chapter

5

Factoring Polynomials

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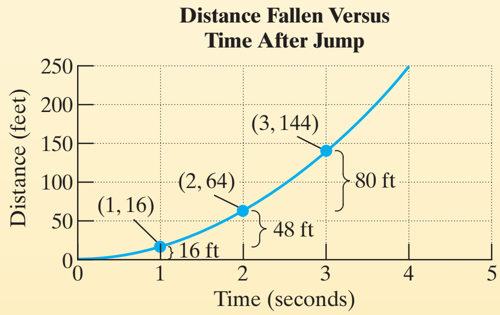
Ignoring air resistance, the distance, d (in feet), that a skydiver falls in t seconds is approximated by the formula:

$$d = 16t^2$$

After 1 second, the skydiver will have fallen 16 ft. After 2 seconds the skydiver will have fallen 64 ft, and after 3 seconds, the skydiver will have fallen 144 ft. Notice that each consecutive 1-second interval results in a larger increase in the distance fallen as shown in the graph.



To graph this and other quadratic relationships, see the Technology Connections in MathZone at

www.mhhe.com/miller_oneill

Concepts

1. Introduction to Factoring
2. Greatest Common Factor of Two Integers
3. GCF of Two or More Monomials
4. Factoring out the Greatest Common Factor
5. Factoring out a Negative Factor
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section **5.1**

Greatest Common Factor and Factoring by Grouping

1. Introduction to Factoring

Chapter 5 is devoted to a mathematical operation called **factoring**. To factor an integer means to write the integer as a product of two or more integers. To factor a polynomial means to express the polynomial as a product of two or more polynomials.

In the product $2 \cdot 5 = 10$, for example, 2 and 5 are factors of 10.

In the product $(3x + 4)(2x - 1) = 6x^2 + 5x - 4$, the quantities $(3x + 4)$ and $(2x - 1)$ are factors of $6x^2 + 5x - 4$.

2. Greatest Common Factor of Two Integers

We begin our study of factoring by factoring integers. The number 20 for example can be factored as $1 \cdot 20$, $2 \cdot 10$, $4 \cdot 5$, or $2 \cdot 2 \cdot 5$. The product $2 \cdot 2 \cdot 5$ (or equivalently $2^2 \cdot 5$) consists only of prime numbers and is called the **prime factorization**.

The **greatest common factor** (denoted **GCF**) of two or more integers is the greatest factor common to each integer. To find the greatest common factor of two integers, it is often helpful to express the numbers as a product of prime factors as shown in Example 1.

example 1 Identifying the GCF of Two Integers

Find the greatest common factor of each pair of integers.

a. 24 and 36 b. 105 and 40

Solution:

First find the prime factorization of each number. Then find the product of common factors.

a.

$$\begin{array}{r|l} 2 & 24 \\ 2 & 12 \\ 2 & 6 \\ & 3 \end{array} \qquad \begin{array}{r|l} 2 & 36 \\ 2 & 18 \\ 3 & 9 \\ & 3 \end{array}$$

Factors of 24 $= 2 \cdot 2 \cdot 2 \cdot 3$

Factors of 36 $= 2 \cdot 2 \cdot 3 \cdot 3$

The numbers 24 and 36 share two factors of 2 and one factor of 3. Therefore, the greatest common factor is $2 \cdot 2 \cdot 3 = 12$.

b.

$$\begin{array}{r|l} 5 & 105 \\ 3 & 21 \\ & 7 \end{array} \qquad \begin{array}{r|l} 5 & 40 \\ 2 & 8 \\ 2 & 4 \\ & 2 \end{array}$$

Factors of 105 $= 3 \cdot 7 \cdot 5$

Factors of 40 $= 2 \cdot 2 \cdot 2 \cdot 5$

The greatest common factor is 5.

3. GCF of Two or More Monomials

example 2

Identifying the Greatest Common Factor of Two or More Polynomials

Find the GCF among each group of terms.

a. $7x^3, 14x^2, 21x^4$ b. $15a^4b, 25a^3b^2$ c. $8c^2d^7e, 6c^3d^4$

Solution:

List the factors of each term.

a.
$$\begin{aligned} 7x^3 &= 7 \cdot x \cdot x \cdot x \\ 14x^2 &= 2 \cdot 7 \cdot x \cdot x \\ 21x^4 &= 3 \cdot 7 \cdot x \cdot x \cdot x \cdot x \end{aligned}$$
The GCF is $7x^2$.

b.
$$\begin{aligned} 15a^4b &= 3 \cdot 5 \cdot a \cdot a \cdot a \cdot a \cdot b \\ 25a^3b^2 &= 5 \cdot 5 \cdot a \cdot a \cdot a \cdot b \cdot b \end{aligned}$$
The GCF is $5a^3b$.

Tip: Notice that the expressions $15a^4b$ and $25a^3b^2$ share factors of 5, a, and b. The GCF is the product of the common factors, where each factor is raised to the lowest power to which it occurs in the original expressions.

$$\left.\begin{aligned} 15a^4b &= 3 \cdot 5a^4b \\ 25a^3b^2 &= 5^2a^3b^2 \end{aligned}\right\} \quad \left.\begin{array}{ll} \text{Lowest power of 5 is 1:} & 5^1 \\ \text{Lowest power of } a \text{ is 3:} & a^3 \\ \text{Lowest power of } b \text{ is 1:} & b^1 \end{array}\right\} \quad \text{The GCF is } 5a^3b.$$

c.
$$\left.\begin{aligned} 8c^2d^7e &= 2^3c^2d^7e \\ 6c^3d^4 &= 2 \cdot 3c^3d^4 \end{aligned}\right\} \quad \text{The common factors are 2, } c\text{, and } d.$$

$$\left.\begin{array}{ll} \text{The lowest power of 2 is 1:} & 2^1 \\ \text{The lowest power of } c \text{ is 2:} & c^2 \\ \text{The lowest power of } d \text{ is 4:} & d^4 \end{array}\right\} \quad \text{The GCF is } 2c^2d^4.$$

Sometimes two polynomials share a common binomial factor as shown in Example 3.

example 3

Finding the Greatest Common Binomial Factor

Find the greatest common factor between the terms: $3x(a + b)$ and $2y(a + b)$

Solution:

$$\left.\begin{matrix} 3x(a+b) \\ 2y(a+b) \end{matrix}\right\}$$

The only common factor is the binomial $(a + b)$. The GCF is $(a + b)$.

4. Factoring out the Greatest Common Factor

The process of factoring a polynomial is the reverse process of multiplying polynomials. Both operations use the distributive property: $ab + ac = a(b + c)$.

Multiply

$$\begin{aligned} 5y(y^2 + 3y + 1) &= 5y(y^2) + 5y(3y) + 5y(1) \\ &= 5y^3 + 15y^2 + 5y \end{aligned}$$

Factor

$$\begin{aligned} 5y^3 + 15y^2 + 5y &= 5y(y^2) + 5y(3y) + 5y(1) \\ &= 5y(y^2 + 3y + 1) \end{aligned}$$

Steps to Removing the Greatest Common Factor

1. Identify the GCF of all terms of the polynomial.
2. Write each term as the product of the GCF and another factor.
3. Use the distributive property to remove the GCF.

Note: To check the factorization, multiply the polynomials to remove parentheses.

example 4

Factoring out the Greatest Common Factor

Factor out the GCF.

a. $4x - 20$ b. $6w^2 + 3w$ c. $15y^3 + 12y^4$ d. $9a^4b - 18a^5b + 27a^6b$

Solution:

a. $4x - 20$ The GCF is 4.

$= 4(x) - 4(5)$ Write each term as the product of the GCF and another factor.

$= 4(x - 5)$ Use the distributive property to factor out the GCF.

Tip: Any factoring problem can be checked by multiplying the factors:

Check: $4(x - 5) = 4x - 20$ ✔

Avoiding Mistakes

In Example 4(b) the GCF, $3w$, is equal to one of the terms of the polynomial. In such a case, you must leave a 1 in place of that term after the GCF is factored out.

b. $6w^2 + 3w$ — The GCF is $3w$.

$= 3w(2w) + 3w(1)$ — Write each term as the product of $3w$ and another factor.

$= 3w(2w + 1)$ — Use the distributive property to factor out the GCF.

c. $15y^3 + 12y^4$ — The GCF is $3y^3$.

$= 3y^3(5) + 3y^3(4y)$ — Write each term as the product of $3y^3$ and another factor.

$= 3y^3(5 + 4y)$ — Use the distributive property to factor out the GCF.

d. $9a^4b - 18a^5b + 27a^6b$ — The GCF is $9a^4b$.

$= 9a^4b(1) - 9a^4b(2a) + 9a^4b(3a^2)$ — Write each term as the product of $9a^4b$ and another factor.

$= 9a^4b(1 - 2a + 3a^2)$ — Use the distributive property to factor out the GCF.

5. Factoring out a Negative Factor

Sometimes it is advantageous to factor out the *opposite* of the GCF when the leading coefficient of the polynomial is negative. This is demonstrated in Example 5. Notice that this *changes the signs* of the remaining terms inside the parentheses.

example 5 Factoring out a Negative Factor

Factor out the quantity $-4pq$ from the polynomial $-12p^3q - 8p^2q^2 + 4pq^3$

Solution:

$-12p^3q - 8p^2q^2 + 4pq^3$ — The GCF is $4pq$. However, in this case, we will factor out the *opposite* of the GCF, $-4pq$.

$= -4pq(3p^2) + (-4pq)(2pq) + (-4pq)(-q^2)$ — Write each term as the product of $-4pq$ and another factor.

$= -4pq[3p^2 + 2pq + (-q^2)]$ — Factor out $-4pq$.

$= -4pq(3p^2 + 2pq - q^2)$ — Notice that each sign within the trinomial has changed.

Tip: To verify that this is the correct factorization and that the signs are correct, multiply the factors.

6. Factoring out a Binomial Factor

The distributive property may also be used to factor out a common factor that consists of more than one term as shown in Example 6.

example 6

Factoring out a Binomial Factor

Factor out the greatest common factor: $2w(x + 3) - 5(x + 3)$

Solution:

$2w(x + 3) - 5(x + 3)$	The greatest common factor is the quantity $(x + 3)$.
$= (x + 3)(2w) - (x + 3)(5)$	Write each term as the product of $(x + 3)$ and another factor.
$= (x + 3)(2w - 5)$	Use the distributive property to factor out the GCF.

7. Factoring by Grouping

When two binomials are multiplied, the product before simplifying contains four terms. For example:

$$\begin{aligned}(x + 4)(3a + 2b) &= (x + 4)(3a) + (x + 4)(2b)\\ &= (x + 4)(3a) + (x + 4)(2b)\\ &= 3ax + 12a + 2bx + 8b\end{aligned}$$

In Example 7, we learn how to reverse this process. That is, given a four-term polynomial, we will factor it as a product of two binomials. The process is called **factoring by grouping**.

example 7

Factoring by Grouping

Factor by grouping: $3ax + 12a + 2bx + 8b$

Solution:

$3ax + 12a + 2bx + 8b$	**Step 1:** Identify and factor out the GCF from all four terms. In this case, the GCF is 1.
$= 3ax + 12a \mid + 2bx + 8b$	Group the first pair of terms and the second pair of terms.

$= 3a(x + 4) + 2b(x + 4)$ **Step 2:** Factor out the GCF from each pair of terms. *Note*: The two terms now share a common binomial factor of $(x + 4)$.

$= (x + 4)(3a + 2b)$ **Step 3:** Factor out the common binomial factor.

Check: $(x + 4)(3a + 2b) = 3ax + 2bx + 12a + 8b$ ✔

Note: Step 2 results in two terms with a common binomial factor. If the two binomials are different, step 3 cannot be performed. In such a case, the original polynomial may not be factorable by grouping.

Tip: One frequently asked question when factoring is whether the order can be switched between the factors. The answer is yes. Because multiplication is commutative, the order in which the factors are written does not matter.

$$(x + 4)(3a + 2b) = (3a + 2b)(x + 4)$$

Steps to Factoring by Grouping

To factor a four-term polynomial by grouping:

1. Identify and factor out the GCF from all four terms.
2. Factor out the GCF from the first pair of terms. Factor out the GCF from the second pair of terms. (Sometimes it is necessary to factor out the opposite of the GCF.)
3. If the two terms share a common binomial factor, factor out the binomial factor.

example 8 Factoring by Grouping

Factor the polynomials by grouping.

a. $ax + ay - bx - by$ b. $16w^4 - 40w^3 - 12w^2 + 30w$

Solution:

a. $ax + ay - bx - by$ **Step 1:** Identify and factor out the GCF from all four terms. In this case, the GCF is 1.

$= ax + ay \mid - bx - by$ Group the first pair of terms and the second pair of terms.

Avoiding Mistakes

In step 2, the expression $a(x + y) - b(x + y)$ is not yet factored because it is a *difference*, not a product. To factor the expression, you must carry it one step further.

$$a(x + y) - b(x + y)$$
$$= (x + y)(a - b)$$

The factored form must be represented as a product.

$= a(x + y) - b(x + y)$ **Step 2:** Factor out a from the first pair of terms.

Factor out $-b$ from the second pair of terms. (This causes sign changes within the second parentheses.)

$= (x + y)(a - b)$ **Step 3:** Factor out the common binomial factor.

b. $16w^4 - 40w^3 - 12w^2 + 30w$ **Step 1:** Identify and factor out the GCF from all four terms. In this case, the GCF is $2w$.

$= 2w(8w^3 - 20w^2 - 6w + 15)$

$= 2w[8w^3 - 20w^2 \mid - 6w + 15]$ Group the first pair of terms and the second pair of terms.

$= 2w[4w^2(2w - 5) - 3(2w - 5)]$ **Step 2:** Factor out $4w^2$ from the first pair of terms.

Factor out -3 from the second pair of terms. (This causes sign changes within the second parentheses.)

$= 2w[(2w - 5)(4w^2 - 3)]$ **Step 3:** Factor out the common binomial factor.

section 5.1 Practice Exercises

For Exercises 1–12, identify the greatest common factor between each pair of terms.

1. 28, 63
2. 24, 40
3. 42, 30
4. 18, 52
5. $2a^2b, 3ab^2$
6. $3x^3y^2, 5xy^4$
7. $12w^3z, 16w^2z$
8. $20cd, 15c^3d$
9. $7(x - y), 9(x - y)$
10. $(2a - b), 3(2a - b)$
11. $14(3x + 1)^2, 7(3x + 1)$
12. $a^2(w + z), a^3(w + z)^2$
13. a. Use the distributive property to multiply $3(x - 2y)$.
    b. Use the distributive property to factor $3x - 6y$.
14. a. Use the distributive property to multiply $a^2(5a + b)$.
    b. Use the distributive property to factor $5a^3 + a^2b$.

For Exercises 15–38, factor out the GCF.

15. $4p + 12$
16. $3q - 15$
17. $5c^2 - 10c$
18. $24d + 16d^3$
19. $x^5 + x^3$
20. $y^2 - y^4$
21. $t^4 - 4t$
22. $7r^3 - r^5$
23. $2ab + 4a^3b$
24. $5u^3v^2 - 5uv$
25. $38x^2y - 19x^2y^4$
26. $100a^5b^3 + 16a^2b$
27. $42p^3q^2 + 14pq^2 - 7p^4q^4$

28. $8m^2n^3 - 24m^2n^2 + 4m^3n$
29. $t^5 + 2rt^3 - 3t^4 + 4r^2t^2$
30. $u^2v + 5u^3v^2 - 2u^2 + 8uv$
31. $13(a + 6) - 4b(a + 6)$
32. $7(x^2 + 1) - y(x^2 + 1)$
33. $8v(w^2 - 2) + (w^2 - 2)$
34. $t(r + 2) + (r + 2)$
35. $21x(x + 3) + 7x^2(x + 3)$
36. $5y^3(y - 2) - 20y(y - 2)$
37. $6(z - 1)^3 + 7z(z - 1)^2 - (z - 1)$
38. $4(q + 5)^2 + 5q(q + 5) - (q + 5)$
39. For the polynomial $-2x^3 - 4x^2 + 8x$
    a. Factor out $-2x$.
    b. Factor out $2x$.
40. For the polynomial $-9y^5 + 3y^3 - 12y$
    a. Factor out $-3y$.
    b. Factor out $3y$.
41. Factor out -1 from the polynomial $-8t^2 - 9t - 2$.
42. Factor out -1 from the polynomial $-6x^3 - 2x - 5$.
43. Factor out -1 from the polynomial $-4y^3 + 5y - 7$.
44. Factor out -1 from the polynomial $-w^2 + w - 5$.

For Exercises 45–52, factor out the opposite of the greatest common factor.

45. $-15p^3 - 30p^2$
46. $-24m^3 - 12m^4$
47. $-q^4 + 2q^2 - 9q$
48. $-r^3 + 9r^2 - 5r$
49. $-7x - 6y - 2z$
50. $-4a + 5b - c$
51. $-3(2c + 5) - 4c(2c + 5)$
52. $-6n(4n - 1) - 7(4n - 1)$

For Exercises 53–68, factor by grouping.

53. $8a^2 - 4ab + 6ac - 3bc$
54. $4x^3 + 3x^2y + 4xy^2 + 3y^3$
55. $3q + 3p + qr + pr$
56. $xy - xz + 7y - 7z$
57. $6x^2 + 3x + 4x + 2$
58. $4y^2 + 8y + 7y + 14$
59. $2t^2 + 6t - 5t - 15$
60. $2p^2 - p - 6p + 3$
61. $6y^2 - 2y - 9y + 3$
62. $5a^2 + 30a - 2a - 12$
63. $b^4 + b^3 - 4b - 4$
64. $8w^5 + 12w^2 - 10w^3 - 15$
65. $3j^2k + 15k + j^2 + 5$
66. $2ab^2 - 6ac + b^2 - 3c$
67. $14w^6x^6 + 7w^6 - 2x^6 - 1$
68. $18p^4q - 9p^5 - 2q + p$

For Exercises 69–74, factor out the GCF first. Then factor by grouping.

69. $15x^4 + 15x^2y^2 + 10x^3y + 10xy^3$
70. $2a^3b - 4a^2b + 32ab - 64b$
71. $4abx - 4b^2x - 4ab + 4b^2$
72. $p^2q - pq^2 - rp^2q + rpq^2$
73. $6st^2 - 18st - 6t^4 + 18t^3$
74. $15j^3 - 10j^2k - 15j^2k^2 + 10jk^3$
75. The formula $P = 2l + 2w$ represents the perimeter, P, of a rectangle given the length, l, and the width, w. Factor out the GCF and write an equivalent formula in factored form.
76. The formula $P = 2a + 2b$ represents the perimeter, P, of a parallelogram given the base, b, and an adjacent side, a. Factor out the GCF and write an equivalent formula in factored form.

77. The formula $S = 2\pi r^2 + 2\pi rh$ represents the surface area, S, of a cylinder with radius, r, and height, h. Factor out the GCF and write an equivalent formula in factored form.

78. The formula $A = P + Prt$ represents the total amount of money, A, in an account that earns simple interest at a rate, r, for t years. Factor out the GCF and write an equivalent formula in factored form.

Expanding Your Skills

79. Factor out $\frac{1}{7}$ from $\frac{1}{7}x^2 + \frac{3}{7}x - \frac{5}{7}$.

80. Factor out $\frac{1}{5}$ from $\frac{6}{5}y^2 - \frac{4}{5}y + \frac{1}{5}$.

81. Factor out $\frac{1}{4}$ from $\frac{5}{4}w^2 + \frac{3}{4}w + \frac{9}{4}$.

82. Factor out $\frac{1}{6}$ from $\frac{1}{6}p^2 - \frac{3}{6}p + \frac{5}{6}$.

83. Factor out $\frac{1}{12}$ from $\frac{1}{12}z^2 + \frac{1}{3}z + \frac{1}{2}$.
(*Hint:* Write each coefficient as an equivalent fraction with a common denominator of 12.)

84. Factor out $\frac{1}{15}$ from $\frac{1}{5}t^2 - \frac{2}{3}t - \frac{4}{15}$.

85. Factor out $\frac{1}{6}$ from $\frac{5}{6}q^2 + \frac{1}{3}q - 2$.

86. Factor out $\frac{1}{8}$ from $\frac{3}{8}x^2 - \frac{1}{2}x - 1$

87. Write a polynomial that has a GCF of $3x$. (Answers may vary.)

88. Write a polynomial that has a GCF of $7y$. (Answers may vary.)

89. Write a polynomial that has a GCF of $4p^2q$. (Answers may vary.)

90. Write a polynomial that has a GCF of $2ab^2$. (Answers may vary.)

Concepts

1. **Grouping Method to Factor Trinomials**
2. **Factoring Trinomials with a Leading Coefficient of 1**
3. **Prime Polynomials**

section

5.2 Factoring Trinomials: Grouping Method

We have already learned how to factor out the GCF from a polynomial and how to factor a four-term polynomial by grouping. As we work through this chapter, we will expand our knowledge of factoring by learning how to factor trinomials and binomials.

There are two commonly used methods for factoring trinomials. The grouping method (or "ac" method) is presented here and the trial-and-error method is presented in the next section.

1. Grouping Method to Factor Trinomials

The product of two binomials results in a four-term expression that can sometimes be simplified to a trinomial. To factor the trinomial, we want to reverse the process.

Multiply

$$(2x+3)(x+2) = \xrightarrow{\text{Multiply the binomials.}} 2x^2 + 4x + 3x + 6 = \xrightarrow{\text{Add the middle terms.}} 2x^2 + 7x + 6$$

Factor

$$2x^2 + 7x + 6 = \xrightarrow[\text{a sum or difference of terms.}]{\text{Rewrite the middle term as}} 2x^2 + 4x + 3x + 6 = \xrightarrow[\text{Factor by grouping.}]{} (2x+3)(x+2)$$

To factor a trinomial, $ax^2 + bx + c$, by the grouping method, we rewrite the middle term, bx, as a sum or difference of terms. The goal is to produce a four-term polynomial that can be factored by grouping. The process is outlined in the following box.

Grouping Method Factor $ax^2 + bx + c$ $(a \neq 0)$

1. Multiply the coefficients of the first and last terms (ac).
2. Find two integers whose product is ac and whose sum is b. (If no pair of integers can be found, then the trinomial cannot be factored further and is called a **prime polynomial**.)
3. Rewrite the middle term bx as the sum of two terms whose coefficients are the integers found in step 2.
4. Factor by grouping.

The grouping method for factoring trinomials is illustrated in Example 1. However, before we begin, keep these two important guidelines in mind:

- For any factoring problem you encounter, always factor out the GCF from all terms first.
- To factor a trinomial, write the trinomial in descending order, $ax^2 + bx + c$.

example 1 Factoring a Trinomial by the Grouping Method

Factor the trinomial by the grouping method: $2x^2 + 7x + 6$

Solution:

$2x^2 + 7x + 6$		Factor out the GCF from all terms. In this case, the GCF is 1.
$2x^2 + 7x + 6$	**Step 1:**	The trinomial is written in the form $ax^2 + bx + c$.
$a = 2, b = 7, c = 6$		Find the product $ac = (2)(6) = 12$.

Tip: Note that the negative factors of 12 do not need to be considered in this case because we are trying to form a sum of positive 7.

12	12
$1 \cdot 12$	$(-1)(-12)$
$2 \cdot 6$	$(-2)(-6)$
$3 \cdot 4$	$(-3)(-4)$

Step 2: List all the factors of ac and search for the pair whose sum equals the value of b.

That is, list the factors of 12 and find the pair whose sum equals 7.

The numbers 3 and 4 satisfy both conditions: $3 \cdot 4 = 12$ and $3 + 4 = 7$.

$2x^2 + 7x + 6$

$= 2x^2 + 3x + 4x + 6$

Step 3: Write the middle term of the trinomial as the sum of two terms whose coefficients are the selected pair of numbers: 3 and 4.

$= 2x^2 + 3x + 4x + 6$

$= x(2x + 3) + 2(2x + 3)$

$= (2x + 3)(x + 2)$

Step 4: Factor by grouping.

<u>Check</u>: $(2x + 3)(x + 2) = 2x^2 + 4x + 3x + 6$
$= 2x^2 + 7x + 6$ ✔

Tip: One frequently asked question is whether the order matters when we rewrite the middle term of the trinomial as two terms (step 3). The answer is no. From the previous example, the two middle terms in step 3 could have been reversed to obtain the same result:

$$\begin{aligned} &2x^2 + 7x + 6 \\ &\quad = 2x^2 + 4x + 3x + 6 \\ &\quad = 2x(x + 2) + 3(x + 2) \\ &\quad = (x + 2)(2x + 3) \end{aligned}$$

This example also points out that the order in which two factors are written does not matter. The expression $(x + 2)(2x + 3)$ is equivalent to $(2x + 3)(x + 2)$ because multiplication is a commutative operation.

example 2 Factoring Trinomials by the Grouping Method

Factor the trinomial by the grouping method: $-2x + 8x^2 - 3$

Solution:

$-2x + 8x^2 - 3$

First rewrite the polynomial in the form $ax^2 + bx + c$.

$= 8x^2 - 2x - 3$

The GCF is 1.

$a = 8, b = -2, c = -3$	**Step 1:** Find the product $ac = (8)(-3) = -24$.
$\underline{-24}$ $\quad$ $\underline{-24}$ $-1 \cdot 24$ $\quad$ $-6 \cdot 4$ $-2 \cdot 12$ $\quad$ $-8 \cdot 3$ $-3 \cdot 8$ $\quad$ $-12 \cdot 2$ $-4 \cdot 6$ $\quad$ $-24 \cdot 1$	**Step 2:** List all the factors of -24 and find the pair of factors whose sum equals -2. The numbers -6 and 4 satisfy both conditions: $(-6)(4) = -24$ and $-6 + 4 = -2$.
$= 8x^2 - 2x - 3$ $= 8x^2 - 6x + 4x - 3$	**Step 3:** Write the middle term of the trinomial as two terms whose coefficients are the selected pair of numbers, -6 and 4.
$= 8x^2 - 6x \mid + 4x - 3$	**Step 4:** Factor by grouping.
$= 2x(4x - 3) + 1(4x - 3)$	
$= (4x - 3)(2x + 1)$	

<u>Check</u>: $(4x - 3)(2x + 1) = 8x^2 + 4x - 6x - 3$
$= 8x^2 - 2x - 3$ ✔

example 3 Factoring a Trinomial by the Grouping Method

Factor the trinomial by the grouping method: $10x^3 - 85x^2 + 105x$

Solution:

$10x^3 - 85x^2 + 105x$	The GCF is $5x$.
$= 5x(2x^2 - 17x + 21)$	The trinomial is in the form $ax^2 + bx + c$.
$a = 2, b = -17, c = 21$	**Step 1:** Find the product $ac = (2)(21) = 42$.
$\underline{42}$ $\quad$ $\underline{42}$ $1 \cdot 42$ $\quad$ $(-1)(-42)$ $2 \cdot 21$ $\quad$ $(-2)(-21)$ $3 \cdot 14$ $\quad$ $(-3)(-14)$ $6 \cdot 7$ $\quad$ $(-6)(-7)$	**Step 2:** List all the factors of 42 and find the pair whose sum equals -17. The numbers -3 and -14 satisfy both conditions: $(-3)(-14) = 42$ and $-3 + (-14) = -17$.
$= 5x(2x^2 - 17x + 21)$ $= 5x(2x^2 - 3x - 14x + 21)$	**Step 3:** Write the middle term of the trinomial as two terms whose coefficients are the selected pair of numbers, -3 and -14.

Tip: Note that the positive factors of 42 do not need to be considered in this case because we are trying to form a sum of -17.

$= 5x(2x^2 - 3x \mid - 14x + 21)$ **Step 4:** Factor by grouping.

$= 5x[x(2x - 3) - 7(2x - 3)]$

$= 5x[(2x - 3)(x - 7)]$

$= 5x(2x - 3)(x - 7)$

Tip: Notice when the GCF is removed from the original trinomial, the new trinomial has smaller coefficients. This makes the factoring process simpler because the product ac is smaller. It is much easier to list the factors of 42 than the factors of 1050.

Original trinomial	**With the GCF factored out**
$10x^3 - 85x^2 + 105x$	$5x(2x^2 - 17x + 21)$
$ac = (10)(105) = 1050$	$ac = (2)(21) = 42$

In most cases it is easier to factor a trinomial with a positive leading coefficient. If the leading coefficient is negative, a factor of -1 can be removed to change the sign of the leading coefficient as well as the coefficients of the remaining terms.

example 4 Factoring a Trinomial by the Grouping Method

Factor the trinomial by the grouping method: $-a^2 - 7ab + 18b^2$

Solution:

$-a^2 - 7ab + 18b^2$

$= -1(a^2 + 7ab - 18b^2)$ Factor out -1.

Step 1: Find the product $ac = -18$.

Step 2: The numbers 9 and -2 have a product of -18 and a sum of 7.

$= -1(a^2 + 9ab - 2ab - 18b^2)$ **Step 3:** Rewrite the middle term $7ab$ as $9ab - 2ab$.

$= -1(a^2 + 9ab \mid - 2ab - 18b^2)$ **Step 4:** Factor by grouping.

$= -1[a(a + 9b) - 2b(a + 9b)]$

$= -1[(a + 9b)(a - 2b)]$

<u>Check</u>: $-1[(a + 9b)(a - 2b)] = -1[a^2 - 2ab + 9ab - 18b^2]$

$= -1[a^2 + 7ab - 18b^2]$

$= -a^2 - 7ab + 18b^2$ ✔

2. Factoring Trinomials with a Leading Coefficient of 1

The grouping method is a general method to factor trinomials of the form $ax^2 + bx + c$. If the leading coefficient, a, is equal to 1, then the process can be simplified. First note that if $a = 1$, then the product $ac = 1c = c$. Therefore, in step 2, we find two integers whose product is c and whose sum is b. Then, after rewriting the middle term and factoring by grouping, we have two binomial factors whose leading terms are x and whose constant terms are the integers found in step 2.

In the next example, we will factor trinomials of the form $x^2 + bx + c$ (leading coefficient of 1) to illustrate this process.

example 5 Factoring Trinomials with a Leading Coefficient of 1

Factor the trinomials.

a. $w^2 + 10w + 9$ b. $x^2 - 2x - 15$

Solution:

a. $1w^2 + 10w + 9$ — The trinomial is in the form $ax^2 + bx + c$, where $a = 1$.

Step 1: Because $a = 1$, the product $ac = c = 9$.

Step 2: Find two numbers whose product is 9 and whose sum is 10. The numbers are 9 and 1.

$= w^2 + 9w + 1w + 9$ — **Step 3:** Rewrite the middle term as a sum of terms.

$= w(w + 9) + 1(w + 9)$ — **Step 4:** Factor by grouping.

$= (w + 9)(w + 1)$ — The check is left to the reader.

Tip: The constants in the binomial factors are the two integers found in step 2.

b. $1x^2 - 2x - 15$ — The trinomial is in the form $ax^2 + bx + c$, where $a = 1$.

Step 1: Because $a = 1$, the product $ac = c = -15$.

$= (x - 5)(x + 3)$ — **Step 2:** Find two numbers whose product is -15 and whose sum is -2. The numbers are -5 and 3. These values are the constants in the binomial factors.

3. Prime Polynomials

Note that not every trinomial is factorable by the methods presented in this text.

example 6 Factoring a Trinomial by the Grouping Method

Factor the trinomial by the grouping method: $2p^2 - 8p + 3$

Solution:

$2p^2 - 8p + 3$ **Step 1:** The GCF is 1.

Step 2: The product $ac = 6$.

$\underline{6}$	$\underline{6}$
$1 \cdot 6$	$(-1)(-6)$
$2 \cdot 3$	$(-2)(-3)$

Step 3: List the factors of 6. Notice that no pair of factors has a sum of -8. Therefore, the trinomial cannot be factored.

The trinomial $2p^2 - 8p + 3$ is prime.

section 5.2 Practice Exercises

For Exercises 1–6, factor out the GCF.

1. $8p^9 + 24p^3$
2. $5q^4 - 10q^5$
3. $9x^2y + 12xy^2 - 15x^2y^2$
4. $15ab^3 - 10a^2bc + 25b^2c^3$
5. $5x(x - 2) - 2(x - 2)$
6. $8(y + 5) + 9y(y + 5)$

For Exercises 7–10, factor by grouping.

7. $p^2 - 2pq - pq + 2q^2$
8. $2u - 10 - uv + 5v$
9. $6a^2 + 24a - 12a - 48$
10. $5b^2 + 30b - 10b - 60$
11. What is a prime polynomial?
12. How do you determine if a trinomial, $ax^2 + bx + c$, is prime?

For Exercises 13 – 20, find the pair of integers whose product and sum are given.

13. Product: 12 Sum: 13
14. Product: 12 Sum: 7
15. Product: 8 Sum: -9
16. Product: -4 Sum: -3
17. Product: -6 Sum: 5
18. Product: -18 Sum: 7
19. Product: -72 Sum: -6
20. Product: 36 Sum: -12

For Exercises 21–62, factor the trinomials using the grouping method.

21. $3x^2 + 13x + 4$
22. $2y^2 + 7y + 6$
23. $4w^2 - 9w + 2$
24. $2p^2 - 3p - 2$
25. $2m^2 + 5m - 3$
26. $6n^2 + 7n - 3$
27. $8k^2 - 6k - 9$
28. $9h^2 - 12h + 4$
29. $4k^2 - 20k + 25$
30. $16h^2 + 24h + 9$
31. $5x^2 + x + 7$
32. $4y^2 - y + 2$
33. $4p^2 + 5pq - 6q^2$
34. $6u^2 - 19uv + 10v^2$
35. $15m^2 + mn - 2n^2$
36. $12a^2 + 11ab - 5b^2$
37. $3r^2 - rs - 14s^2$
38. $3h^2 + 19hk - 14k^2$
39. $2x^2 - 13xy + y^2$
40. $3p^2 + 20pq - q^2$

41. $q^2 - 11q + 10$
42. $a^2 + 7a - 18$
43. $r^2 - 6r - 40$
44. $s^2 - 10s - 24$
45. $x^2 + 6x - 7$
46. $y^2 + 5y - 24$
47. $m^2 - 13m + 42$
48. $n^2 - 9n + 20$
49. $a^2 + 9a + 20$
50. $b^2 + 13b + 42$
51. $t^2 + 5t + 5$
52. $s^2 - 6s + 3$
53. $p^2 + 20pq + 100q^2$
54. $c^2 - 14cd + 49d^2$
55. $x^2 - xy - 42y^2$
56. $a^2 - 13ab + 40b^2$
57. $r^2 + 8rs + 15s^2$
58. $u^2 + 2uv - 15v^2$
59. $9z^2 - 21z + 10$
60. $4x^2 + 13x - 12$
61. $7y^2 + 25y + 12$
62. $20p^2 - 19p + 3$
63. Is the expression $(2x + 4)(x - 7)$ factored completely? Explain why or why not.
64. Is the expression $(3x + 1)(5x - 10)$ factored completely? Explain why or why not.

For Exercises 65–78, first factor out a common factor. Then factor the trinomials by using the grouping method.

65. $72x^2 + 18x - 2$
66. $20y^2 - 78y - 8$
67. $p^3 - 6p^2 - 27p$
68. $w^5 - 11w^4 + 28w^3$
69. $6x^2 + 20x + 14$
70. $4r^3 + 3r^2 - 10r$
71. $2p^3 - 38p^2 + 120p$
72. $4q^3 - 4q^2 - 80q$
73. $x^2y^2 + 14x^2y + 33x^2$
74. $a^2b^2 + 13ab^2 + 30b^2$
75. $-k^2 - 7k - 10$
76. $-m^2 - 15m + 34$
77. $-3n^2 - 3n + 90$
78. $-2h^2 + 28h - 90$

For Exercises 79–86, write the polynomials in descending order. Then factor the polynomials.

79. $3 - 14z + 16z^2$
80. $10w + 1 + 16w^2$
81. $b^2 + 16 - 8b$
82. $1 + q^2 - 2q$
83. $25x - 5x^2 - 30$
84. $20a - 18 - 2a^2$
85. $-6 - t + t^2$
86. $-6 + m + m^2$

Concepts

1. **Trial-and-Error Method for Factoring Trinomials**
2. **Identifying the Signs When Using the Trial-and-Error Method**
3. **Factoring Trinomials with a Leading Coefficient of 1**
4. **Greatest Common Factor and Factoring Trinomials**

section

5.3 Factoring Trinomials: Trial-and-Error Method

In Section 5.2, the grouping method was presented for factoring trinomials. In this section, we offer another method to factor trinomials, called the trial-and-error method. You and your instructor must determine which method is best for you.

1. Trial-and-Error Method for Factoring Trinomials

To understand the basis of factoring trinomials of the form $ax^2 + bx + c$, first consider the multiplication of two binomials:

$$(2x + 3)(1x + 2) = \overset{\text{Product of } 2 \cdot 1}{2}x^2 + \underbrace{\mathbf{4x + 3x}}_{\substack{\text{Sum of products of inner}\\ \text{terms and outer terms}}} + \overset{\text{Product of } 3 \cdot 2}{6} = 2x^2 + 7x + 6$$

To factor the trinomial, $2x^2 + 7x + 6$ this operation is reversed. Hence:

Factors of 2

$$2x^2 + 7x + 6 = (\square x \quad \square)(\square x \quad \square)$$

Factors of 6

We need to fill in the blanks so that the product of the first terms in the binomials is $2x^2$ and the product of the last terms in the binomials is 6. Furthermore, the factors of $2x^2$ and 6 must be chosen so that the sum of the products of the inner terms and outer terms equals $7x$.

To produce the product $2x^2$ we might try the factors $2x$ and x within the binomials:

$$(2x \quad \square)(x \quad \square)$$

To produce a product of 6, the remaining terms in the binomials must either both be positive or both be negative. To produce a positive middle term, we will try positive factors of 6 in the remaining blanks until the correct product is found. The possibilities are $1 \cdot 6$, $2 \cdot 3$, $3 \cdot 2$, and $6 \cdot 1$.

$(2x + 1)(x + 6) = 2x^2 + 12x + 1x + 6 = 2x^2 + 13x + 6$	Wrong middle term
$(2x + 2)(x + 3) = 2x^2 + 6x + 2x + 6 = 2x^2 + 8x + 6$	Wrong middle term
$(2x + 3)(x + 2) = 2x^2 + 4x + 3x + 6 = 2x^2 + 7x + 6$	Correct!
$(2x + 6)(x + 1) = 2x^2 + 2x + 6x + 6 = 2x^2 + 8x + 6$	Wrong middle term

The correct factorization of $2x^2 + 7x + 6$ is $(2x + 3)(x + 2)$. ✔

As this example shows, we factor a trinomial of the form $ax^2 + bx + c$ by shuffling the factors of a and c within the binomials until the correct product is obtained. However, sometimes it is not necessary to test all the possible combinations of factors. In the previous example, the GCF of the original trinomial is 1. Therefore, any binomial factor that shares a common factor *greater than 1* does not need to be considered. In this case the possibilities $(2x + 2)(x + 3)$ and $(2x + 6)(x + 1)$ cannot work.

$$\underbrace{(2x + 2)}_{\substack{\text{Common}\\ \text{factor of 2}}}(x + 3) \qquad \underbrace{(2x + 6)}_{\substack{\text{Common}\\ \text{factor of 2}}}(x + 1).$$

The steps to factor a trinomial by the trial-and-error method are outlined in the following box.

Trial-and-Error Method to Factor $ax^2 + bx + c$

1. Factor out the GCF.
2. List all pairs of positive factors of a and pairs of positive factors of c. Consider the reverse order for either list of factors.
3. Construct two binomials of the form:

$$\overbrace{(\square x \quad \square)(\square x}^{\text{Factors of } a} \quad \square) \qquad (\square x \quad \underbrace{\square)(\square x \quad \square}_{\text{Factors of } c})$$

   Test each combination of factors and signs until the correct product is found. If no combination of factors produces the correct product, the trinomial cannot be factored further and is called a **prime polynomial**.

Before we begin our next example, keep these two important guidelines in mind.

- For any factoring problem you encounter, always factor out the GCF from all terms first.
- To factor a trinomial, write the trinomial in the form $ax^2 + bx + c$.

example 1

Factoring a Trinomial by the Trial-and-Error Method

Factor the trinomial by the trial-and-error method: $10x^2 - 9x - 1$

Solution:

$10x^2 - 9x - 1$ **Step 1:** Factor out the GCF from all terms. In this case, the GCF is 1.

The trinomial is written in the form $ax^2 + bx + c$.

To factor $10x^2 - 9x - 1$, two binomials must be constructed in the form:

$$(\overbrace{\square x \quad \square)(\square}^{\text{Factors of 10}} x \quad \square)$$

Factors of 10: first terms; Factors of -1: last terms

$(\square x \quad \square)(\square x \quad \square)$ **Step 2:** To produce the product $10x^2$, we might try $5x$ and $2x$ or $10x$ and $1x$. To produce a product of -1, we will try the factors $1(-1)$ and $-1(1)$.

Step 3: Construct all possible binomial factors using different combinations of the factors of $10x^2$ and -1.

$(5x + 1)(2x - 1) = 10x^2 - 5x + 2x - 1 = 10x^2 - 3x - 1$ Wrong middle term

$(5x - 1)(2x + 1) = 10x^2 + 5x - 2x - 1 = 10x^2 + 3x - 1$ Wrong middle term

Because the numbers 1 and -1 did not produce the correct trinomial when coupled with $5x$ and $2x$, try using $10x$ and $1x$.

$(10x - 1)(1x + 1) = 10x^2 + 10x - 1x - 1 = 10x^2 + 9x - 1$ Wrong middle term

$(10x + 1)(1x - 1) = 10x^2 - 10x + 1x - 1 = 10x^2 - 9x - 1$ Correct!

Hence $10x^2 - 9x - 1 = (10x + 1)(x - 1)$.

2. Identifying the Signs When Using the Trial-and-Error Method

In Example 1, the factors of -1 must have opposite signs to produce a negative product. Therefore, one binomial factor is a sum and one is a difference. Determining the correct signs is an important aspect of factoring trinomials. We suggest the following guidelines:

Sign Rules for the Trial-and-Error Method

Given the trinomial $ax^2 + bx + c$, $(a > 0)$ the signs can be determined as follows:

1. If *c is positive*, then the signs in the binomials must be the same (either both positive or both negative). The correct choice is determined by the middle term. If the middle term is positive, then both signs must be positive. If the middle term is negative, then both signs must be negative.

$$20x^2 + 43x + \overset{c \text{ is positive}}{\overset{\downarrow}{21}} \qquad 20x^2 - 43x + \overset{c \text{ is positive}}{\overset{\downarrow}{21}}$$

$$(4x \underbrace{+ 3)(5x +}_{\text{Same signs}} 7) \qquad (4x \underbrace{- 3)(5x -}_{\text{Same signs}} 7)$$

2. If *c is negative*, then the signs in the binomial must be different. The middle term in the trinomial determines which factor gets the positive sign and which gets the negative sign.

$$x^2 + 3x - \overset{c \text{ is negative}}{\overset{\downarrow}{28}} \qquad x^2 - 3x - \overset{c \text{ is negative}}{\overset{\downarrow}{28}}$$

$$(x \underbrace{+ 7)(x -}_{\text{Different signs}} 4) \qquad (x \underbrace{- 7)(x +}_{\text{Different signs}} 4)$$

example 2 Factoring a Trinomial

Factor the trinomial: $8y^2 + 13y - 6$

Solution:

$8y^2 + 13y - 6$ **Step 1:** The GCF is 1.

$(\square y \quad \square)(\square y \quad \square)$

Factors of 8	Factors of 6	
$1 \cdot 8$	$1 \cdot 6$	**Step 2:** List the positive factors of 8 and positive factors of 6. Consider the reverse order in one list of factors.
$2 \cdot 4$	$2 \cdot 3$	
	$3 \cdot 2$	(reverse order)
	$6 \cdot 1$	(reverse order)

Step 3: Construct all possible binomial factors using different combinations of the factors of 8 and 6.

$$\left.\begin{array}{l}(2y \quad 1)(4y \quad 6)\\ (2y \quad 2)(4y \quad 3)\\ (2y \quad 3)(4y \quad 2)\\ (2y \quad 6)(4y \quad 1)\\ (1y \quad 1)(8y \quad 6)\\ (1y \quad 3)(8y \quad 2)\end{array}\right\}$$

Without regard to signs, these factorizations cannot work because the terms in a binomial share a common factor greater than 1.

Test the remaining factorizations. Keep in mind that to produce a product of -6, the signs within the parentheses must be opposite (one positive and one negative). Also, the sum of the products of the inner terms and outer terms must be combined to form $13y$.

$(1y \quad 6)(8y \quad 1)$ *Incorrect.* Wrong middle term.
Regardless of signs, the product of inner terms, $48y$, and the product of outer terms, $1y$, cannot be combined to form the middle term $13y$.

$(1y \quad 2)(8y \quad 3)$ *Correct.* The terms $16y$ and $3y$ can be combined to form the middle term $13y$, provided the signs are applied correctly. We require $+16y$ and $-3y$.

$(y + 2)(8y - 3)$

$= 8y^2 - 3y + 16y - 6$ Verify by multiplication.

$= 8y^2 + 13y - 6$

Hence, the correct factorization is $(y + 2)(8y - 3)$.

3. Factoring Trinomials with a Leading Coefficient of 1

If a trinomial has a leading coefficient of 1, the factoring process simplifies significantly. Consider the trinomial $x^2 + bx + c$. To produce a leading term of x^2, we can construct binomials of the form $(x + \quad)(x + \quad)$. The remaining terms may be satisfied by two numbers p and q whose product is c and whose sum is b:

Factors of c

$$(x + p)(x + q) = x^2 + px + qx + pq = x^2 + \underbrace{(p + q)}_{\text{Sum} = b}x + \underbrace{pq}_{\text{Product} = c}$$

This process is demonstrated in Example 3.

example 3 **Factoring a Trinomial with a Leading Coefficient of 1**

Factor the trinomial: $x^2 - 10x + 16$

Solution:

$x^2 - 10x + 16$	Factor out the GCF from all terms. In this case, the GCF is 1.
$= (x \quad)(x \quad)$	The trinomial is written in the form $x^2 + bx + c$. To form the product x^2, use the factors x and x.
	Next look for two numbers whose product is 16 and whose sum is -10. Because the middle term is negative, we will consider only the negative factors of 16.

Factors of 16	**Sum**
$-1(-16)$	$-1 + (-16) = -17$
$-2(-8)$	$-2 + (-8) = -10$
$-4(-4)$	$-4 + (-4) = -8$

The numbers are -2 and -8.

Hence $x^2 - 10x + 16 = (x - 2)(x - 8)$.

example 4 Factoring Trinomials with a Leading Coefficient of 1

Factor.

a. $t^2 + 34t + 33$ b. $c^2 - 7cd - 30d^2$

Solution:

a. $t^2 + 34t + 33$	Factor out the GCF from all terms. In this case, the GCF is 1.
$= (t \quad)(t \quad)$ $= (t + 1)(t + 33)$	To complete the factorization, we need two numbers whose product is 33 and whose sum is 34. The numbers are 1 and 33.
b. $c^2 - 7cd - 30d^2$	Factor out the GCF from all terms. In this case, the GCF is 1.
$= (c \quad d)(c \quad d)$ $= (c - 10d)(c + 3d)$	The presence of two variables c and d, does not change the factoring process. We will still look for two numbers whose product is -30 and whose sum is -7. The numbers are -10 and 3. These will be the coefficients on the d terms.

4. Greatest Common Factor and Factoring Trinomials

Remember that the first step in any factoring problem is to remove the GCF. By removing the GCF, the remaining terms of the trinomial will be simpler and may have smaller coefficients.

example 5 Factoring a Trinomial by the Trial-and-Error Method

Tip: Notice when the GCF, $2x$, is removed from the original trinomial, the new trinomial has smaller coefficients. This makes the factoring process simpler. It is easier to list the factors of 20 and 5 rather than the factors of 40 and 10.

Factor the trinomial by the trial-and-error method: $40x^3 - 104x^2 + 10x$

Solution:

$40x^3 - 104x^2 + 10x$

$= 2x(20x^2 - 52x + 5)$ — **Step 1:** The GCF is $2x$.

$= 2x(\square x \quad \square)(\square x \quad \square)$ — **Step 2:** List the factors of 20 and factors of 5. Consider the reverse order in one list of factors.

Factors of 20	Factors of 5
$1 \cdot 20$	$1 \cdot 5$
$2 \cdot 10$	$5 \cdot 1$
$4 \cdot 5$	

Step 3: Construct all possible binomial factors using different combinations of the factors of 20 and factors of 5. The signs in the parentheses must both be negative.

$$\left.\begin{aligned} &= 2x(1x - 1)(20x - 5) \\ &= 2x(2x - 1)(10x - 5) \\ &= 2x(4x - 1)(5x - 5) \end{aligned}\right\}$$

Incorrect. The binomials contain a GCF greater than 1.

$= 2x(1x - 5)(20x - 1)$ — *Incorrect.* Wrong middle term.

$$\begin{aligned} &2x(x - 5)(20x - 1) \\ &\quad = 2x(20x^2 - 1x - 100x + 5) \\ &\quad = 2x(20x^2 - 101x + 5) \end{aligned}$$

$= 2x(4x - 5)(5x - 1)$ — *Incorrect.* Wrong middle term.

$$\begin{aligned} &2x(4x - 5)(5x - 1) \\ &\quad = 2x(20x^2 - 4x - 25x + 5) \\ &\quad = 2x(20x^2 - 29x + 5) \end{aligned}$$

$= 2x(2x - 5)(10x - 1)$ — *Correct.*

$$\begin{aligned} &2x(2x - 5)(10x - 1) \\ &\quad = 2x(20x^2 - 2x - 50x + 5) \\ &\quad = 2x(20x^2 - 52x + 5) \\ &\quad = 40x^3 - 104x^2 + 10x \end{aligned}$$

The correct factorization is $2x(2x - 5)(10x - 1)$.

Often it is easier to factor a trinomial when the leading coefficient is positive. If the leading coefficient is negative, consider factoring out the opposite of the GCF.

example 6 Factoring a Trinomial by the Trial-and-Error Method

Factor the trinomial by the trial-and-error method: $-w^2 - 7w + 18$

Solution:

$-w^2 - 7w + 18$

$= -1(w^2 + 7w - 18)$ Factor out -1. The resulting trinomial has a leading coefficient of 1.

$= -1[(w \quad)(w \quad)]$ To complete the factorization, we need two numbers whose product is -18 and whose sum is 7. The numbers are 9 and -2.

$= -1[(w + 9)(w - 2)]$

$= -(w + 9)(w - 2)$

Avoiding Mistakes

Do not forget to write the GCF as part of the final answer.

Note that not every trinomial is factorable by the methods presented here.

example 7 Factoring a Trinomial by the Trial-and-Error Method

Factor the trinomial by the trial-and-error method: $2p^2 - 8p + 3$

Solution:

$2p^2 - 8p + 3$ **Step 1:** The GCF is 1.

$(1p \quad \square)(2p \quad \square)$ **Step 2:** List the factors of 2 and the factors of 3.

Factors of 2	Factors of 3
$1 \cdot 2$	$1 \cdot 3$
	$3 \cdot 1$

Step 3: Construct all possible binomial factors using different combinations of the factors of 2 and 3. Because the third term in the trinomial is positive, both signs in the binomial must be the same. Because the middle term coefficient is negative, both signs will be negative.

$(p - 1)(2p - 3) = 2p^2 - 3p - 2p + 3$
$= 2p^2 - 5p + 3$ *Incorrect.* Wrong middle term.

$(p - 3)(2p - 1) = 2p^2 - p - 6p + 3$
$= 2p^2 - 7p + 3$ *Incorrect.* Wrong middle term.

Because none of the combinations of factors results in the correct product, we say that the trinomial $2p^2 - 8p + 3$ is prime and cannot be factored.

section 5.3 PRACTICE EXERCISES

For Exercises 1–6, factor out the greatest common factor.

1. $7a^9 + 28a^3$
2. $r^4 - 9r^5$
3. $12w^2 - 4w$
4. $15x^2 + 3x$
5. $21a^2b^2 + 12ab^2 - 15a^2b$
6. $5uv^2 - 10u^2v + 25u^2v^2$

For Exercises 7–10, assume a, b, and c represent positive integers.

7. When factoring a polynomial of the form $ax^2 + bx - c$, should the signs in the binomials be both positive, both negative, or different?
8. When factoring a polynomial of the form $ax^2 - bx - c$, should the signs in the binomials be both positive, both negative, or different?
9. When factoring a polynomial of the form $ax^2 - bx + c$, should the signs in the binomials be both positive, both negative, or different?
10. When factoring a polynomial of the form $ax^2 + bx + c$, should the signs in the binomials be both positive, both negative, or different?

For Exercises 11–14, complete the factorization.

11. $x^2 + x - 56 = (x - 7)(\quad)$
12. $y^2 + y - 30 = (y - 5)(\quad)$
13. $x^2 - x - 56 = (x + 7)(\quad)$
14. $y^2 - y - 30 = (y + 5)(\quad)$
15. What is a prime polynomial?
16. How do you determine if a trinomial is prime?

For Exercises 17–58, factor the trinomial using the trial-and-error method.

17. $2y^2 - 3y - 2$
18. $2w^2 + 5w - 3$
19. $9x^2 - 12x + 4$
20. $3n^2 + 13n + 4$
21. $2a^2 + 7a + 6$
22. $8b^2 - 6b - 9$
23. $6t^2 + 7t - 3$
24. $4p^2 - 9p + 2$
25. $4m^2 - 20m + 25$
26. $16r^2 + 24r + 9$
27. $5c^2 - c + 2$
28. $7s^2 + 2s + 9$
29. $6x^2 - 19xy + 10y^2$
30. $15p^2 + pq - 2q^2$
31. $12m^2 + 11mn - 5n^2$
32. $4a^2 + 5ab - 6b^2$
33. $6r^2 + rs - 2s^2$
34. $18x^2 - 9xy - 2y^2$
35. $4s^2 - 8st + t^2$
36. $6u^2 - 10uv + 5v^2$
37. $x^2 + 7x - 18$
38. $y^2 - 6y - 40$
39. $a^2 - 10a - 24$
40. $b^2 + 6b - 7$
41. $r^2 + 5r - 24$
42. $t^2 + 20t + 100$
43. $w^2 - 14w + 49$
44. $h^2 - 11h + 10$
45. $k^2 + 5k + 4$
46. $u^2 + 9u - 22$
47. $v^2 - 4v + 1$
48. $x^2 + 5x + 2$
49. $m^2 - 13mn + 40n^2$
50. $r^2 - rs - 42s^2$
51. $a^2 + 9ab + 8b^2$
52. $y^2 - yz - 12z^2$
53. $x^2 + 9xy + 20y^2$
54. $p^2 - 13pq + 36q^2$
55. $10t^2 - 23t - 5$
56. $16n^2 + 14n + 3$
57. $14w^2 + 13w - 12$
58. $12x^2 - 16x + 5$
59. Is the expression $(3x + 6)(x - 5)$ factored completely? Explain why or why not.
60. Is the expression $(5x + 1)(4x - 12)$ factored completely? Explain why or why not.

For Exercises 61–74, first factor out the GCF. Then factor by using the trial-and-error method if possible.

61. $2m^2 - 12m - 80$
62. $3c^2 - 33c + 72$
63. $2y^5 + 13y^4 + 6y^3$
64. $3u^8 - 13u^7 + 4u^6$
65. $5d^3 + 3d^2 - 10d$
66. $3y^3 - y^2 + 12y$
67. $4b^3 - 4b^2 - 80b$
68. $2w^2 + 20w + 42$
69. $x^2y^2 - 13xy^2 + 30y^2$
70. $p^2q^2 - 14pq^2 + 33q^2$
71. $-a^2 - 15a + 34$
72. $-j^2 - 7j - 10$
73. $-2u^2 + 28u - 90$
74. $-3v^2 - 3v + 90$

For Exercises 75–82, write the polynomial in descending order. Then factor the polynomial.

75. $10x + 1 + 16x^2$

76. $k^2 + 16 - 8k$

77. $1 + c^2 - 2c$

78. $3 - 14t + 16t^2$

79. $20z - 18 - 2z^2$

80. $25t - 5t^2 - 30$

81. $42 - 13q + q^2$

82. $-5w - 24 + w^2$

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section **5.4**

Factoring Perfect Square Trinomials and the Difference of Squares

Concepts

1. Factoring Perfect Square Trinomials
2. Factoring a Difference of Squares
3. Analyzing a Sum of Squares
4. Factoring Using Multiple Methods

1. Factoring Perfect Square Trinomials

Recall from Section 4.6 that the square of a binomial always results in a perfect square trinomial:

$$(a + b)^2 = (a + b)(a + b) = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2$$

$$(a - b)^2 = (a - b)(a - b) = a^2 - ab - ab + b^2 = a^2 - 2ab + b^2$$

For example, $(3x + 5)^2 = (3x)^2 + 2(3x)(5) + (5)^2 = 9x^2 + 30x + 25$

$a = 3x \quad b = 5$

To factor the trinomial $9x^2 + 30x + 25$, the grouping method or the trial-and-error method can be used. However, if we recognize that the trinomial is a perfect square trinomial, we can use one of the following patterns to reach a quick solution.

Factored Form of a Perfect Square Trinomial

$$a^2 + 2ab + b^2 = (a + b)^2$$

$$a^2 - 2ab + b^2 = (a - b)^2$$

Checking for a Perfect Square Trinomial

1. Check if the first and third terms are both perfect squares with positive coefficients.
2. If this is the case, identify a and b, and determine if the middle term equals $2ab$.

example 1 **Factoring Perfect Square Trinomials**

Factor the trinomials completely.

a. $x^2 + 14x + 49$

b. $25y^2 - 20y + 4$

c. $18c^3 - 48c^2d + 32cd^2$

d. $5w^2 + 50w + 45$

Solution:

a. $x^2 + 14x + 49$

The GCF is 1.

- The first and third terms are positive.
- The first term is a perfect square: $x^2 = (x)^2$.
- The third term is a perfect square: $49 = (7)^2$.
- The middle term is twice the product of x and 7: $14x = 2(x)(7)$

Perfect squares

$x^2 + 14x + 49$

$= (x)^2 + 2(x)(7) + (7)^2$

Hence, the trinomial is in the form $a^2 + 2ab + b^2$, where $a = x$ and $b = 7$.

$= (x + 7)^2$

Factor as $(a + b)^2$.

b. $25y^2 - 20y + 4$

The GCF is 1.

- The first and third terms are positive.
- The first term is a perfect square: $25y^2 = (5y)^2$.
- The third term is a perfect square: $4 = (2)^2$.
- The middle term is twice the product of $5y$ and 2: $20y = 2(5y)(2)$

Perfect squares

$25y^2 - 20y + 4$

$= (5y)^2 - 2(5y)(2) + (2)^2$

$= (5y - 2)^2$

Factor as $(a - b)^2$.

c. $18c^3 - 48c^2d + 32cd^2$

$= 2c(9c^2 - 24cd + 16d^2)$

The GCF is $2c$.

- The first and third terms are positive.
- The first term is a perfect square: $9c^2 = (3c)^2$.
- The third term is a perfect square: $16d^2 = (4d)^2$.
- The middle term is twice the product of $3c$ and $4d$: $24cd = 2(3c)(4d)$

Perfect squares

$= 2c(9c^2 - 24cd + 16d^2)$

$= 2c[(3c)^2 - 2(3c)(4d) + (4d)^2]$

$= 2c(3c - 4d)^2$

Factor as $(a - b)^2$.

d. $5w^2 + 50w + 45$

$= 5(w^2 + 10w + 9)$	The GCF is 5.
Perfect squares	The first and third terms are perfect squares. $w^2 = (w)^2$ and $9 = (3)^2$
$= 5(w^2 + 10w + 9)$	However, the middle term is not 2 times the product of w and 3. Therefore, this is not a perfect square trinomial. $10w \neq 2(w)(3)$
$= 5(w + 9)(w + 1)$	To factor, use either the grouping method or the trial-and-error method.

Tip: To help you identify a perfect square trinomial, it is recommended that you familiarize yourself with the first several perfect squares.

$1 \cdot 1 = \mathbf{2}$	$4 \cdot 4 = \mathbf{16}$	$7 \cdot 7 = \mathbf{49}$	$10 \cdot 10 = \mathbf{100}$	$13 \cdot 13 = \mathbf{169}$
$2 \cdot 2 = \mathbf{4}$	$5 \cdot 5 = \mathbf{25}$	$8 \cdot 8 = \mathbf{64}$	$11 \cdot 11 = \mathbf{121}$	$14 \cdot 14 = \mathbf{196}$
$3 \cdot 3 = \mathbf{9}$	$6 \cdot 6 = \mathbf{36}$	$9 \cdot 9 = \mathbf{81}$	$12 \cdot 12 = \mathbf{144}$	$15 \cdot 15 = \mathbf{225}$

If you do not recognize that a trinomial is a perfect square trinomial, you may still use either the trial-and-error method or the grouping method to factor the trinomial.

2. Factoring a Difference of Squares

Up to this point, we have learned several methods of factoring, including:

- Factoring out the greatest common factor from a polynomial
- Factoring a four-term polynomial by grouping
- Recognizing and factoring perfect square trinomials
- Factoring trinomials by the grouping method or by the trial-and-error method

Next, we will learn how to factor binomials that fit the pattern of a difference of squares. Recall from Section 4.6 that the product of two conjugates results in a difference of squares:

$$(a + b)(a - b) = a^2 - b^2$$

Therefore, to factor a difference of squares, the process is reversed. Identify a and b and construct the conjugate factors.

Factored Form of a Difference of Squares

$$a^2 - b^2 = (a + b)(a - b)$$

In addition to recognizing numbers that are perfect squares, it is helpful to recognize that a variable expression is a perfect square if its exponent is a multiple of 2. For example:

Perfect Squares

$x^2 = (x)^2$

$x^4 = (x^2)^2$

$x^6 = (x^3)^2$

$x^8 = (x^4)^2$

$x^{10} = (x^5)^2$

example 2 **Factoring Differences of Squares**

Factor the binomials.

a. $y^2 - 25$ b. $49s^2 - 4t^4$ c. $18w^2z - 2z$

Solution:

a. $y^2 - 25$	The binomial is a difference of squares.
$= (y)^2 - (5)^2$	Write in the form: $a^2 - b^2$, where $a = y$, $b = 5$.
$= (y + 5)(y - 5)$	Factor as $(a + b)(a - b)$.
b. $49s^2 - 4t^4$	The binomial is a difference of squares.
$= (7s)^2 - (2t^2)^2$	Write in the form $a^2 - b^2$, where $a = 7s$ and $b = 2t^2$.
$= (7s + 2t^2)(7s - 2t^2)$	Factor as $(a + b)(a - b)$.
c. $18w^2z - 2z$	The GCF is $2z$.
$= 2z(9w^2 - 1)$	$(9w^2 - 1)$ is a difference of squares.
$= 2z[(3w)^2 - (1)^2]$	Write in the form: $a^2 - b^2$, where $a = 3w$, $b = 1$.
$= 2z(3w + 1)(3w - 1)$	Factor as $(a + b)(a - b)$.

3. Analyzing a Sum of Squares

Suppose a and b share no common factors. Then the difference of squares $a^2 - b^2$ can be factored as $(a + b)(a - b)$. However, the sum of squares $a^2 + b^2$ cannot be factored over the real numbers. To see why, consider the expression $a^2 + b^2$. The factored form would require two binomials of the form:

$$(a \quad b)(a \quad b) \stackrel{?}{=} a^2 + b^2$$

If all possible combinations of signs are considered, none produces the correct product.

$(a + b)(a - b) = a^2 - b^2$	Wrong sign
$(a + b)(a + b) = a^2 + 2ab + b^2$	Wrong middle term
$(a - b)(a - b) = a^2 - 2ab + b^2$	Wrong middle term

After exhausting all possibilities, we see that if a and b share no common factors, then the sum of squares $a^2 + b^2$ is a prime polynomial.

4. Factoring Using Multiple Methods

Some factoring problems require more than one method of factoring. In general, when factoring a polynomial, be sure to factor completely.

example 3 Factoring Polynomials

Factor completely.

a. $w^4 - 16$ b. $4x^3 + 4x^2 - 25x - 25$ c. $8p^3 + 24p^2q + 18pq^2$

Solution:

a. $w^4 - 16$	The GCF is 1. $w^4 - 16$ is a difference of squares.
$= (w^2)^2 - (4)^2$	Write in the form: $a^2 - b^2$, where $a = w^2$, $b = 4$.
$= (w^2 + 4)(w^2 - 4)$	Factor as $(a + b)(a - b)$.
$= (w^2 + 4)\overbrace{(w + 2)(w - 2)}$	Note that $w^2 - 4$ can be factored further as a difference of squares. (The binomial $w^2 + 4$ is a sum of squares and cannot be factored further.)
b. $4x^3 + 4x^2 - 25x - 25$	The GCF is 1.
$= 4x^3 + 4x^2 \mid - 25x - 25$	The polynomial has four terms. Factor by grouping.
$= 4x^2(x + 1) - 25(x + 1)$	
$= (x + 1)(4x^2 - 25)$	$4x^2 - 25$ is a difference of squares.
$= (x + 1)\overbrace{(2x + 5)(2x - 5)}$	
c. $8p^3 + 24p^2q + 18pq^2$	The GCF is $2p$.
$= 2p(4p^2 + 12pq + 9q^2)$	$4p^2 + 12pq + 9q^2$ is a perfect square trinomial.
$= 2p(2p + 3q)^2$	

section 5.4 Practice Exercises

For Exercises 1–10, factor the polynomials.

1. $3x^2 + x - 10$
2. $6a^2b + 3a^3b$
3. $x^2yz^2 + 6y^2z + yz$
4. $2x^2 - x - 1$
5. $12x^2 - 34x + 10$
6. $3x^2 - 3xy - 6y^2$
7. $ax + ab - 6x - 6b$
8. $2xy - 3x - 4y + 6$
9. $x^2 + 6x + 9$
10. $y^2 - 4y + 4$
11. What perfect square trinomial factors to $(2x + 3)^2$?
12. What perfect square trinomial factors to $(3k + 5)^2$?
13. What perfect square trinomial factors to $(6h - 1)^2$?
14. What perfect square trinomial factors to $(4y - 5)^2$?
15. a. Identify which trinomial is a perfect square trinomial:

    $x^2 + 4x + 4$ or $x^2 + 5x + 4$

    b. Factor both of these trinomials.
16. a. Identify which trinomial is a perfect square trinomial:

    $x^2 + 13x + 36$ or $x^2 + 12x + 36$

    b. Factor both of these trinomials.
17. a. Identify which trinomial is a perfect square trinomial:

    $4x^2 - 25x + 25$ or $4x^2 - 20x + 25$

    b. Factor both of these trinomials.
18. a. Identify which trinomial is a perfect square trinomial:

    $9x^2 + 12x + 4$ or $9x^2 + 15x + 4$

    b. Factor both of these trinomials.

For Exercises 19–40, factor the trinomials, if possible.

19. $y^2 - 10y + 25$
20. $t^2 - 16t + 64$
21. $m^2 + 6m + 9$
22. $n^2 + 18n + 81$
23. $r^2 - 2r + 36$
24. $s^2 - 4s + 100$
25. $49q^2 - 28q + 4$
26. $64y^2 - 80y + 25$
27. $9p^2 + 42p + 49$
28. $4x^2 + 36x + 81$
29. $25h^2 + 50h + 16$
30. $4w^2 - 20w + 9$
31. $16a^2 + 8ab + b^2$
32. $25m^2 + 10mn + n^2$
33. $16q^2 + 40qr + 25r^2$
34. $u^2 - 2uv + v^2$
35. $a^2 + 2ab + b^2$
36. $49h^2 - 14hk + k^2$
37. $k^2 - k + \frac{1}{4}$
38. $v^2 + \frac{2}{3}v + \frac{1}{9}$
39. $9x^2 + x + \frac{1}{36}$
40. $4y^2 - y + \frac{1}{16}$
41. What binomial factors as $(x - 5)(x + 5)$?
42. What binomial factors as $(n - 3)(n + 3)$?
43. What binomial factors as $(2w - 3)(2w + 3)$?
44. What binomial factors as $(7y - 4)(7y + 4)$?

For Exercises 45–64, factor the binomials, if possible.

45. $x^2 - 36$
46. $r^2 - 81$
47. $w^2 - 100$
48. $t^2 - 49$
49. $4a^2 - 121b^2$
50. $9x^2 - y^2$
51. $49m^2 - 16n^2$
52. $100a^2 - 49b^2$
53. $9q^2 + 16$
54. $36 + s^2$
55. $c^6 - 25$
56. $z^6 - 4$
57. $25 - 16t^2$
58. $64 - h^2$
59. $p^2 - \frac{1}{9}$
60. $q^2 - \frac{1}{36}$
61. $m^2 + \frac{100}{81}$
62. $n^2 + \frac{25}{4}$
63. $\frac{4}{9} - w^2$
64. $\frac{16}{25} - x^2$

65. a. Write a polynomial that represents the area of the shaded region in the figure.
    b. Factor the expression from part (a).

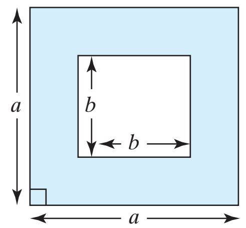


Figure for Exercise 65

66. a. Write a polynomial that represents the area of the shaded region in the figure.
    b. Factor the expression from part (a).

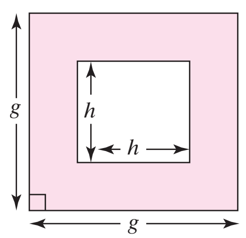


Figure for Exercise 66

For Exercises 67–88, factor the polynomials completely.

67. $3w^2 - 27$
68. $6y^2 - 6$
69. $50p^4 - 2$
70. $18q^2 - 98n^2$
71. $2x^2 + 24x + 72$
72. $3x^2y - 66xy + 363y$
73. $2t^3 - 10t^2 - 2t + 10$
74. $9a^3 + 27a^2 - 4a - 12$
75. $100y^4 + 25x^2$
76. $36a^2 + 9b^4$
77. $4a^2b - 40ab^2 + 100b^3$
78. $18u^2 + 24uv + 8v^2$
79. $2x^3 + 3x^2 - 2x - 3$
80. $3x^3 + x^2 - 12x - 4$
81. $81y^4 - 16$
82. $u^4 - 256$
83. $81k^2 + 30k + 1$
84. $9h^2 - 15h + 4$
85. $k^3 + 4k^2 - 9k - 36$
86. $w^3 - 2w^2 - 4w + 8$
87. $4m^{14} - 20m^7 + 25$
88. $9n^{12} + 24n^6 + 16$

Expanding Your Skills

For Exercises 89–100, factor the difference of squares.

89. $0.36x^2 - 0.01$
90. $0.81p^2 - 0.25q^2$
91. $\frac{1}{4}w^2 - \frac{1}{9}v^2$
92. $\frac{4}{9}c^2 - \frac{9}{16}d^2$
93. $(y - 3)^2 - 9$
94. $(x - 2)^2 - 4$
95. $(2p + 1)^2 - 36$
96. $(4q + 3)^2 - 25$
97. $16 - (t + 2)^2$
98. $81 - (a + 5)^2$
99. $100 - (2b - 5)^2$
100. $49 - (3k - 7)^2$

Concepts

1. Factoring the Sum and Difference of Cubes
2. Factoring Binomials: A Summary

section

5.5 Factoring the Sum and Difference of Cubes

1. Factoring the Sum and Difference of Cubes

In Section 5.4, you learned that a binomial $a^2 - b^2$ is a difference of squares and can be factored as $(a - b)(a + b)$. Furthermore, if a and b share no common factors, then a sum of squares $a^2 + b^2$ is not factorable over the real numbers. In this section, we will learn that both a difference of cubes, $a^3 - b^3$, and a sum of cubes $a^3 + b^3$ are factorable.

Factoring a Sum and Difference of Cubes

Sum of Cubes: $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

Difference of Cubes: $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Multiplication can be used to confirm the formulas for factoring a sum or difference of cubes:

$$(a + b)(a^2 - ab + b^2) = a^3 - \cancel{a^2b} + \cancel{ab^2} + \cancel{a^2b} - \cancel{ab^2} + b^3 = a^3 + b^3 \checkmark$$

$$(a - b)(a^2 + ab + b^2) = a^3 + \cancel{a^2b} + \cancel{ab^2} - \cancel{a^2b} - \cancel{ab^2} - b^3 = a^3 - b^3 \checkmark$$

To help you remember the formulas for factoring a sum or difference of cubes, keep the following guidelines in mind.

- The factored form is the product of a binomial and a trinomial.
- The first and third terms in the trinomial are the squares of the terms within the binomial factor.
- Without regard to signs, the middle term in the trinomial is the product of terms in the binomial factor.

Square the first term of the binomial. Product of terms in the binomial

$$x^3 + 8 = (x)^3 + (2)^3 = (x + 2)[(x)^2 - (x)(2) + (2)^2]$$

Square the last term of the binomial.

- The sign within the binomial factor is the same as the sign of the original binomial.
- The first and third terms in the trinomial are always positive.
- The sign of the middle term in the trinomial is opposite the sign within the binomial.

Same sign Positive

$$x^3 + 8 = (x)^3 + (2)^3 = (x + 2)[(x)^2 - (x)(2) + (2)^2]$$

Opposite signs

To help you recognize a sum or difference of cubes, we recommend that you familiarize yourself with the first several perfect cubes:

Perfect Cube	Perfect Cube
$1 = (1)^3$	$216 = (6)^3$
$8 = (2)^3$	$343 = (7)^3$
$27 = (3)^3$	$512 = (8)^3$
$64 = (4)^3$	$729 = (9)^3$
$125 = (5)^3$	$1000 = (10)^3$

It is also helpful to recognize that a variable expression is a perfect cube if its exponent is a multiple of 3. For example,

Perfect Cube

$$x^3 = (x)^3$$
$$x^6 = (x^2)^3$$
$$x^9 = (x^3)^3$$
$$x^{12} = (x^4)^3$$

example 1

Factoring a Sum of Cubes

Factor: $w^3 + 64$

Solution:

$w^3 + 64$ — w^3 and 64 are perfect cubes.

$= (w)^3 + (4)^3$ — Write as $a^3 + b^3$, where $a = w$, $b = 4$.

$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ — Apply the formula for a sum of cubes.

$(w)^3 + (4)^3 = (w + 4)[(w)^2 - (w)(4) + (4)^2]$

$= (w + 4)(w^2 - 4w + 16)$ — Simplify.

Check: $(w + 4)(w^2 - 4w + 16) = w^3 - 4w^2 + 16w + 4w^2 - 16w + 64$
$= w^3 + 64$ ✔

example 2

Factoring a Difference of Cubes

Factor: $27p^3 - q^6$

Solution:

$27p^3 - q^6$ — $27p^3$ and q^6 are perfect cubes.

$(3p)^3 - (q^2)^3$ — Write as $a^3 - b^3$, where $a = 3p$, $b = q^2$.

$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ — Apply the formula for a difference of cubes.

$(3p)^3 - (q^2)^3 = (3p - q^2)[(3p)^2 + (3p)(q^2) + (q^2)^2]$

$= (3p - q^2)(9p^2 + 3pq^2 + q^4)$ — Simplify.

2. Factoring Binomials: A Summary

After removing the GCF, the next step in any factoring problem is to recognize what type of pattern it follows. Exponents that are divisible by 2 are perfect squares and those divisible by 3 are perfect cubes. The formulas for factoring binomials are summarized in the following box:

Factoring Binomials

1. Difference of Squares: $a^2 - b^2 = (a + b)(a - b)$
2. Difference of Cubes: $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$
3. Sum of Cubes: $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

example 3

Factoring Binomials

Factor completely: a. $27y^3 + 1$ b. $m^2 - \frac{1}{4}$

c. $3y^4 - 48$ d. $z^6 - 8w^3$

Solution:

a. $27y^3 + 1$ — Sum of cubes: $27y^3 = (3y)^3$ and $1 = (1)^3$.

$= (3y)^3 + (1)^3$ — Write as $a^3 + b^3$, where $a = 3y$ and $b = 1$.

$= (3y + 1)[(3y)^2 - (3y)(1) + (1)^2]$ — Apply the formula $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.

$= (3y + 1)(9y^2 - 3y + 1)$ — Simplify.

b. $m^2 - \frac{1}{4}$ — Difference of squares

$= (m)^2 - \left(\frac{1}{2}\right)^2$ — Write as $a^2 - b^2$, where $a = m$ and $b = \frac{1}{2}$.

$= \left(m + \frac{1}{2}\right)\left(m - \frac{1}{2}\right)$ — Apply the formula $a^2 - b^2 = (a + b)(a - b)$.

c. $3y^4 - 48$

$= 3(y^4 - 16)$ — Factor out the GCF. The binomial is a difference of squares.

$= 3[(y^2)^2 - (4)^2]$ — Write as $a^2 - b^2$, where $a = y^2$ and $b = 4$.

$= 3(y^2 + 4)(y^2 - 4)$ Apply the formula $a^2 - b^2 = (a + b)(a - b)$.

$y^2 + 4$ is a sum of squares and cannot be factored.

$= 3(y^2 + 4)(y + 2)(y - 2)$ $y^2 - 4$ is a difference of squares and can be factored further.

d. $z^6 - 8w^3$ Difference of cubes: $z^6 = (z^2)^3$ and $8w^3 = (2w)^3$

$= (z^2)^3 - (2w)^3$ Write as $a^3 - b^3$, where $a = z^2$ and $b = 2w$.

$= (z^2 - 2w)[(z^2)^2 + (z^2)(2w) + (2w)^2]$ Apply the formula $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.

$= (z^2 - 2w)(z^4 + 2z^2w + 4w^2)$ Simplify.

Each of the factorizations in Example 3 can be checked by multiplying.

example 4 Factoring Binomials

Factor the binomial $x^6 - y^6$ as

a. A difference of cubes
b. A difference of squares

Solution:

a. $x^6 - y^6$

Difference of cubes

$= (x^2)^3 - (y^2)^3$ Write as $a^3 - b^3$, where $a = x^2$ and $b = y^2$.

$= (x^2 - y^2)[(x^2)^2 + (x^2)(y^2) + (y^2)^2]$ Apply the formula $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

$= (x^2 - y^2)(x^4 + x^2y^2 + y^4)$ Factor $x^2 - y^2$ as a difference of squares.

$= \overbrace{(x + y)(x - y)}(x^4 + x^2y^2 + y^4)$

b. $x^6 - y^6$

Difference of squares

$= (x^3)^2 - (y^3)^2$ Write as $a^2 - b^2$, where $a = x^3$ and $b = y^3$.

$$= (x^3 + y^3)(x^3 - y^3)$$

Apply the formula $a^2 - b^2 = (a + b)(a - b)$.

Factor $x^3 + y^3$ as a sum of cubes.

Factor $x^3 - y^3$ as a difference of cubes.

$$= \overbrace{(x + y)(x^2 - xy + y^2)}^{\text{Sum of cubes}}\overbrace{(x - y)(x^2 + xy + y^2)}^{\text{Difference of cubes}}$$

Notice that the expressions x^6 and y^6 are both perfect squares and perfect cubes because the exponents are both multiples of 2 and of 3. Consequently, $x^6 - y^6$ can be factored initially as either the difference of squares or as the difference of cubes. In such a case, it is recommended that you factor the expression as a difference of squares first because it factors more completely into polynomials of lower degree. Hence:

$$x^6 - y^6 = (x + y)(x^2 - xy + y^2)(x - y)(x^2 + xy + y^2)$$

section 5.5 Practice Exercises

1. Multiply the polynomials: $(x - y)(x^2 + xy + y^2)$
2. Multiply the polynomials: $(x + y)(x^2 - xy + y^2)$
3. Identify the expressions that are perfect cubes:
   $\{x^3, 8, 9, y^6, a^4, b^2, 3p^3, 27q^3, w^{12}, r^3s^6\}$
4. Identify the expressions that are perfect cubes:
   $\{z^9, -81, 30, 8, 6x^3, y^{15}, 27a^3, b^2, p^3q^2, -1\}$
5. How do you determine if a binomial is a sum of cubes?
6. How do you determine if a binomial is a difference of cubes?
7. From memory, write the formula to factor a sum of cubes:
   $a^3 + b^3 =$ ____________
8. From memory, write the formula to factor a difference of cubes:
   $a^3 - b^3 =$ ____________

For Exercises 9–28, factor the sums and differences of cubes.

9. $y^3 - 8$
10. $x^3 + 27$
11. $1 - p^3$
12. $q^3 + 1$
13. $w^3 + 64$
14. $8 - t^3$
15. $1000a^3 + 27$
16. $216b^3 - 125$
17. $x^3 - 1000$
18. $8y^3 - 27$
19. $64t^3 + 1$
20. $125r^3 + 1$
21. $n^3 - \dfrac{1}{8}$
22. $\dfrac{8}{27} + m^6$
23. $a^3 + b^6$
24. $u^6 - v^3$
25. $x^9 + 64y^3$
26. $125w^3 - z^9$
27. $25m^{12} + 16$
28. $36p^6 + 49q^4$
29. From memory, write the formula to factor a difference of squares.
    $a^2 - b^2 =$ ____________
30. Write a short paragraph explaining a strategy to factor binomials.

For Exercises 31–48, factor the binomials completely, if possible.

31. $x^4 - 4$
32. $b^4 - 25$
33. $a^2 + 9$
34. $w^2 + 36$
35. $t^3 + 64$
36. $u^3 + 27$

37. $g^3 - 4$
38. $h^3 - 25$
39. $4b^3 + 108$
40. $3c^3 - 24$
41. $5p^2 - 125$
42. $2q^4 - 8$
43. $\frac{1}{64} - 8h^3$
44. $\frac{1}{125} + k^6$
45. $x^4 - 16$
46. $p^4 - 81$
47. $q^6 - 64$
48. $a^6 - 1$

For Exercises 49–68, factor completely using the techniques learned in Sections 5.1–5.5.

49. $4b + 16$
50. $2a^2 - 162$
51. $y^2 + 4y + 3$
52. $6w^2 - 6w$
53. $16z^4 - 81$
54. $3t^2 + 13t + 4$
55. $5r^3 + 5$
56. $3ac + ad - 3bc - bd$
57. $7p^2 - 29p + 4$
58. $3q^2 - 9q - 12$
59. $-2x^2 + 8x - 8$
60. $18a^2 + 12a$
61. $54 - 2y^3$
62. $4t^2 - 100$
63. $4t^2 - 31t - 8$
64. $10c^2 + 10c + 10$
65. $2xw - 10x + 3yw - 15y$
66. $x^3 + 0.001$
67. $4q^2 - 9$
68. $64 + 16k + k^2$
69. What trinomial multiplied by $(x - 2)$ gives a difference of cubes?
70. What trinomial multiplied by $(p + 3)$ gives a sum of cubes?
71. Write a binomial that when multiplied by $(4x^2 - 2x + 1)$ produces a sum of cubes.
72. Write a binomial that when multiplied by $(9y^2 + 15y + 25)$ produces a difference of cubes.

Expanding Your Skills

For Exercises 73–76, factor the sum and difference of cubes.

73. $\frac{64}{125}p^3 - \frac{1}{8}q^3$
74. $\frac{1}{1000}r^3 + \frac{8}{27}s^3$
75. $a^{12} + b^{12}$
76. $a^9 - b^9$

Use Exercises 77–80, to investigate the relationship between division and factoring.

77. a. Use long division to divide $x^3 - 8$ by $(x - 2)$.
    b. Factor $x^3 - 8$.
78. a. Use long division to divide $y^3 + 27$ by $(y + 3)$.
    b. Factor $y^3 + 27$.
79. a. Use long division to divide $m^3 + 1$ by $(m + 1)$.
    b. Factor $m^3 + 1$.
80. a. Use long division to divide $n^3 - 64$ by $(n - 4)$.
    b. Factor $n^3 - 64$.

Concepts

1. A Factoring Strategy
2. Part I: General Factoring Review
3. Part II: Additional Factoring Strategies
4. Factoring Using Substitution
5. Factoring by Grouping—Rearranging Terms
6. Other Grouping Techniques for Factoring Four-Term Polynomials

section

5.6 General Factoring Summary

1. A Factoring Strategy

We now review the techniques of factoring presented thus far along with a general strategy for factoring polynomials.

Factoring Strategy

1. Factor out the greatest common factor, GCF. (Section 5.1)
2. Identify whether the polynomial has two terms, three terms, or more than three terms.
3. If the polynomial has two terms, determine if it fits the pattern for a difference of squares, difference of cubes, or sum of cubes. (Sections 5.4 or 5.5)
4. If the polynomial has three terms, check first for a perfect square trinomial (Section 5.4). Otherwise, factor the trinomial with the grouping method or the trial-and-error method. (Sections 5.2 or 5.3)
5. If the polynomial has more than three terms, try factoring by grouping. (Sections 5.1 or 5.6)
6. Be sure to factor the polynomial completely.
7. Check by multiplying.

2. Part I: General Factoring Review

example 1

Factoring Polynomials

Factor out the GCF and identify the number of terms and type of factoring pattern represented by the polynomial. Then factor the polynomial completely.

a. $abx^2 - 3ax + 5bx - 15$ b. $20y^2 - 110y - 210$
c. $4p^3 + 20p^2 + 25p$ d. $w^3 + 1000$
e. $t^3 - 25t$

Solution:

a. $abx^2 - 3ax + 5bx - 15$ The GCF is 1. The polynomial has four terms. Therefore, factor by grouping.

$abx^2 - 3ax \mid + 5bx - 15$

$= ax(bx - 3) + 5(bx - 3)$

$= (bx - 3)(ax + 5)$

Check: $(bx - 3)(ax + 5) = abx^2 + 5bx - 3ax - 15$ ✔

b. $20y^2 - 110y - 210$
$= 10(2y^2 - 11y - 21)$
$= 10(2y + 3)(y - 7)$

The GCF is 10. The polynomial has three terms. The trinomial is not a perfect square trinomial. Use either the grouping method or the trial-and-error method.

<u>Check</u>: $10(2y + 3)(y - 7) = 10(2y^2 - 14y + 3y - 21)$
$= 10(2y^2 - 11y - 21)$
$= 20y^2 - 110y - 210$ ✔

c. $4p^3 + 20p^2 + 25p$
$= p(4p^2 + 20p + 25)$

The GCF is p. The polynomial has three terms and is a perfect square trinomial, $a^2 + 2ab + b^2$, where $a = 2p$ and $b = 5$.

$= p(2p + 5)^2$

Apply the formula $a^2 + 2ab + b^2 = (a + b)^2$.

<u>Check</u>: $p(2p + 5)^2 = p[(2p)^2 + 2(2p)(5) + (5)^2]$
$= p(4p^2 + 20p + 25)$
$= 4p^3 + 20p^2 + 25p$ ✔

d. $w^3 + 1000$
$= (w)^3 + (10)^3$

The GCF is 1. The polynomial has two terms. The binomial is a sum of cubes, $a^3 + b^3$, where $a = w$ and $b = 10$.

$= (w + 10)(w^2 - 10w + 100)$

Apply the formula $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.

<u>Check</u>: $(w + 10)(w^2 - 10w + 100) = w^3 - \cancel{10w^2} + \cancel{100w} + \cancel{10w^2} - \cancel{100w} + 1000$
$= w^3 + 100$ ✔

e. $t^3 - 25t$
$= t(t^2 - 25)$
$= t[(t)^2 - (5)^2]$

The GCF is t. The polynomial has two terms. The binomial is a difference of squares, $a^2 - b^2$, where $a = t$ and $b = 5$.

$= t(t + 5)(t - 5)$

Apply the formula $a^2 - b^2 = (a + b)(a - b)$.

<u>Check</u>: $t(t + 5)(t - 5) = t(t^2 - \cancel{5t} + \cancel{5t} - 25)$
$= t(t^2 - 25)$
$= t^3 - 25t$ ✔

3. Part II: Additional Factoring Strategies

Some factoring problems may require more than one type of factoring. We also may encounter polynomials that require slight variations on the factoring techniques already learned. These are demonstrated in Examples 2–7.

example 2 Factoring Binomials with Multiple Steps

Factor completely. $d^4 - \dfrac{1}{16}$

Solution:

$$d^4 - \frac{1}{16}$$

$$= (d^2)^2 - \left(\frac{1}{4}\right)^2$$ d^4 and $\frac{1}{16}$ are perfect squares.

$$= \left(d^2 + \frac{1}{4}\right)\left(d^2 - \frac{1}{4}\right)$$ Factor as a difference of squares.

$$= \left(d^2 + \frac{1}{4}\right)\underbrace{\left(d - \frac{1}{2}\right)\left(d + \frac{1}{2}\right)}$$ The binomial $d^2 - \frac{1}{4}$ is also a difference of squares.

Avoiding Mistakes

Remember that a sum of squares such as $d^2 + \frac{1}{4}$ cannot be factored over the real numbers.

example 3 Factoring a Trinomial Involving Fractional Coefficients

Factor completely: $\frac{1}{9}x^2 + \frac{1}{3}x + \frac{1}{4}$

Solution:

$$\frac{1}{9}x^2 + \frac{1}{3}x + \frac{1}{4}$$

$$= \left(\frac{1}{3}x\right)^2 + 2\left(\frac{1}{3}x\right)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2$$

$$= \left(\frac{1}{3}x + \frac{1}{2}\right)^2$$

The fractions may make this polynomial look difficult to factor. However, notice that both $\frac{1}{9}x^2$ and $\frac{1}{4}$ are perfect squares. Furthermore, the middle term $\frac{1}{3}x = 2(\frac{1}{3}x)(\frac{1}{2})$. Therefore, the trinomial is a perfect square trinomial.

4. Factoring Using Substitution

Sometimes it is convenient to use substitution to convert a polynomial into a simpler form before factoring.

example 4 Using Substitution to Factor a Trinomial

Factor completely: $(2x - 7)^2 - 3(2x - 7) - 40$

Solution:

$$(2x - 7)^2 - 3(2x - 7) - 40$$

$$= u^2 - 3u - 40$$ Substitute $u = 2x - 7$. The trinomial is simpler in form.

$= (u - 8)(u + 5)$	Factor the trinomial.
$= [(2x - 7) - 8][(2x - 7) + 5]$	Reverse substitute. Replace u by $2x - 7$.
$= (2x - 7 - 8)(2x - 7 + 5)$	Simplify.
$= (2x - 15)(2x - 2)$	The second binomial has a GCF of 2.
$= (2x - 15)(2)(x - 1)$	Factor out the GCF from the second binomial.
$= 2(2x - 15)(x - 1)$	

5. Factoring by Grouping—Rearranging Terms

Sometimes it is necessary to rearrange terms when factoring by grouping.

example 5 Factoring by Grouping Where Rearranging Terms Is Necessary

Factor completely: $4x + 6pa - 8a - 3px$

Solution:

$4x + 6pa - 8a - 3px$	The GCF of all four terms is 1.
$= 4x + 6pa \mid - 8a - 3px$	Try factoring by grouping.
$= 2(2x + 3pa) - 1(8a + 3px)$	The binomial factors in each term are different.
$= 4x - 8a \mid - 3px + 6pa$	*Try rearranging the original four terms* in such a way that the first pair of coefficients is in the same ratio as the second pair of coefficients. Notice that the ratio 4 to -8 is the same as the ratio -3 to 6.
$= 4(x - 2a) - 3p(x - 2a)$	Factor out the GCF from the first pair of terms. Factor out the GCF from the second pair of terms.
$= (x - 2a)(4 - 3p)$	Factor out the common binomial factor, $(x - 2a)$.

6. Other Grouping Techniques for Factoring Four-Term Polynomials

example 6 Factoring a Four-Term Polynomial by Grouping Three Terms

Factor completely: $25w^2 + 90w + 81 - p^2$

Solution:

With a four-term polynomial, we recommend "2-by-2" grouping—that is, to group the first pair of terms and the second pair of terms. However, in this case, there is no common binomial factor shared by each pair of terms. (Even rearranging terms does not help.)

Since this polynomial is not factorable with "2-by-2" grouping, try grouping three terms. Notice that the first three terms constitute a perfect square trinomial. Hence, we will use "3-by-1" grouping.

$\underbrace{25w^2 + 90w + 81} \mid - p^2$	Group "3 by 1."
	Factor $25w^2 + 90w + 81 = (5w + 9)^2$.
$= (5w + 9)^2 - p^2$	$(5w + 9)^2 - p^2$ is a difference of squares $a^2 - b^2$, where $a = (5w + 9)$ and $b = p$.
$= [(5w + 9) + p][(5w + 9) - p]$	Factor as $a^2 - b^2 = (a + b)(a - b)$.
$= (5w + 9 + p)(5w + 9 - p)$	Simplify.

example 7 Factoring a Four-Term Polynomial by Grouping Three Terms

Factor completely: $x^2 - y^2 - 6y - 9$

Solution:

Grouping "2 by 2" will not work to factor this polynomial. However, if we factor out -1 from the last three terms, the resulting trinomial will be a perfect square trinomial.

$x^2 \mid - y^2 - 6y - 9$	Group the last three terms.
$= x^2 - 1(y^2 + 6y + 9)$	Factor out -1 from the last three terms.
$= x^2 - (y + 3)^2$	Factor the perfect square trinomial $y^2 + 6y + 9$ as $(y + 3)^2$.
	The quantity $x^2 - (y + 3)^2$ is a difference of squares, $a^2 - b^2$, where $a = x$ and $b = (y + 3)$.
$= [x - (y + 3)][x + (y + 3)]$	Factor as $a^2 - b^2 = (a + b)(a - b)$.
$= (x - y - 3)(x + y + 3)$	Apply the distributive property to clear the inner parentheses.

Avoiding Mistakes

When factoring the expression $x^2 - (y + 3)^2$ as a difference of squares, be sure to use parentheses around the quantity $(y + 3)$. This will help you remember to "distribute the negative" in the expression $[x - (y + 3)]$.

$$[x - (y + 3)] = (x - y - 3)$$

section 5.6 Practice Exercises

Part I:
For Exercises 1–84, factor completely using the strategy found on page 347. Identify any polynomials that are prime.

1. $x^2 - 6x - 16$
2. $a^2 - a - 42$
3. $20b^2 - 11b - 3$
4. $6y^2 - 29y + 20$
5. $100 - 9u^2$
6. $144 - x^2$
7. $p^3 - 216$
8. $125 + q^3$
9. $x^2y^3 + x^5y^2$
10. $3a^3b + 6a^2$
11. $2y - 22 + 9xy - 99x$
12. $21b + 7 - 3ab - a$
13. $w^2 - 40w + 400$
14. $z^2 + 28z + 196$
15. $x^3 - 3x^2 - 10x$
16. $x^3 - x^2 - 12x$
17. $3y^2 + 21y + 36$
18. $2w^2 + 16w + 24$
19. $p^2 + 12pq + 36q^2$
20. $m^2 - 8mn + 16n^2$
21. $x^2 + 3x + 8$
22. $x^2 - 4x + 10$
23. $2x^2 + 13x - 24$
24. $3x^2 - 17x - 6$
25. $u^2 - 25v^2$
26. $16p^2 - 1$
27. $a^3b^3 - 36ab$
28. $x^3y^3 - xy$
29. $2x^3 - 20x^2 + 18x$
30. $3x^3 - 30x^2 + 63x$
31. $-3a^2b^2 + 3ab^3 - 6ab^2$
32. $-5p^4q - 5p^3q^3 + 10p^3q$
33. $11r(s + 4) - 6(s + 4)$
34. $10a(b - 7) + 7(b - 7)$
35. $100 + t^2$
36. $81 + 4x^2$
37. $x^4 - 8x$
38. $5u - 5u^4$
39. $m^3 + 16$
40. $n^3 - 9$
41. $5xy - 3y + 15x - 9$
42. $2mn + 2m - 5n - 5$
43. $2pq - 14p - 8q + 56$
44. $3ab - 12a - 24b + 96$
45. $4x^2 - 16x + 16$
46. $5x^2 + 10x + 5$
47. $50 + p^2 - 15p$
48. $3q - 88 + q^2$
49. $24xy + 16x^2 + 9y^2$
50. $9p^2 + 4q^2 + 12pq$
51. $2x^5 + 6x^3 - 10x^4 - 30x^2$
52. $35a^2 + 14a - 15a - 6$
53. $-x^2 + 16x - 63$
54. $-x^2 - 7x + 60$
55. $6x^2 - 21x - 45$
56. $20y^2 - 14y + 2$
57. $5a^2bc^3 - 7abc^2$
58. $8a^2 - 50$
59. $t^2 + 2t - 63$
60. $b^2 + 2b - 80$
61. $ab + ay - b^2 - by$
62. $6x^3y^4 + 3x^2y^5$
63. $14u^2 - 11uv + 2v^2$
64. $9p^2 - 36pq + 4q^2$
65. $4q^2 - 8q - 6$
66. $9w^2 + 3w - 15$
67. $9m^2 + 16n^2$
68. $5b^2 - 30b + 45$
69. $6r^2 + 11r + 3$
70. $4s^2 + 4s - 15$
71. $81u^2 - 90uv + 25v^2$
72. $4x^2 + 16$
73. $2ax - 6ay + 4bx - 12by$
74. $8m^3 - 10m^2 - 3m$
75. $21x^4y + 41x^3y + 10x^2y$
76. $2m^4 - 128$
77. $8uv - 6u + 12v - 9$
78. $4t^2 - 20t + st - 5s$
79. $12x^2 - 12x + 3$
80. $p^2 + 2pq + q^2$
81. $6n^3 + 5n^2 - 4n$
82. $4k^3 + 4k^2 - 3k$
83. $64 - y^2$
84. $36b - b^3$

Part II:
For Exercises 85–112, factor completely using the strategy found on page 347 and any additional techniques of factoring illustrated in Examples 2–7.

85. $x^2(x + y) - y^2(x + y)$
86. $u^2(u - v) - v^2(u - v)$

87. $(a + 3)^4 + 6(a + 3)^5$

88. $(4 - b)^4 - 2(4 - b)^3$

89. $24(3x + 5)^3 - 30(3x + 5)^2$

90. $10(2y + 3)^2 + 15(2y + 3)^3$

91. $16p^4 - q^4$

92. $s^4t^4 - 81$

93. $y^3 + \frac{1}{64}$

94. $z^3 + \frac{1}{125}$

95. $6a^3 + a^2b - 6ab^2 - b^3$

96. $4p^3 + 12p^2q - pq^2 - 3q^3$

97. $\frac{1}{9}t^2 + \frac{1}{6}t + \frac{1}{16}$

98. $\frac{1}{25}y^2 + \frac{1}{5}y + \frac{1}{4}$

99. $x^2 + 12x + 36 - a^2$

100. $a^2 + 10a + 25 - b^2$

101. $p^2 + 2pq + q^2 - 81$

102. $m^2 - 2mn + n^2 - 9$

103. $b^2 - (x^2 + 4x + 4)$

104. $p^2 - (y^2 - 6y + 9)$

105. $4 - u^2 + 2uv - v^2$

106. $25 - a^2 - 2ab - b^2$

107. $6ax - by + 2bx - 3ay$

108. $5pq - 12 - 4q + 15p$

109. $u^6 - 64$ [*Hint*: Factor first as a difference of squares, $(u^3)^2 - (8)^2$.]

110. $1 - v^6$

111. $x^8 - 1$

112. $y^8 - 256$

Expanding Your Skills

For Exercises 113–116, factor the polynomial in part (a). Then use substitution to help factor the polynomials in parts (b) and (c).

113. a. $u^2 - 10u + 25$
 b. $x^4 - 10x^2 + 25$
 c. $(a + 1)^2 - 10(a + 1) + 25$

114. a. $u^2 + 12u + 36$
 b. $y^4 + 12y^2 + 36$
 c. $(b - 2)^2 + 12(b - 2) + 36$

115. a. $u^2 + 11u - 26$
 b. $w^6 + 11w^3 - 26$
 c. $(y - 4)^2 + 11(y - 4) - 26$

116. a. $u^2 + 17u + 30$
 b. $z^6 + 17z^3 + 30$
 c. $(x + 3)^2 + 17(x + 3) + 30$

For Exercises 117–120, use substitution to factor the expressions.

117. $(5x^2 - 1)^2 - 4(5x^2 - 1) - 5$

118. $(x^3 + 4)^2 - 10(x^3 + 4) + 24$

119. $2(3w - 5)^2 - 19(3w - 5) + 35$

120. $3(2y + 3)^2 + 23(2y + 3) - 8$

For Exercises 121–124, factor completely. Then check by multiplying.

121. $a^2 - b^2 + a + b$

122. $25c^2 - 9d^2 + 5c - 3d$

123. $5wx^3 + 5wy^3 - 2zx^3 - 2zy^3$

124. $3xu^3 - 3xv^3 - 5yu^3 + 5yv^3$

Concepts

1. Definition of a Quadratic Equation
2. Zero Product Rule
3. Solving Quadratic Equations
4. Solving Higher Degree Polynomial Equations
5. Applications of Quadratic Equations
6. Pythagorean Theorem

section **5.7**

Solving Quadratic Equations Using the Zero Product Rule

1. Definition of a Quadratic Equation

In Section 2.1 we solved linear equations in one variable. These are equations of the form $ax + b = 0$ $(a \neq 0)$. A linear equation in one variable is sometimes called a first-degree polynomial equation because the highest degree of all its terms is 1. A second-degree polynomial equation in one variable is called a quadratic equation.

Definition of a Quadratic Equation in One Variable

If a, b, and c are real numbers such that $a \neq 0$, then a **quadratic equation** is an equation that can be written in the form

$$ax^2 + bx + c = 0$$

The following equations are quadratic because they can each be written in the form $ax^2 + bx + c = 0$, $(a \neq 0)$.

$$\begin{array}{ccc}
-4x^2 + 4x = 1 & x(x-2) = 3 & (x-4)(x+4) = 9 \\
-4x^2 + 4x - 1 = 0 & x^2 - 2x = 3 & x^2 - 16 = 9 \\
 & x^2 - 2x - 3 = 0 & x^2 - 25 = 0 \\
 & & x^2 + 0x - 25 = 0
\end{array}$$

2. Zero Product Rule

One method for solving a quadratic equation is to factor the equation and apply the zero product rule. The **zero product rule** states that if the product of two factors is zero, then one or both of its factors is zero.

Zero Product Rule

If $ab = 0$, then $a = 0$ or $b = 0$

For example, the quadratic equation $x^2 - x - 12 = 0$ can be written in factored form as $(x - 4)(x + 3) = 0$. By the zero product rule, one or both factors must be zero. Hence, either $x - 4 = 0$ or $x + 3 = 0$. Therefore, to solve the quadratic equation, set each factor equal to zero and solve for x.

$$\begin{array}{lclll}
(x-4)(x+3) = 0 & & & & \text{Apply the zero product rule.} \\
x - 4 = 0 & \text{or} & x + 3 = 0 & & \text{Set each factor equal to zero.} \\
x = 4 & \text{or} & x = -3 & & \text{Solve each equation for } x.
\end{array}$$

3. Solving Quadratic Equations

Quadratic equations, like linear equations, arise in many applications in mathematics, science, and business. The following steps summarize the factoring method for solving a quadratic equation.

Steps for Solving a Quadratic Equation by Factoring

1. Write the equation in the form: $ax^2 + bx + c = 0$.
2. Factor the equation completely.
3. Apply the zero product rule. That is, set each factor equal to zero and solve the resulting equations.

Note: The solution(s) found in step 3 may be checked by substitution into the original equation.

example 1 Solving Quadratic Equations

Solve the quadratic equations.

a. $2x^2 - 9x = 5$ b. $4x^2 + 24x = 0$ c. $5x(5x + 2) = 10x + 9$

Solution:

a. $2x^2 - 9x = 5$

$2x^2 - 9x - 5 = 0$ Write the equation in the form $ax^2 + bx + c = 0$.

$(2x + 1)(x - 5) = 0$ Factor the polynomial completely.

$2x + 1 = 0$ or $x - 5 = 0$ Set each factor equal to zero.

$2x = -1$ or $x = 5$ Solve each equation.

$x = -\dfrac{1}{2}$ or $x = 5$ There are two solutions, $x = -\frac{1}{2}$ and $x = 5$.

<u>Check</u>: $x = -\dfrac{1}{2}$

$$2x^2 - 9x = 5$$
$$2\left(-\frac{1}{2}\right)^2 - 9\left(-\frac{1}{2}\right) \stackrel{?}{=} 5$$
$$2\left(\frac{1}{4}\right) + \frac{9}{2} \stackrel{?}{=} 5$$
$$\frac{1}{2} + \frac{9}{2} \stackrel{?}{=} 5$$
$$\frac{10}{2} \stackrel{?}{=} 5 \checkmark$$

<u>Check</u>: $x = 5$

$$2x^2 - 9x = 5$$
$$2(5)^2 - 9(5) \stackrel{?}{=} 5$$
$$2(25) - 45 \stackrel{?}{=} 5$$
$$50 - 45 \stackrel{?}{=} 5 \checkmark$$

b. $4x^2 + 24x = 0$ — The equation is already in the form $ax^2 + bx + c = 0$ (Note that $c = 0$).

$4x(x + 6) = 0$ — Factor completely.

$4x = 0 \quad \text{or} \quad x + 6 = 0$ — Set each factor equal to zero.

$x = 0 \quad \text{or} \quad x = -6$ — Each solution checks in the original equation.

c. $5x(5x + 2) = 10x + 9$

$25x^2 + 10x = 10x + 9$ — Clear parentheses.

$25x^2 + 10x - 10x - 9 = 0$ — Set the equation equal to zero.

$25x^2 - 9 = 0$ — The equation is in the form $ax^2 + bx + c = 0$ (Note that $b = 0$).

$(5x - 3)(5x + 3) = 0$ — Factor completely.

$5x - 3 = 0 \quad \text{or} \quad 5x + 3 = 0$ — Set each factor equal to zero.

$5x = 3 \quad \text{or} \quad 5x = -3$ — Solve each equation.

$$\frac{5x}{5} = \frac{3}{5} \quad \text{or} \quad \frac{5x}{5} = \frac{-3}{5}$$

$$x = \frac{3}{5} \quad \text{or} \quad x = -\frac{3}{5}$$

4. Solving Higher Degree Polynomial Equations

The zero product rule can be used to solve higher degree polynomial equations provided the equations can be set to zero and written in factored form.

example 2 **Solving Higher Degree Polynomial Equations**

Solve the equations.

a. $-6(y + 3)(y - 5)(2y + 7) = 0$ b. $w^3 + 5w^2 - 9w - 45 = 0$

Solution:

a. $-6(y + 3)(y - 5)(2y + 7) = 0$ — The equation is already in factored form and equal to zero.

Set each factor equal to zero. Solve each equation for y.

$$-6 \cancel{=} 0 \quad \text{or} \quad y + 3 = 0 \quad \text{or} \quad y - 5 = 0 \quad \text{or} \quad 2y + 7 = 0$$

No solution, $\quad y = -3 \quad \text{or} \quad y = 5 \quad \text{or} \quad y = -\frac{7}{2}$

Notice that when the constant factor is set equal to zero, the result is a contradiction $-6 = 0$. The constant factor does not produce a solution to the equation. Therefore, the only solutions are $y = -3$, $y = 5$, and $y = -\frac{7}{2}$. Each solution can be checked in the original equation.

b. $w^3 + 5w^2 - 9w - 45 = 0$ — This is a higher degree polynomial equation.

$w^3 + 5w^2 \mid - 9w - 45 = 0$ — The equation is already set equal to zero. Now factor.

$$w^2(w + 5) - 9(w + 5) = 0$$
$$(w + 5)(w^2 - 9) = 0$$

Because there are four terms, try factoring by grouping.

$$(w + 5)(w - 3)(w + 3) = 0$$

$w^2 - 9$ is a difference of squares and can be factored further.

$w + 5 = 0$ or $w - 3 = 0$ or $w + 3 = 0$ — Set each factor equal to zero.

$w = -5$ or $w = 3$ or $w = -3$ — Solve each equation.

Each solution checks in the original equation.

5. Applications of Quadratic Equations

example 3 Using a Quadratic Equation in an Application

The base of a triangle is 3 m more than the height. The area is 35 m^2. Find the base and height of the triangle.

Solution:

Let x represent the height of the triangle.

Then $x + 3$ represents the base (Figure 5-1).

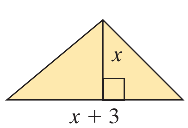


Figure 5-1

To set up an equation to solve for x, use $A = \frac{1}{2}bh$.

$\text{Area} = \dfrac{1}{2}(\text{base})(\text{height})$ — Verbal equation

$35 = \dfrac{1}{2}(x + 3)(x)$ — Algebraic equation

$2 \cdot 35 = \cancel{2} \cdot \dfrac{1}{\cancel{2}}(x + 3)(x)$ — Multiply both sides by 2 to clear fractions.

$70 = (x + 3)(x)$

$70 = x^2 + 3x$	Clear parentheses.
$0 = x^2 + 3x - 70$	Write the equation in the form $ax^2 + bx + c = 0$.
$0 = (x + 10)(x - 7)$	Factor the equation.
$x + 10 = 0 \quad \text{or} \quad x - 7 = 0$	Set each factor equal to zero.
$x \cancel{=} -10 \quad \text{or} \quad x = 7$	Because x represents the height of a triangle, reject the negative solution.

The variable x represents the height of the triangle. Therefore, the height is 7 m.

The expression $x + 3$ represents the base of the triangle. Therefore, the base is 10 m.

example 4 Using Translations to Set up a Quadratic Equation

The product of two consecutive integers is 48 more than the larger integer. Find the integers.

Solution:

Let x represent the first (smaller) integer.

Then $x + 1$ represents the second (larger) integer.	Label the variables.
(First integer)(second integer) = (second integer) + 48	Verbal model
$x(x + 1) = (x + 1) + 48$	Algebraic equation
$x^2 + x = x + 49$	Simplify.
$x^2 + x - x - 49 = 0$	Set the equation equal to zero.
$x^2 - 49 = 0$	
$(x - 7)(x + 7) = 0$	Factor.
$x - 7 = 0 \quad \text{or} \quad x + 7 = 0$	Set each factor equal to zero.
$x = 7 \quad \text{or} \quad x = -7$	Solve for x.

Recall that x represents the smaller integer. Therefore, there are two possibilities for the pairs of consecutive integers.

If $x = 7$, then the larger integer is $x + 1$ or $7 + 1 = 8$.

If $x = -7$, then the larger integer is $x + 1$ or $-7 + 1 = -6$.

The integers are 7 and 8 or -7 and -6.

Tip: To check your answer in Example 4, verify that each pair of integers satisfies the requirements that the product of integers is equal to 48 more than the larger integer:

Product	**Larger Integer + 48**
$(7)(8) = 56$	$8 + 48 = 56$
$(-7)(-6) = 42$	$-6 + 48 = 42$

example 5 Using a Quadratic Equation in an Application

A stone is dropped off a 64-ft cliff and falls into the ocean below. The height of the stone above sea level is given by the equation

$h = -16t^2 + 64$ where h is the stone's height in feet, and t is the time in seconds.

Find the time required for the stone to hit the water.

Solution:

When the stone hits the water, its height is zero. Therefore, substitute $h = 0$ into the equation.

$h = -16t^2 + 64$	The equation is quadratic.
$0 = -16t^2 + 64$	Substitute $h = 0$.
$0 = -16(t^2 - 4)$	Factor out the GCF.
$0 = -16(t - 2)(t + 2)$	Factor as a difference of squares.
$-16 \neq 0$ or $t - 2 = 0$ or $t + 2 = 0$	Set each factor to zero.
No solution, $t = 2$ or $t \neq -2$	Solve for t.

The negative value of t is rejected because the stone cannot fall for a negative time. Therefore, the stone hits the water after 2 seconds.

In Example 5, we can analyze the path of the stone as it falls from the cliff. Compute the height values at various times between 0 and 2 sec (Table 5-1 and Table 5-2). The ordered pairs can be graphed where t is used in place of x and h is used in place of y.

Table 5-1

Time, t (sec)	Height, h (ft)
0.0	
0.5	
1.0	
1.5	
2.0	

$$h = -16(0.0)^2 + 64 = 64$$
$$h = -16(0.5)^2 + 64 = 60$$
$$h = -16(1.0)^2 + 64 = 48$$
$$h = -16(1.5)^2 + 64 = 28$$
$$h = -16(2.0)^2 + 64 = 0$$

Table 5-2

Time, t (sec)	Height, h (ft)
0.0	64
0.5	60
1.0	48
1.5	28
2.0	0

The graph of the height of the stone versus time is shown in Figure 5-2. From the graph, we can verify that the stone hits the water after 2 sec.

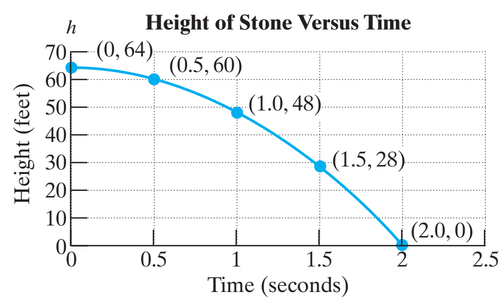


Figure 5-2

6. Pythagorean Theorem

Recall that a right triangle is a triangle that contains a 90º angle. Furthermore, the sum of the squares of the two legs (the shorter sides) of a right triangle equals the square of the hypotenuse (the longest side). This important fact is known as the Pythagorean theorem. The Pythagorean theorem is an enduring landmark of mathematical history from which many mathematical ideas have been built. Although the theorem is named after Pythagoras (sixth century B.C.E.), a Greek mathematician and philosopher, it is thought that the ancient Babylonians were familiar with the principle more than a thousand years earlier.

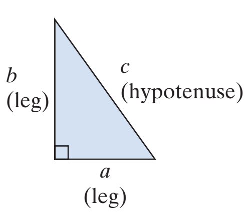


Figure 5-3

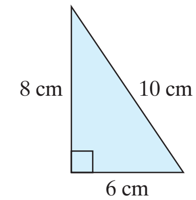
For the right triangle shown in Figure 5-3, the **Pythagorean theorem** is stated as:

$$a^2 + b^2 = c^2.$$

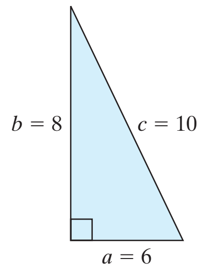
In this formula, a and b are the legs of the right triangle and c is the hypotenuse. Notice that the hypotenuse is the longest side of the right triangle and is opposite the 90º angle.

example 6 Applying the Pythagorean Theorem

Show that the lengths of the sides of the right triangle in the figure satisfy the Pythagorean theorem.



Solution:



Label the triangle.

$a^2 + b^2 = c^2$	Apply the Pythagorean theorem.
$(6)^2 + (8)^2 \stackrel{?}{=} (10)^2$	$a = 6, b = 8, c = 10$
$36 + 64 \stackrel{?}{=} 100$	
$100 = 100$ ✔	

example 7 Using a Quadratic Equation in an Application

A 13-ft board is used as a ramp to unload furniture off a loading platform. If the distance between the top of the board and the ground is 7 ft less than the distance between the bottom of the board and the base of the platform, find both distances.

Solution:

Let x represent the distance between the bottom of the board and the base of the platform. Then $x - 7$ represents the distance between the top of the board and the ground (Figure 5-4).

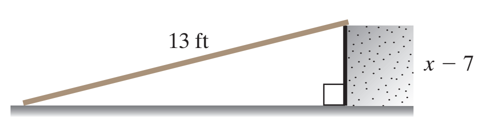


Figure 5-4

$a^2 + b^2 = c^2$	Pythagorean theorem
$x^2 + (x - 7)^2 = (13)^2$	
$x^2 + [(x)^2 - 2(x)(7) + (7)^2] = 169$	
$x^2 + x^2 - 14x + 49 = 169$	
$2x^2 - 14x + 49 = 169$	Combine *like* terms.
$2x^2 - 14x + 49 - 169 = 0$	Set the equation equal to zero.
$2x^2 - 14x - 120 = 0$	Write the equation in the form $ax^2 + bx + c = 0$.
$2(x^2 - 7x - 60) = 0$	Factor.
$2(x - 12)(x + 5) = 0$	
$2 \cancel{=} 0$ or $x - 12 = 0$ or $x + 5 = 0$	Set each factor equal to zero.
$x = 12$ or $x \cancel{=} -5$	Solve both equations for x.

Avoiding Mistakes

Recall that the square of a binomial results in a perfect square trinomial.

$(a - b)^2 = a^2 - 2ab + b^2$

Don't forget the middle term.

Recall that x represents the distance between the bottom of the board and the base of the platform. We reject the negative value of x because a distance cannot be negative. Therefore, the distance between the bottom of the board and the base of the platform is 12 ft. The distance between the top of the board and the ground is $x - 7 = 5$ ft.

section 5.7 Practice Exercises

For Exercises 1–10, factor completely.

1. $4x - 2 + 2bx - b$
2. $6a - 8 - 3ab + 4b$
3. $4b^2 - 44b + 120$
4. $8u^2v^2 - 4uv$
5. $16w^2 - 1$
6. $3x^2 + 10x - 8$
7. $12k + 16$
8. $3h^2 - 75$
9. $2y^2 + 3y - 44$
10. $4x^2 + 16y^2$

For Exercises 11–18, identify the polynomials as linear, quadratic, or neither.

11. $4 - 5x$
12. $5x^3 + 2$
13. $3x - 6x^2$
14. $1 - x + 2x^2$
15. $7x^4 + 8$
16. $3x + 2$
17. $6x^2 - 7x - 2$
18. $4x^2 - 1$

19. State the zero product rule.

For Exercises 20–27, solve the equations using the zero product rule.

20. $(x - 5)(x + 1) = 0$
21. $(x + 3)(x - 1) = 0$
22. $(3x - 2)(3x + 2) = 0$
23. $(2x - 7)(2x + 7) = 0$
24. $2(x - 7)(x - 7) = 0$
25. $3(x + 5)(x + 5) = 0$
26. $x(x - 4)(2x + 3) = 0$
27. $x(3x + 1)(x + 1) = 0$

28. For a quadratic equation of the form $ax^2 + bx + c = 0$, what must be done before applying the zero product rule?

For Exercises 29–40, solve the equations.

29. $p^2 - 2p - 15 = 0$
30. $y^2 - 7y - 8 = 0$
31. $z^2 + 10z - 24 = 0$
32. $w^2 - 10w + 16 = 0$
33. $2q^2 - 7q - 4 = 0$
34. $4x^2 - 11x - 3 = 0$
35. $0 = 9x^2 - 4$
36. $4a^2 - 49 = 0$
37. $2k^2 - 28k + 96 = 0$
38. $0 = 2t^2 + 20t + 50$
39. $0 = 2m^3 - 5m^2 - 12m$
40. $3n^3 + 4n^2 + n = 0$
41. What are the requirements to use the zero product rule to solve a quadratic equation or higher degree polynomial equation?

For Exercises 42–63, solve the equations.

42. $x^2 + 10x = 24$
43. $x^2 - 10x = -16$
44. $9d^2 = 4$
45. $4p^2 = 49$
46. $2(c^2 - 14c) = -96$
47. $2(q^2 + 10q) = -50$
48. $12x = 2x^3 - 5x^2$
49. $-x = 3x^3 + 4x^2$
50. $3(a^2 + 2a) = 2a^2 - 9$
51. $9(k - 1) = -4k^2$
52. $2n(n + 2) = 6$
53. $3p(p - 1) = 18$
54. $27q^2 = 9q$
55. $21w^2 = 14w$
56. $3(c^2 - 2c) = 0$
57. $2(4d^2 + d) = 0$
58. $y^3 - 3y^2 - 4y + 12 = 0$
59. $t^3 + 2t^2 - 16t - 32 = 0$
60. $(x - 1)(x + 2) = 18$
61. $(w + 5)(w - 3) = 20$
62. $(p + 2)(p + 3) = 1 - p$
63. $(k - 6)(k - 1) = -k - 2$
64. If 11 is added to the square of a number, the result is 60. Find all such numbers.
65. If a number is added to 2 times its square, the result is 36. Find all such numbers.
66. If 12 is added to 6 times a number, the result is 28 less than the square of the number. Find all such numbers.
67. The square of a number is equal to 20 more than the number. Find all such numbers.
68. The product of two consecutive odd integers is 63. Find all such integers.
69. The product of two consecutive even integers is 48. Find all such integers.
70. The sum of the squares of two consecutive integers is one more than 10 times the larger number. Find all such integers.
71. The sum of the squares of two consecutive integers is 9 less than 10 times the sum of the integers. Find all such integers.
72. The length of a rectangular room is 5 yd more than the width. If 300 yd^2 of carpeting cover the room, what are the dimensions of the room?

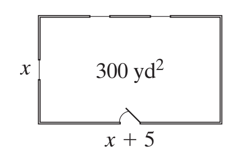


Figure for Exercise 72

73. The width of a rectangular painting is 2 in. less than the length. The area is 120 $in.^2$ Find the length and width.

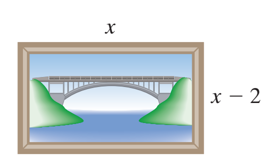


Figure for Exercise 73

74. The width of a rectangular slab of concrete is 3 m less than the length. If the area is 28 m^2,
    a. What are the dimensions of the rectangle?
    b. What is the perimeter of the rectangle?
75. The width of a rectangular picture is 7 in. less than the length. If the area of the picture is 78 $in.^2$,
    a. What are the dimensions of the rectangle?
    b. What is the perimeter of the rectangle?

76. The base of a triangle is 1 ft less than twice the height. The area is 14 ft^2. Find the base and height of the triangle.

77. The height of a triangle is 5 cm less than 3 times the base. The area is 125 cm^2. Find the base and height of the triangle.

78. In a physics experiment, a ball is dropped off a 144-ft platform. The height of the ball above the ground is given by the equation

$h = -16t^2 + 144$ where h is the ball's height in feet and t is the time in seconds after the ball is dropped ($t \geq 0$).

Find the time required for the ball to hit the ground. (*Hint*: Let $h = 0$)

79. A stone is dropped off a 64-ft cliff. The height of the stone above the ground is given by the equation

$h = -16t^2 + 64$ where h is the stone's height in feet, and t is the time in seconds after the stone is dropped ($t \geq 0$).

Find the time required for the stone to hit the ground.

80. An object is shot straight up into the air from ground level with initial speed of 24 ft/sec. The height of the object (in feet) is given by the equation

$h = -16t^2 + 24t$ where t is the time in seconds after launch ($t \geq 0$).

Find the time(s) when the object is at ground level.

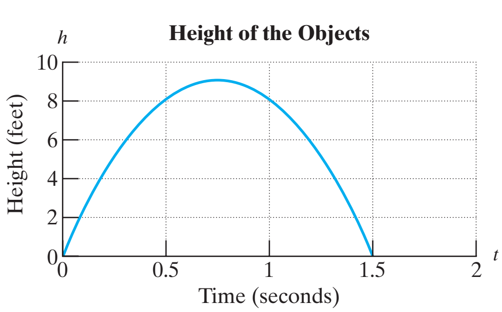


Figure for Exercise 80

81. A rocket is launched straight up into the air from the ground with initial speed of 64 ft/sec. The height of the rocket (in feet) is given by the equation

$h = -16t^2 + 64t$ where t is the time in seconds after launch ($t \geq 0$).

Find the time(s) when the rocket is at ground level.

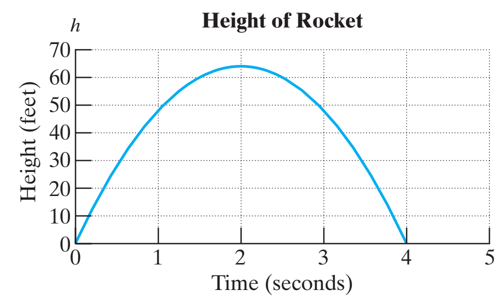


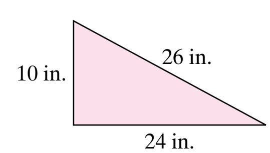
Figure for Exercise 81

82. Draw a right triangle and label the sides with the words *leg* and *hypotenuse.*

83. State the Pythagorean theorem.

For Exercises 84–87, use the Pythagorean theorem to determine whether the triangle could be a right triangle.

84.



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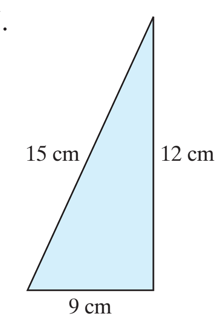

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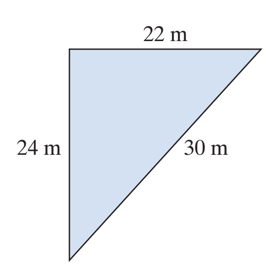
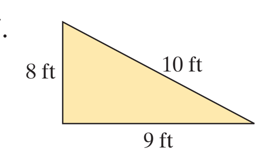


87.



88. Darcy holds the end of a kite string 3 ft (1 yd) off the ground and wants to estimate the height of the kite. Her friend Jenna is 24 yd away from her, standing directly under the kite as shown in

the figure. If Darcy has 30 yd of string out, find the height of the kite (ignore the sag in the string).

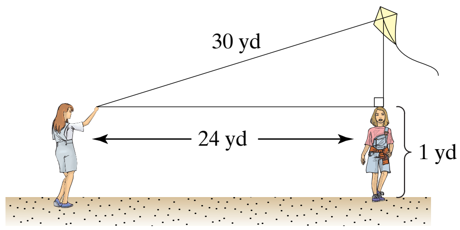


Figure for Exercise 88

89. A 17-ft ladder rests against the side of a house. The distance between the top of the ladder and the ground is 7 ft more than the distance between the base of the ladder and the bottom of the house. Find both distances.

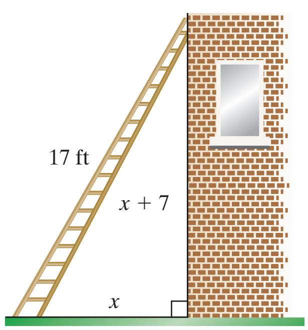


Figure for Exercise 89

90. Two boats leave a marina. One travels east, and the second travels south. After 30 min, the second boat has traveled 1 mile farther than the first boat and the distance between the boats is 5 miles. Find the distance each boat traveled.

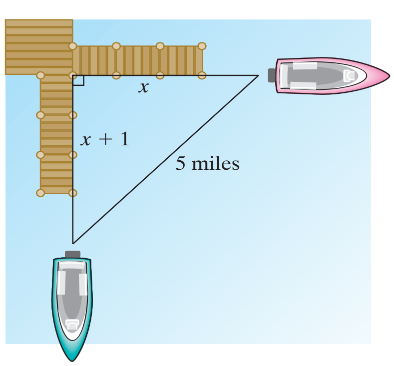


Figure for Exercise 90

91. One leg of a right triangle is 4 m less than the hypotenuse. The other leg is 2 m less than the hypotenuse. Find the length of the hypotenuse.

92. The longer leg of a right triangle is 1 cm less than twice the shorter leg. The hypotenuse is 1 cm greater than twice the shorter leg. Find the length of the shorter leg.

Expanding Your Skills

93. The formula

$$N = \frac{x(x - 3)}{2}$$

gives the number of diagonals, N, for a polygon with x sides, where $x \geq 3$.

a. Find the number of diagonals for a four-sided polygon.

b. Find the number of diagonals for a five-sided polygon.

c. Find the number of sides of a polygon if the polygon has 35 diagonals.

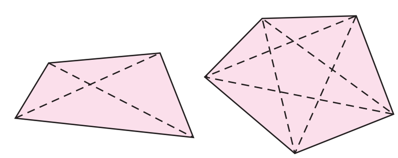


Figure for Exercise 93

94. A cardboard box is to be constructed from a square piece of cardboard by cutting out 2-in. squares from the corners and folding up the sides. If the volume of the box is 128 in.^{3}, find the original dimensions of the square piece of cardboard.

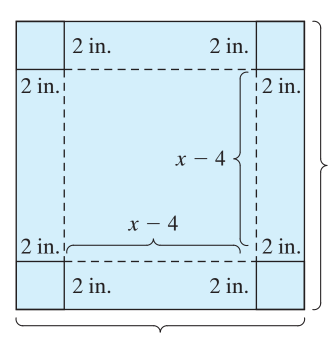


Figure for Exercise 94

For Exercises 95–98, solve the equations.

95. $(a + 2)^2 - 4(a + 2) - 21 = 0$

96. $(t - 3)^2 + 8(t - 3) + 15 = 0$

97. $2(w - 1)^2 - 7(w - 1) - 4 = 0$

98. $3(p + 4)^2 - (p + 4) - 4 = 0$

chapter 5 SUMMARY

SECTION 5.1—GREATEST COMMON FACTOR AND FACTORING BY GROUPING

KEY CONCEPTS:

The greatest common factor (GCF) is the greatest factor common to all terms of a polynomial. To factor out the GCF from a polynomial, use the distributive property.

A four-term polynomial may be factorable by grouping.

Steps to Factoring by Grouping

1. Identify and factor out the GCF from all four terms.
2. Factor out the GCF from the first pair of terms. Factor out the GCF or its opposite from the second pair of terms.
3. If the two terms share a common binomial factor, factor out the binomial factor.

KEY TERMS:

factoring
factoring by grouping
greatest common factor (GCF)
prime factorization

EXAMPLES:

Factor out the GCF:

$$\begin{aligned}&3x^2(a + b) - 6x(a + b)\\&\quad = 3x(a + b)x - 3x(a + b)(2)\\&\quad = 3x(a + b)(x - 2)\end{aligned}$$

Factor by Grouping:

$$\begin{aligned}&60xa - 30xb - 80ya + 40yb\\&\quad = 10[6xa - 3xb - 8ya + 4yb]\\&\quad = 10[3x(2a - b) - 4y(2a - b)]\\&\quad = 10[(2a - b)(3x - 4y)]\\&\quad = 10(2a - b)(3x - 4y)\end{aligned}$$

SECTION 5.2—FACTORING TRINOMIALS: GROUPING METHOD

KEY CONCEPTS:

Grouping Method for Factoring Trinomials of the Form $ax^2 + bx + c$ (where $a \neq 0$)

1. Factor out the GCF from all terms.
2. Find the product ac.
3. Find two integers whose product is ac and whose sum is b. (If no pair of integers can be found, then the trinomial is prime.)

EXAMPLES:

Factor: $10y^2 + 35y - 20$
$= 5(2y^2 + 7y - 4)$

Note: $ac = (2)(-4) = -8$
$b = 7$

4. Rewrite the middle term (bx) as the sum of two terms whose coefficients are the numbers found in step 3.
5. Factor the polynomial by grouping.

Key Term:

prime polynomial

Find two integers whose product is -8 and whose sum is 7. The numbers are 8 and -1.

$$\begin{aligned} &5[2y^2 + 8y - 1y - 4] \\ &= 5[2y(y + 4) - 1(y + 4)] \\ &= 5(y + 4)(2y - 1) \end{aligned}$$

Section 5.3—Factoring Trinomials: Trial-and-Error Method

Key Concepts:

Trial-and-Error Method for Factoring Trinomials in the Form $ax^2 + bx + c$

1. Factor out the GCF from all terms.
2. List the pairs of factors of a and the pairs of factors of c. Consider the reverse order in either list.
3. Construct two binomials of the form

$$(\overbrace{\square x \quad \square)(\square}^{\text{Factors of } a} x \quad \square)$$

Factors of c

4. Test each combination of factors and signs until the product forms the correct trinomial.
5. If no combination of factors produces the correct product, then the trinomial is prime.

Key Term:

prime polynomial

Examples:

Factor: $10y^2 + 35y - 20$
$= 5(2y^2 + 7y - 4)$

The pairs of factors of 2 are: $2 \cdot 1$
The pairs of factors of -4 are:

$$\begin{array}{ll} -1 \cdot 4 & 1(-4) \\ -2 \cdot 2 & 2(-2) \\ -4 \cdot 1 & 4(-1) \end{array}$$

$$\begin{array}{ll} (2y - 2)(y + 2) = 2y^2 + 2y - 4 & \text{No} \\ (2y - 4)(y + 1) = 2y^2 - 2y - 4 & \text{No} \\ (2y + 1)(y - 4) = 2y^2 - 7y - 4 & \text{No} \\ (2y + 2)(y - 2) = 2y^2 - 2y - 4 & \text{No} \\ (2y + 4)(y - 1) = 2y^2 + 2y - 4 & \text{No} \\ (2y - 1)(y + 4) = 2y^2 + 7y - 4 & \text{Yes} \end{array}$$

The complete factorization is $5(2y - 1)(y + 4)$.

Section 5.4—Factoring Perfect Square Trinomials and the Difference of Squares

Key Concepts:

The factored form of a perfect square trinomial is the square of a binomial:

$$a^2 + 2ab + b^2 = (a + b)^2$$

$$a^2 - 2ab + b^2 = (a - b)^2$$

Difference of Squares

$$a^2 - b^2 = (a + b)(a - b)$$

Examples:

Factor: $9w^2 - 30wz + 25z^2$
$= (3w)^2 - 2(3w)(5z) + (5z)^2$

$= (3w - 5z)^2$

Factor: $25z^2 - 4y^2$
$= (5z + 2y)(5z - 2y)$

Section 5.5—Factoring the Sum and Difference of Cubes

Key Concepts:

Factoring a Difference of Cubes

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Factoring a Sum of Cubes

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

Key Terms:

difference of cubes
sum of cubes

Examples:

Factor: $m^3 - 64$
$= (m)^3 - (4)^3$

This is a difference of cubes: $a = m$ and $b = 4$. Apply the formula:

$$m^3 - 64 = (m)^3 - (4)^3 = (m - 4)(m^2 + 4m + 16)$$

Factor: $x^6 + 8y^3$
$= (x^2)^3 + (2y)^3$

This is a sum of cubes: $a = x^2$ and $b = 2y$. Apply the formula:

$$x^6 + 8y^3 = (x^2)^3 + (2y)^3 = (x^2 + 2y)(x^4 - 2x^2y + 4y^2)$$

Section 5.6—General Factoring Summary

Key Concepts:

Factoring Strategy

1. Factor out the greatest common factor, GCF. (Section 5.1)
2. Identify whether the polynomial has two terms, three terms, or more than three terms.
3. If the polynomial has two terms, determine if it fits the pattern for a difference of squares, difference of cubes, or sum of cubes. (Sections 5.4 or 5.5)
4. If the polynomial has three terms, check first for a perfect square trinomial. (Section 5.4) Otherwise, factor the trinomial with the grouping method or the trial-and-error method. (Sections 5.2 or 5.3)
5. If the polynomial has more than three terms, try factoring by grouping. (Sections 5.1 or 5.6)
6. Be sure to factor the polynomial completely.
7. Check by multiplying.

Examples:

Part I: Factoring using the factoring strategy.

1. $9x^2 - 4x + 9x^3$

$= x(9x - 4 + 9x^2)$ Factor out the GCF.

$= x(9x^2 + 9x - 4)$ Descending order.

$= x(3x + 4)(3x - 1)$ Factor the trinomial.

Part II: Factoring with multiple techniques.

2. $4a^2 - 12ab + 9b^2 - c^2$

$= 4a^2 - 12ab + 9b^2 \mid - c^2$ Group 3 by 1.

$= (2a - 3b)^2 - c^2$ Perfect square trinomial.

$= (2a - 3b - c)(2a - 3b + c)$ Difference of squares.

Key Term:

factoring strategy

Factor completely.

3. $3w^6 - 192$

$= 3(w^6 - 64)$ Factor out the GCF.

$= 3[(w^3)^2 - (8)^2]$ Difference of squares.

$= 3(w^3 - 8)(w^3 + 8)$ Sum/difference of cubes.

$= 3(w - 2)(w^2 + 2w + 4)(w + 2)(w^2 - 2w + 4)$

Section 5.7—Solving Quadratic Equations Using the Zero Product Rule

Key Concepts:

An equation of the form: $ax^2 + bx + c = 0$, where $a \neq 0$ is a quadratic equation.

The zero product rule states that if $ab = 0$, then either $a = 0$ or $b = 0$. The zero product rule can be used to solve a quadratic equation or a higher degree polynomial equation that is factored and set to zero.

Key Terms:

Pythagorean theorem
quadratic equation
zero product rule

Examples:

The equation $2x^2 - 17x + 30 = 0$ is a *quadratic equation*.

Solve: $3w(w - 4)(2w + 1) = 0$

$3w = 0$ or $w - 4 = 0$ or $2w + 1 = 0$

$w = 0$ or $w = 4$ or $w = -\frac{1}{2}$

Solve: $4x^2 = 34x - 60$

$4x^2 - 34x + 60 = 0$
$2(2x^2 - 17x + 30) = 0$
$2(2x - 5)(x - 6) = 0$

$2 \cancel{=} 0$ or $2x - 5 = 0$ or $x - 6 = 0$

$x = \frac{5}{2}$ or $x = 6$

chapter 5 REVIEW EXERCISES

Section 5.1

For Exercises 1–6, identify the greatest common factor between each pair of terms.

1. 24, 18
2. $16x^2, 20x^3$
3. $15a^2b^4, 22ab^5$
4. $3(x + 5), x(x + 5)$
5. $2c^3(3c - 5), 4c(3c - 5)$
6. $-2wyz, -4xyz$

For Exercises 7–14, factor out the greatest common factor.

7. $6x^2 + 2x^3 - 8x$
8. $11w^3 - 44w^2$
9. $32y^2 - 48$
10. $5a^3 + 9a^2 + 2a$

11. $-t^2 + 5t$

12. $-6u^2 - u$

13. $3b(b + 2) - 7(b + 2)$

14. $2(5x + 9) + 8x(5x + 9)$

For Exercises 15–20, factor by grouping.

15. $7w^2 + 14w + wb + 2b$

16. $b^2 - 2b + 5b - 10$

17. $x^2 - 6x - 4x + 24$

18. $18p^2 + 12pq - 3p - 2q$

19. $60y^2 - 45y - 12y + 9$

20. $6a - 3a^2 - 2ab + a^2b$

Section 5.2

For Exercises 21–26, find a pair of integers whose product and sum are given.

21. Product: -6 sum: -5

22. Product: 12 sum: 13

23. Product: 24 sum: 11

24. Product: -60 sum: 17

25. Product: -5 sum: 4

26. Product: 15 sum: -8

For Exercises 27–40, factor the trinomial using the grouping method.

27. $3c^2 - 5c - 2$

28. $4y^2 + 13y + 3$

29. $2t^2 + 11st + 12s^2$

30. $4x^3 + 17x^2 - 15x$

31. $w^3 + 4w^2 - 5w$

32. $p^2 - 8pq + 15q^2$

33. $40v^2 + 22v - 6$

34. $40s^2 + 30s - 100$

35. $x^2 + 9x - 22$

36. $y^2 - 9y + 8$

37. $a^3b - 10a^2b^2 + 24ab^3$

38. $2z^6 + 8z^5 - 42z^4$

39. $3m + 9m^2 - 2$

40. $10 + 6p^2 + 19p$

Section 5.3

For Exercises 41–44, let a, b, and c represent positive integers.

41. When factoring a polynomial of the form $ax^2 - bx - c$, should the signs in the binomials be both positive, both negative, or different?

42. When factoring a polynomial of the form $ax^2 - bx + c$, should the signs in the binomials be both positive, both negative, or different?

43. When factoring a polynomial of the form $ax^2 + bx + c$, should the signs in the binomials be both positive, both negative, or different?

44. When factoring a polynomial of the form $ax^2 + bx - c$, should the signs in the binomials be both positive, both negative, or different?

For Exercises 45–58, factor the trinomial using the trial-and-error method.

45. $2y^2 - 5y - 12$

46. $4w^2 - 5w - 6$

47. $2p^2 - 4p - 48$

48. $3c^2 + 18c - 21$

49. $10z^2 + 29z + 10$

50. $8z^2 + 6z - 9$

51. $2p^2 - 5p + 1$

52. $5r^2 - 3r + 7$

53. $10w^2 - 60w - 270$

54. $3y^2 - 18y - 48$

55. $9c^2 - 30cd + 25d^2$

56. $121m^2 + 154mn + 49n^2$

57. $v^4 - 2v^2 - 3$

58. $x^4 + 7x^2 + 10$

Section 5.4

For Exercises 59–64, determine if the trinomial is a perfect square trinomial. If it is, factor the trinomial. If the trinomial is not a perfect square trinomial, explain why.

59. $4x^2 - 20x + 25$

60. $y^2 + 12y + 36$

61. $c^2 - 6c + 9$

62. $9b^2 + 6b + 1$

63. $t^2 + 8t + 49$

64. $k^2 - 10k + 64$

For Exercises 65–72, determine if the binomial is a difference of two squares. If it is, factor the binomial. If the binomial is not a difference of squares, explain why.

65. $a^2 - 49$

66. $d^2 - 64$

67. $h - 25$
68. $c - 9$
69. $100 - 81t^2$
70. $4 - 25k^2$
71. $x^2 + 16$
72. $y^2 + 121$

For Exercises 73–78, factor completely.

73. $2c^4 - 18$
74. $72x^2 - 2y^2$
75. $8x^2 + 24x + 18$
76. $48t^2 - 24t + 3$
77. $p^3 + 3p^2 - 16p - 48$
78. $4k - 8 - k^3 + 2k^2$

Section 5.5

79. Write the formula for factoring the sum of cubes: $a^3 + b^3$

80. Write the formula for factoring the difference of cubes: $a^3 - b^3$

For Exercises 81–88, factor the sums and differences of cubes.

81. $z^3 - w^3$
82. $r^3 + s^3$
83. $64 + a^3$
84. $125 - b^3$
85. $p^6 + 8$
86. $q^6 - \frac{1}{27}$
87. $6x^3 - 48$
88. $7y^3 + 7$

Match the polynomials in Exercises 89–91 with its factored form.

89. $a^2 - b^2$
90. $a^3 - b^3$
91. $a^3 + b^3$

i. $(a - b)^3$
ii. $(a - b)(a^2 + ab + b^2)$
iii. $(a + b)(a^2 - ab + b^2)$
iv. $(a + b)^3$
v. $(a - b)(a + b)$

For Exercises 92–97, factor the binomials completely.

92. $y^2 - 81$
93. $216w^3 - 1$
94. $4a^2 + b^2$
95. $128 + 2v^6$
96. $p^6 + 1$
97. $q^6 - 1$

For Exercises 98–107, factor completely.

98. $6y^2 - 11y - 2$
99. $3p^2 - 6p + 3$
100. $x^3 - 36x$
101. $k^2 - 13k + 42$
102. $7ac - 14ad - bc + 2bd$
103. $q^4 - 64q$
104. $8h^2 + 20$
105. $2t^2 + t + 3$
106. $m^2 - 8m$
107. $x^3 + 4x^2 - x - 4$

Section 5.6

For Exercises 108–121, factor completely using the factoring strategy found on page 347.

108. $12s^3t - 45s^2t^2 - 12st^3$
109. $5p^4q - 20q^3$
110. $4d^2(3 + d) - (3 + d)$
111. $(y - 4)^3 + 4(y - 4)^2$
112. $49x^2 + 36 - 84x$
113. $80z + 32 + 50z^2$
114. $18a^2 + 39a - 15$
115. $w^4 + w^3 - 56w^2$
116. $8n + n^4$
117. $14m^3 - 14$
118. $b^2 + 16b + 64 - 25c^2$
119. $a^2 - 6a + 9 - 16x^2$
120. $(9w + 2)^2 + 4(9w + 2) - 5$
121. $(4x + 3)^2 - 12(4x + 3) + 36$

Section 5.7

122. For which of the following equations can the zero product rule be applied directly? Explain.

$(x - 3)(2x + 1) = 0$ or
$(x - 3)(2x + 1) = 6$

For Exercises 123–136, solve the equation using the zero product rule.

123. $(4x - 1)(3x + 2) = 0$
124. $(a - 9)(2a - 1) = 0$
125. $3w(w + 3)(5w + 2) = 0$

126. $6u(u - 7)(4u - 9) = 0$

127. $7k^2 - 9k - 10 = 0$

128. $4h^2 - 23h - 6 = 0$

129. $q^2 - 144 = 0$

130. $r^2 = 25$

131. $5v^2 - v = 0$

132. $x(x - 6) = -8$

133. $36t^2 + 60t = -25$

134. $9s^2 + 12s = -4$

135. $3(y^2 + 4) = 20y$

136. $2(p^2 - 66) = -13p$

137. The base of a parallelogram is 1 ft more than twice the height. If the area is 78 ft^2 what are the base and height of the parallelogram?

138. A ball is tossed into the air from ground level with initial speed of 16 ft/sec. The height of the ball is given by the equation

$h = -16x^2 + 16x \quad (x \geq 0)$ where h is the ball's height in feet and x is the time in seconds

Find the time(s) when the ball is at ground level.

139. Using the Pythagorean theorem determine whether the triangle could be a right triangle.

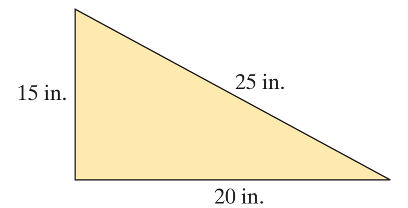


Figure for Exercise 139

140. A right triangle has one leg that is 2 ft more than the other leg. The hypotenuse is 2 ft less than twice the shorter leg. Find the lengths of all sides of the triangle.

141. If the square of a number is subtracted from 60, the result is -4. Find all such numbers.

142. The product of two consecutive integers is 44 more than 14 times their sum.

143. The base of a triangle is 1 m more than twice the height. If the area of the triangle is 18 m^2, find the base and height.

chapter 5 TEST

1. Factor out the GCF: $15x^4 - 3x + 6x^3$

2. Factor by grouping: $7a - 35 - a^2 + 5a$

3. Factor the trinomial: $6w^2 - 43w + 7$

4. Factor the difference of squares: $169 - p^2$

5. Factor the perfect square trinomial: $q^2 - 16q + 64$

6. Factor the sum of cubes: $8 + t^3$

For Exercises 7–16, factor completely.

7. $3a^2 + 27ab + 54b^2$

8. $c^4 - 1$

9. $xy - 7x + 3y - 21$

10. $49 + p^2$

11. $-10u^2 + 30u - 20$

12. $12t^2 - 75$

13. $5y^2 - 50y + 125$

14. $21q^2 + 14q$

15. $2x^3 + x^2 - 8x - 4$

16. $y^3 - 125$

17. $x^2 + 8x + 16 - y^2$

18. $r^6 - 256r^2$

19. $12a - 6ac + 2b - bc$

For Exercises 20–23, solve the equation.

20. $(2x - 3)(x + 5) = 0$

21. $x^2 - 7x = 0$

22. $x^2 - 6x = 16$

23. $x(5x + 4) = 1$

24. A tennis court has an area of 312 yd^2. If the length is 2 yd more than twice the width, find the dimensions of the court.

25. The hypotenuse of a right triangle is 2 ft less than three times the shorter leg. The longer leg is 3 ft less than three times the shorter leg. Find the length of the shorter leg.

CUMULATIVE REVIEW EXERCISES, CHAPTERS 1–5

For Exercises 1–2, simplify completely.

1. $\dfrac{|4 - 25 \div (-5) \cdot 2|}{\sqrt{8^2 + 6^2}}$

2. $-\dfrac{2}{3} - \dfrac{1}{3}(3^2 + \sqrt{81})$

3. Solve for x: $-3.5 - 2.5x = 1.5(x - 3)$

4. Solve for t: $5 - 2(t + 4) = 3t + 12$

5. Solve for y: $3x - 2y = 8$

6. The circumference of a circular fountain is 50 ft. Find the radius of the fountain. Round to the nearest tenth of a foot.

7. A child's piggy bank has \$3.80 in quarters, dimes, and nickels. The number of nickels is two more than the number of quarters. The number of dimes is three less than the number of quarters. Find the number of each type of coin in the bank.

8. Solve the inequality. Graph the solution and write the solution set in interval notation.

$$-\frac{5}{12}x \le \frac{5}{3}$$

For Exercises 9–11, perform the indicated operations.

9. $2\left(\dfrac{1}{3}y^3 - \dfrac{3}{2}y^2 - 7\right) - \left(\dfrac{2}{3}y^3 + \dfrac{1}{2}y^2 + 5y\right)$

10. $(4p^2 - 5p - 1)(2p - 3)$

11. $(2w - 7)^2$

12. Divide using long division: $(r^4 + 2r^3 - 5r + 1) \div (r - 3)$

For Exercises 13–15, simplify the expressions. Write the final answer using positive exponents only.

13. $\dfrac{c^{12}c^{-5}}{c^3}$

14. $\left(\dfrac{2a^2b^{-3}}{c}\right)^{-2}$

15. $\left(\dfrac{1}{2}\right)^0 - \left(\dfrac{1}{4}\right)^{-2}$

16. Divide. Write the final answer in scientific notation: $\dfrac{8.0 \times 10^{-3}}{5.0 \times 10^{-6}}$

For Exercises 17–24, factor completely.

17. $w^4 - 16$

18. $2ax + 10bx - 3ya - 15yb$

19. $4a^2 - 12a + 9$

20. $4x^2 - 8x - 5$

21. $y^3 - 27$

22. $p^6 + q^6$

23. $(a - 2)^2 + 5(a - 2) + 6$

24. $x^2 - (z^2 + 2z + 1)$

For Exercises 25–26, solve the equation.

25. $4x(2x - 1)(x + 5) = 0$

26. $x(x + 2) = 35$