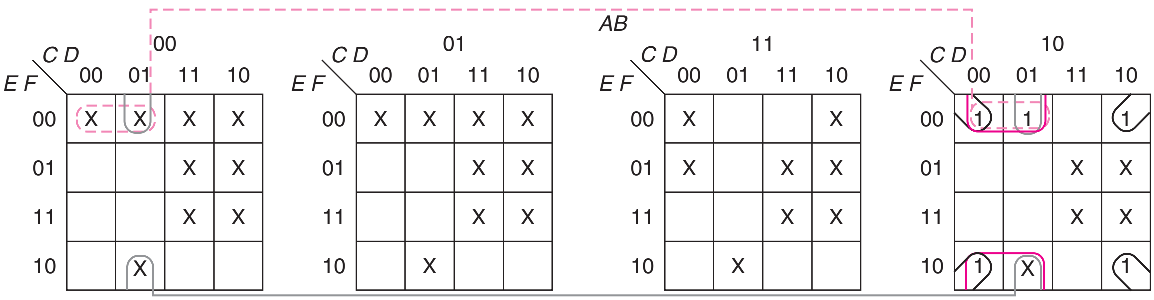
10. Find the four minimum sum of products expressions for the following six-variable function

$$G(A, B, C, D, E, F) = \Sigma m(0, 4, 6, 8, 9, 11, 12, 13, 15, 16, 20, 22, 24, 25, 27, 28, 29, 31, 32, 34, 36, 38, 40, 41, 42, 43, 45, 47, 48, 49, 54, 56, 57, 59, 61, 63)$$

On the first map, the three essential prime implicants, $ABD'E'$, CF, and $C'DEF'$, are circled in black. The first is on just the third layer. The other two include 1's on all four layers (and thus do not involve the variable A and B). Also circled (in brown) is a group of eight, $A'E'F'$, that is not essential (since each of the 1's is part of some other prime implicant). If that is not used, however, at least two terms would be needed to cover those 1's.



On the next map, the 1's that have been covered are shown as don't cares. The remaining 1's are all on the bottom (10) layer. The four corners, $AB'D'F'$, covers four of the five remaining 1's. Then, either $AB'C'F'$ (on the bottom layer) or $B'C'E'F'$ or $B'C'DF'$ (both half on the top layer and half on the bottom) can be used to cover the remaining 1's. These terms are circled below.



Also, as shown on the map below, $AB'C'F'$ could be used with $AB'CD'$.

AB

$AB = 00$

$EF \backslash CD$	00	01	11	10
00	X	X	X	X
01			X	X
11			X	X
10		X		

$AB = 01$

$EF \backslash CD$	00	01	11	10
00	X	X	X	X
01			X	X
11			X	X
10		X		

$AB = 11$

$EF \backslash CD$	00	01	11	10
00	X			X
01	X		X	X
11			X	X
10		X		

$AB = 10$

$EF \backslash CD$	00	01	11	10
00	1	1		1
01			X	X
11			X	X
10	1	X		1

Thus, we have the following four solutions

$$H = ABD'E' + CF + C'DEF' + A'E'F' + AB'D'F' + AB'C'F'$$

$$H = ABD'E' + CF + C'DEF' + A'E'F' + AB'D'F' + B'C'E'F'$$

$$H = ABD'E' + CF + C'DEF' + A'E'F' + AB'D'F' + B'C'DF'$$

$$H = ABD'E' + CF + C'DEF' + A'E'F' + AB'C'F' + AB'CD'$$

11. Find a minimum two-level circuit (corresponding to sum of products expressions) using AND gates and one OR gate per function for each of the following sets of functions:

a. $f(a, b, c, d) = \Sigma m(0, 1, 2, 3, 5, 7, 8, 10, 11, 13)$
$g(a, b, c, d) = \Sigma m(0, 2, 5, 8, 10, 11, 13, 15)$
(7 gates, 19 inputs)

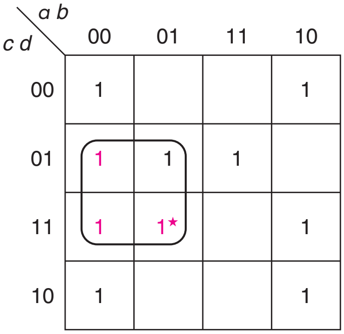
b. $f(a, b, c, d) = \Sigma m(1, 2, 4, 5, 6, 9, 11, 13, 15)$
$g(a, b, c, d) = \Sigma m(0, 2, 4, 8, 9, 11, 12, 13, 14, 15)$
(8 gates, 23 inputs)

c. $F(W, X, Y, Z) = \Sigma m(2, 3, 6, 7, 8, 9, 13)$
$G(W, X, Y, Z) = \Sigma m(2, 3, 6, 7, 9, 10, 13, 14)$
$H(W, X, Y, Z) = \Sigma m(0, 1, 4, 5, 9, 10, 13, 14)$
(8 gates, 22 inputs)

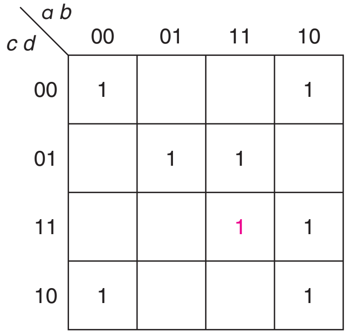
d. $f(a, b, c, d) = \Sigma m(0, 2, 3, 8, 9, 10, 11, 12, 13, 15)$
$g(a, b, c, d) = \Sigma m(3, 5, 7, 12, 13, 15)$
$h(a, b, c, d) = \Sigma m(0, 2, 3, 4, 6, 8, 10, 14)$
(10 gates, 28 inputs)

e. $f(a, b, c, d) = \Sigma m(0, 3, 5, 7) + \Sigma d(10, 11, 12, 13, 14, 15)$
$g(a, b, c, d) = \Sigma m(0, 5, 6, 7, 8) + \Sigma d(10, 11, 12, 13, 14, 15)$
(7 gates, 19 inputs)

a. The maps below show the prime implicant $a'd$ in f, that covers a 1 not part of the other function.

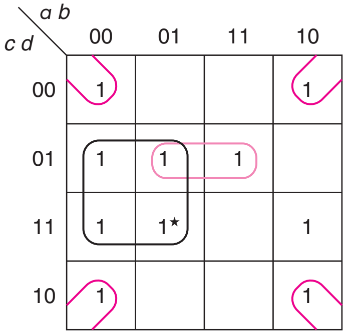


f

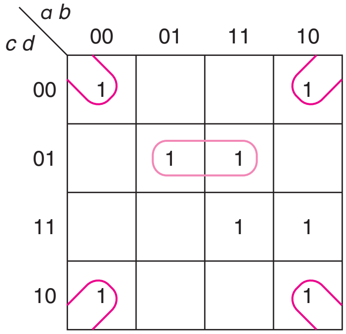


g

No other 1 (of either f or g) that is not shared makes a prime implicant essential (m_1 or m_3 in f or m_{15} in g). Two other terms, $b'd'$ and $bc'd$, are essential prime implicants of both f and g and have been thus chosen in the maps below.

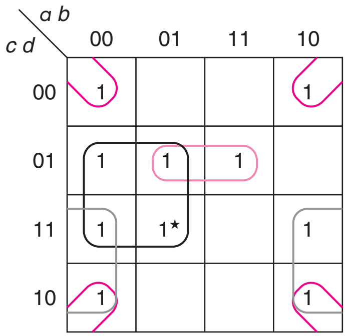


f

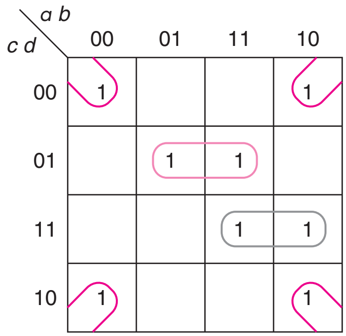


g

Although the term $ab'c$ could be shared, another term would be needed for g (either abd or acd). This would require seven gates and 20 gate inputs (one input too many). But, if acd is used for g, we could then complete covering both functions using $b'c$ for f as shown on the maps below.



f



g

Thus,

$$f = a'd + b'd' + bc'd + b'c$$
$$g = b'd' + bc'd + acd$$

requiring seven gates and 19 inputs.

b. Scanning each function for 1's that are not part of the other function, we find m_1, m_5, and m_6 in f and m_0, m_8, m_{12}, and m_{14} in g. The only ones that make a prime implicant essential are indicated on the map below.

cd \ ab	00	01	11	10
00		1		
01	1*	1	1	1
11			1	1
10	1	1		

f

cd \ ab	00	01	11	10
00	1	1	1	1
01			1	1
11			1	1
10	1		1*	

g

Next, we note that *ad* is an essential prime implicant of both functions, producing the following maps:

cd \ ab	00	01	11	10
00		1		
01	1*	1	1	1
11			1	1
10	1	1		

f

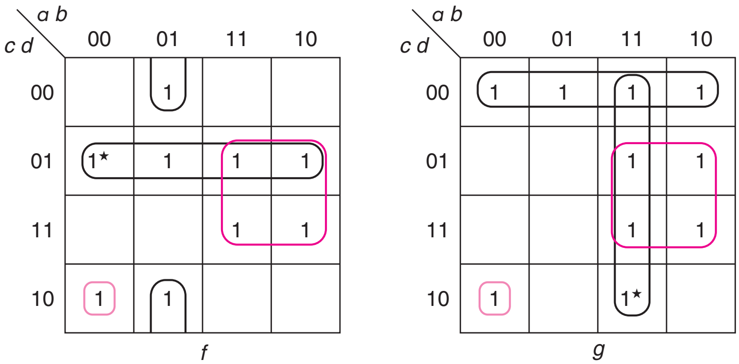
cd \ ab	00	01	11	10
00	1	1	1	1
01			1	1
11			1	1
10	1		1*	

g

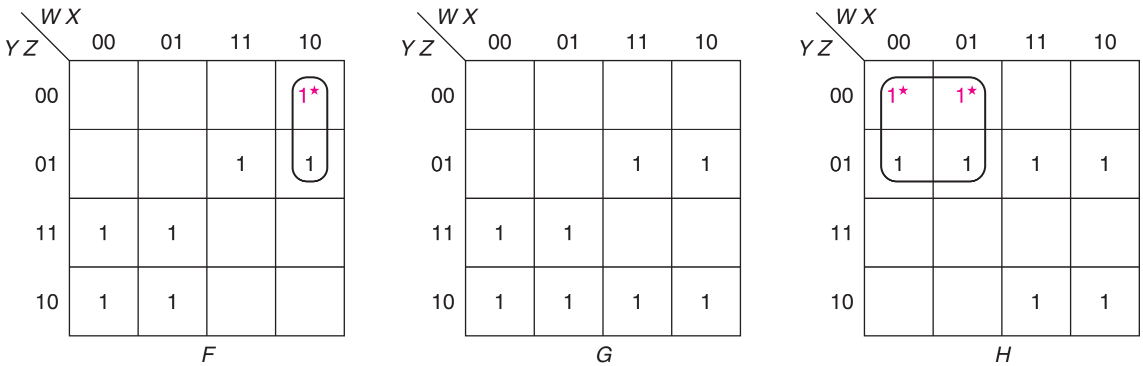
Unless we choose $c'd'$ to cover the remaining three 1's in the first row of g, we will need an extra term. Once we have done that, we see that the last 1 (m_2) of g can be covered by the minterm and shared with f. That leaves just two 1's of f that can be covered with the term $a'bd'$. The functions and the maps are shown next:

$$f = c'd + ad + a'b'cd' + a'bd'$$
$$g = ab + ad + c'd' + a'b'cd'$$

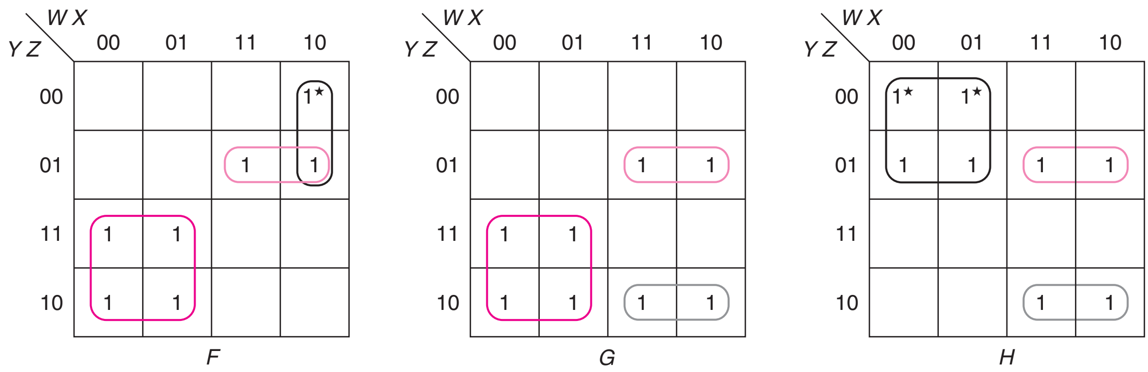
for a total of eight gates and 23 inputs.



c. When minimizing three functions, we still look for 1's that are only included in one of the functions and that make a prime implicant essential. In this problem, the only ones that satisfy these conditions are m_8 in F and m_0 and m_4 in H, as shown on the map below.

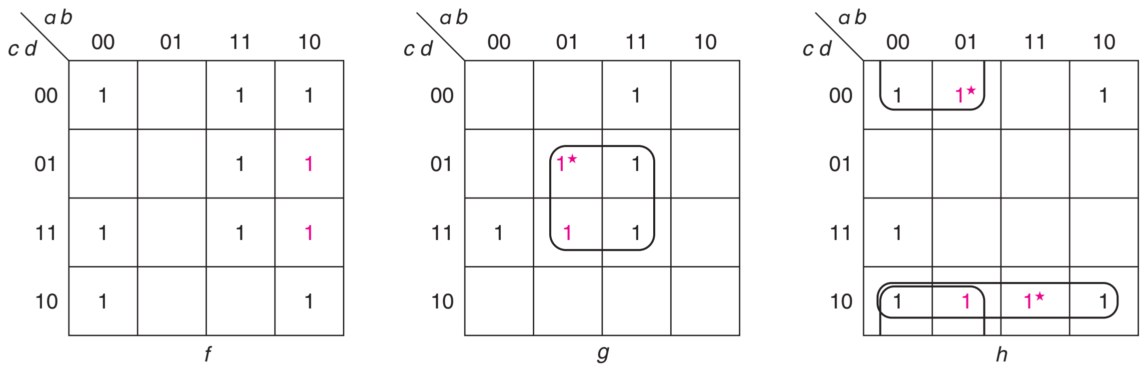


Next, notice that $W'Y$ is an essential prime implicant of both F and G. Once that is chosen, the term $WY'Z$ covers the remaining 1 of F and two 1's in G and H. (That term would be used for both F and G in any case since it is an essential prime implicant of both and is shareable. It is used for H since the remaining 1's in the prime implicant $Y'Z$ are already covered.) Finally, WYZ', an essential prime implicant of H, finishes the cover of G and H. The maps and functions below show the final solution, utilizing eight gates and 22 inputs.

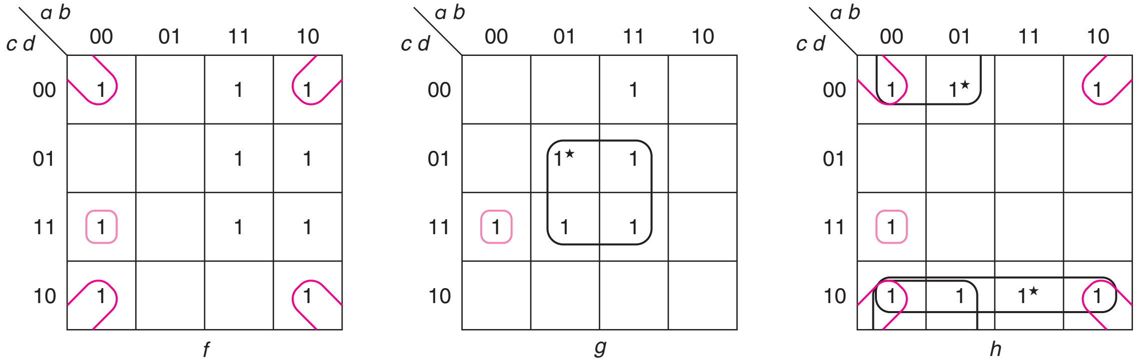


$$F = WX'Y' + W'Y + WY'Z$$
$$G = W'Y + WY'Z + WYZ'$$
$$H = W'Y' + WY'Z + WYZ'$$

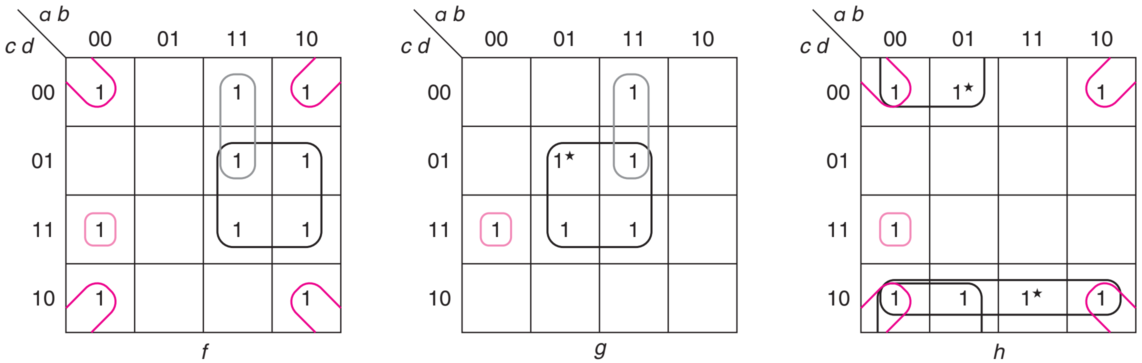
d. On the maps below, the essential prime implicants that cover 1's not part of any other function are circled. In f, m_9 and m_{11} can be covered with any of three prime implicants.



Next, we note that m_8 can only be covered by $b'd'$ in h and that $b'd'$ is also an essential prime implicant of f. That leaves only m_3 uncovered in h; by using the minterm for that, it can be shared with both f and g. (Otherwise, a new term would be required in each of those functions.) The resulting maps are shown below.

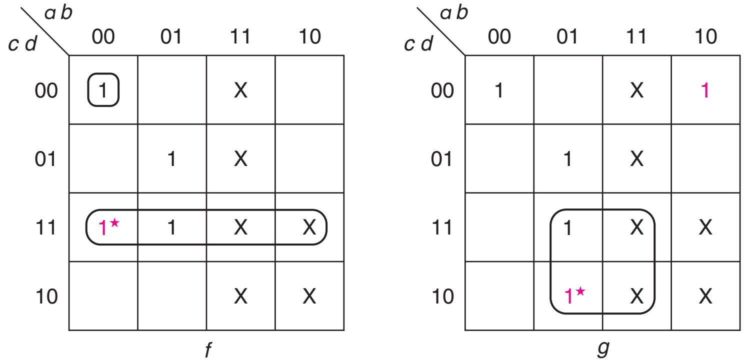


The only uncovered 1 in g is m_{12}. By using abc' for both that and for f, we can cover the three remaining 1's in f with ad, yielding the maps and equations below.

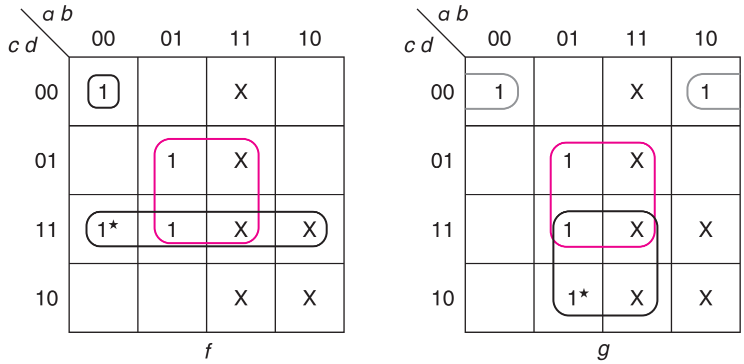


$$f = b'd' + a'b'cd + abc' + ad$$
$$g = bd + a'b'cd + abc'$$
$$h = a'd' + cd' + b'd' + a'b'cd$$

e. This example includes a number of don't cares, but that does not change the process significantly. There are two essential prime implicants, cd in f and bc in g, that cover 1's that cannot be shared. In addition, $a'b'c'd'$ must be used in f since it is the only prime implicant that covers m_0. (If a minterm is a prime implicant, we have no choice but to use it.) The following maps show these terms circled.



Next, we use *bd* to cover m_5 in both functions, and complete the cover of *f*. The obvious choice is to use $b'c'd'$ for the remaining 1's of *g*, producing the following maps and equations:

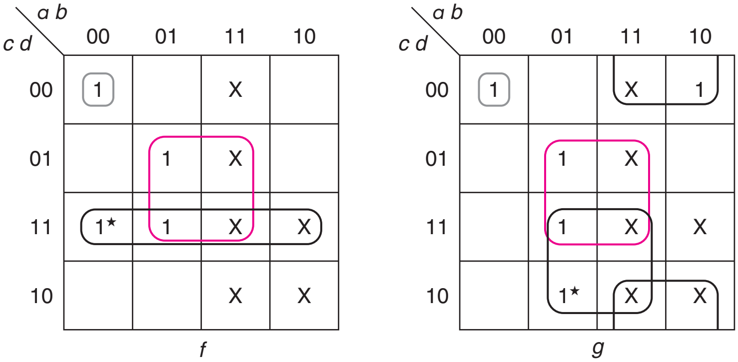


$$f = cd + a'b'c'd' + bd$$
$$g = bc + bd + b'c'd'$$

But, there is another solution, as illustrated below. By using $a'b'c'd'$ to cover m_0 in *g* (we already needed that term for *f*), we can cover the remaining 1 in *g* with a group of four, ad', producing the solution

$$f = cd + a'b'c'd' + bd$$
$$g = bc + bd + a'b'c'd' + ad'$$

as shown on the following maps. Both solutions require seven gates and 19 inputs.



3.8 EXERCISES

1. Plot the following functions on the Karnaugh map:

a. $f(a, b, c) = \Sigma m(1, 2, 3, 4, 6)$

*b. $g(w, x, y, z) = \Sigma m(1, 3, 5, 6, 7, 13, 14) + \Sigma d(8, 10, 12)$

c. $F = WX'Y'Z + W'XYZ + W'X'Y'Z' + W'XY'Z + WXYZ$

*d. $g = a'c + a'bd' + bc'd + ab'd + ab'cd'$

e. $h = x + yz' + x'z$

2. For each of the following, find all minimum sum of products expressions. (If there is more than one solution, the number of solutions is given in parentheses.)

a. $f(a, b, c) = \Sigma m(1, 2, 3, 6, 7)$

*b. $g(w, x, y) = \Sigma m(0, 1, 5, 6, 7)$ (2 solutions)

c. $h(a, b, c) = \Sigma m(0, 1, 2, 5, 6, 7)$ (2 solutions)

d. $f(a, b, c, d) = \Sigma m(1, 2, 3, 5, 6, 7, 8, 11, 13, 15)$

*e. $G(W, X, Y, Z) = \Sigma m(0, 2, 5, 7, 8, 10, 12, 13)$

f. $h(a, b, c, d) = \Sigma m(2, 4, 5, 6, 7, 8, 10, 12, 13, 15)$ (2 solutions)

g. $f(a, b, c, d) = \Sigma m(1, 3, 4, 5, 6, 11, 12, 13, 14, 15)$ (2 solutions)

h. $g(w, x, y, z) = \Sigma m(2, 3, 6, 7, 8, 10, 11, 12, 13, 15)$ (2 solutions)

*i. $h(p, q, r, s) = \Sigma m(0, 2, 3, 4, 5, 8, 11, 12, 13, 14, 15)$ (3 solutions)

j. $F(W, X, Y, Z) = \Sigma m(0, 2, 3, 4, 5, 8, 10, 11, 12, 13, 14, 15)$ (4 solutions)

k. $f(w, x, y, z) = \Sigma m(0, 1, 2, 4, 5, 6, 9, 10, 11, 13, 14, 15)$ (2 solutions)

l. $g(a, b, c, d) = \Sigma m(0, 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15)$

*m. $H(W, X, Y, Z) = \Sigma m(0, 2, 3, 5, 7, 8, 10, 12, 13)$ (4 solutions)

★n. $f(a, b, c, d) = \Sigma m(0, 1, 2, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15)$
(6 solutions)

o. $g(w, x, y, z) = \Sigma m(0, 1, 2, 3, 5, 6, 7, 8, 9, 10, 13, 14, 15)$
(6 solutions)

★p. $f(a, b, c, d) = \Sigma m(0, 3, 5, 6, 7, 9, 10, 11, 12, 13, 14)$
(32 solutions)

3. For the following functions,

i. List all prime implicants, indicating which are essential.

ii. Show the minimum sum of products expression(s).

a. $f(a, b, c, d) = \Sigma m(0, 3, 4, 5, 8, 11, 12, 13, 14, 15)$

★b. $g(w, x, y, z) = \Sigma m(0, 3, 4, 5, 6, 7, 8, 9, 11, 13, 14, 15)$

4. Map each of the following functions and find the minimum sum of products expression:

a. $F = AD + AB + A'CD' + B'CD + A'BC'D'$

★b. $g = w'yz + xy'z + wy + wxy'z' + wz + xyz'$

5. For each of the following, find all minimum sum of products expressions. (If there is more than one solution, the number of solutions is given in parentheses.) Label the solutions $f_1, f_2, \ldots$.

a. $f(w, x, y, z) = \Sigma m(1, 3, 6, 8, 11, 14) + \Sigma d(2, 4, 5, 13, 15)$
(3 solutions)

b. $f(a, b, c, d) = \Sigma m(0, 3, 6, 9, 11, 13, 14) + \Sigma d(5, 7, 10, 12)$

★c. $f(a, b, c, d) = \Sigma m(0, 2, 3, 5, 7, 8, 9, 10, 11) + \Sigma d(4, 15)$
(3 solutions)

d. $f(w, x, y, z) = \Sigma m(0, 2, 4, 5, 10, 12, 15) + \Sigma d(8, 14)$
(2 solutions)

e. $f(a, b, c, d) = \Sigma m(5, 7, 9, 11, 13, 14) + \Sigma d(2, 6, 10, 12, 15)$
(4 solutions)

★f. $f(a, b, c, d) = \Sigma m(0, 2, 4, 5, 6, 7, 8, 9, 10, 14) + \Sigma d(3, 13)$
(3 solutions)

g. $f(w, x, y, z) = \Sigma m(1, 2, 5, 10, 12) + \Sigma d(0, 3, 4, 8, 13, 14, 15)$
(7 solutions)

6. For each of the functions of problem 5, indicate which solutions are equal.

7. For each of the following functions, find all of the minimum sum of products expressions and all of the minimum product of sums expressions:

★a. $f(A, B, C, D) = \Sigma m(1, 4, 5, 6, 7, 9, 11, 13, 15)$

b. $f(W, X, Y, Z) = \Sigma m(2, 4, 5, 6, 7, 10, 11, 15)$

c. $f(A, B, C, D) = \Sigma m(1, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15)$
(1 SOP and 2 POS solutions)

*d. $f(a, b, c, d) = \Sigma m(0, 2, 4, 6, 7, 9, 11, 12, 13, 14, 15)$
(2 SOP and 1 POS solutions)

e. $f(w, x, y, z) = \Sigma m(0, 4, 6, 9, 10, 11, 14) + \Sigma d(1, 3, 5, 7)$

f. $f(a, b, c, d) = \Sigma m(0, 1, 2, 5, 7, 9) + \Sigma d(6, 8, 11, 13, 14, 15)$
(4 SOP and 2 POS solutions)

g. $f(w, x, y, z) = \Sigma m(4, 6, 9, 10, 11, 13) + \Sigma d(2, 12, 15)$
(2 SOP and 2 POS solutions)

h. $f(a, b, c, d) = \Sigma m(0, 1, 4, 6, 10, 14) + \Sigma d(5, 7, 8, 9, 11, 12, 15)$
(13 SOP and 3 POS solutions)

*i. $f(w, x, y, z) = \Sigma m(1, 3, 7, 11, 13, 14) + \Sigma d(0, 2, 5, 8, 10, 12, 15)$
(6 SOP and 1 POS solutions)

j. $f(a, b, c, d) = \Sigma m(0, 1, 6, 15) + \Sigma d(3, 5, 7, 11, 14)$
(1 SOP and 2 POS solutions)

8. Label the solutions of each part of problem 7 as $f_1, f_2, \ldots$ and indicate which solutions are equal.

9. For each of the following five-variable functions, find all minimum sum of products expressions. (If there is more than one solution, the number of solutions is given in parentheses.)

a. $F(A, B, C, D, E) = \Sigma m(0, 1, 5, 7, 8, 9, 10, 11, 13, 15, 18, 20, 21, 23, 26, 28, 29, 31)$

b. $G(A, B, C, D, E) = \Sigma m(0, 1, 2, 4, 5, 6, 10, 13, 14, 18, 21, 22, 24, 26, 29, 30)$

*c. $H(A, B, C, D, E) = \Sigma m(5, 8, 12, 13, 15, 17, 19, 21, 23, 24, 28, 31)$

d. $F(V, W, X, Y, Z) = \Sigma m(2, 4, 5, 6, 10, 11, 12, 13, 14, 15, 16, 17, 18, 21, 24, 25, 29, 30, 31)$

e. $G(V, W, X, Y, Z) = \Sigma m(0, 1, 4, 5, 8, 9, 10, 15, 16, 18, 19, 20, 24, 26, 28, 31)$

*f. $H(V, W, X, Y, Z) = \Sigma m(0, 1, 2, 3, 5, 7, 10, 11, 14, 15, 16, 18, 24, 25, 28, 29, 31)$ (2 solutions)

g. $F(A, B, C, D, E) = \Sigma m(0, 4, 6, 8, 12, 13, 14, 15, 16, 17, 18, 21, 24, 25, 26, 28, 29, 31)$ (6 solutions)

h. $G(A, B, C, D, E) = \Sigma m(0, 3, 5, 7\ 12, 13, 14, 15, 19, 20, 21, 22, 23, 25, 26, 29, 30)$ (3 solutions)

*i. $H(A, B, C, D, E) = \Sigma m(0, 1, 5, 6, 7, 8, 9, 14, 17, 20, 21, 22, 23, 25, 28, 29, 30)$ (3 solutions)

j. $F(V, W, X, Y, Z) = \Sigma m(0, 4, 5, 7, 10, 11, 14, 15, 16, 18, 20, 21, 23, 24, 25, 26, 29, 31)$ (4 solutions)

k. $G(V, W, X, Y, Z) = \Sigma m(0, 2, 5, 6, 8, 10, 11, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 24, 26, 29, 31)$
(3 solutions)

l. $H(V, W, X, Y, Z) = \Sigma m(0, 1, 2, 3, 5, 8, 9, 10, 13, 17, 18, 19, 20, 21, 26, 28, 29)$ (3 solutions)

m. $F(A, B, C, D, E) = \Sigma m(1, 2, 5, 8, 9, 10, 12, 13, 14, 15, 16, 18, 21, 22, 23, 24, 26, 29, 30, 31)$ (18 solutions)

*n. $G(V, W, X, Y, Z) = \Sigma m(0, 1, 5, 7, 8, 13, 24, 25, 29, 31) + \Sigma d(9, 15, 16, 17, 23, 26, 27, 30)$ (2 solutions)

o. $H(A, B, C, D, E) = \Sigma m(0, 4, 12, 15, 27, 29, 30) + \Sigma d(1, 5, 9, 10, 14, 16, 20, 28, 31)$ (4 solutions)

p. $F(A, B, C, D, E) = \Sigma m(8, 9, 11, 14, 28, 30) + d(0, 3, 4, 6, 7, 12, 13, 15, 20, 22, 27, 29, 31)$ (8 solutions)

10. For each of the following six-variable functions, find all minimum sum of products expressions. (The number of terms and literals, and, if there is more than one solution, the number of solutions is given in parentheses.)

a. $G(A, B, C, D, E, F) = \Sigma m(4, 5, 6, 7, 8, 10, 13, 15, 18, 20, 21, 22, 23, 26, 29, 30, 31, 33, 36, 37, 38, 39, 40, 42, 49, 52, 53, 54, 55, 60, 61)$ (6 terms, 21 literals)

*b. $G(A, B, C, D, E, F) = \Sigma m(2, 3, 6, 7, 8, 12, 14, 17, 19, 21, 23, 25, 27, 28, 29, 30, 32, 33, 34, 35, 40, 44, 46, 49, 51, 53, 55, 57, 59, 61, 62, 63)$ (8 terms, 30 literals)

c. $G(A, B, C, D, E, F) = \Sigma m(0, 1, 2, 4, 5, 6, 7, 9, 13, 15, 17, 19, 21, 23, 26, 27, 29, 30, 31, 33, 37, 39, 40, 42, 44, 45, 46, 47, 49, 53, 55, 57, 59, 60, 61, 62, 63)$ (8 terms, 28 literals, 2 solutions)

11. Find a minimum two-level circuit (corresponding to sum of products expressions) using AND and one OR gate per function for each of the following sets of functions.

*a. $f(a, b, c, d) = \Sigma m(1, 3, 5, 8, 9, 10, 13, 14)$
$g(a, b, c, d) = \Sigma m(4, 5, 6, 7, 10, 13, 14)$ (7 gates, 21 inputs)

b. $f(a, b, c, d) = \Sigma m(0, 1, 2, 3, 4, 5, 8, 10, 13)$
$g(a, b, c, d) = \Sigma m(0, 1, 2, 3, 8, 9, 10, 11, 13)$ (6 gates, 16 inputs)

c. $f(a, b, c, d) = \Sigma m(5, 8, 9, 12, 13, 14)$
$g(a, b, c, d) = \Sigma m(1, 3, 5, 8, 9, 10)$ (3 solutions, 8 gates, 25 inputs)

d. $f(a, b, c, d) = \Sigma m(1, 3, 4, 5, 10, 11, 12, 14, 15)$
$g(a, b, c, d) = \Sigma m(0, 1, 2, 8, 10, 11, 12, 15)$
(9 gates, 28 inputs)

*e. $F(W, X, Y, Z) = \Sigma m(1, 5, 7, 8, 10, 11, 12, 14, 15)$
$G(W, X, Y, Z) = \Sigma m(0, 1, 4, 6, 7, 8, 12)$ (8 gates, 23 inputs)

f. $F(W, X, Y, Z) = \Sigma m(0, 2, 3, 7, 8, 9, 13, 15)$
$G(W, X, Y, Z) = \Sigma m(0, 2, 8, 9, 10, 12, 13, 14)$
(2 solutions, 8 gates, 23 inputs)

g. $f(a, b, c, d) = \Sigma m(1, 3, 5, 7, 8, 9, 10)$
$g(a, b, c, d) = \Sigma m(0, 2, 4, 5, 6, 8, 10, 11, 12)$
$h(a, b, c, d) = \Sigma m(1, 2, 3, 5, 7, 10, 12, 13, 14, 15)$
(2 solutions, 12 gates, 33 inputs)

*h. $f(a, b, c, d) = \Sigma m(0, 3, 4, 5, 7, 8, 12, 13, 15)$
$g(a, b, c, d) = \Sigma m(1, 5, 7, 8, 9, 10, 11, 13, 14, 15)$
$h(a, b, c, d) = \Sigma m(1, 2, 4, 5, 7, 10, 13, 14, 15)$
(2 solutions, 11 gates, 33 inputs)

i. $f(a, b, c, d) = \Sigma m(0, 2, 3, 4, 6, 7, 9, 11, 13)$
$g(a, b, c, d) = \Sigma m(2, 3, 5, 6, 7, 8, 9, 10, 13)$
$h(a, b, c, d) = \Sigma m(0, 4, 8, 9, 10, 13, 15)$
(2 solutions for f and g, 10 gates, 32 inputs)

*j. $f(a, c, b, d) = \Sigma m(0, 1, 2, 3, 4, 9) + \Sigma d(10, 11, 12, 13, 14, 15)$
$g(a, c, b, d) = \Sigma m(1, 2, 6, 9) + \Sigma d(10, 11, 12, 13, 14, 15)$
(3 solutions for f, 6 gates, 15 inputs)

k. $f(a, c, b, d) = \Sigma m(5, 6, 11) + \Sigma d(0, 1, 2, 4, 8)$
$g(a, c, b, d) = \Sigma m(6, 9, 11, 12, 14) + \Sigma d(0, 1, 2, 4, 8)$
(2 solutions for g, 7 gates, 18 inputs)

12. In each of the following sets, the functions have been minimized individually. Find a minimum two-level circuit (corresponding to sum of products expressions) using AND and one OR gate per function for each.

a. $F = B'D' + CD' + AB'C$
$G = BC + ACD$ (6 gates, 15 inputs)

*b. $F = A'B'C'D + BC + ACD + AC'D'$
$G = A'B'C'D' + A'BC + BCD'$
$H = B'C'D' + BCD + AC' + AD$
(2 solutions for H, 10 gates, 35 inputs)

c. $f = a'b' + a'd + b'c'd'$
$g = b'c'd' + bd + acd + abc$
$h = a'd' + a'b + bc'd + b'c'd'$ (10 gates, 31 inputs)

3.9 CHAPTER 3 TEST (100 MINUTES, OR TWO 50-MINUTE TESTS)

1. Map each of the following functions (be sure to label the maps):

a. $f(x, y, z) = \Sigma m(1, 2, 7) + \Sigma d(4, 5)$

	00	01	11	10
0				
1				

b. $g = a'c + ab'c'd + a'bd + abc'$

Circle each of the terms.

	00	01	11	10
00				
01				
11				
10				

2. Find the minimum sum of products expression for each of the following functions (that is, circle the terms on the map and write the algebraic expressions).

a.

yz \ wx	00	01	11	10
00		1		1
01		1		
11		1	1	1
10		1		

b.

cd \ ab	00	01	11	10
00	1	1	1	
01	1	1	1	
11			1	1
10	1	1	1	

3. Find all four minimum sum of products expressions for the following function. (Two copies of the map are given for your convenience.)

cd \ ab	00	01	11	10
00	1	1	1	
01	1		1	1
11	1		1	1
10		1	1	1

cd \ ab	00	01	11	10
00	1	1	1	
01	1		1	1
11	1		1	1
10		1	1	1

4. For the following function (three copies of the map are shown),

a. List all prime implicants, indicating which, if any, are essential.

b. Find all four minimum solutions.

yz \ wx	00	01	11	10
00		1	1	X
01	X	X		X
11	X			1
10			1	1

yz \ wx	00	01	11	10
00		1	1	X
01	X	X		X
11	X			1
10			1	1

yz \ wx	00	01	11	10
00		1	1	X
01	X	X		X
11	X			1
10			1	1

5. For the following four-variable function, *f*, find both minimum sum of products expressions and both minimum product of sums expressions.

cd \ ab	00	01	11	10
00			X	
01	X	1	X	1
11	1	1		X
10		X		

6. For the following function, *f*, find all four minimum sum of products expressions and all four minimum product of sums expressions.

yz \ wx	00	01	11	10
00	X		1	
01	X	1	1	
11	X		X	1
10	X		X	

7. For the following five-variable problem, find both minimum sum of products expressions.

A = 0

DE \ BC	00	01	11	10
00	1		1	
01	1	1		
11	1			
10	1			

A = 1

DE \ BC	00	01	11	10
00			1	
01		1	1	
11		1	1	1
10				1

8. For the following five-variable problem, find both minimum sum of products expressions. (5 terms, 15 literals)

A = 0

DE \ BC	00	01	11	10
00	1			1
01				1
11		1	1	1
10	1			1

A = 1

DE \ BC	00	01	11	10
00	1	1	1	
01	1			1
11	1	1	1	1
10	1	1	1	

9. a. For the following two functions, find the minimum sum of products expression for each (treating them as two separate problems).

$yz \backslash wx$	00	01	11	10
00		1	1	1
01				1
11				1
10			1	1

f

$yz \backslash wx$	00	01	11	10
00				
01	1	1		1
11	1	1		1
10		1		

g

b. For the same two functions, find a minimum sum of products solution (corresponding to minimum number of gates, and among those with the same number of gates, minimum number of gate inputs). (7 gates, 19 inputs)

10. Consider the three functions, the maps of which are shown below.

$yz \backslash wx$	00	01	11	10
00				1
01	1	1		
11	1	1	1	1
10	1	1		1

f

$yz \backslash wx$	00	01	11	10
00		1	1	1
01				
11			1	
10	1	1	1	

g

$yz \backslash wx$	00	01	11	10
00		1		1
01	1	1		
11	1	1	1	
10		1		

h

a. Find the minimum sum of products expression (individually) for each of the three functions. Indicate which, if any, prime implicants can be shared.

b. Find a minimum two-level NAND gate solution. Full credit for a solution using 10 gates and 32 inputs. All variables are available both uncomplemented and complemented. Show the equations *and* a block diagram.