CHAPTER 2

Review of Signals and Linear Systems

Communication systems transfer information using signals. **Signals** are functions of time that convey information from the transmitter to the receiver at the other end of the transmission medium. In electrical communication systems, signals take the form of electromagnetic waves that can be transmitted over wired or wireless media. Examples of wired media include twisted wire pair, coaxial cable, and optical fiber in which the signal energy is contained and guided within the medium. In wireless media, on the other hand, the signal energy propagates in the form of unguided electromagnetic waves. Radio, microwave, and infrared are examples of wireless media.

A **system** is an interconnection of devices and subsystems chosen to perform a desired function on signals. In this chapter, we review representative signal types and system models frequently encountered in modern communication systems. We next consider frequency domain representation of signals and linear, time-invariant systems. Many practical communication subsystems and channels can be closely modeled by this important subclass of systems. Thus it is useful to understand the effect of linear, time-invariant systems on transmission of signals.

The chapter is organized into the following sections:

2.11 FOURIER TRANSFORMS FOR DISCRETE-TIME SIGNALS.

We introduce two alternative Fourier transform representations for discrete-time signals that lead to efficient computational algorithms.

The chapter concludes with final remarks and a selected list of readings.

2.1 BASIC SIGNAL CONCEPTS

A signal $x(t)$ is called a **continuous-time (CT) signal** if it is defined for every instant of time t in the range $-\infty$ to ∞. On the other hand, a **discrete-time (DT) signal** is defined for discrete instants of time, and hence it is a sequence of numbers, called **samples.** It is denoted by $\{x[n],\ n = \text{integer in the range } -\infty \text{ to } \infty\}$. In this book, we will use the notation $x[n]$ to denote the entire sequence as well as the nth sample or number in the sequence. The intended meaning will be obvious from the context. It is important to recognize that the sequence $x[n]$ is defined only for integer values of n. Figure 2.1(a) and (b) display examples of CT and DT signals.

Analog and Digital Signals

A continuous-time signal that assumes a continuum of amplitude values between given maximum and minimum is called an **analog signal.** Most signals we encounter in the real world are analog in nature. Examples include speech, music, image, and video signals. **Digital signals,** on the other hand, can change values at discrete instants of time, assuming one of a finite number of amplitude levels. Figure 2.1(c) shows examples of binary and quaternary digital signals.

So far we have considered real signals for which the amplitude of the signal takes its values from the set of real numbers, that is, $x(t) \in \mathbb{R}, -\infty < t < \infty$. A **complex signal,** on the other hand, takes its values from the set of complex numbers, that is, $x(t) \in \mathbb{C}$, $-\infty < t < \infty$. Complex signals are used to model signals that convey information in both amplitude and phase.

Deterministic and Random Signals

A **deterministic signal** $x(t)$ is completely specified for each value of time t—that is, its amplitude is known either graphically or analytically for all values of t. An example is a simple sinusoidal waveform $\sin(4\pi t)$, which is displayed in Figure 2.2. On the

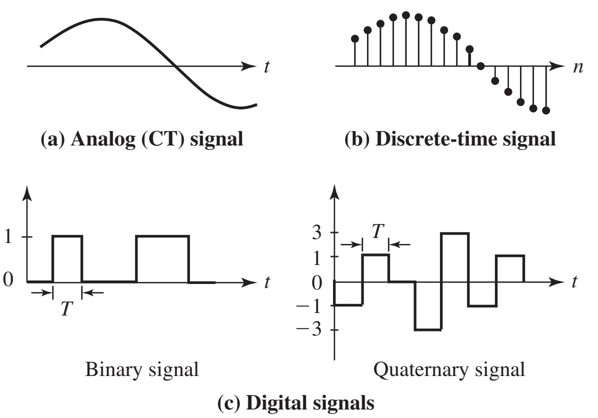


Figure 2.1 Examples of signals.

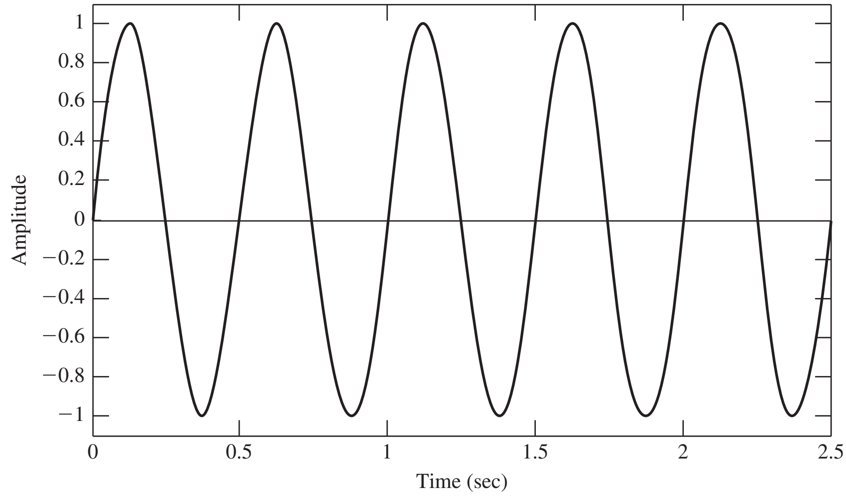


Figure 2.2 Example of a deterministic signal: sine wave.

other hand, a **random signal** is not precisely known for each value of t—it can only be specified in terms of probabilities. This is a very important class of signals that includes noise signals and all information-carrying signals, such as speech and data signals. The variations of these signals are extremely complex, and we have only partial specifications available. Figure 2.3 shows an example of a random signal.

Periodic and Aperiodic Signals

A CT signal $x_p(t)$ is **periodic** with period T if and only if

$$x_p(t) = x_p(t + kT), \ -\infty < t < \infty \quad T > 0 \tag{2.1}$$

where k is any integer. From equation (2.1) it is obvious that $x_p(t)$ repeats its values at integer multiples of its period T. The minimum value of the period $T > 0$ that satisfies (2.1) is called the **fundamental period** of the signal and is denoted as T_o. A signal not

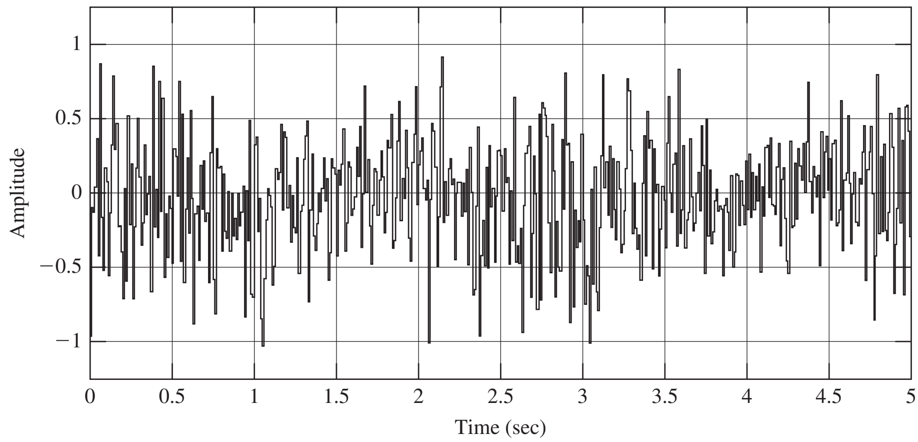


Figure 2.3 Example of a random signal.

satisfying the periodicity condition (2.1) is called an **aperiodic signal.** The sinusoidal waveform displayed in Figure 2.2 is an example of a continuous-time periodic signal with fundamental period $T_o = 0.5$ sec.

A discrete-time signal (sequence) $x_p[n]$ satisfying

$$x_p[n] = x_p[n + kN], \; N > 0 \tag{2.2}$$

is called a **periodic sequence** with period N where k is any integer. The smallest value of the period N that satisfies (2.2) is called the fundamental period N_o of the sequence. Figure 2.4 shows an example of a discrete-time periodic signal.

A sequence not satisfying the periodicity condition (2.2) is called an **aperiodic sequence.**

2.1.1 Some Useful Basic Signals

In the study of communication systems, certain signal types occur recurrently. These signals are defined next.

The Sinusoidal Signal

The most common real-valued signal is the sinusoidal waveform

$$x(t) = A\cos(2\pi f_o t + \phi) \tag{2.3}$$

where A, f_o, and ϕ are its amplitude, frequency, and phase, respectively. A sinusoidal signal $x(t) = 5\sin(4\pi t + \pi/4)$ is shown in Figure 2.5. Sinusoidal signals are important

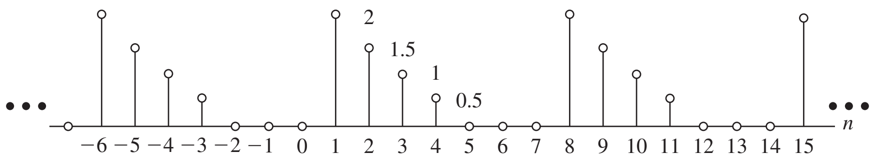


Figure 2.4 Discrete-time periodic signal with fundamental period $N_o = 7$.

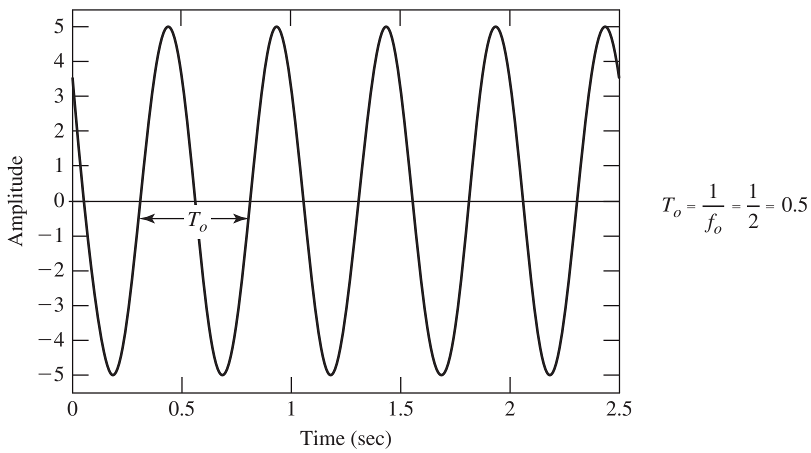


Figure 2.5 Sinusoidal signal where $A = 5$, $f_o = 2$, and $\phi = \pi/4$.

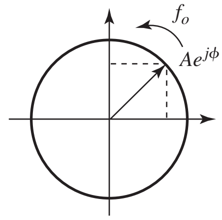


Figure 2.6 Interpretation of a complex exponential signal as a rotating phasor.

because they can be used to synthesize many waveforms. An arbitrary signal defined over a finite interval can be expressed as a sum of sinusoidal signals with different frequencies, amplitudes, and phases. Pure musical notes are essentially sinusoidal signals at different frequencies.

The Complex Exponential Signal

The complex exponential signal is defined by

$$x(t) \triangleq Ae^{j(2\pi f_o t + \phi)} \tag{2.4}$$

where A, f_o, and ϕ are again amplitude, frequency, and phase, respectively. $Ae^{j\phi}$ is known as the signal's complex amplitude or **phasor.** A complex exponential signal can be interpreted as a rotating phasor as illustrated in Figure 2.6. The frequency f_o of the complex exponential signal corresponds to the number of times the phasor rotates per second. Its horizontal and vertical projections at any time correspond to the real and imaginary parts of $x(t)$, respectively. It should be noted that the sinusoidal signal in equation (2.3) is a real part of the complex exponential signal.

The Unit Step Signal

This signal is defined as

$$u(t) \triangleq \begin{cases} 1, & t \geq 0 \\ 0, & \text{otherwise} \end{cases} \tag{2.5}$$

The corresponding signal in the discrete-time domain is called a **unit step sequence.** It is defined as

$$u[n] \triangleq \begin{cases} 1, & n \geq 0 \\ 0, & \text{otherwise} \end{cases} \tag{2.6}$$

Figure 2.7 displays the unit step signal $u(t)$ and its discrete-time version $u[n]$.

A signal $x(t)$ is called **causal** if $x(t) = 0$ for all $t < 0$. Otherwise, the signal is called **noncausal.** For a noncausal signal $x(t)$, we can generate a causal version of it by multiplying it with $u(t)$. That is,

$$x(t)u(t) = \begin{cases} x(t), & t \geq 0 \\ 0, & \text{otherwise} \end{cases} \tag{2.7}$$

Example 2.1

The sinusoidal signal $x(t) = A\cos(2\pi f_c t + \phi)$ is a noncausal signal. However, $A\cos(2\pi f_c t + \phi)u(t)$ is a causal signal.

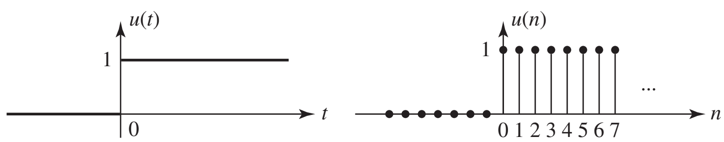


Figure 2.7 Continuous- and discrete-time versions of the unit step signal.

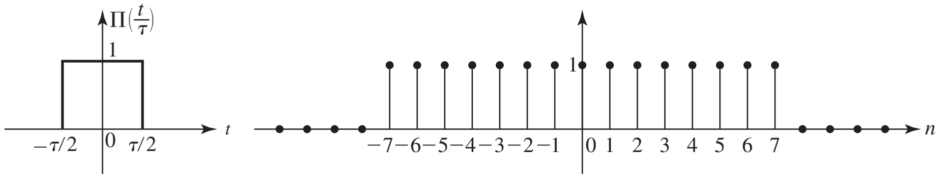


Figure 2.8 Rectangular pulse and corresponding discrete-time sequence.

The Rectangular Pulse

The rectangular pulse $\Pi\left(\frac{t}{\tau}\right)$ is a pulse of unit amplitude and width τ centered at $t = 0$.

$$\Pi\left(\frac{t}{\tau}\right) \triangleq \begin{cases} 1, & -\tau/2 \le t \le \tau/2 \\ 0, & \text{otherwise} \end{cases} \tag{2.8}$$

Figure 2.8 displays the rectangular pulse $\Pi\left(\frac{t}{\tau}\right)$ and a discrete-time version of it.

The triangular pulse $\Lambda\left(\frac{t}{\tau}\right)$ is defined by

$$\Lambda\left(\frac{t}{\tau}\right) \triangleq \begin{cases} 1 + \dfrac{2t}{\tau}, & -\tau/2 \le t \le 0 \\ 1 - \dfrac{2t}{\tau}, & 0 \le t \le +\tau/2 \\ 0, & \text{otherwise} \end{cases} \tag{2.9}$$

Figure 2.9 displays the triangular pulse $\Lambda\left(\frac{t}{\tau}\right)$.

The Impulse Signal

The impulse signal $\delta(t)$ is defined by the equations

$$\int_{-\infty}^{\infty} \delta(t)dt = 1 \tag{2.10}$$

and

$$\delta(t) = 0, \ t \neq 0$$

Thus the impulse signal $\delta(t)$ is zero everywhere except at the origin, and it has unit area or **weight.** Note that the impulse signal is defined by its properties rather than its values. We depict the impulse signal as a vertical arrow as illustrated in Figure 2.10(a) where the number beside the arrow indicates its weight. In mathematics, $\delta(t)$ is referred to as Dirac delta function or functional. The impulse signal can be viewed as a narrow pulse with large amplitude and having a unit area. In the limit, as the width of the pulse approaches zero, its amplitude increases such that the area of the pulse remains unity. Figure 2.10(b) shows the impulse signal as a limit of the narrowing rectangular pulse.

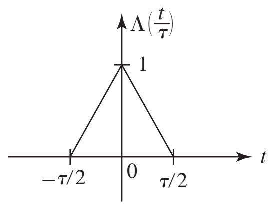


Figure 2.9 Triangular pulse.

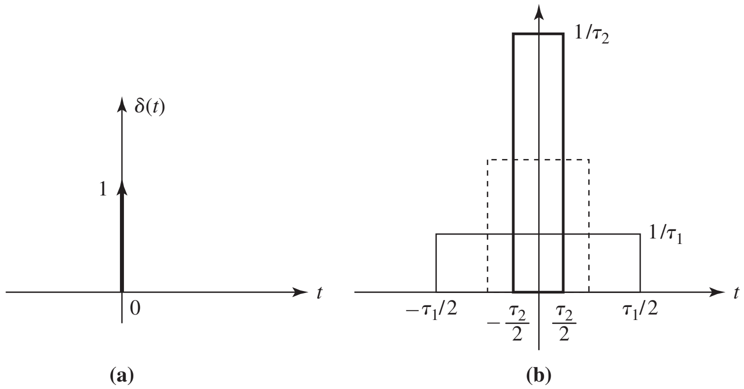


Figure 2.10 (a) Impulse signal; (b) approximating the impulse signal with narrowing rectangular pulses.

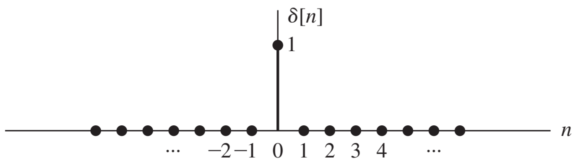


Figure 2.11 Unit impulse sequence.

The impulse signal is a mathematical representation for excitation or action that is highly localized in time.

In discrete-time domain, a **unit impulse** or **sample sequence** is defined by

$$\delta[n] \triangleq \begin{cases} 1, & n = 0 \\ 0, & \text{otherwise} \end{cases} \tag{2.11}$$

Figure 2.11 illustrates a unit impulse sequence.

The following properties of the impulse signal can be derived from its definition:

1. $x(t)\delta(t - t_o) = x(t_o)\delta(t - t_o)$ (2.12)

Equation (2.12) follows from the fact that $\delta(t - t_o) = 0$ everywhere except at $t = t_o$.

2. $\delta(\alpha t) = \dfrac{1}{|\alpha|}\delta(t)$ (2.13)

3. $\displaystyle\int_{-\infty}^{\infty} x(t)\delta(t - t_o)dt = x(t_o)$ (2.14)

Equation (2.14) is obtained by substituting (2.12) into the integral as follows:

$$\int_{-\infty}^{\infty} x(t)\delta(t - t_o)dt = \int_{-\infty}^{\infty} x(t_o)\delta(t - t_o)dt = x(t_o)\int_{-\infty}^{\infty} \delta(t - t_o)dt = x(t_o)$$

This is called the **sampling** or **sifting** property of the impulse signal.

4. $x(t) \otimes \delta(t) = \displaystyle\int_{-\infty}^{\infty} x(\tau)\delta(t - \tau)d\tau = x(t)$ (2.15)

Equation (2.15) is obtained by using the property $\delta(t) = \delta(-t)$ and a change of variables as follows:

$$x(t) \otimes \delta(t) = \int_{-\infty}^{\infty} x(\tau)\delta(\tau - t)d\tau = x(t)\int_{-\infty}^{\infty} \delta(\tau - t)d\tau = x(t)$$

Similarly,

$$x(t) \otimes \delta(t - t_o) = \int_{-\infty}^{\infty} x(\tau)\delta(t - t_o - \tau)d\tau = \int_{-\infty}^{\infty} x(\tau)\delta[\tau - (t - t_o)]d\tau = x(t - t_o) \quad (2.16)$$

Thus, the convolution of an arbitrary signal with the impulse signal yields the signal itself. Further, the convolution of an arbitrary signal with a shifted impulse signal yields the signal shifted by the same amount.

Example 2.2

Evaluate the following expressions.

a. $\sin(3500\pi t)\delta(t)$
b. $\cos(\pi t)\delta(4t - 1)$
c. $\int_{-\infty}^{\infty}[t^2 - \cos(\pi t)]\delta(t - 2)dt$
d. $\int_{-\infty}^{\infty}[e^{-t} + \cos(10\pi t)]\delta(2t - 4)dt$

Solution

a. Applying Property 1 of the impulse signal yields

$$\sin(3500\pi t)\delta(t) = \sin(0)\delta(t) = 0 \times \delta(t) = 0$$

b. Using Property 2, we obtain

$$\cos(\pi t)\delta(4t - 1) = \frac{1}{4}\cos(\pi t)\delta\left(t - \frac{1}{4}\right)$$

Now applying Property 1 yields

$$\cos(\pi t)\delta(4t - 1) = \frac{1}{4}\cos\left(\frac{\pi}{4}\right)\delta\left(t - \frac{1}{4}\right) = \frac{1}{4\sqrt{2}}\delta\left(t - \frac{1}{4}\right)$$

c. Using Property 3, we can write

$$\int_{-\infty}^{\infty}[t^2 - \cos(\pi t)]\delta(t - 2)dt = [t^2 - \cos(\pi t)]\big|_{t=2} = 4 - 1 = 3$$

d. Applying Property 2 yields

$$\int_{-\infty}^{\infty}[e^{-t} + \cos(10\pi t)]\delta(2t - 4)dt = \int_{-\infty}^{\infty}[e^{-t} + \cos(10\pi t)]\delta[2(t - 2)]dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}[e^{-t} + \cos(10\pi t)]\delta(t - 2)dt$$

Now using Property 3, we obtain

$$\int_{-\infty}^{\infty}[e^{-t}+\cos(10\pi t)]\delta(2t-4)dt = \left.\frac{e^{-t}+\cos(10\pi t)}{2}\right|_{t=2} = \frac{1}{2}[e^{-2}+1] = \frac{1.135}{2} = 0.567$$

Sinc Signal

The sinc signal is defined as

$$\mathrm{sinc}(t) \triangleq \frac{\sin(\pi t)}{\pi t} \tag{2.17}$$

The waveform of the sinc signal is displayed in Figure 2.12. We observe from Figure 2.12 that the sinc signal undergoes zero crossings at $t = \pm 1, \pm 2, \pm 3, \ldots$ The sinc signal assumes a maximum value of 1 at $t = 0$ (obtained as a limit using L'Hopital's rule).

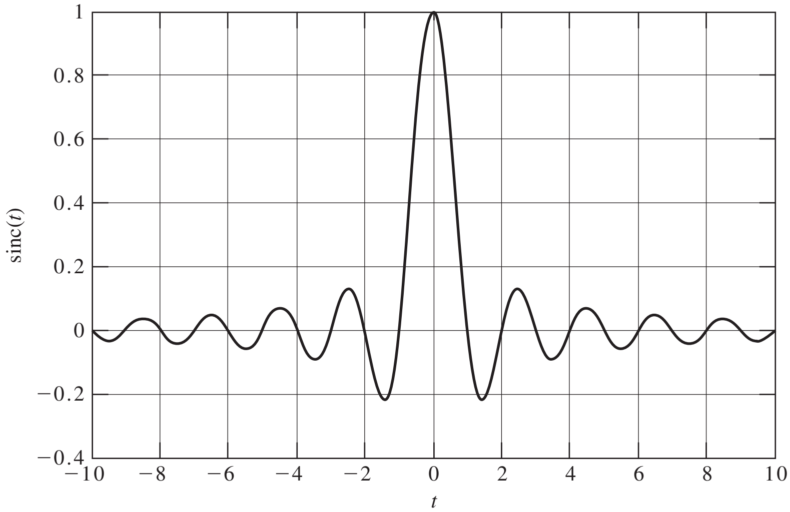


Figure 2.12 The sinc signal.

Sign or Signum Signal

The sign signal sgn(t) is defined as

$$\mathrm{sgn}(t) \triangleq \begin{cases} 1, & t > 0 \\ -1, & t < 0 \\ 0, & \text{otherwise} \end{cases} \tag{2.18}$$

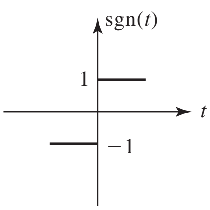


Figure 2.13 The sign or signum signal.

Note that sgn(t) denotes the sign of the independent variable t. Figure 2.13 depicts the sign signal.

2.1.2 Energy and Power Signals

Energy and power are useful parameters of a signal. The **normalized energy** of a signal $x(t)$ is defined as the energy dissipated by a voltage $x(t)$ applied across a 1-ohm resistor (or a current $x(t)$ passing though a 1-ohm resistor).

$$E_x \triangleq \lim_{T\to\infty} \int_{-T/2}^{T/2} |x(t)|^2 dt = \int_{-\infty}^{\infty} |x(t)|^2 dt \tag{2.19}$$

The energy of a signal is meaningful only if the limit in (2.19) exists (that is, finite). Such signals are called **energy signals.**

Example 2.3

Find the energy of a rectangular pulse $x(t) = A\Pi(t/T_b)$.

$$x(t) = \begin{cases} A, & |t| \leq T_b/2 \\ 0, & \text{otherwise} \end{cases}$$

Solution

Using (2.19), the energy is given by

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-T_b/2}^{T_b/2} A^2 dt = A^2 T_b$$

Example 2.4

Find the energy of the carrier pulse $x(t) = A\Pi(t/T_b)\cos(2\pi f_o t)$.

$$x(t) = \begin{cases} A\cos(2\pi f_o t), & |t| \leq T_b/2 \\ 0, & \text{otherwise} \end{cases}$$

Solution

Substituting into (2.19) yields

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = A^2 \int_{-T_b/2}^{T_b/2} \cos^2(2\pi f_o t) dt = \frac{A^2}{2} \int_{-T_b/2}^{T_b/2} [1 + \cos(4\pi f_o t)] dt = \frac{A^2 T_b}{2}$$

where we have used the trigonometric identity $2\cos^2(\theta) = 1 + \cos(2\theta)$. The integral of the second term is zero because $f_o >> 1/T_b$ has been assumed. The energy content of the signal becomes infinite in the limit as $T_b \to \infty$.

Example 2.5

Find the energy of the sinusoidal waveform $x(t) = A\cos(2\pi f_o t)$.

Solution

$$E_x = \lim_{T\to\infty} \int_{-T/2}^{T/2} |x(t)|^2 dt = A^2 \lim_{T\to\infty} \int_{-T/2}^{T/2} \cos^2(2\pi f_o t) dt = \lim_{T\to\infty} \frac{A^2 T}{2} \to \infty$$

Therefore, this signal is not an energy signal. In such cases, the concept of power of a signal is meaningful.

The normalized power of a signal $x(t)$ is the power dissipated by a voltage $x(t)$ applied across a 1-ohm resistor (or a current $x(t)$ passing though a 1-ohm resistor). The **normalized average power** of a signal $x(t)$ is defined as

$$P_x \triangleq \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt \tag{2.20}$$

The normalized average power of a signal is meaningful only if the limit in (2.20) exists (that is, finite). Such signals are called **power** signals. For a periodic signal $x_p(t)$ with fundamental period T_o, (2.20) simplifies to

$$P_x = \frac{1}{T_o} \int_{-T_o/2}^{T_o/2} |x_p(t)|^2 dt \tag{2.21}$$

A signal cannot be both power- and energy-type, because $P_x = 0$ for energy signals and $E_x = \infty$ for power signals. A signal may be neither energy-type nor power-type.

Example 2.6

Calculate the power of sinusoidal signal $x_p(t) = A\cos(2\pi f_o t + \phi)$.

Solution

Substituting $A\cos(2\pi f_o t + \phi)$ into (2.21) yields

$$P_x = \frac{A^2}{T_o} \int_{-T_o/2}^{T_o/2} \cos^2(2\pi f_o t + \phi) dt$$

$$= \frac{A^2}{2T_o} \left[\int_{-T_o/2}^{T_o/2} dt + \int_{-T_o/2}^{T_o/2} \cos(4\pi f_o t + 2\phi) dt \right] = \frac{A^2}{2} + 0 = \frac{A^2}{2}$$

The second integral is zero because it evaluates the integrand $\cos(4\pi f_o t + 2\phi)$ over two complete periods.

2.1.3 Logarithmic Power Calculations

In communication systems, it is often convenient to work with power levels and component losses (gains) in logarithmic units. Engineers prefer to express power levels as dB above or below 1 milliWatt (mW) and call it **dBm.** The power level in dBm is defined as

$$\text{Power level in dBm} = 10 \log_{10} \frac{P}{1\text{mW}} \tag{2.22}$$

where P is power level in mW. Thus, 1 mW is 0 dBm and 100 mW equals 20 dBm. To convert from dBm to mW, the following formula can be used:

$$\text{Power level in mW} = 10^{(\text{dBm}/10)} \tag{2.23}$$

The power level at any point in a transmission link can now be calculated by adding the algebraic sum of gains (in dB) up to that point to the input level in dBm. The following example illustrates the advantage of using logarithmic units.

Example 2.7

A semiconductor laser couples 5 mW into an optical fiber link. The optical signal travels through a group of components (e.g., cable, connectors, splitters) with gains specified in Figure 2.14. Compute the power input to the optical receiver.

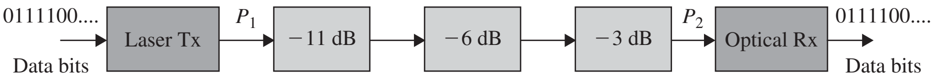


Figure 2.14 Optical fiber link.

Solution

Power coupled into the optical fiber by the laser transmitter (Tx) $P_1 = 5$ mW

Input optical power P_1 in dBm $= 10 \log_{10} \dfrac{5 \text{ mW}}{1 \text{ mW}} = 7$ dBm

Total losses in the optical fiber link $= -11 - 6 - 3 = -20$ dB

Output optical power P_2 in dBm $= P_1\,(\text{dBm}) + \text{Loss (dB)}$

$= 7 - 20 = -13$ dBm

Power input to the optical receiver $= -13$ dBm

Power input to the optical receiver in micro Watts $= 10^{(-13/10)} = 50$

2.1.4 Some Basic Operations on Signals

We consider four basic operations on signals in time-domain. These include time reversal, time shifting, time scaling, and amplitude scaling.

Time Reversal

In time reversal we create a new signal $x_1(t)$ by flipping the original signal $x(t)$ around vertical axis.

$$x_1(t) = x(-t) \tag{2.24}$$

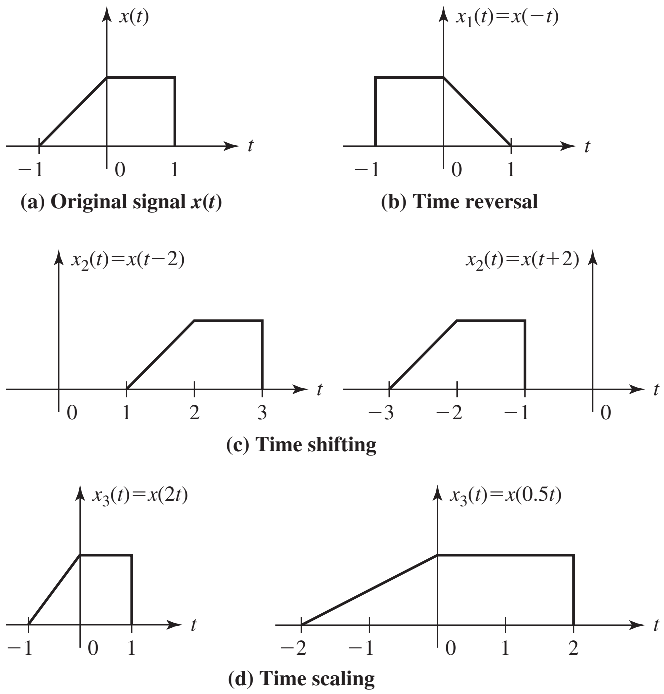


Figure 2.15 Transformations of signals in time-domain.

Figure 2.15(b) illustrates time reversal operation. Note that the resultant signal $x_1(t)$ is mirror image of the original signal $x(t)$.

Time Shifting

Given a CT signal $x(t)$, a time-shifted version of this signal is

$$x_2(t) = x(t - t_o), \; t_o = \text{constant} \tag{2.25}$$

If t_o is positive, the time-shifted signal is delayed in time. The resultant signal $x_2(t)$ is shifted by t_o to the right of the original signal. On the other hand, if t_o is negative, the time-shifted signal is advanced in time. The resultant signal $x_2(t)$ is shifted by t_o to the left of the original signal. Figure 2.15(c) illustrates time-shifting operation.

Time Scaling

Given a CT signal $x(t)$, a time-scaled version of this signal is

$$x_3(t) = x(\alpha t), \; \alpha = \text{constant} \tag{2.26}$$

Time scaling results in either an expanded or compressed version of the original signal $x(t)$. If $\alpha > 1$, the resultant signal $x_3(t)$ is compressed or contracted in time. On the other hand, the signal $x_3(t)$ is expanded in time for $\alpha < 1$. Figure 2.15(d) illustrates time scaling operation.

Amplitude Scaling

Given a CT signal $x(t)$, an amplitude-scaled version of this signal is

$$x_4(t) = Ax(t) + B, \; A, B = \text{constants} \tag{2.27}$$

If $A > 1$, it indicates amplification of the original signal $x(t)$. The nonzero value of B shifts the DC level of the resultant signal $x_4(t)$.

In general, a combination of the above operations may be involved in generating the new signal.

Example 2.8

Plot the following signals.

a. $x_1(t) = \Pi\left(\frac{t}{100}\right) + \Pi\left(\frac{t}{50}\right)$

b. $x_2(t) = 2\Pi\left(\frac{t}{12}\right) + 2\Pi\left(\frac{t}{6}\right) + 2\Lambda\left(\frac{t}{6}\right)$

c. $x_3(t) = \Lambda\left(\frac{t}{2} - 1\right) + \Lambda\left(\frac{t}{2} + 1\right)$

Solution

The waveforms are illustrated in Figure 2.16.

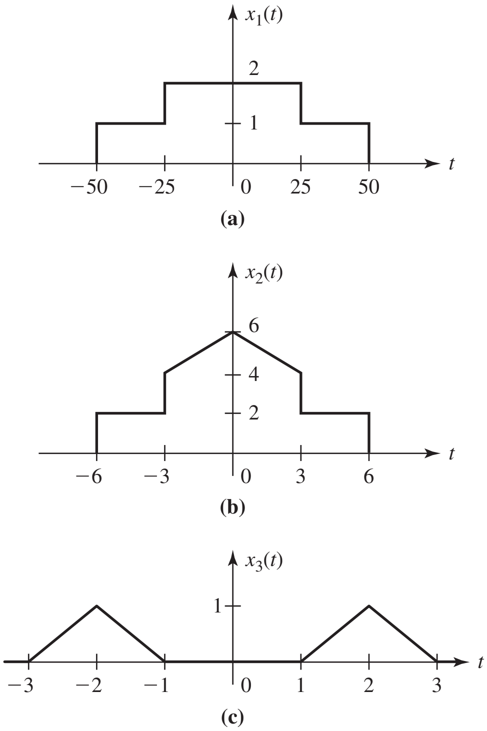


Figure 2.16 Example 2.8 signal waveforms.

2.2 BASIC SYSTEM CONCEPTS

A continuous-time system operates on a continuous-time input signal $x(t)$, according to some well-defined rule or **transformation** $\mathcal{T}$, to produce a continuous-time output signal $y(t)$ as a result of it. We will use the following notation to denote the action of a system:

$$\text{Continuous-time (CT) system:} \quad x(t) \xrightarrow{\mathcal{T}} y(t) \tag{2.28}$$

Note that the output $y(t)$ for any value of t may depend on $x(t)$ for all values of t. Similarly, a discrete-time system accepts an input sequence $x[n]$ to produce an output sequence $y[n]$

$$\text{Discrete-time (DT) system:} \quad x[n] \xrightarrow{\mathcal{T}} y[n] \tag{2.29}$$

As in the case of continuous-time systems, the output $y[n]$ for any value of n may depend on $x[n]$ for all values of n. Figure 2.17 shows a block diagram representation of the CT and DT systems.

In the context of communication systems, the system entity may represent the effect of transmission media or signal processing operations on signals. An example is the attenuation and distortion of the output signal produced by the twisted copper wire pair. Another example modeled by a system entity is the filtering action produced by an interconnection of circuit elements (ICs, resistors, capacitors, etc.) on the received signal with the intent to remove out-of-band noise.

Example 2.9

The square law device is a simple example of a continuous-time system. It is defined by the input-output relation

$$y(t) = x^2(t)$$

which states that the output signal value at time t is equal to the square of the input signal value at that same time.

Example 2.10

Another simple example is the ideal delay system defined by

$$y(t) = x(t - t_o)$$

where t_o is the time delay introduced by the system.

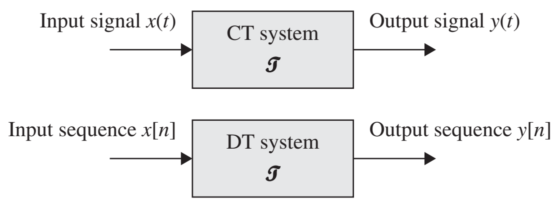


Figure 2.17 Block diagram of a system.

Example 2.11

The **accumulator**

$$y[n] = \sum_{l=-\infty}^{n} x[\ell] = \sum_{l=-\infty}^{n-1} x[\ell] + x[n] = y[n-1] + x[n]$$

is a discrete-time system. The output at time instant n is the sum of the current input sample $x[n]$ and the previous output $y[n-1]$. $y[n-1]$, in turn, is the sum of all previous input sample values from $-\infty$ to $n-1$. The system cumulatively adds, that is, it accumulates all input sample values.

In all the aforementioned cases, the input–output relation defining the system lends itself to simple mathematical definition. The rule for obtaining the output from the input can alternatively be defined by a graph or a table (e.g., in a read-only memory [ROM]).

2.2.1 Classification of Systems

There are various ways to classify systems based on the input–output relation of the system.

Linear Systems

The most widely used system model, and the one that we frequently use in this text, is a **linear** system for which the **superposition principle** always holds. More precisely, a system $x(t) \xrightarrow{\mathcal{T}} y(t)$ is linear, if $x_1(t)$ results in the output $y_1(t)$ and $x_2(t)$ results in the output $y_2(t)$, then the response due to the input is

$$x(t) = \alpha x_1(t) + \beta x_2(t) \tag{2.30}$$

is given by

$$y(t) = \alpha y_1(t) + \beta y_2(t) \tag{2.31}$$

The superposition property must hold for any arbitrary constants α and β, and for all possible inputs $x_1(t)$ and $x_2(t)$. This property makes it feasible to compute the response to a complex signal that can be decomposed as a weighted combination of some fundamental signals, such as unit impulse or complex exponential signals. In this case, the desired output is given by a similarly weighted combination of outputs to the constituent fundamental signals. A system that does not satisfy the superposition property is called a **nonlinear** system.

Similarly, a DT system $x[n] \xrightarrow{\mathcal{T}} y[n]$ is linear if and only if

$$x_1[n] \xrightarrow{\mathcal{T}} y_1[n]$$

$$x_2[n] \xrightarrow{\mathcal{T}} y_2[n]$$

then

$$\alpha x_1[n] + \beta x_2[n] \xrightarrow{\mathcal{T}} \alpha y_1[n] + \beta y_2[n] \tag{2.32}$$

for any arbitrary constants α and β.

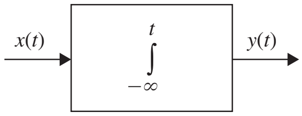


Figure 2.18 Integrator.

Example 2.12

The integrator in Figure 2.18 is a linear system. The output of the integrator is related to its input by

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

To show this, let $x(t) = \alpha x_1(t) + \beta x_2(t)$ be the system input. The corresponding output is

$$\begin{aligned} y(t) &= \int_{-\infty}^{t} x(\tau)d\tau = \int_{-\infty}^{t} [\alpha x_1(\tau) + \beta x_2(\tau)]d\tau \\ &= \alpha \int_{-\infty}^{t} x_1(\tau)d\tau + \beta \int_{-\infty}^{t} x_2(\tau)d\tau \\ &= \alpha y_1(t) + \beta y_2(t) \end{aligned}$$

Thus, the integrator is a linear system.

Example 2.13

The square law device $y(t) = x^2(t)$ is a nonlinear system. To see this, let $x(t) = \alpha x_1(t) + \beta x_2(t)$ be the system input. The corresponding output is

$$\begin{aligned} y(t) &= x^2(t) = [\alpha x_1(t) + \beta x_2(t)]^2 \\ &= \alpha^2[x_1(t)]^2 + 2\alpha\beta x_1(t)x_2(t) + \beta^2[x_2(t)]^2 \\ &\neq \alpha y_1(t) + \beta y_2(t) \end{aligned}$$

where $y_1(t) = [x_1(t)]^2$ and $y_2(t) = [x_2(t)]^2$. Therefore the system is nonlinear.

Example 2.14

The accumulator $y[n] = \sum_{l=-\infty}^{n} x[\ell]$ is a linear system. Let

$$y_1[n] = \sum_{l=-\infty}^{n} x_1[\ell], \quad y_2[n] = \sum_{l=-\infty}^{n} x_2[\ell]$$

For an input $x[n] = \alpha x_1[n] + \beta x_2[n]$, the output is

$$\begin{aligned} y[n] &= \sum_{l=-\infty}^{n} (\alpha x_1[\ell] + \beta x_2[\ell]) \\ &= \alpha \sum_{l=-\infty}^{n} x_1[\ell] + \beta \sum_{l=-\infty}^{n} x_2[\ell] \\ &= \alpha y_1[n] + \beta y_2[n] \end{aligned}$$

Hence, the system is linear.

Memoryless Systems

A memoryless system is one whose current value of the output depends only on the current value of the input; that is, the current value of the output does not depend on either past values or future values of the input. The integrator $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$ in Example 2.12 is an example of a system with memory. The integrator's current output depends on the history of its input. The square law device $y(t) = x^2(t)$ in Example 2.13 is an example of a memoryless system. Its current output depends on its current input value only.

Time-Invariant Systems

A system $x(t) \xrightarrow{\mathcal{T}} y(t)$ is said to be **time-invariant** if the delayed input $x_1(t) = x(t - t_o)$ results in the delayed output. That is,

$$y_1(t) = y(t - t_o) \tag{2.33}$$

for all t_o.

Similarly, a DT system $x[n] \xrightarrow{\mathcal{T}} y[n]$ is said to be **shift-invariant** if the delayed input sequence $x_1[n] = x[n - n_o]$ results in the delayed output sequence. That is,

$$y_1[n] = y[n - n_o] \tag{2.34}$$

for all n_o.

For a system to be time- (shift-) invariant, this relationship between the input and output must hold for any arbitrary input signal (sequence) and the corresponding output signal (sequence). Time (shift) invariance ensures that the same input signal always generates the same output signal, regardless of the time when the input signal is applied to the system. Of course the output signal is a delayed replica corresponding to the delay in the input signal.

Example 2.15

The integrator is a time-invariant system.

To show this, let $x_1(t) = x(t - t_o)$ be the system input. The corresponding output is

$$y_1(t) = \int_{-\infty}^{t} x_1(\tau)d\tau = \int_{-\infty}^{t} x(\tau - t_o)d\tau = \int_{-\infty}^{t-t_o} x(v)dv = y(t - t_o)$$

Therefore, the integrator is time-invariant.

Example 2.16

The amplitude modulator (Figure 2.19) defined by $y(t) = x(t)\cos(2\pi f_c t)$ is a time-varying system.

To show this, let $x_1(t) = x(t - t_o)$ be the system input. The corresponding output is

$$y_1(t) = x_1(t)\cos(2\pi f_c t) = x(t - t_o)\cos(2\pi f_c t) \neq y(t - t_o)$$

Therefore the amplitude modulator is not time-invariant.

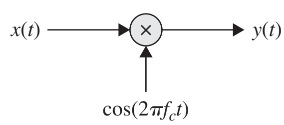


Figure 2.19 Amplitude modulator.

Example 2.17

The M-point moving average system $y[n] = \frac{1}{M}\sum_{k=0}^{M-1} x[n-k]$ is shift-invariant. To prove it, let $x_1[n] = x[n-n_o]$. Then

$$y_1[n] = \frac{1}{M}\sum_{k=0}^{M-1} x[n-n_o-k]$$

Also,

$$y[n-n_o] = \frac{1}{M}\sum_{k=0}^{M-1} x[n-n_o-k]$$

Because $y_1[n] = y[n-n_o] = \frac{1}{M}\sum_{k=0}^{M-1} x[n-n_o-k]$, the DT system is shift-invariant.

A system satisfying both the linearity and the time-invariance properties is called a **linear time-invariant (LTI)** system. LTI systems are mathematically easy to analyze, and consequently, easy to design.

2.2.2 Characterization of LTI Systems

We will next derive a very important result that the impulse response completely determines the behavior of an LTI system. For this purpose, we will start with an approximation of an arbitrary CT signal $x(t)$ by a sum of shifted, scaled pulses as shown in Figure 2.20(a). That is,

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta \tag{2.35}$$

where $\delta_\Delta(t)$ is unit area pulse in Figure 2.20(b). In the limit as $\Delta \to 0$, the summation approaches the integral (2.15).

$$\hat{x}(t) \xrightarrow{\Delta \to 0} \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \tag{2.36}$$

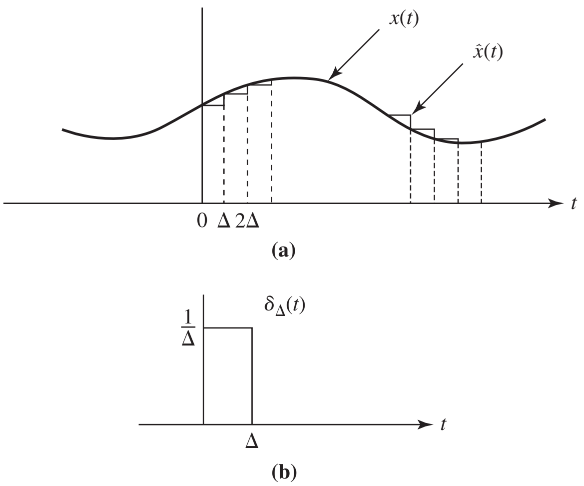


Figure 2.20 Staircase approximation to a CT signal.

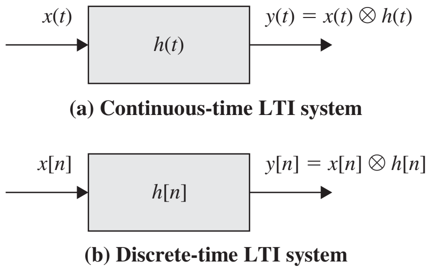


Figure 2.21 Characterization of LTI system.

Let $h_\Delta(t)$ be the response of the system to the pulse $\delta_\Delta(t)$. That is,

$$\delta_\Delta(t) \xrightarrow{\mathcal{T}} h_\Delta(t) \tag{2.37}$$

Then, using linearity and time-invariance properties, we obtain

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t - k\Delta)\Delta \xrightarrow{\mathcal{T}} \hat{y}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)h_\Delta(t - k\Delta)\Delta \tag{2.38}$$

In the limit as $\Delta \to 0$, $\delta_\Delta(t) \xrightarrow{\Delta \to 0} \delta(t)$ and $h_\Delta(t) \xrightarrow{\Delta \to 0} h(t)$. Therefore, we can write

$$y(t) = \lim_{\Delta \to 0} \hat{y}(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau = x(t) \otimes h(t) \tag{2.39}$$

where

$$\delta(t) \xrightarrow{\mathcal{T}} h(t) \tag{2.40}$$

$h(t)$ is called the **impulse response** of the LTI system. Equation (2.40) states that the response of the system to an arbitrary input $x(t)$ is the convolution of $x(t)$ with the system impulse response $h(t)$. Similarly, it can be shown that for a DT system, the output sequence $y[n]$ is the convolution sum of input sequence $x[n]$ with the system impulse response $h[n]$. Summarizing

$$\text{CT system: } y(t) = x(t) \otimes h(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t - \tau)d\tau \tag{2.41}$$

$$\text{DT system: } y[n] = x[n] \otimes h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n - k] = \sum_{k=-\infty}^{\infty} x[n - k]h[k] \tag{2.42}$$

Figure 2.21 displays these relationships for LTI systems. As a consequence of (2.41) and (2.42), an LTI system is **completely characterized** by its impulse response. If the system is nonlinear or time-variant, its impulse response describes only part of the system's characteristics.

Example 2.18

The impulse response of an integrator is a unit step function. To show this, let $x(t) = \delta(t)$ be the system input. The corresponding output is

$$h(t) = \int_{-\infty}^{t} \delta(\tau)d\tau$$

Now

$$\int_{-\infty}^{t} \delta(\tau)d\tau = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases} \tag{2.43}$$

But the right-hand side of (2.43) is identical to the definition of the unit step signal in (2.5). So

$$h(t) = \int_{-\infty}^{t} \delta(\tau)d\tau = u(t) \tag{2.44}$$

Equation (2.44) states that impulse response function of the integrator is a unit step function.

Example 2.19

The output of a **unit delay** system is $x(t-1)$ to an input $x(t)$. Using the property (2.16) of the unit impulse signal, we can write

$$x(t) \otimes \delta(t-1) = x(t-1)$$

Therefore, the impulse response of a unit delay system is $h(t) = \delta(t-1)$.

Example 2.20

The impulse response of the accumulator $y[n] = \sum_{l=-\infty}^{n} x[\ell]$ is obtained by setting $x[n] = \delta[n]$ resulting in $h[n] = \sum_{l=-\infty}^{n} \delta[\ell]$.

Now

$$\sum_{l=-\infty}^{n} \delta[\ell] = \begin{cases} 1, & n \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

is a unit step sequence $u[n]$. Thus, impulse response sequence of the accumulator is unit step sequence.

Causal Systems

A system is said to be **causal** if its current output depends only on its current and past inputs. If the output depends on future inputs, the system is said to be **noncausal** or **anticipatory.** In a CT causal system, for every choice of t_o, the output signal value $y(t_o)$ depends only on the input signal values $x(t)$ for $t \leq t_o$ and does not depend on input signal values for $t > t_o$. A causal system does not anticipate the future. No physical system has such a capability. Thus every physical system is causal, and causality is a necessary condition for a system to be realizable in the real world.

For an LTI system, it is possible to derive a very simple condition for causality in terms of its impulse response $h(t)$. A CT LTI system is causal iff its impulse response satisfies the following condition:

$$h(t) = 0, \ t < 0 \tag{2.45}$$

The equivalent condition for a DT system is that its impulse response $h[n]$ satisfies the following condition for causality.

$$h[n] = 0, \ n < 0 \tag{2.46}$$

Example 2.21

The integrator is a causal system because its impulse response is a unit step function and thus satisfies the property (2.45).

Example 2.22

The impulse response of a unit delay system is given by $h(t) = \delta(t - 1)$. It is causal because it satisfies the property (2.45).

Stable Systems

A system is **stable** if for every bounded input, the output is bounded. This implies that, if the input $|x(t)| \leq B$ for all values of t, then the output of the system $|y(t)| \leq C$ for all values of t, where B and C are finite constants. This type of stability is referred to as **bounded input, bounded output (BIBO)** stability.

For an LTI system, it is possible to derive a very simple condition for stability in terms of its impulse response. A CT LTI system is stable if its impulse response $h(t)$ is absolutely integrable, that is,

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty \tag{2.47}$$

The equivalent condition for a DT system is that its impulse response $h[n]$ is absolutely summable for BIBO stability.

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty \tag{2.48}$$

One important consequence of (2.48) is that a DT system whose impulse response is of finite length (“FIR system”), the stability condition is always satisfied as long as $|h[n]| < \infty$.

Example 2.23

The integrator system is unstable. The impulse response of the integrator is $h(t) = u(t)$, so applying the criterion (2.47) leads to

$$\int_{-\infty}^{\infty} |h(t)| dt = \int_{-\infty}^{\infty} |u(t)| dt = \int_{0}^{\infty} dt \to \infty$$

Therefore, the integrator is an unstable system.

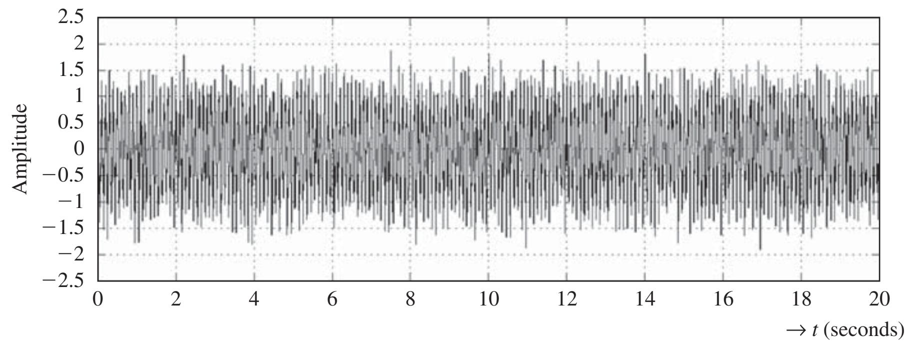
2.3 FREQUENCY DOMAIN REPRESENTATION

Although electrical signals used in communication systems are commonly viewed as functions of time, it is very useful to think of them in terms of their frequency content. Certain characteristics of signals are easier to analyze and measure in the frequency domain. In addition, the frequency domain analysis of many important operations on signals leads to unique and valuable insights toward understanding their effects. That is why the frequency domain representation and analysis of signals and systems is an integral part of design tools for communication systems.

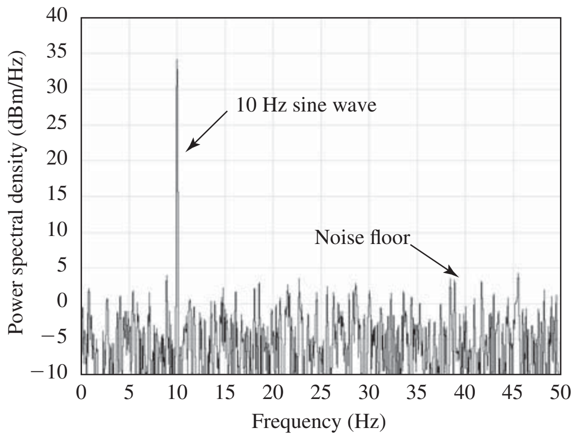
Figure 2.22(a) shows the time domain representation of a 10 Hz sine wave embedded in noise. It is difficult to identify a simple 10 Hz waveform in the presence of wideband ("white") noise by simply looking at it on an oscilloscope. However, if we look at the same signal and noise in the frequency domain using a spectrum analyzer, it is very easy to identify the 10 Hz tone. We observe from Figure 2.22(b) that the noise is spread out over all frequencies and forms the floor of the spectrum analyzer display. In more complex situations, the composite signal may consist of hundreds of channels or carriers. An example is a CATV system where a few hundred channels or signals are present. Analyzing such a complex signal in the time domain is not very useful. The frequency domain analysis, however, provides valuable insight into the effects of system impairments and noise.

In this chapter, we will consider two useful frequency domain representations of continuous-time signals:

1. Fourier series (FS)
2. Fourier transform (FT)



(a)



(b)

Figure 2.22 (a) Time and (b) frequency domain representation of a sine wave embedded in noise.

Pioneers in the Field

Jean Baptiste Joseph Fourier was born in Auxerre, France in 1768. He was orphaned at age eight and grew up with his aunt and uncle in the same town. On the recommendation of the Bishop of Auxerre, Fourier was offered a place at the nearby École Royale Militaire. He demonstrated such proficiency in mathematics in his early years that he later became a teacher there. When the École Normale was founded in 1794 in Paris, Fourier was among its first students, and, in 1795, he began teaching there. That same year, Fourier joined the faculty at the brand new École Polytechnique and became a colleague of Gaspard Monge and other mathematicians.

Fourier accompanied Napoléon Bonaparte on his Egyptian expedition in 1798. He was appointed governor of Lower Egypt and secretary of the Institut d'Égypte. While there, he organized ammunition workshops to support the French army during the war with the English. He also contributed several mathematical papers to the Egyptian Institute (also called the Cairo Institute), which Napoléon founded in Cairo, with a view of weakening English influence in the East. After the French defeat in 1801, Fourier returned to France and became prefect of Isère. It was here that he carried out his investigation of propagation of heat in the solid bodies. These experiments led to the development of the Fourier series and Fourier integral. Fourier claimed that an arbitrary function defined in a finite interval can be expressed as a sum of sinusoids. He submitted his initial work on heat transfer to the Institut de France in 1807. The judges, including the great French mathematicians Laplace, Lagrange, Monge, and LaCroix admitted the originality and significance of Fourier's work, but criticized its lack of mathematical rigor. Fourier believed the criticism was unjustified but was unable to defend his claim because the tools required for operations with infinite series were not available at the time. Although three of the four judges were in favor of publication, the paper was rejected for publication because of the forceful opposition by Lagrange. Fourier was elected to the Académie des Sciences in 1817 and became its secretary in 1822. Fifteen years after he presented the results, the Académie des Sciences published his prize-winning essay *Théorie analytique de la chaleur* in 1822. The book is now considered a classic.

The Fourier series can be used to represent periodic signals in the frequency domain. It expresses such a signal as a **superposition** of an infinite ($\approx$ large) number of complex exponential waveforms. The Fourier transform, however, is applicable to aperiodic waveforms in a strict mathematical sense. Both provide a simpler description of signals in terms of magnitudes and phases of the constituent frequency components.

2.4 FOURIER SERIES

A CT periodic function $x_p(t)$ with period T_o can be represented by an **exponential Fourier series (FS)**

$$x_p(t) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi n f_o t} \tag{2.49}$$

The series coefficients C_n are related to $x_p(t)$ by

$$C_n = \frac{1}{T_o}\int_{T_o} x_p(t) e^{-j2\pi n f_o t} dt \tag{2.50}$$

where $f_o = 1/T_o$ is called the **fundamental frequency** of the periodic signal $x_p(t)$. From (2.49), it can be observed that the Fourier series expands a periodic function as an infinite sum of complex phasor signals $C_n e^{j2\pi n f_o t}$. The term C_0 corresponding to $n = 0$ in (2.49) equals the **average** or **DC** component of the signal, and is given by

$$C_0 = \frac{1}{T_o}\int_{T_o} x_p(t) dt \tag{2.51}$$

The phasor signal $C_1 e^{j2\pi f_o t}$ represents the fundamental frequency (f_o) component in the periodic signal $x_p(t)$. The terms in summation (2.49) for $n \geq 2$ consists of phasor signals at **harmonic frequencies** $f = nf_o$, $n = 0, 1, 2, 3, \ldots$ in the FS expansion of the signal $x_p(t)$. They are called its **frequency** or **spectral components**. Each phasor term in (2.49) can be written as

$$C_n e^{j2\pi n f_o t} = \underbrace{|C_n|}_{\substack{\text{Magnitude of the frequency}\\ \text{component at } f = nf_o}} e^{j(2\pi n f_o t + \overbrace{\measuredangle C_n}^{\substack{\text{Phase of the frequency}\\ \text{component at } f = nf_o}})} \tag{2.52}$$

Plots of $|C_n|$ and $\measuredangle C_n$ as function of frequency are called the **magnitude** and the **phase** spectrum of the signal, respectively. Because the magnitude and phase spectra of periodic signals contain spectral components at discrete frequencies $f = nf_o$, $n = 0, 1, 2, 3, \ldots$, these are called **line spectra.** For $x_p(t)$ real function of time, we have

$$C_{-n} = \frac{1}{T_o}\int_{T_0} x_p(t) e^{j2\pi n f_o t} dt = \left(\frac{1}{T_o}\int_{T_o} x_p(t) e^{-j2\pi n f_o t} dt \right)^* = C_n^* \tag{2.53}$$

From (2.53) it follows that for a real signal, the magnitude spectrum is an even function, and the phase spectrum is an odd function of frequency.

2.4.1 Trigonometric Fourier Series

We next derive a second form of the Fourier series for real signals. For this purpose, we write (2.49) as

$$\begin{aligned}
x_p(t) &= C_o + \sum_{n=1}^{\infty} C_{-n} e^{-j2\pi n f_o t} + \sum_{n=1}^{\infty} C_n e^{j2\pi n f_o t} = C_o + \sum_{n=1}^{\infty} C_n^* e^{-j2\pi n f_o t} + \sum_{n=1}^{\infty} C_n e^{j2\pi n f_o t} \\
&= C_0 + \sum_{n=1}^{\infty} |C_n| [e^{-j(2\pi n f_o t + \measuredangle C_n)} + e^{j(2\pi n f_o t + \measuredangle C_n)}] \\
&= C_0 + 2\sum_{n=1}^{\infty} |C_n| \cos(2\pi n f_o t + \measuredangle C_n) \qquad (2.54)
\end{aligned}$$

Equation (2.54) is called the **trigonometric Fourier series.** We can write an alternative form of (2.54) by expanding the cosine function as follows:

$$x_p(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(2\pi n f_o t) + \sum_{n=1}^{\infty} B_n \sin(2\pi n f_o t) \tag{2.55}$$

where

$$A_n = 2|C_n|\cos(\measuredangle C_n) = C_n + C_n^* = \frac{2}{T_o}\int_{T_o} x_p(t)\cos(2\pi n f_o t) dt \tag{2.56}$$

$$B_n = -2|C_n|\sin(\measuredangle C_n) = -(C_n - C_n^*)/j = \frac{2}{T_o}\int_{T_o} x_p(t)\sin(2\pi n f_o t) dt \tag{2.57}$$

The coefficients of the exponential and trigonometric forms of the Fourier series are related by

$$C_o = A_o;\ 2C_n = A_n - jB_n \tag{2.58}$$

We can conclude from (2.56) and (2.57) that if

1. $x_p(t)$ is an even function of time, its Fourier series expansion will contain only cosine terms, i.e., $B_n = 0, n = 1, 2, \ldots$
2. $x_p(t)$ is an odd function of time, its Fourier series expansion will contain only sine terms, i.e., $A_n = 0, n = 1, 2, \ldots$

Example 2.24

Determine the Fourier series expansion of a periodic pulse train $g_{T_o}(t) = \sum_{n=-\infty}^{\infty} \Pi\left[\frac{(t - nT_o)}{\tau}\right]$ of rectangular pulses shown in Figure 2.23.

Solution

Each pulse in Figure 2.23 has unity amplitude and duration τ. The Fourier coefficients are given by

$$C_n = \frac{1}{T_o}\int_{T_o} g_{T_o}(t)e^{-j2\pi nf_o t}\,dt = \frac{1}{T_o}\int_{-\tau/2}^{\tau/2} e^{-j2\pi nf_o t}\,dt$$

$$= -\frac{1}{j2\pi nf_o T_o}\left[e^{-j\pi nf_o\tau} - e^{j\pi nf_o\tau}\right]$$

$$= \frac{\tau}{T_o}\frac{\sin(\pi nf_o\tau)}{\pi nf_o\tau} = \frac{\tau}{T_o}\,\mathrm{sinc}(nf_o\tau) \tag{2.59}$$

The magnitude spectrum, given by $|C_n| = \frac{\tau}{T_o}|\mathrm{sinc}(nf_o\tau)|$, is shown in Figure 2.24 for the case $\frac{\tau}{T_o} = 0.25$. Because the sinc function is always real, the phase spectrum in Figure 2.24 assumes values 0° or 180°, depending on the sign of the $\mathrm{sinc}(nf_o\tau)$ function. Note the following points about the magnitude spectrum of the periodic pulse train:

a. The value of the DC coefficient is τ/T_o.
b. The frequency spacing between adjacent spectral components is $f_o = 1/T_o$ Hz.
c. The zero crossings of the envelope occur at integral multiples of $4f_o = 1/\tau$ Hz.

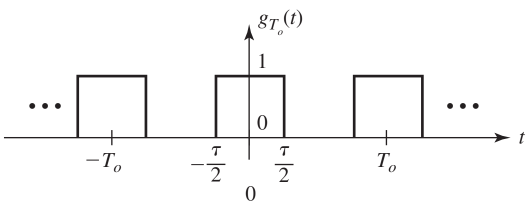


Figure 2.23 Rectangular pulse train.

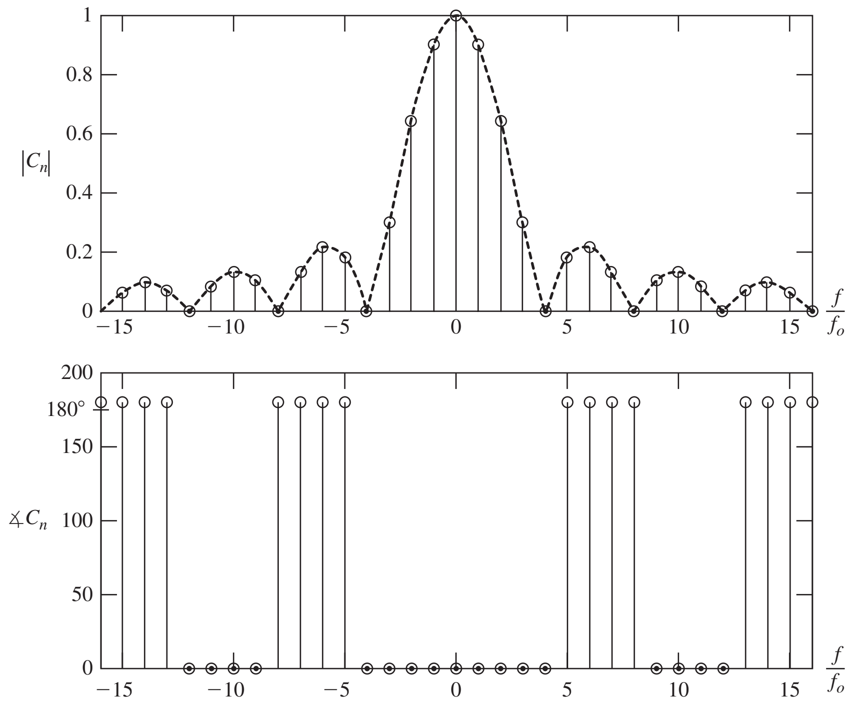


Figure 2.24 Magnitude and phase spectra of a rectangular pulse train.

Example 2.25

Suppose that a binary information source sends a repetition of pattern 01111110 at a rate of 8 Kbps. A binary 1 is transmitted by sending a rectangular pulse of 1 V with a width of 0.125 ms, and a 0 is transmitted by sending a pulse of −1 V. Find the Fourier series expansion of the periodic waveform $x_p(t)$ shown in Figure 2.25.

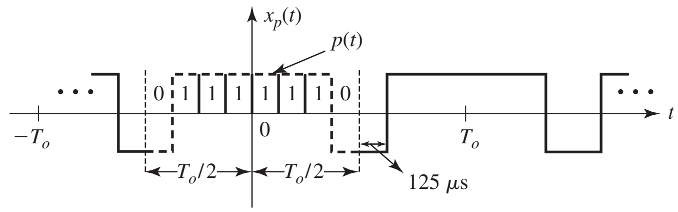


Figure 2.25 Periodic waveform $x_p(t)$ corresponding to repeated pattern 01111110.

Solution

The periodic waveform $x_p(t)$ in Figure 2.25 can be expressed as

$$x_p(t) = \sum_{k=-\infty}^{\infty} p(t - kT_o)$$

where $p(t)$ is defined in Figure 2.25. Because $x_p(t)$ is an even function of time, its Fourier series expansion will contain only cosine terms. That is,

$$x_p(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(2\pi n f_o t)$$

where

$$f_0 = \frac{1}{T_o} = 1000 \text{ Hz}$$

$$A_0 = \frac{1}{T_o}\int_{T_o} x_p(t)dt = \frac{1}{T_o}\int_0^{T_o} p(t)\,dt = \frac{1}{T_o}\left[\frac{6T_o}{8} - \frac{2T_o}{8}\right] = 0.5$$

$$A_n = \frac{2}{T_o}\int_{T_o} x_p(t)\cos(2\pi n f_o t)\,dt = \frac{4}{T_o}\int_0^{T_o/2} p(t)\cos(2\pi n f_o t)\,dt$$

$$= \frac{4}{T_o}\left\{\int_0^{3T_o/8}\cos(2\pi n f_o t)dt - \int_{3T_o/8}^{T_o/2}\cos(2\pi n f_o t)dt\right\} = \frac{4}{\pi n}\sin(3\pi n/4)$$

The FS representation for $x_p(t)$ can now be written as

$$x_p(t) = A_0 + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(3\pi n/4)}{n}\cos(2\pi n f_o t)$$

$$= 0.5 + \frac{4}{\pi}\sin\left(\frac{3\pi}{4}\right)\cos(2\pi \times 1000t) + \frac{4}{2\pi}\sin\left(\frac{3\pi}{2}\right)\cos(2\pi \times 2000t)$$

$$+ \frac{4}{3\pi}\sin\left(\frac{9\pi}{4}\right)\cos(2\pi \times 3000t) + \frac{1}{\pi}\sin(3\pi)\cos(2\pi \times 4000t) - \ \ldots$$

Figure 2.26 displays the plot of Fourier coefficients A_n as a function of frequency.

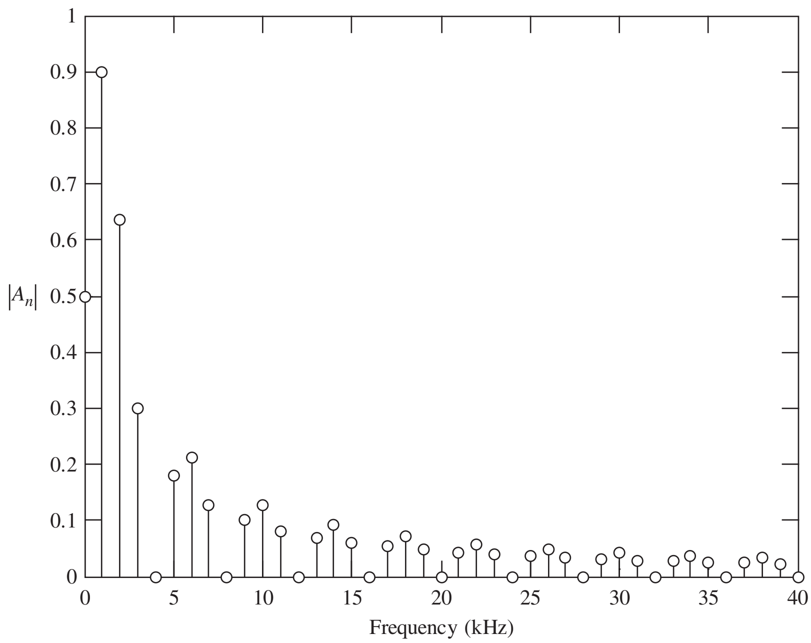


Figure 2.26 One-sided magnitude spectrum for $x_p(t)$.

2.4.2 Parseval's Theorem

The normalized average power P_x of a periodic signal is given from (2.21) as

$$P_x = \frac{1}{T_o}\int_{T_o} |x_p(t)|^2 dt = \frac{1}{T_o}\int_{T_o} x_p(t)x_p^*(t)dt \tag{2.21}$$

Substituting the FS expansion for $x_p^*(t)$ from (2.49) into (2.21), we get

$$\begin{aligned} P_x &= \frac{1}{T_o}\int_{T_o} x_p(t)\left[\sum_{n=-\infty}^{\infty} C_n^* e^{-j2\pi n f_o t}\right] dt \\ &= \sum_{n=-\infty}^{\infty}\left[\frac{1}{T_o}\int_{T_o} x_p(t)e^{-j2\pi n f_o t}dt\right]C_n^* \\ &= \sum_{n=-\infty}^{\infty} C_n C_n^* = \sum_{n=-\infty}^{\infty} \underbrace{|C_n|^2}_{\substack{\text{Average power in}\\ \text{the frequency}\\ \text{component at } f = nf_o}} \end{aligned} \tag{2.60}$$

The normalized average power of a periodic signal is the sum of the average power of its frequency components.

Bandwidth of a Signal

The **bandwidth** of a signal is a measure of the range of significant frequency components present in the signal. The term significant here implies inclusion of those frequencies that represent the signal with acceptable distortion. The latter is determined by the relevance in a given application. If the significant energy of the signal lies in the range of positive frequencies $f_1 < f < f_2$, the bandwidth would be $f_2 - f_1$. There are many definitions of bandwidth, depending on how frequencies f_1 and f_2 are defined. If f_1 and f_2 are chosen so that the spectrum of the signal is zero outside the frequency band $f_1 < f < f_2$, the quantity $f_2 - f_1$ is called the **absolute bandwidth.**

Example 2.26

Determine the absolute bandwidth of a periodic pulse train of rectangular pulses shown in Example 2.24.

The magnitude spectrum of a periodic pulse train of rectangular pulses, shown in Figure 2.23, is given by $|C_n| = \frac{\tau}{T_o}|\mathrm{sinc}(nf_o\tau)|$. The values of the sinc function become smaller and smaller as $n \to \infty$, yet they remain nonzero. Therefore, the absolute bandwidth of this signal is infinite.

In another popular definition of bandwidth, the frequencies f_1 and f_2 are chosen so that 99% of the power resides in the frequency band $f_1 < f < f_2$. In this case the quantity $f_2 - f_1$ is called the **99% power bandwidth.**

Example 2.27

Determine the 99% power bandwidth of the periodic pulse train of rectangular pulses shown in Example 2.24. Assume $T_o = 1\ \mu$sec and $\tau/T_o = 0.5$.

Solution

From Example 2.24, the FS expansion of the periodic pulse train $x_p(t)$ can be expressed as

$$x_p(t) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi n f_o t}$$

where

$$C_n = \frac{\tau}{T_o}\,\text{sinc}(nf_o\tau)$$

Now

$$f_o = \frac{1}{T_o} = 1 \text{ MHz}$$

$$C_0 = \frac{\tau}{T_o} = 0.5$$

$$C_n = \frac{\tau}{T_o}\,\text{sinc}(nf_o\tau) = 0.5\,\text{sinc}(n/2)$$

Using Parseval's theorem, we can write

$$\begin{aligned} P_x &= \sum_{n=-\infty}^{\infty} |C_n|^2 = C_0^2 + 2\sum_{n=1}^{\infty}|C_n|^2 \\ &= 0.25 + 0.5\sum_{n=1}^{\infty}|\text{sinc}(n/2)|^2 \end{aligned}$$

Table 2.1 displays Fourier coefficients and the accumulated power up to and including frequency $f = nf_o$. As the table shows, we need to include 21 Fourier coefficients to get 99% power in the signal. Because each spectral component is separated by 1 MHz, the 99% power bandwidth of the periodic pulse train is approximately 21 MHz.

Table 2.1 Power in the Fourier Components of the Rectangular Pulse Train

n	C_n	Accumulated Power (Up to and Including Frequency $f = nf_o$)
0	0.25	0.25
1	0.6366	0.4526
3	−0.212	0.4752
5	0.1273	0.4833
7	−0.091	0.4874
9	0.0707	0.4899
11	−0.058	0.4916
13	0.0490	0.4928
15	−0.0424	0.4937
17	0.0374	0.4944
19	−0.0335	0.4949
21	0.0303	0.4954

We observe that the normalized average power in the periodic pulse train is 0.5 W as shown below:

$$P_x = \frac{1}{T_o}\int_{-T_o/2}^{T_o/2} |g_{T_o}(t)|^2\,dt = \frac{1}{T_o}\int_{-T_o/2}^{T_o/2} \Pi^2\!\left(\frac{t}{\tau}\right)dt = \frac{1}{T_o}\int_{-\tau/2}^{\tau/2} dt = \frac{\tau}{T_o} = 0.5$$

2.4.3 Convergence of Fourier Series

It is interesting to consider the sequence of signals that we obtain as we incorporate finite number of terms into the Fourier series of a signal. The partial sum representing the FS approximation to $x_p(t)$ can be expressed as

$$S_N(t) = \sum_{n=-N}^{N} C_n e^{j2\pi n f_o t} \tag{2.61}$$

It is quite reasonable to expect that as more and more terms are included in the partial sum (2.61), the approximation should get better and better, yielding zero approximation error as $n \to \infty$. The fundamental result on the convergence of Fourier series, due to Dirichlet, states that the approximation error

$$\varepsilon_N = \max_{t\in[0,\, T_o]} |x_p(t) - S_N(t)| \tag{2.62}$$

decreases to zero as $n \to \infty$ for all values of t where the function is continuous. However, if the function is discontinuous at a point t_o, the partial sum $S_N(t)$ at $t = t_o$ converges to the value $\frac{x_p(t_o^+) + x_p(t_o^-)}{2}$. Thus the maximum error is always half the size of the jump in the waveform at the discontinuity point. However, on each side of the discontinuity, $S_N(t)$ has oscillatory **overshoot** with peak value of about 9% of the size of the discontinuity as shown in Figure 2.27. This behavior is independent of value of N except that the period of oscillation changes to $T_o/2N$. This is also known as **Gibbs phenomenon** in the theory of Fourier series.

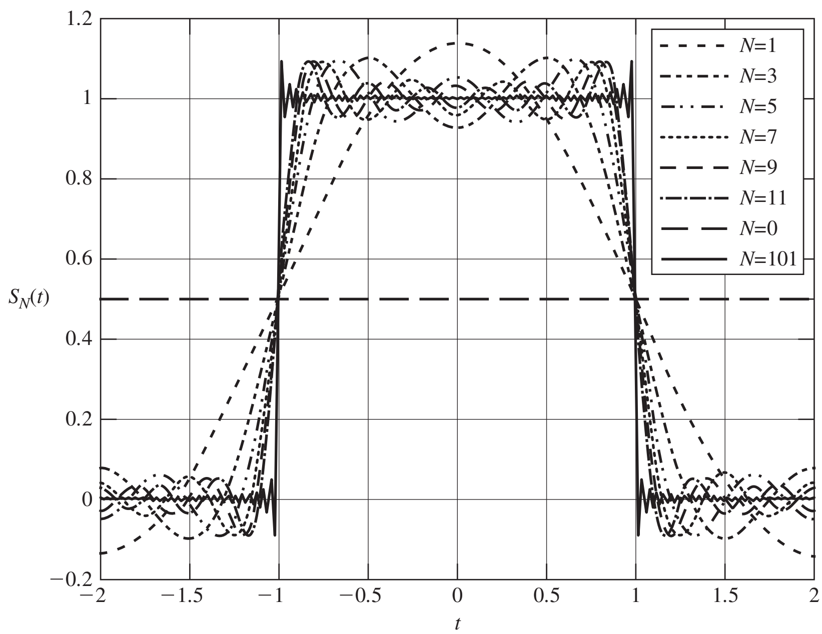


Figure 2.27 FS approximations to a rectangular pulse train.

2.5 FOURIER TRANSFORM

The Fourier coefficients of the rectangular pulse train are spaced at a discrete set of frequencies $nf_o = n/T_o$. The component frequencies are separated by $1/T_o$ Hz. Note that as the period T_o gets larger, the frequency separation gets smaller. Obviously, as the period approaches infinity, the frequency separation tends to zero. Thus a nonperiodic function, which can be viewed as a periodic function with infinite period, contains all frequencies in its Fourier expansion rather than just a discrete set. The Fourier expansion of a nonperiodic function is defined in terms of an integral rather than the infinite sum in (2.49). That is,

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}\,df \tag{2.63}$$

where $X(f) = \Im\{x(t)\}$ is the **Fourier transform (FT)** of the signal $x(t)$. It is defined by the following formula

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\,dt \tag{2.64}$$

We will use the following notation to denote the FT and its inverse operation:

FT operation

$$x(t) \xrightarrow{\Im} X(f)\colon X(f) = \Im\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\,dt$$

Inverse FT operation

$$X(f) \xrightarrow{\Im^{-1}} x(t)\colon x(t) = \Im^{-1}\{X(f)\} = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}\,df$$

Most frequently, the notation $\overset{\Im}{\longleftrightarrow}$ will be used to denote either the FT operation or its inverse. The meaning will be obvious from the context. Equations (2.63) and (2.64) form the Fourier transform pair.

We observe from (2.64) that $X(f)$ is defined over all frequencies f and plays the same role for nonperiodic signals as Fourier coefficients C_n do for periodic signals. $X(f)$ is called the **frequency spectrum** of the nonperiodic signal $x(t)$. It is a continuous spectrum as opposed to the line spectrum produced by coefficients C_n for a periodic signal. In general, $X(f)$ is a complex function of the real variable f and can be written as

$$X(f) = |X(f)|e^{j\measuredangle X(f)} \tag{2.65}$$

where $|X(f)|$ and $\measuredangle X(f)$ are, respectively, called the **magnitude** and the **phase spectrum** of the signal $x(t)$.

$$\text{For real } x(t),\ X(-f) = \int_{-\infty}^{\infty} x(t)e^{j2\pi ft}dt = X^*(f) \tag{2.66}$$

Comparing magnitude and phase responses of both sides of (2.66) yields

$$|X(-f)| = |X(f)| \tag{2.67}$$

$$\measuredangle X(-f) = -\measuredangle X(f) \tag{2.68}$$

Thus $|X(f)|$ and $\measuredangle X(f)$ are even and odd functions of f, respectively.

2.5.1 Fourier Transforms of Some Common Signals

Rectangular Pulse

$$x(t) = A\Pi(t/\tau)$$

$$X(f) = A\int_{-\infty}^{\infty} e^{-j2\pi ft}dt = A\int_{-\tau/2}^{\tau/2} e^{-j2\pi ft}dt = A\frac{e^{-j\pi f\tau} - e^{j\pi f\tau}}{-j2\pi f} = A\frac{\sin(\pi f\tau)}{\pi f}$$
$$= A\tau\mathrm{sinc}(f\tau) \qquad (2.69)$$

The magnitude and phase spectra of the rectangular pulse are plotted in Figure 2.28. It is interesting to note that the width of the mainlobe increases as the pulse width τ narrows.

Unit Impulse Signal

$$x(t) = \delta(t)$$

$$X(f) = \int_{-\infty}^{\infty} \delta(t)e^{-j2\pi ft}\,dt = \int_{-\infty}^{\infty} \delta(t)dt = 1 \qquad (2.70)$$

Equation (2.70) states that the unit impulse signal contains all frequencies with equal magnitudes as shown in Figure 2.29.

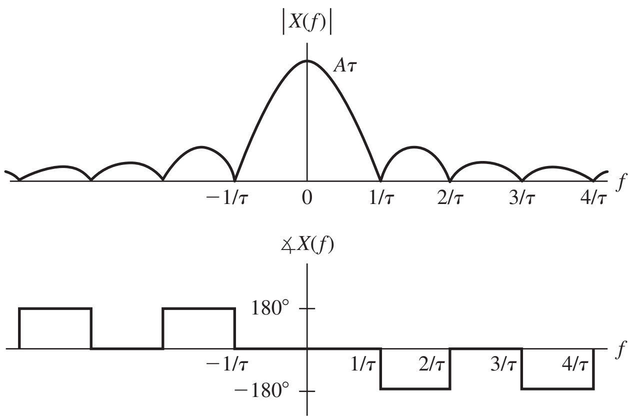


Figure 2.28 Magnitude and phase spectra of the rectangular pulse.

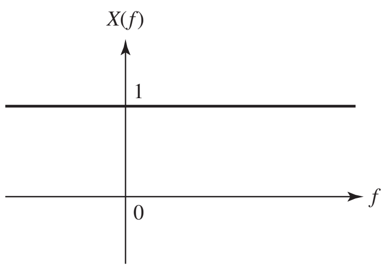


Figure 2.29 Fourier transform of the unit impulse signal.

Complex Exponential Signal

The Fourier transform of the complex exponential signal $e^{j2\pi f_c t}$ is $\delta(f - f_c)$. This can be verified by substituting $\delta(f - f_c)$ in the inverse Fourier transform formula as follows:

$$\int_{-\infty}^{\infty} \delta(f - f_c) e^{j2\pi ft}\, df = \int_{-\infty}^{\infty} \delta(f - f_c) e^{j2\pi f_c t}\, df = e^{j2\pi f_c t} \tag{2.71}$$

Figure 2.30 displays the result. It is intuitively satisfying in that it affirms that the spectrum of a complex sinusoid $e^{j2\pi f_c t}$ contains energy at only single frequency f_c.

Substituting $f_c = 0$ into (2.71), we obtain the FT of a DC signal as

$$1 \xleftrightarrow{\Im} \delta(f).$$

Signum Signal

The signum signal sgn(t) in (2.18) can be expressed as

$$\operatorname{sgn}(t) = \begin{cases} 1, & t \geq 0 \\ -1, & t \leq 0 \end{cases} = \lim_{\alpha \to 0} \begin{cases} e^{-\alpha t}, & t \geq 0 \\ -e^{\alpha t}, & t \leq 0 \end{cases} \tag{2.72}$$

The FT of sgn(t) can now be written as

$$\begin{aligned} \Im\{\operatorname{sgn}(t)\} &= \lim_{\alpha \to 0} \left[-\int_{-\infty}^{0} e^{\alpha t} e^{-j2\pi ft}\, dt + \int_{0}^{\infty} e^{-\alpha t} e^{-j2\pi ft}\, dt \right] \\ &= \lim_{\alpha \to 0} \left[-\int_{-\infty}^{0} e^{(\alpha - j2\pi f)t}\, dt + \int_{0}^{\infty} e^{-(\alpha + j2\pi f)t}\, dt \right] \\ &= \lim_{\alpha \to 0} \frac{-4j\pi f}{\alpha^2 + 4\pi^2 f^2} = \frac{1}{j\pi f} \end{aligned} \tag{2.73}$$

Unit Step Signal

The step function $u(t)$ can be expressed as

$$u(t) = \frac{1}{2} + \frac{1}{2}\operatorname{sgn}(t) \tag{2.74}$$

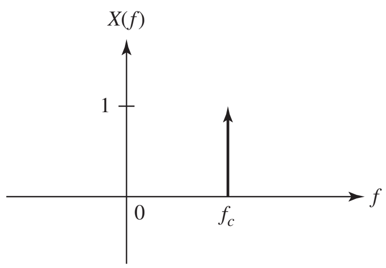


Figure 2.30 Fourier transform of a complex exponential signal.

Taking the FT of both sides of (2.74) yields

$$U(f) = \frac{1}{2}\delta(f) + \frac{1}{j2\pi f} \tag{2.75}$$

Table 2.2 tabulates Fourier transforms of frequently used signal types.

2.5.2 Properties of Fourier Transform

There are a number of important properties of the Fourier transform that are useful in the analysis and design of communication systems.

Linearity

$$\alpha x(t) + \beta y(t) \stackrel{\Im}{\longleftrightarrow} \alpha X(f) + \beta Y(f) \tag{2.76}$$

where α and β are arbitrary real or complex constants.

Table 2.2 Basic Fourier Transform Pairs

Time Function $x(t)$	Fourier Transform $X(f)$
DC signal A	$A\delta(f)$
Rectangular pulse $\Pi(t/\tau)$	$\tau \mathrm{sinc}(f\tau)$
Triangular pulse $\Lambda(t/\tau)$	$\frac{\tau}{2}\mathrm{sinc}^2\left(\frac{f\tau}{2}\right)$
Decaying exponential $e^{-\alpha t}u(t)$	$\frac{1}{\alpha + j2\pi f}$
$e^{-\alpha\|t\|}$	$\frac{2\alpha}{\alpha^2 + (2\pi f)^2}$
Unit impulse $\delta(t)$	1
$\delta(t - t_o)$	$e^{-j2\pi f t_o}$
Sinc pulse $\mathrm{sinc}(2Wt)$	$\frac{1}{2W}\Pi(f/2W)$
Complex sinusoid $e^{j2\pi f_c t}$	$\delta(f - f_c)$
Sinusoid $\sin(2\pi f_c t)$	$\frac{1}{2j}[\delta(f - f_c) - \delta(f + f_c)]$
Sinusoid $\cos(2\pi f_c t)$	$\frac{1}{2}[\delta(f - f_c) + \delta(f + f_c)]$
Gaussian pulse $e^{-\pi t^2}$	$e^{-\pi f^2}$
$\mathrm{sgn}(t)$	$\frac{1}{j\pi f}$
Unit step $u(t)$	$\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$
$\frac{1}{\pi t}$	$-j\mathrm{sgn}(f)$
$\delta'(t)$	$j2\pi f$

To prove this property, let us take the FT of the left-hand side to obtain

$$\alpha x(t) + \beta y(t) \overset{\Im}{\longleftrightarrow} \int_{-\infty}^{\infty} [\alpha x(t) + \beta y(t)] e^{-j2\pi f t} dt = \alpha \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt + \beta \int_{-\infty}^{\infty} y(t) e^{-j2\pi f t} dt$$
$$= \alpha X(f) + \beta Y(f)$$

Time Shifting

$$x(t - t_o) \overset{\Im}{\longleftrightarrow} X(f) e^{-j2\pi f t_o} \tag{2.77}$$

This can be proved by using the inverse Fourier transform formula as follows:

$$x(t - t_o) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f (t - t_o)} df = \int_{-\infty}^{\infty} [X(f) e^{-j2\pi f t_o}] e^{j2\pi f t} df = \Im^{-1}\{X(f) e^{-j2\pi f t_o}\}$$

The Fourier transform of a signal $x(t)$ represents the magnitude and phase of all the frequency components in it. Now, if $x(t)$ is shifted in time by t_o seconds, it is equivalent to shifting all the component sinusoids by the same amount. This does not change their magnitudes, so the magnitude of $X(f)$ remains unchanged with a time shift. However, the phase of each constituent sinusoid does change with a time shift, and the higher the frequency of the sinusoid, the larger the phase change.

Frequency Translation

$$x(t) e^{j2\pi f_c t} \overset{\Im}{\longleftrightarrow} X(f - f_c) \tag{2.78}$$

Taking the Fourier transform of the left-hand side yields

$$\int_{-\infty}^{\infty} x(t) e^{j2\pi f_c t} e^{-j2\pi f t} dt = \int_{-\infty}^{\infty} x(t) e^{-j2\pi (f - f_c) t} dt = X(f - f_c)$$

This property states that multiplication of a signal $x(t)$ by $e^{j2\pi f_c t}$ translates its frequency spectrum $X(f)$ by the amount f_c (to the right on a graph). Communication systems often use frequency translation to assign frequency slots within a shared frequency spectrum to individual users on a demand basis—as in cellular telephone networks, for example.

Convolution

$$x(t) \otimes y(t) \overset{\Im}{\longleftrightarrow} X(f) Y(f) \tag{2.79}$$

To prove this property, recall that $x(t) \otimes y(t) = \int_{-\infty}^{\infty} x(\tau) y(t - \tau) d\tau$. If we take the Fourier transform of the right-hand side and exchange the order of integration, we get

$$\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau) y(t - \tau) d\tau \right] e^{-j2\pi f t} dt = \int_{-\infty}^{\infty} d\tau x(\tau) \left[\int_{-\infty}^{\infty} y(t - \tau) e^{-j2\pi f t} dt \right]$$
$$= \int_{-\infty}^{\infty} x(\tau) Y(f) e^{-j2\pi f \tau} d\tau$$

where we have used the time-shifting property of the Fourier transform on $y(t - \tau)$. Now taking $Y(f)$ outside the integral yields

$$\int_{-\infty}^{\infty} x(\tau)y(t - \tau)d\tau \overset{\Im}{\longleftrightarrow} Y(f)\int_{-\infty}^{\infty} x(\tau)e^{-j2\pi f\tau}d\tau = Y(f)X(f)$$

Equation (2.79) states that the convolution operation in the time domain is *equivalent* to multiplication in the frequency domain. This is a very useful result.

Time/Frequency Scaling

$$x(at) \overset{\Im}{\longleftrightarrow} \frac{1}{|a|}X\left(\frac{f}{a}\right) \tag{2.80}$$

To prove this property, we first assume $a > 0$. Using the change of variables, $u = at$, we have

$$x(at) \overset{\Im}{\longleftrightarrow} \int_{-\infty}^{\infty} x(at)e^{-j2\pi ft}dt = \frac{1}{a}\int_{-\infty}^{\infty} x(u)e^{-j2\pi(f/a)u}\,du = \frac{1}{a}X\left(\frac{f}{a}\right)$$

Now with $a < 0$, substituting $u = -|a|t$ yields

$$x(at) \overset{\Im}{\longleftrightarrow} \int_{-\infty}^{\infty} x(-|a|t)e^{-j2\pi ft}dt = \frac{1}{|a|}\int_{-\infty}^{\infty} x(u)e^{j2\pi(f/|a|)u}\,du = \frac{1}{|a|}X\left(-\frac{f}{|a|}\right) = \frac{1}{|a|}X\left(\frac{f}{a}\right)$$

The function $x(at)$, for $a > 0$, is a time-compressed (by a factor a) version of $x(t)$. On the other hand, a function $X(f/a)$ represents a function $X(f)$ expanded by the same factor a. The scaling property therefore states that compressing a signal in the time domain will stretch its Fourier transform. Similarly, stretching a time signal will compress its Fourier transform. The result is intuitively satisfying because compression in time by the factor a means that the function is varying rapidly in time by the same amount. Consequently, the extent of its frequency spectrum will be increased by the factor a. The converse can also be justified by a similar argument.

Duality

$$\text{If } x(t) \overset{\Im}{\longleftrightarrow} X(f), \text{ then } X(t) \overset{\Im}{\longleftrightarrow} x(-f) \tag{2.81}$$

To prove this property, we begin with

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df$$

Making a change of variable $f = -v$ yields

$$x(t) = \int_{-\infty}^{\infty} X(-v)e^{-j2\pi vt}dv$$

If we set $t = -f$, we get

$$x(-f) = \int_{-\infty}^{\infty} X(-v)e^{j2\pi fv}dv$$

Finally, substituting t for $-v$, we get

$$x(-f) = \int_{-\infty}^{\infty} X(t)e^{-j2\pi ft}dt = \Im\{X(t)\}$$

Example 2.28

Calculate the Fourier transform of the sinc pulse $2W\text{sinc}(t2W)$.

Solution

From Table 2.2, the Fourier transform of a rectangular pulse is a sinc function in the frequency domain.

$$\Pi(t/\tau) \stackrel{\Im}{\longleftrightarrow} \tau\ \text{sinc}(f\tau)$$

Using the duality property, we obtain

$$2W\text{sinc}(t2W) \stackrel{\Im}{\longleftrightarrow} \Pi(f/2W)$$

Thus the Fourier transform of a sinc pulse is a rectangular function in frequency.

Differentiation Property

$$\frac{d}{dt}x(t) \stackrel{\Im}{\longleftrightarrow} j2\pi f X(f) \tag{2.82}$$

To prove this, we have

$$\frac{d}{dt}x(t) = \frac{d}{dt}\int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df = \int_{-\infty}^{\infty} X(f)\left(\frac{d}{dt}e^{j2\pi ft}\right)df = \int_{-\infty}^{\infty} [j2\pi fX(f)]e^{j2\pi ft}df \tag{2.83}$$

From (2.83) we conclude that

$$\Im^{-1}\{j2\pi fX(f)\} = \frac{d}{dt}x(t)$$

or

$$\Im\left\{\frac{d}{dt}x(t)\right\} = j2\pi fX(f)$$

Equation (2.82) states that the differentiation in the time domain is equivalent to multiplication by $j2\pi f$ in the frequency domain. With repeated application of the differentiation property, we obtain the following relation

$$\Im\left\{\frac{d^n}{dt^n}x(t)\right\} = (j2\pi f)^n X(f) \tag{2.84}$$

Differentiation in Frequency Domain

$$tx(t) \overset{\Im}{\longleftrightarrow} \frac{j}{2\pi}\frac{d}{df}X(f) \tag{2.85}$$

The proof follows the same basic steps as involved in proving the differentiation theorem.

Integration Property

$$\int_{-\infty}^{t} x(\tau)d\tau \overset{\Im}{\longleftrightarrow} \frac{X(f)}{j2\pi f} + \frac{1}{2}X(0)\delta(f) \tag{2.86}$$

To prove this, we first note that

$$\int_{-\infty}^{t} x(\tau)d\tau = x(t) \otimes u(t)$$

Using the convolution property of the FT, we can write

$$\Im\left\{\int_{-\infty}^{t} x(\tau)d\tau\right\} = X(f)U(f) \tag{2.87}$$

Substituting (2.75) into (2.87) yields

$$\Im\left\{\int_{-\infty}^{t} x(\tau)d\tau\right\} = X(f)\left\{\frac{1}{j2\pi f} + \frac{1}{2}\delta(f)\right\} = \frac{X(f)}{j2\pi f} + \frac{1}{2}X(0)\delta(f)$$

Parseval's Relation

$$\int_{-\infty}^{\infty} x(t)y^*(t)dt = \int_{-\infty}^{\infty} X(f)Y^*(f)df \tag{2.88}$$

To prove this, we substitute

$$y^*(t) = \int_{-\infty}^{\infty} Y^*(f)e^{-j2\pi ft}df$$

into the left-hand side of (2.88) and exchanging the order of integration yields

$$\int_{-\infty}^{\infty} x(t)y^*(t)dt = \int_{-\infty}^{\infty} x(t)\left[\int_{-\infty}^{\infty} Y^*(f)e^{-j2\pi ft}df\right]dt$$

$$= \int_{-\infty}^{\infty} Y^*(f)\left[\int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt\right]df = \int_{-\infty}^{\infty} Y^*(f)X(f)df$$

If we let $y(t) = x(t)$ in Parseval's formula, we get the well-known relationship for the energy of a signal in time and frequency domains.

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2dt = \int_{-\infty}^{\infty} |X(f)|^2df \tag{2.89}$$

Equation (2.89) states that the energy of a signal is given by the area under the $|X(f)|^2$ curve. $|X(f)|^2$ is called the **energy density spectrum** of $x(t)$. Note that the quantity $|X(f_o)|^2\ \Delta f$ represents the energy contained in a spectral band of Δf Hz centered at frequency f_o. Thus $|X(f)|^2$ may be interpreted as the energy per Hz of bandwidth contained in spectral components of $x(t)$ centered at frequency f. It is specified in units of Joules/Hz.

Table 2.3 summarizes important properties of the Fourier transform. The following example illustrates how the properties in Table 2.3 may be used to calculate the FT of other signals not listed in Table 2.2.

Table 2.3 Fourier Transform Properties

Property	Time Function $x(t)$ $y(t)$	Fourier Transform $X(f)$ $Y(f)$
Linearity	$\alpha x(t) + \beta y(t)$	$\alpha X(f) + \beta Y(f)$
Time-shifting	$x(t - t_o)$	$X(f)e^{-j2\pi ft_o}$
Frequency translation	$x(t)e^{j2\pi f_c t}$	$X(f - f_c)$
Convolution	$x(t) \otimes y(t)$	$X(f)Y(f)$
Multiplication	$x(t)y(t)$	$X(f) \otimes Y(f)$
Time/Frequency scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{f}{a}\right)$
Duality	$X(t)$	$x(-f)$
Differentiation in time	$\frac{d}{dt}x(t)$	$j2\pi f X(f)$
Differentiation in frequency	$tx(t)$	$\frac{j}{2\pi}\frac{d}{df}X(f)$
Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{X(f)}{j2\pi f} + \frac{1}{2}X(0)\delta(f)$
Parseval's relation	$\int_{-\infty}^{\infty} x(t)y^*(t)dt = \int_{-\infty}^{\infty} X(f)Y^*(f)df$	

Example 2.29

Calculate the FT of the signals in Figure 2.31(a) and (b).

Solution

a. $x_1(t) = -\Pi\left[\frac{2(t + \tau/4)}{\tau}\right] + \Pi\left[\frac{(t - \tau/2)}{\tau}\right]$

Now

$$\Pi\left(\frac{2t}{\tau}\right) \overset{\Im}{\longleftrightarrow} \frac{\tau}{2}\,\mathrm{sinc}\left(\frac{f\tau}{2}\right)$$

Applying the time-shifting property of the FT yields

$$\Pi\left[\frac{2(t + \tau/4)}{\tau}\right] \overset{\Im}{\longleftrightarrow} \frac{\tau}{2}\,\mathrm{sinc}\left(\frac{f\tau}{2}\right)e^{j\pi f\tau/2}$$

and

$$\Pi\left[\frac{(t - \tau/2)}{\tau}\right] \overset{\Im}{\longleftrightarrow} \tau\,\mathrm{sinc}(f\tau)e^{-j\pi f\tau}$$

Adding

$$X_1(f) = -\frac{\tau}{2}\,\mathrm{sinc}\left(\frac{f\tau}{2}\right)e^{j\pi f\tau/2} + \tau\,\mathrm{sinc}(f\tau)e^{-j\pi f\tau}$$

b. As shown in Figure 2.31(c), differentiating $x_2(t)$ yields

$$\frac{dx_2(t)}{dt} = \Pi(t + 1/2) - \delta(t - 1)$$

Taking the FT of both sides and using the differentiation property, we obtain

$$j2\pi f X_2(f) = \Im\{\Pi(t + 1/2) - \delta(t - 1)\}$$

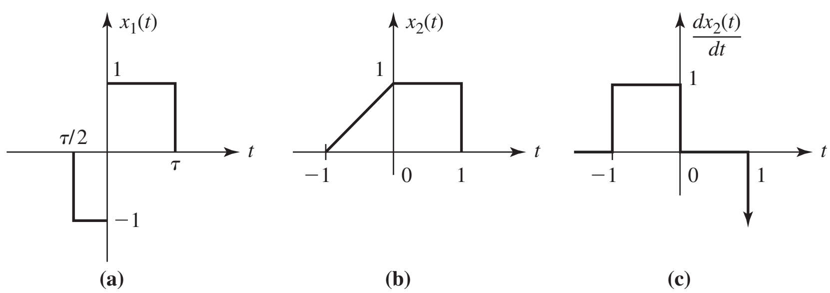


Figure 2.31 Waveforms: Example 2.29.

Now $\Pi(t) \overset{\Im}{\longleftrightarrow} \mathrm{sinc}(f)$. Applying the time-shifting property of the FT, we can write

$$\Pi(t + 1/2) \overset{\Im}{\longleftrightarrow} \mathrm{sinc}(f)e^{j\pi f}$$

Also $\delta(t) \overset{\Im}{\longleftrightarrow} 1$. Again, using the time-shifting property of the FT, we get

$$\delta(t - 1) \overset{\Im}{\longleftrightarrow} e^{-j2\pi f}$$

Adding

$$j2\pi f X_2(f) = \mathrm{sinc}(f)e^{j\pi f} - e^{-j2\pi f}$$

or

$$X_2(f) = \frac{1}{j2\pi f}[\mathrm{sinc}(f)e^{j\pi f} - e^{-j2\pi f}]$$

2.5.3 Fourier Transforms of Periodic Signals

The Fourier transform is strictly defined for finite energy signals. However, the Fourier transform of a real sinusoidal signal exists, as is evident from Table 2.2. Thus it is possible to formally determine the Fourier transform of a periodic signal by taking the Fourier transform of its complex Fourier series term by term. The FS expansion for a periodic function $x_p(t)$ can be written from (2.49) as

$$x_p(t) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi n f_o t}$$

Taking the Fourier transform of both sides, we have

$$X_p(f) = \Im\left\{\sum_{n=-\infty}^{\infty} C_n e^{j2\pi n f_o t}\right\} = \sum_{n=-\infty}^{\infty} C_n \Im\{e^{j2\pi n f_o t}\} = \sum_{n=-\infty}^{\infty} C_n \delta(f - nf_o) \quad (2.90)$$

Equation (2.90) states that the FT of a periodic signal consists of impulses located at harmonic frequencies of the signal. The weight of the impulse located at $f = nf_o$ in the FT $X_p(f)$, denoted by $X_p(nf_o)$, is equal to the corresponding coefficient in the exponential FS expansion of $x_p(t)$. That is,

$$X_p(nf_o) = C_n \quad (2.91)$$

Example 2.30

Calculate the Fourier transform of a cosine wave $A\cos(2\pi f_c t + \phi)$.

Solution

$$x(t) = A\cos(2\pi f_c t + \phi) = \frac{A}{2}e^{j(2\pi f_c t + \phi)} + \frac{A}{2}e^{-j(2\pi f_c t + \phi)}$$

Taking the FT of both sides and using Table 2.2, we obtain

$$X(f) = \frac{A}{2}e^{j\phi}\delta(f - f_c) + \frac{A}{2}e^{-j\phi}\delta(f + f_c) \tag{2.92}$$

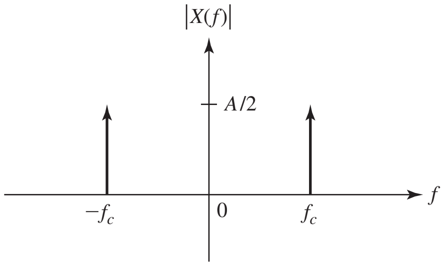


Figure 2.32 Fourier transform of $A\cos(2\pi f_c t + \phi)$.

Figure 2.32 displays the magnitude of the Fourier transform of $A\cos(2\pi f_c t + \phi)$. Similarly, it can be shown that

$$A\sin(2\pi f_c t + \phi) \overset{\Im}{\longleftrightarrow} \frac{A}{2j}e^{j\phi}\delta(f - f_c) - \frac{A}{2j}e^{-j\phi}\delta(f + f_c) \tag{2.93}$$

Example 2.31

Determine the FT of the periodic impulse train displayed in Figure 2.33.

Solution

The periodic impulse train with period T_o is given by

$$\delta_p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_o) \tag{2.94}$$

The FS expansion for this signal can be expressed as

$$\delta_p(t) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi n f_o t} \tag{2.95}$$

where

$$f_o = \frac{1}{T_o}$$

$$C_n = \frac{1}{T_o}\int_{T_o} \delta_p(t)e^{-j2\pi n f_o t}dt = \frac{1}{T_o}\int_{-T_o/2}^{T_o/2} \delta(t)e^{-j2\pi n f_o t}dt = \frac{1}{T_o} = f_o \tag{2.96}$$

Substituting into (2.95) yields

$$\delta_p(t) = f_o \sum_{n=-\infty}^{\infty} e^{j2\pi n f_o t} \tag{2.97}$$

Taking the Fourier transform of both sides of (2.97), we obtain

$$\Delta_p(f) = \Im\{\delta_p(t)\} = f_o \sum_{n=-\infty}^{\infty} \delta(f - nf_o) \tag{2.98}$$

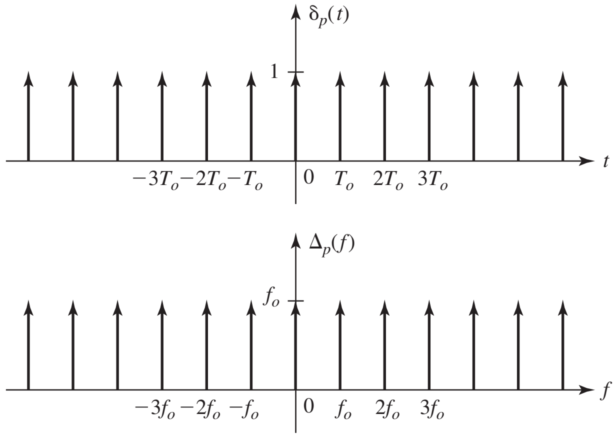


Figure 2.33 Periodic impulse train and its Fourier transform.

Equation (2.98) states that the Fourier transform $\Delta_p(f)$ of a periodic impulse train with period T_o is also a periodic impulse train but with period $f_o = \dfrac{1}{T_o}$. The weight of each impulse in $\Delta_p(f)$ is f_o as displayed in Figure 2.33.

Poisson Sum Formula

Consider the signal $y(t)$ obtained by convolving an energy signal $x(t)$ with the periodic impulse train in (2.94).

$$y(t) = \delta_p(t) \otimes x(t) = x(t) \otimes \sum_{n=-\infty}^{\infty} \delta(t - nT_o) = \sum_{n=-\infty}^{\infty} x(t - nT_o) \qquad (2.99)$$

It is obvious from (2.99) that $y(t)$ is a periodic and is referred to as a **periodic extension** of $x(t)$. Now using the convolution property of Fourier transform and (2.98), we obtain

$$\sum_{n=-\infty}^{\infty} x(t - nT_o) = x(t) \otimes \sum_{n=-\infty}^{\infty} \delta(t - nT_o) \stackrel{\mathfrak{F}}{\longleftrightarrow} X(f)\Delta_p(f)$$

$$= X(f)f_o \sum_{n=-\infty}^{\infty} \delta(f - nf_o) = f_o \sum_{n=-\infty}^{\infty} X(nf_o)\delta(f - nf_o)$$

That is,

$$\sum_{n=-\infty}^{\infty} x(t - nT_o) \stackrel{\mathfrak{F}}{\longleftrightarrow} f_o \sum_{n=-\infty}^{\infty} X(nf_o)\delta(f - nf_o) \qquad (2.100)$$

Now taking the inverse Fourier transform of the right-hand side of (2.100) yields

$$\mathfrak{F}^{-1}\left\{ f_o \sum_{n=-\infty}^{\infty} X(nf_o)\delta(f - nf_o) \right\} = f_o \sum_{n=-\infty}^{\infty} X(nf_o)e^{j2\pi nf_o t} \qquad (2.101)$$

Combining (2.100) and (2.101) yields the Poisson sum formula

$$\sum_{n=-\infty}^{\infty} x(t - nT_o) = \frac{1}{T_o} \sum_{n=-\infty}^{\infty} X(nf_o) e^{j2\pi n f_o t} \tag{2.102}$$

Equation (2.102) states that the sample values $X(nf_o)$ of the Fourier transform of $x(t)$ are the FS coefficients of the periodic signal $T_o \sum_{n=-\infty}^{\infty} x(t - nT_o)$. As a special case, setting $t = 0$ in (2.102), we obtain

$$\sum_{n=-\infty}^{\infty} x(nT_o) = \frac{1}{T_o} \sum_{n=-\infty}^{\infty} X(nf_o) \tag{2.103}$$

2.6 TIME-BANDWIDTH PRODUCT

Recall the scaling property of the Fourier transform, which states that the compression in the time domain is equivalent to the expansion in the frequency domain, and vice versa. Thus, the frequency- and time-domain behaviors of a signal are **inversely** related. This important relationship is captured in the statement that the time-bandwidth product of a signal is constant.

$$\text{Time Duration} \times \text{Frequency Bandwidth} = k \tag{2.104}$$

where k is some constant determined by the precise definitions of **duration** in the time domain and **bandwidth** in the frequency domain. For example,

1. The unit impulse signal, which has zero duration, has a Fourier transform with infinite extent.
2. The sinc signal, which has infinite time duration, has a Fourier transform with finite bandwidth.

Thus, a signal cannot be both duration-limited and bandwidth-limited. We provide the proof of (2.104) by choosing the pulse duration definition as the width Δt of a rectangle whose height matches its peak value (say $x(0)$ for convenience), and whose area is the same as that under the pulse $x(t)$. This is illustrated in Figure 2.34.

$$\Delta t = \frac{\int_{-\infty}^{\infty} x(t)dt}{x(0)} \tag{2.105}$$

We define the bandwidth Δf in a similar manner as

$$\Delta f = \frac{\int_{-\infty}^{\infty} X(f)df}{X(0)} \tag{2.106}$$

The product of these two is

$$\Delta f \Delta t = \frac{\int_{-\infty}^{\infty} X(f)df}{X(0)} \frac{\int_{-\infty}^{\infty} x(t)dt}{x(0)} \tag{2.107}$$

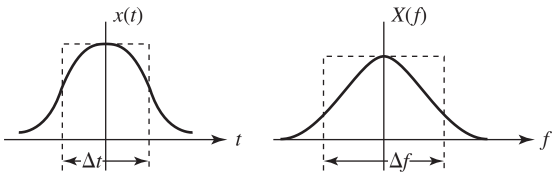


Figure 2.34 Definition of pulse width and bandwidth.

Now from the definition of Fourier transform pair, we have

$$X(0) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt\big|_{f=0} = \int_{-\infty}^{\infty} x(t)dt \tag{2.108}$$

$$x(0) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df\big|_{t=0} = \int_{-\infty}^{\infty} X(f)df \tag{2.109}$$

Substituting into (2.107) yields

$$\Delta f \Delta t = 1 \tag{2.110}$$

(2.110) states that the product of the pulse duration and its bandwidth is unity. If we instead use **root-mean-square (RMS)** definitions for signal duration and bandwidth, it can be shown that

$$\Delta f_{rms} \Delta t_{rms} \geq \frac{1}{4\pi} \tag{2.111}$$

where

$$\Delta t_{rms} = \frac{\int_{-\infty}^{\infty} t^2 |x(t)|^2 dt}{\int_{-\infty}^{\infty} |x(t)|^2 dt} \tag{2.112}$$

$$\Delta f_{rms} = \frac{\int_{-\infty}^{\infty} f^2 |X(f)|^2 df}{\int_{-\infty}^{\infty} {}^2|X(f)|^2 df} \tag{2.113}$$

For a Gaussian pulse $e^{-\pi t^2}$, (2.111) is satisfied with the equality sign. Again, the time duration and bandwidth of a signal have an inverse relationship. This leads to the general conclusion that the time duration of a signal and its bandwidth cannot both be made arbitrarily small. The relationship (2.111) is also known as the **uncertainty principle** in quantum physics, where Δf_{rms} and Δt_{rms} are interpreted as resolutions in frequency and time, respectively. Frequency resolution means the ability to clearly identify signal components that are concentrated at particular frequencies, and time resolution implies the precision to clearly identify signal events that manifest during a short time interval. The uncertainty principle sets a fundamental limit on resolution in both time and frequency.

2.7 TRANSMISSION OF SIGNALS THROUGH LTI SYSTEMS

An LTI system is completely characterized in the time domain by its impulse response function $h(t)$. Recall that the input–output relationship in the time domain of an LTI system with impulse response function $h(t)$ is given by the convolution integral of (2.41) and is of the form

$$y(t) = x(t) \otimes h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \tag{2.41}$$

where $y(t)$ and $x(t)$ are, respectively, the output and the input signals. We now consider the response of an LTI system to complex exponential signal $e^{j2\pi f_c t}$. From (2.41), the output is given by

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} e^{j2\pi f_c \tau} h(t-\tau)d\tau = \int_{-\infty}^{\infty} e^{j2\pi f_c (t-u)} h(u)du = e^{j2\pi f_c t} \underbrace{\int_{-\infty}^{\infty} e^{-j2\pi f_c u} h(u)du}_{H(f_c)} \\ &= e^{j2\pi f_c t} H(f_c) \end{aligned} \tag{2.114}$$

where

$$H(f) \stackrel{\Im}{\longleftrightarrow} h(t)$$

Note that $H(f)$ is FT of the impulse response function $h(t)$ and provides frequency domain description of the system. It is called **frequency response function** of the LTI system. (2.114) states that for a complex exponential input of frequency f_c, the output $y(t)$ is also a complex exponential signal of the *same* frequency but *scaled* by the complex weight $H(f_c)$. We can write (2.114) as

$$y(t) = |H(f_c)| e^{j[2\pi f_c t + \measuredangle H(f_c)]} \tag{2.115}$$

It is obvious from (2.115) that the value of $H(f)$ at f_c determines the magnitude and phase shifts introduced by the system in passing the input complex exponential signal from input to output. Equation (2.115) implies that the response of the system to a real sinusoidal input signal $x(t) = \cos(2\pi f_c t)$ of frequency f_c is given by

$$y(t) = |H(f_c)| \cos[2\pi f_c t + \measuredangle H(f_c)] \tag{2.116}$$

Similarly, the response of the system to sinusoidal input signal $x(t) = \sin(2\pi f_c t)$ is

$$y(t) = |H(f_c)| \sin[2\pi f_c t + \measuredangle H(f_c)] \tag{2.117}$$

For a periodic input signal $x_p(t)$ represented by its Fourier series (2.49), the output of an LTI system can be obtained by applying (2.114) to each discrete frequency component as follows:

$$y_p(t) = \sum_{n=-\infty}^{\infty} C_n H(nf_o) e^{j2\pi n f_o t} \tag{2.118}$$

Equation (2.118) states that the output of an LTI system to a periodic input is *also* periodic and FS coefficients of the output signal are given by $C_n H(nf_o)$.

Example 2.32

The frequency response $H(f)$ of an LTI system is given by

$$H(f) = \frac{2}{1 + j0.0025\pi f}$$

Determine the output $y(t)$ for an input $x(t) = \sin(800\pi t) + 2\sin(2000\pi t + \pi/4)$.

Solution

The response of the system to input sinusoidal signal $\sin(800\pi t)$ is given by

$$H(400) = 0.184 - j0.578$$

This can be represented in the magnitude-phase form as

$$|H(400)| = 0.6066$$

$$\measuredangle H(400) = -72.34°$$

The output of the system for an input $\sin(800\pi t)$ can now be expressed using (2.117) as $0.6066\sin(800\pi t - 72.34°)$.

The response of the system to input sinusoidal signal $2\sin(2000\pi t + \pi/4)$ is

$$H(1000) = 0.032 - j0.2506$$

This can be represented in the magnitude-phase form as

$$|H(1000)| = 0.2526$$

$$\measuredangle |H(1000)| = -82.75^o$$

The output of the system for an input $2\sin(2000\pi t + \pi/4)$ can now be expressed using (2.117) as $0.5052\sin(2000\pi t - 82.75^o + \pi/4)$.

Therefore, the combined output is

$$y(t) = 0.6066\sin(800\pi t - 72.34^o) + 0.5052\sin(2000\pi t - 82.75^o + \pi/4)$$

For any arbitrary input, the frequency domain response of an LTI system can be obtained by applying the FT to both sides of (2.41) and using the convolution property (2.79).

$$Y(f) = X(f)H(f) \tag{2.119}$$

Equation (2.119) states that the output of the system in the frequency domain is given by multiplying the Fourier transform of the input by the system frequency response $H(f)$. $H(f)$, in general, is a complex function of f and can be expressed in the magnitude-phase form as

$$H(f) = |H(f)|e^{j\measuredangle H(f)} \tag{2.120}$$

where $|H(f)|$ and $\measuredangle H(f)$ are, respectively, called the **magnitude** and the **phase responses** of the system.

Design specifications for the LTI system, in many applications, are given in terms of the magnitude and phase responses. If $h(t)$ is a real function of time, then it follows from (2.67) and (2.68) that $|H(f)|$ is an even function of f

$$|H(f)| = |H(-f)| \tag{2.121}$$

and the phase function $\measuredangle H(f)$ is an odd function of f

$$\measuredangle H(f) = -\measuredangle H(-f) \tag{2.122}$$

In the frequency domain, the magnitude and the phase of the system input and output are related by

$$|Y(f)| = |X(f)||H(f)| \tag{2.123}$$

$$\measuredangle Y(f) = \measuredangle X(f) + \measuredangle H(f) \tag{2.124}$$

The magnitude and the phase effects represented by (2.123) and (2.124) can be either *intentional* or *undesirable.* In the latter case, these effects of an LTI system on a signal are referred to as magnitude and phase distortions, respectively.

Example 2.33

Determine the magnitude and phase response of the RC low-pass filter shown in Figure 2.35.

Solution

The transfer function of the RC low-pass filter is obtained from Figure 2.35 as

$$H(f) = \frac{Y(f)}{X(f)} = \frac{1/j2\pi fC}{R + 1/j2\pi fC} = \frac{1}{1 + j2\pi fRC} = \frac{1}{1 + j(f/f_{3\mathrm{dB}})} \tag{2.125}$$

where

$$f_{3\mathrm{dB}} = \frac{1}{2\pi RC} \tag{2.126}$$

The magnitude and phase responses of the RC low-pass filter can now be expressed as

$$|H(f)| = \frac{1}{\sqrt{1 + (f/f_{3\mathrm{dB}})^2}} \tag{2.127}$$

$$\measuredangle H(f) = -\tan^{-1}(f/f_{3\mathrm{dB}}) \tag{2.128}$$

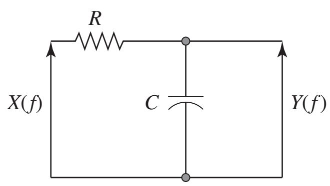


Figure 2.35 RC low-pass filter.

$f_{3\mathrm{dB}}$ is called the **3-dB cutoff frequency** or **3-dB bandwidth** of the low-pass filter because its magnitude-squared response drops by 3 dB (i.e., one-half the power) at $f = f_{3\mathrm{dB}}$ as shown below:

$$10\log_{10}|H(f_{3\mathrm{dB}})|^2 = 20\log_{10}\frac{1}{\sqrt{2}} = -3\ \mathrm{dB} \tag{2.129}$$

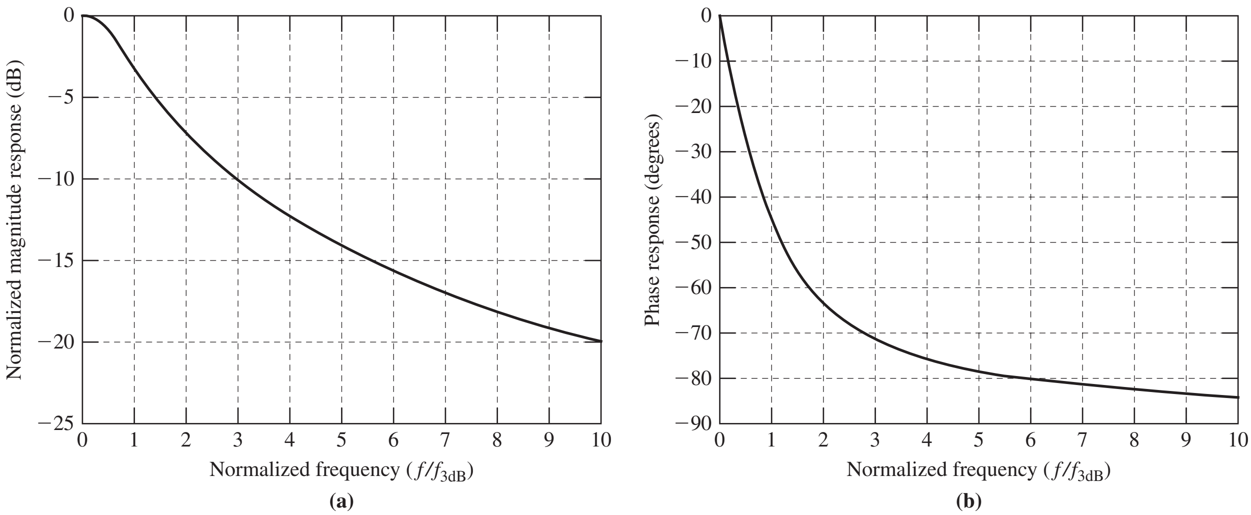


Figure 2.36 (a) Magnitude and (b) phase responses of the RC low-pass filter.

Equation (2.129) states that the input signal frequency component at $f = f_{3dB}$ is attenuated by 3 dB compared with that at $f = 0$ in the filter output waveform. Figure 2.36 shows the magnitude and phase responses of the low-pass filter with 3-dB bandwidth equal to 1 Hz.

Taking inverse FT of both sides of (2.125), the impulse response of the RC low-pass filter is given by

$$h(t) = 2\pi f_{3dB} e^{-2\pi f_{3dB} t} u(t) = \frac{1}{RC} e^{-t/RC} u(t) \tag{2.130}$$

2.7.1 Distortionless Transmission

In general, both the magnitude and phase of spectral components of the input signal will be altered as the signal passes through an LTI system as indicated by (2.123) and (2.124). This amounts to **distortion** in signal transmission. An LTI system is termed **distortionless** if it introduces the same attenuation to all spectral components and offers linear phase response over the frequency band of interest, that is,

$$H_{ideal}(f) = \begin{cases} H_o e^{-j2\pi f t_o}, & f_1 \le f \le f_2 \\ 0, & \text{otherwise} \end{cases} \tag{2.131}$$

Substituting (2.131) into (2.119) yields

$$Y(f) = X(f) H_{ideal}(f) = H_o X(f) e^{-j2\pi f t_o} \tag{2.132}$$

Taking inverse FT of both sides of (2.132), the output of a distortionless LTI system due to an arbitrary input signal $x(t)$ is given by

$$y(t) = H_o x(t - t_o) \tag{2.133}$$

Consequently, the output of a distortionless LTI system is simply a delayed and scaled replica of the input.

Group Delay

The **group delay** of an LTI system is defined as

$$\tau_g(f) \triangleq -\frac{1}{2\pi}\frac{d\measuredangle H(f)}{df} \tag{2.134}$$

The phase response of a distortionless LTI system from (2.131) is a linear function of frequency as given by

$$\measuredangle H_{ideal}(f) = -2\pi f t_o, \;\; f_1 \leq f \leq f_2 \tag{2.135}$$

where t_o is a constant. For a linear phase LTI system, we obtain

$$\tau_g(f) = -\frac{1}{2\pi}\frac{d(-2\pi f t_o)}{df} = t_o = \text{constant} \tag{2.136}$$

Equation (2.136) states that for a linear phase LTI system, the group delay is constant. We interpret the group delay $\tau_g(f)$ as the time delay that a spectral component at frequency f undergoes as it passes through the LTI system. In this case of a linear phase LTI system, all frequency components of the input signal undergo the same time delay.

Phase Delay

The **phase delay** of an LTI system is defined as

$$\tau_p(f) \triangleq -\frac{1}{2\pi f}\measuredangle H(f) \tag{2.137}$$

For a linear phase LTI system, the phase delay is given by

$$\tau_p(f) = -\frac{1}{2\pi f}(-2\pi f t_o) = t_o = \text{constant} \tag{2.138}$$

2.8 LTI SYSTEMS AS FREQUENCY SELECTIVE FILTERS

One key application of LTI systems is to design **filters** that separate the desired information-bearing signal from unwanted interference and noise. By definition, filters pass certain frequency components in the input signal (sequence) with minimum distortion and block all other frequency components.

2.8.1 Ideal Filters

An ideal filter designed to pass signal components of certain frequencies without any distortion should have a magnitude response that is flat over these frequencies and should totally block signal components at all other frequencies. The range of frequencies

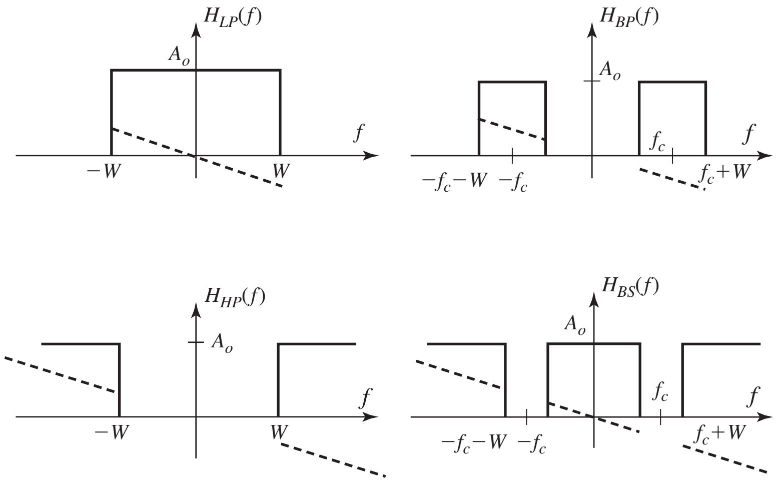


Figure 2.37 Magnitude and phase responses of ideal filters.

where the magnitude response takes the constant value and where the phase response is a linear function of frequency is called the **passband.** The range of frequencies where the frequency response is zero is called the **stopband** of the filter. Figure 2.37 displays the frequency response characteristics of ideal filters.

Ideal Low-Pass Filter

The magnitude response of an ideal **low-pass (LP)** filter is defined as

$$|H_{LP}(f)| = \begin{cases} A_o, & -W \le f \le W \\ 0, & \text{otherwise} \end{cases} \tag{2.139}$$

The range of frequencies $0 \le f \le W$ is the passband of the filter. The range of frequencies $f > W$ is the stopband of the filter.

The frequency response of an ideal LP filter can now be expressed as

$$H_{LP}(f) = A_o \Pi(f/2W) e^{-j2\pi f t_o} \tag{2.140}$$

Taking the inverse Fourier transform yields the impulse response $h_{LP}(t)$ of the ideal LP filter as

$$h_{LP}(t) = 2WA_o \operatorname{sinc}[2W(t - t_o)] \tag{2.141}$$

Figure 2.38 displays the impulse response of an ideal LP filter.

Ideal High-Pass Filter

The magnitude response of an ideal **high-pass (HP)** filter is defined as

$$|H_{HP}(f)| = \begin{cases} 0, & -W \le f \le W \\ A_o, & \text{otherwise} \end{cases} \tag{2.142}$$

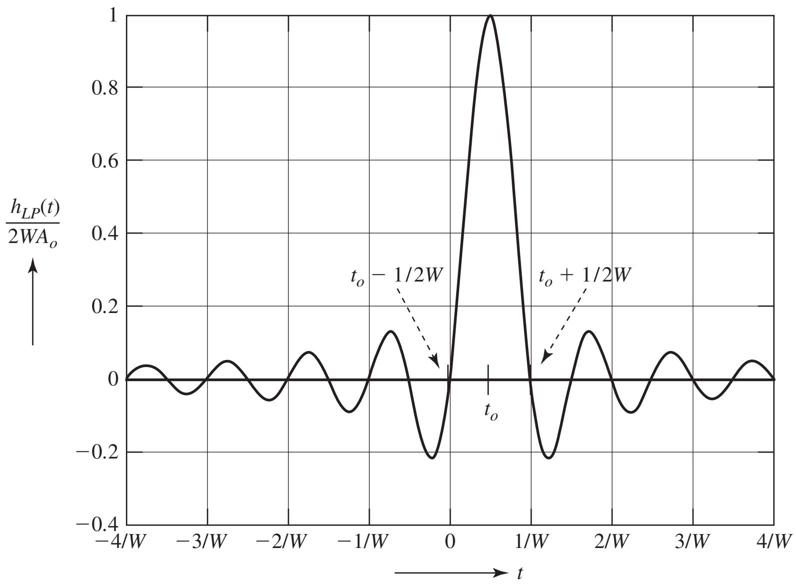


Figure 2.38 Impulse response of an ideal LP filter.

The range of frequencies $f \leq W$ is the stopband of the filter. The range of frequencies $f > W$ is the passband of the filter. The frequency response of an ideal HP filter can now be written as

$$H_{HP}(f) = A_o[1 - \Pi(f/2W)]e^{-j2\pi f t_o} \tag{2.143}$$

Taking the inverse Fourier transform yields the impulse response $h_{HP}(t)$ of the ideal HP filter as

$$h_{HP}(t) = A_o\{\delta(t - t_o) - 2W\,\text{sinc}[2W(t - t_o)]\} \tag{2.144}$$

Ideal Bandpass Filter

The magnitude response of an ideal **bandpass (BP)** filter is defined as

$$|H_{BP}(f)| = \begin{cases} A_o, & f_c - W \leq |f| \leq f_c + W \\ 0, & \text{otherwise} \end{cases} \tag{2.145}$$

The range of frequencies $f_c - W \leq |f| \leq f_c + W$ is the passband of the filter. The range of frequencies $|f| > f_c + W$ and $|f| < f_c - W$ are the stopband regions of the filter. The frequency response of an ideal BP filter can now be written as

$$H_{BP}(f) = H_o(f - f_c) + H_o(f + f_c) \tag{2.146}$$

where

$$H_o(f) = A_o\Pi(f/2W)e^{-j2\pi f t_o} \tag{2.147}$$

is an LP filter with impulse response

$$h_o(t) = 2WA_o\,\text{sinc}[2W(t - t_o)] \tag{2.148}$$

Because

$$H_o(f - f_c) \overset{\mathfrak{F}}{\longleftrightarrow} h_o(t)e^{j2\pi f_c t} \tag{2.149}$$

$$H_o(f + f_c) \overset{\mathfrak{F}}{\longleftrightarrow} h_o(t)e^{-j2\pi f_c t} \tag{2.150}$$

we can now write the impulse response $h_{BP}(t)$ of the bandpass filter as

$$\begin{aligned} h_{BP}(t) &= 4WA_o \operatorname{sinc}[2W(t - t_o)]\left[\frac{e^{j2\pi f_c t} + e^{-j2\pi f_c t}}{2}\right] \\ &= 4WA_o \operatorname{sinc}[2W(t - t_o)] \cos(2\pi f_c t) \end{aligned} \tag{2.151}$$

Thus, the impulse response of the bandpass filter is an oscillatory function. For the important case $f_c >> 2W$, $h_{BP}(t)$ can be viewed as the slowly varying signal $4WA_o \operatorname{sinc}(2Wt)$ shifted by t_o seconds and modulating the high-frequency sinusoidal signal $\cos(2\pi f_c t)$. Figure 2.39 displays the impulse response of the ideal bandpass filter.

Ideal Bandstop Filter

The magnitude response of an ideal **bandstop (BS)** filter is defined as

$$|H_{BS}(f)| = \begin{cases} A_o, & \text{otherwise} \\ 0, & f_c - W \le |f| \le f_c + W \end{cases} \tag{2.152}$$

The range of frequencies $f_c - W \le |f| \le f_c + W$ is the stopband of the filter. The range of frequencies $|f| > f_c + W$ and $|f| < f_c - W$ is the passband region of the filter.

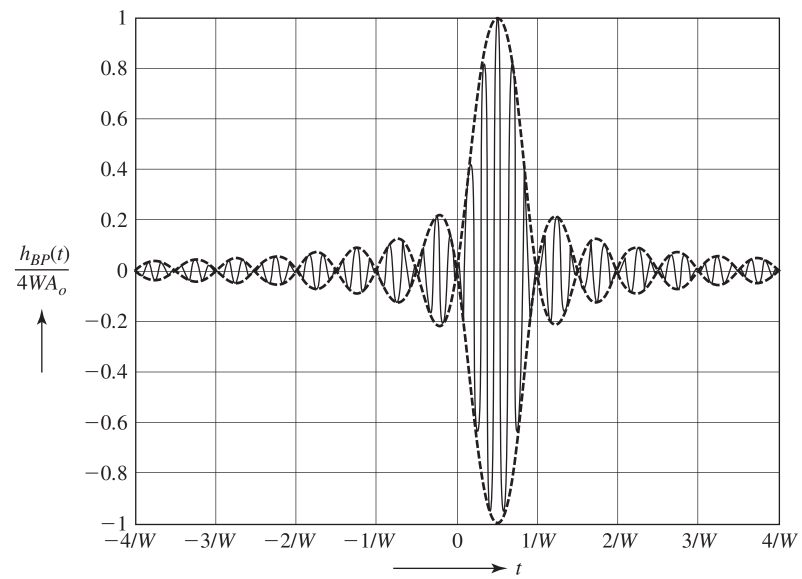


Figure 2.39 Impulse response of an ideal BP filter.

2.8.2 Realizable Approximations to Ideal Filters

In practice, it is impossible to realize a filter with the ideal **brick wall** characteristic of Figure 2.37 because the corresponding impulse response is not causal and extends from $-\infty$ to ∞. In order to develop realizable filter transfer functions, the ideal frequency response specifications of Figure 2.37 are relaxed by including a transition band between the passband and the stopband to permit the magnitude response to decay more gradually from its maximum value in the passband to the zero value in the stopband as shown in Figure 2.40. Moreover, the magnitude response is allowed to vary by a small amount both in the passband and stopband.

The magnitude response specifications for a realizable approximation $|H_a(f)|$ to the ideal brick wall characteristic consists of acceptable tolerances as shown in Figure 2.40. In the **passband** defined by $0 \le f \le f_p$, we require

$$1 - \delta_p \le |H_a(f)| \le 1 + \delta_p \tag{2.153}$$

that is, the magnitude approximates unity within an error of $\pm\delta_p$. In the **stopband,** defined by $f_s \le |f| \le \infty$, we require

$$|H_a(f)| \le \delta_s \tag{2.154}$$

implying that the magnitude approximates zero within an error of δ_s. The frequencies f_p and f_s are, respectively, called the **passband edge frequency** and the **stopband edge frequency.** The limits of the tolerances in the passband and stopband, δ_p and δ_s, are called the **peak ripple values.** The peak passband ripple in dB is

$$R_p = -20\log(1 - \delta_p) \tag{2.155}$$

The minimum stopband attenuation in dB is given by

$$R_s = -20\log(\delta_s) \tag{2.156}$$

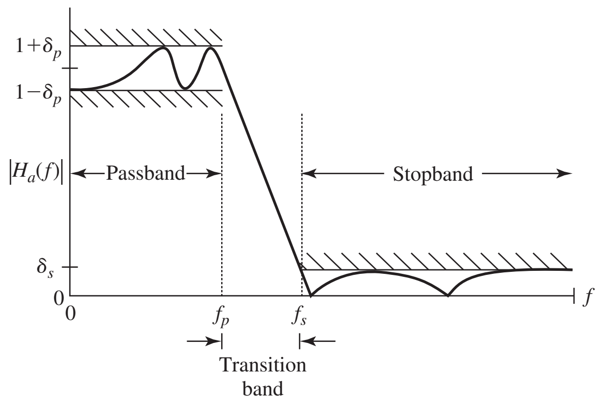


Figure 2.40 Typical magnitude specification of an LP filter.

Now we consider three design approaches to achieve the specifications described in Figure 2.40. These include

1. Butterworth approximation
2. Chebyshev approximation
3. Elliptic approximation

All these methods realize $|H_a(f)|^2$ by an expression of the form

$$|H_a(f)|^2 = \frac{1}{1 + \varepsilon^2 C^2(f)} \tag{2.157}$$

where $C(f)$ is called the **characteristic function,** which is unique to the design approach selected.

Butterworth Approximation

The frequency response of the Butterworth filter is maximally flat (i.e., it has no ripples) in the passband and rolls off toward zero in the stopband. For Butterworth filters,

$$C(f) = (f/f_c)^N \tag{2.158}$$

$$\varepsilon = 1$$

The magnitude-squared response of the Butterworth filter is given by

$$|H_a(f)|^2 = \frac{1}{1 + (f/f_c)^{2N}} \tag{2.159}$$

where f_c is the 3-dB cutoff frequency. Note that as the filter order $N \to \infty$, $|H_a(f)|$ approaches the ideal brick wall characteristic. That is, the passband and the stopband magnitude responses approach the ideal characteristic with a corresponding decrease in the transition band. Butterworth filters have a monotonically decreasing magnitude response with f.

Chebyshev Approximation

Chebyshev filters minimize the peak error between the approximation and the ideal brick wall characteristic over the specified frequency range of the filter. In fact, the magnitude error is equiripple in the passband. For Chebyshev filters,

$$C(f) = T_N(f/f_p) \tag{2.160}$$

where $T_N(x)$ is a Chebyshev polynomial of order N given by the recursion relation

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), n \geq 2 \tag{2.161}$$

and

$$T_1(x) = x, \ T_0(x) = 0 \tag{2.162}$$

The parameter ε specifies peak ripple in the passband. For a Chebyshev filter of fixed order N, there is a trade-off between the ripple and passband width. If one wants a small ripple, then the passband must be narrow. If both a small ripple and a wide passband are required, then a sufficiently large filter order N must be chosen. For a given filter order N, a Chebyshev design provides a sharper transition roll-off than the Butterworth filter. Like the Butterworth, the Chebyshev filter stopband roll-off is monotonic.

Elliptic Approximation

The magnitude-squared response of the Elliptic or Cauer filter of order N is given by

$$|H_a(f)|^2 = \frac{1}{1 + \varepsilon^2 R_N^2(f/f_p)} \tag{2.163}$$

where $R_N(x)$ is an elliptic polynomial of order N and ε determines passband ripple. Although an Elliptic filter achieves faster roll-off than either Butterworth or Chebyshev varieties, it introduces ripple in both the passband and the stopband. Also, the Elliptic filter roll-off is not monotonic, eventually reaching an attenuation limit, called the stopband floor.

Figure 2.41 displays the magnitude response of a sixth-order Elliptic filter designed to achieve 2-dB ripple in the passband (f_p = 2 kHz) and a 50-dB stopband floor (f_s = 2.5 kHz). For comparison, the magnitude responses of the same-order Butterworth and Chebyshev designs are plotted as well. The Elliptic filter has a predictably sharper roll-off characteristic than the other two approximations. However, the faster roll-off in the transition band is accompanied with a nonlinear phase response characteristic as well.

2.8.3 Analog Filter Design Using MATLAB

The **Signal Processing Toolbox** in MATLAB includes a large number of built-in functions to develop analog filter transfer functions for meeting given frequency response specifications. The design procedure consists of two steps:

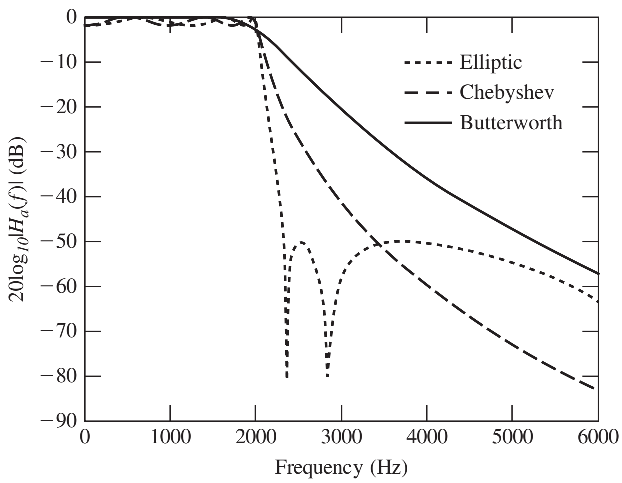


Figure 2.41 Comparison of the frequency responses of three types of analog LP filters.

1. Estimate the order of the filter $H_a(s)$ using any one of the magnitude approximation techniques, that is, Butterworth, Chebyshev, and Elliptic approximations. Specifically, the following M-file functions are available:

```
[N,Wn] = buttord(Wp,Ws,Rp,Rs,'s')
[N,Wn] = cheb1ord(Wp,Ws,Rp,Rs,'s')
[N,Wn] = ellipord(Wp,Ws,Rp,Rs,'s')
```

   where `Wp` and `Ws`, respectively, are passband and stopband edge frequencies in radians/sec with `Wp` < `Ws` for a LP filter. The other two parameters, `Rp` and `Rs`, are the passband ripple and the minimum stopband attenuation in dB, respectively. The outputs of these functions are the filter order `N` and the frequency scaling factor `Wn`. To meet the specified response specifications, `Wn` is a 3-dB angular cutoff frequency in the case of Butterworth design, whereas it is an angular passband edge frequency for Chebyshev and Elliptic filters.
2. Design the LP analog filter $H_a(s)$ using any of the following M-files corresponding to the approximation approach selected:

```
[b,a] = butter(N,Wn,'s')
[b,a] = cheby1(N,Rp,Wn,'s')
[b,a] = ellip(N,Rp,Rs,Wn,'s')
```

   The output data files of these functions are the length $N + 1$ column vectors `b` and `a` providing, respectively, the numerator and denominator coefficients in descending powers of `s`. The form of the transfer function obtained is given by

$$H_a(s) = \frac{B(s)}{A(s)} = \frac{b(1)s^N + b(2)s^{N-1} + \ \ldots \ + b(N)s + b(N+1)}{a(1)s^N + a(2)s^{N-1} + \ \ldots \ + a(N)s + a(N+1)} \tag{2.164}$$

After these coefficients have been calculated, the frequency response can be computed using the M-file function `freqs(b,a,w)`, where `w` is a specified set of angular frequencies (radians/sec). The function `freqs(b,a,w)` generates a complex vector of frequency response samples $H_a(\omega)$ from which magnitude or phase response samples of the filter can be readily computed.

Example 2.34

Design and plot the gain response of an analog Elliptic LP filter with the following specifications:

- Passband frequency $f_p = 800$ Hz
- Stopband frequency $f_s = 1000$ Hz
- Maximum passband ripple $R_p = 1$ dB
- Minimum stopband ripple $R_s = 40$ dB

Solution

To determine the order of the Elliptic filter meeting the specifications, we use the command `[N,Wn] = ellipord(Wp,Ws,Rp,Rs,'s')` with `Wp` $= 2\pi(800)$, `Ws` $= 2\pi(1000)$, `Rp` $= 1$, `Rs` $= 40$. The outputs generated are `N` $= 5$ and `Wn` $= 2\pi(800)$. Next we design the filter using the command `[b,a] = ellip(N,Rp,Rs,Wn,'s')`. Figure 2.42 displays the sample MATLAB code for this example. The magnitude response of the desired filter of order `N` $= 5$ is shown in Figure 2.43.

```
% Program to Design Elliptic Low-pass Filter
%
% Read in passband edge frequency,stopband edge frequency
% passband ripple in dB and minimum stopband
% attenuation in dB
Fp = input('Fp =Passband edge frequency in Hz = ');
Fs = input('Fs =Stopband edge frequency in Hz = ');
Rp = input('Passband ripple in dB = ');
Rs = input('Minimum stopband attenuation in dB = ');
Wp=2*pi*Fp
Ws=2*pi*Fs
%Determine the order of Elliptic filter
[N,Wn] = ellipord(Wp,Ws,Rp,Rs,'s')
%Determine the coefficients of the transfer function
[num,den] = ellip(N,Rp,Rs,Wn,'s');
% Compute and plot the frequency response
omega = [0: 20: 6*Fp*pi];
h = freqs(num,den,omega);
plot (omega/(2*pi),20*log10(abs(h)));
grid on;
title('Magnitude Response of Elliptic LP Filter');
axis([0 3*Fp -80 5])
xlabel('Frequency, Hz');
ylabel('Magnitude Response(dB)');
```

Figure 2.42 MATLAB m-file to design LP Elliptic filter.

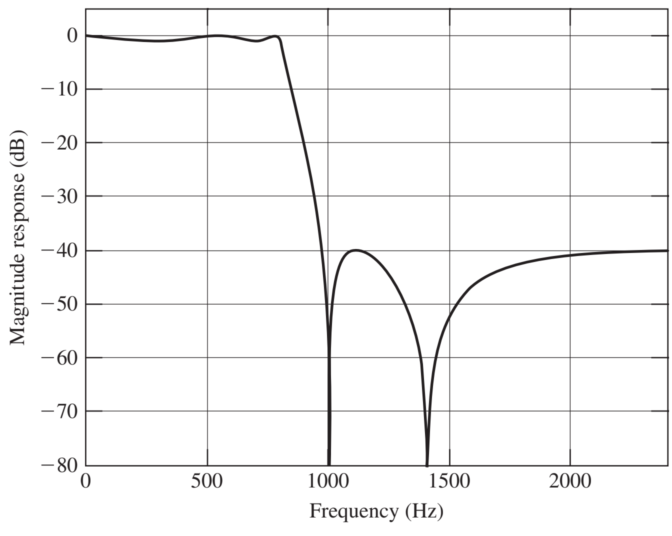


Figure 2.43 Magnitude response of the LP Elliptic filter.

For designing HP, BP, and BS digital filters, the order of the three types of filters can be estimated using the same MATLAB functions as before but with the following differences: For HP filters, `Wp > Ws`. For BP and BS digital filters, `Wp` and `Ws` are vectors of length 2 specifying the transition bandedges. For example, `Wp = [wp1 wp2]` with `wp1<w<wp2`. `wp1` and `wp2` are, respectively, the lower and upper passband edge frequencies. As before, the parameters `Rp` and `Rs` are the passband ripple and the minimum stopband attenuation in dB, respectively. The outputs of these functions are the filter order `N`

and the frequency scaling factor `Wn`. `Wn` is a vector of length 2 for bandpass and bandstop filters. `N` and `Wn` are used as input to the following MATLAB functions for filter design:

```
[b,a] = butter(N,Wn,'filtertype','s')
[b,a] = cheby1(N,Rp,Wn,'filtertype','s')
[b,a] = ellip(N,Rp,Rs,Wn,'filtertype','s')
```

For example, the command

```
[b,a] = butter(N,Wn,'s')
```

is used to design a BP Butterworth filter of order 2N with Wn being a two-element vector. By default, if Wn is a two-element vector, a BP or BS filter is assumed.

For designing high-pass (HP) digital filters, the string `'filtertype'` is `'high'` in all of the preceding commands with Wn being a scalar. For example, the command

```
[b,a] = cheby1(N,Rp,Wn,'high','s')
```

is used to design a HP Chebyshev filter of order N.

Similarly, the string "filtertype" is `'stop'` in all of the preceding commands for designing BS digital filters. For example, the command

```
[b,a] = ellip(N,Rp,Rs,Wn,'stop','s')
```

is used to design a BS Elliptic filter of order 2N with `Wn` being a two-element vector.

Example 2.35

Design and plot the gain response of an Elliptic BP filter with the following specifications:

- Passband edge frequencies $[f_{p1}, f_{p2}] = [4000, 7000]$ Hz
- Stopband edge frequencies $[f_{s1}, f_{s2}] = [3000, 8000]$ Hz
- Maximum passband ripple $R_p = 1$ dB
- Minimum stopband ripple $R_s = 40$ dB

```
% Program to Design Elliptic Bandpass Filter
%
% Read in passband edge frequency,stopband edge frequency
% passband ripple in dB and minimum stopband
% attenuation in dB
Fp = input('Fp =Passband edge frequencies in Hz = ');
Fs = input('Fs =Stopband edge frequencies in Hz = ');
Rp = input('Passband ripple in dB = ');
Rs = input('Minimum stopband attenuation in dB = ');
Wp=2*pi*Fp
Ws=2*pi*Fs
%Determine the order of Elliptic filter
[N,Wn] = ellipord(Wp,Ws,Rp,Rs,'s')
%Determine the coefficients of the transfer function
[num,den] = ellip(N,Rp,Rs,Wn,'s');
% Compute and plot the frequency response
omega = [0: 200: 4*Fp(2)*pi];
h = freqs(num,den,omega);
plot (omega/(2*pi),20*log10(abs(h)));
grid on;
title('Magnitude Response of Elliptic Bandpass Filter');
axis([0 2*Fp(2) -80 5]);
xlabel('Frequency, Hz');
ylabel('Magnitude Response(dB)');
```

Figure 2.44 MATLAB m-file to design BP Elliptic filter.

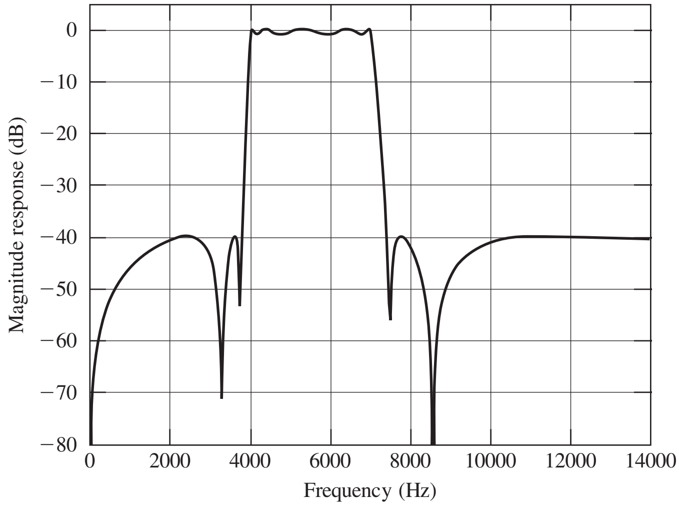


Figure 2.45 Magnitude response of the BP Elliptic filter.

To determine the order of the Elliptic filter meeting the specifications, we use the command `[N,Wn] = ellipord(Wp,Ws,Rp,Rs,'s')` with `Wp` $= 2\pi[4000, 7000]$, `Ws` $= 2\pi[3000, 8000]$, `Rp` $= 1$, `Rs` $= 40$. Next we design the filter using the command `[b,a]=ellip(N,Rp,Rs,Wn,'s')`. Figure 2.44 displays the sample MATLAB code for this example. The magnitude response of the desired filter of order `N` $= 10$ is shown in Figure 2.45.

2.9 POWER SPECTRAL DENSITY

In the design of communication systems, we are interested in power distribution of a power signal in the frequency domain. Recall from Section 2.1 that the normalized average power of a signal $x(t)$ was defined in (2.20) as

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt \tag{2.20}$$

Note that the average power defined here is a **time-average mean-square value** to distinguish it from statistical mean-square value to be discussed in Chapter 6. The problem in dealing with power signals in the frequency domain is that their Fourier transform may not exist as they have infinite energy. To overcome this problem, we define a new function $x_T(t)$ by truncating $x(t)$ outside the interval $|t| > T/2$.

$$x_T(t) = \begin{cases} x(t), & -T/2 \le t \le T/2 \\ 0, & \text{otherwise} \end{cases} \tag{2.165}$$

$x_T(t)$ has finite energy as long as T is finite. We can now write an expression for the energy of $x_T(t)$ by using Parseval's relation (2.89) as follows:

$$E_{x_T} = \int_{-\infty}^{\infty} |x_T(t)|^2 dt = \int_{-\infty}^{\infty} |X_T(f)|^2 df \tag{2.166}$$

where

$$x_T(t) \overset{\mathfrak{I}}{\longleftrightarrow} X_T(f)$$

Because

$$\int_{-\infty}^{\infty} |x_T(t)|^2 dt = \int_{-T/2}^{T/2} |x(t)|^2 dt \tag{2.167}$$

we can write

$$\frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 dt = \int_{-\infty}^{\infty} \frac{|X_T(f)|^2}{T} df \tag{2.168}$$

The normalized average power can now be expressed by substituting (2.168) into (2.20) as

$$P_x = \lim_{T\to\infty} \int_{-\infty}^{\infty} \frac{|X_T(f)|^2}{T} df \tag{2.169}$$

Because $x(t)$ is a power signal, the integral on the right-hand side of (2.169) exists in the limit as $T \to \infty$. Therefore, we can change the order of integration and limit yielding

$$P_x = \lim_{T\to\infty} \int_{-\infty}^{\infty} \frac{|X_T(f)|^2}{T} df = \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{|X_T(f)|^2}{T} df \tag{2.170}$$

The **power spectral density (PSD)** $\mathcal{G}_x(f)$ of power signal $x(t)$ is defined as

$$\mathcal{G}_x(f) \triangleq \lim_{T\to\infty} \frac{|X_T(f)|^2}{T} \tag{2.171}$$

This allows us to express the normalized average power as

$$P_x = \int_{-\infty}^{\infty} \mathcal{G}_x(f) df \tag{2.172}$$

From (2.172), it is obvious $\mathcal{G}_x(f_o)\Delta f$ represents the power contained in a spectral band of Δf Hz centered at frequency f_o. Thus $\mathcal{G}_x(f)$ may be interpreted as the power contained in spectral components of $x(t)$ centered at frequency f per Hz of bandwidth. It is specified in units of W/Hz.

Power Spectral Density of a Periodic Signal

For a periodic signal $x_p(t)$, the normalized average power is given from (2.60) as

$$P_x = \sum_{n=-\infty}^{\infty} |C_n|^2$$

Because $|C_n|^2$ is power contained in the spectral component at $f = nf_o$, the PSD $\mathcal{G}_x(f)$ of a periodic signal can be expressed as

$$\mathcal{G}_x(f) = \sum_{n=-\infty}^{\infty} |C_n|^2 \delta(f - nf_o) \tag{2.173}$$

Substituting (2.91) into (2.173) yields

$$\mathcal{G}_x(f) = \sum_{n=-\infty}^{\infty} |X_p(nf_o)|^2 \delta(f - nf_o) \tag{2.174}$$

2.9.1 Time-Average Autocorrelation Function

The **time-average autocorrelation function** of a power signal $x(t)$ is defined as

$$\mathcal{R}_x(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x(t-\tau)dt \tag{2.175}$$

The normalized average power P_x of $x(t)$ is related to $\mathcal{R}_x(\tau)$ by

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt = \mathcal{R}_x(0) \tag{2.176}$$

It can be shown that the PSD of a power signal $x(t)$ is the Fourier transform of its time-average autocorrelation function.

$$\mathcal{G}_x(f) \overset{\mathfrak{F}}{\longleftrightarrow} \mathcal{R}_x(\tau) \tag{2.177}$$

Example 2.36

Determine the time-average autocorrelation function and PSD of the sinusoidal signal $x(t) = A\cos(2\pi f_o t + \phi)$

Solution

The time-average autocorrelation function is obtained using the definition (2.175) as

$$\mathcal{R}_x(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x(t-\tau)dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} A^2\cos(2\pi f_o t + \phi)\cos[2\pi f_o(t-\tau) + \phi]dt$$

$$= \frac{A^2}{2}\cos(2\pi f_o \tau) + \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \cos[4\pi f_o t - 2\pi f_o \tau + 2\phi]dt$$

The second integral is zero yielding

$$\mathcal{R}_x(\tau) = \frac{A^2}{2}\cos(2\pi f_o \tau) \tag{2.178}$$

Because the PSD of a power signal $x(t)$ is the Fourier transform of its time-average autocorrelation function, we obtain using Table 2.2

$$\mathcal{G}_x(f) = \Im\{\mathcal{R}_x(\tau)\} = \Im\left\{\frac{A^2}{2}\cos(2\pi f_o \tau)\right\} = \frac{A^2}{4}[\delta(f - f_o) + \delta(f + f_o)]$$

The normalized average power may be obtained by using (2.172) as

$$P_x = \int_{-\infty}^{\infty} \mathcal{G}_x(f)df = \int_{-\infty}^{\infty} \frac{A^2}{4}[\delta(f - f_o) + \delta(f + f_o)]df = \frac{A^2}{2}$$

2.9.2 Relationship Between Input and Output Power Spectral Densities

For a linear system with transfer function $H(f)$, the output $y(t)$ of the system in response to the input signal $x(t)$ is given in the frequency domain from (2.119) as

$$Y(f) = X(f)H(f) \tag{2.119}$$

The PSD of a power signal $y(t)$ can be written using (2.169) as

$$\mathcal{G}_y(f) = \lim_{T \to \infty} \frac{|Y_T(f)|^2}{T} \tag{2.179}$$

The relationship in the frequency domain between the truncated versions of $y(t)$ and $x(t)$ is obtained using (2.119) as

$$Y_T(f) = X_T(f)H(f) \tag{2.180}$$

Substituting (2.180) into (2.179) yields

$$\mathcal{G}_y(f) = \lim_{T \to \infty} \frac{|X_T(f)H(f)|^2}{T} = |H(f)|^2 \lim_{T \to \infty} \frac{|X_T(f)|^2}{T} = |H(f)|^2 \mathcal{G}_x(f) \tag{2.181}$$

Equation (2.181) states that the output signal PSD in an LTI system depends on the magnitude of $H(f)$ and is given by $|H(f)|^2$ times the input PSD.

Example 2.37

The periodic pulse train in Example 2.24 is input to a fourth order Butterworth LPF with 3-dB cutoff frequency $f_c = 4$ MHz. Assume $T_o = 1$ μsec and $\tau/T_o = 0.25$. Plot the output PSD.

Solution

The PSD $\mathcal{G}_x(f)$ of periodic pulse train is given by substituting (2.59) into (2.173).

$$\mathcal{G}_x(f) = \left(\frac{\tau}{T_o}\right)^2 \sum_{n=-\infty}^{\infty} \left|\text{sinc}\left(\frac{n\tau}{T_o}\right)\right|^2 \delta(f - nf_o) = \frac{1}{16} \sum_{n=-\infty}^{\infty} |\text{sinc}(n/4)|^2 \delta(f - nf_o) \tag{2.182}$$

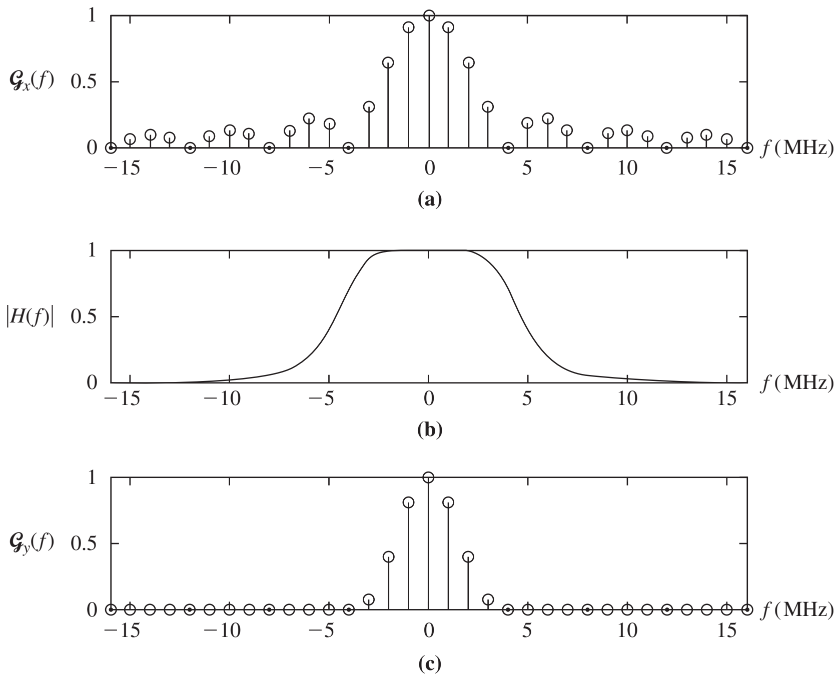


Figure 2.46 Input and output spectral densities.

Substituting (2.182) and (2.181) yields the output PSD $\mathcal{G}_y(f)$ as

$$\mathcal{G}_y(f) = |H(f)|^2\mathcal{G}_x(f) = \frac{1}{16}|H(f)|^2\sum_{n=-\infty}^{\infty}|\text{sinc}(n/4)|^2\delta(f - nf_o)$$

$$= \frac{1}{16}\sum_{n=-\infty}^{\infty}|H(nf_o)|^2|\text{sinc}(n/4)|^2\delta(f - nf_o) \qquad (2.183)$$

Figure 2.46(a) displays the PSD $\mathcal{G}_x(f)$ of the periodic pulse train. The magnitude response of the fourth order Butterworth LPF with 3-dB cutoff frequency $f_c = 4$ MHz is illustrated in Figure 2.46(b). The output PSD $\mathcal{G}_y(f)$ calculated using (2.183) is shown in Figure 2.46(c).

2.10 FREQUENCY RESPONSE CHARACTERISTICS OF TRANSMISSION MEDIA

We consider transmission of signals through widely used wired media such as TWP and coaxial cables.

2.10.1 Twisted Wire Pairs

To obtain frequency domain characterization of twisted wire pairs (TWPs) and coaxial cables, we use transmission line theory concepts. The distributed circuit model of the transmission line consists of a cascade of many transmission line segments of the type shown in Figure 2.47. Each transmission line segment is characterized by an equivalent circuit with lumped-circuit elements *R, L, C,* and *G* where

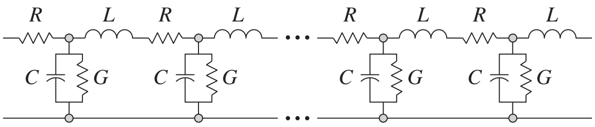


Figure 2.47 Transmission line model of a TWP.

- R = Series resistance per meter
- L = Series inductance per meter
- C = Shunt capacitance per meter
- G = Shunt conductance per meter

The line is **lossless** if $R = G = 0$. R depends on frequency and has the form $R(f) \approx c_o\sqrt{2\pi f}$ at high frequencies because of the **skin effect.** This refers to the tendency of high frequencies in a signal to travel near the surface of a conductor in a layer some tens of microns thick.

If voltage $x_i(t)$ is applied at the input to the transmission line at time $t = 0$, the voltage along the line declines exponentially with distance. At time t its value is given by

$$x(z, t) = x_i(t)e^{-\gamma z} \tag{2.184}$$

where z is the distance in meters. γ is called the **propagation constant** of the TWP. It determines the variation of voltage along the line. As a special case, if the input is a complex sinusoidal signal $x_i(t) = Ae^{j2\pi ft}$ of frequency f Hz, the voltage along the line at distance z is

$$x(z, t) = Ae^{j2\pi ft}e^{-\gamma z} \quad \text{at time } t \tag{2.185}$$

The propagation constant is a complex function of frequency and is given in terms the lumped-circuit model element values as

$$\gamma(f) = \alpha(f) + j\beta(f) = \sqrt{(R + j2\pi fL)(G + j2\pi fC)} \tag{2.186}$$

where α = attenuation coefficient (= 0 for lossless line), and β = phase shift coefficient = $2\pi/\lambda$ (radians/meter). Substituting (2.186) into (2.185) yields

$$x(z, t) = Ae^{-[\alpha(f)+j\beta(f)]z}e^{j2\pi ft} = Ae^{-\alpha(f)z}e^{j[2\pi ft-\beta(f)z]} \tag{2.187}$$

Equation (2.187) states that the entering phasor signal's magnitude decays along the line as $e^{-\alpha(f)z}$. Further, a phase shift of $-\beta(f)z$ radians is introduced in the input phasor signal. Another important parameter of the transmission line is its **characteristic impedance,** Z_o. It is defined as the input impedance of an infinite line or that of a finite line terminated with a load impedance, $Z_L = Z_o$. It is given in terms the lumped-circuit model element values as

$$Z_o = \sqrt{\frac{R + j2\pi fL}{G + j2\pi fC}} \tag{2.188}$$

For a transmission line terminated with its characteristic impedance, the transfer function $H_{TWP}(f, \ell)$ is given by

$$H_{TWP}(f, \ell) = e^{-\gamma(f)\ell} \tag{2.189}$$

where ℓ is line length. The **attenuation** or **insertion loss** is defined as the reduction or loss in signal power as it is transferred across the transmission medium. It is determined by the magnitude of its transfer function, which is given by

$$|H_{TWP}(f, \ell)| = e^{-\alpha(f)\ell} \tag{2.190}$$

where $\alpha(f)$ = real part of the propagation constant in (2.186). The attenuation of a TWP is usually expressed in dB as

$$\text{Insertion Loss} = -20\log_{10}|H_{TWP}(f, \ell)| = -20\log_{10}e^{-\alpha(f)\ell} = 8.686\alpha(f)\ell \text{ dB} \tag{2.191}$$

The parameter $\alpha(f)$ has the form

$$\alpha(f) = c_1\sqrt{f} + c_2 f \tag{2.192}$$

where f is in Hz. For $f \geq 300$ kHz, $\alpha(f) \approx c_1\sqrt{f}$ Substituting in (2.191) allows us to write the following simplified expression for the attenuation of a TWP:

$$\text{Insertion Loss} = 8.686c_1\sqrt{f}\ell \text{ dB}, \; f \geq 300 \text{ kHz} \tag{2.193}$$

where f and ℓ are specified in Hz and miles, respectively. Attenuation of the TWP increases both with the length and the frequency of operation. The increase in attenuation is linear with length and is proportional to $\sqrt{f}$ at high frequencies because of the skin effect. The parameters c_1 and c_2 for popular TWP cables are listed in Table 2.4.

Figure 2.48 shows insertion losses that are produced using the parameters from Table 2.4 for a length of 1 mile. The attenuation for a TWP, measured in dB/mile, can range from a few dB/mile at 1 kHz to 15 to 30 dB/mile at 500 kHz, depending on the AWG of the wire. Because the insertion loss of a TWP increases linearly with distance, the bandwidth decreases correspondingly with the length of TWP drop.

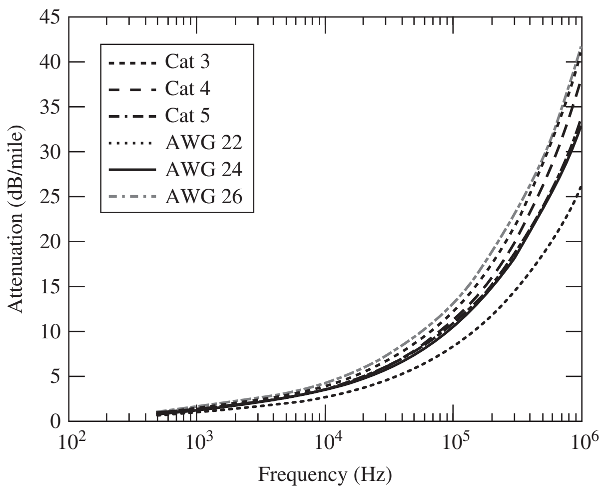


Figure 2.48 Attenuation characteristics of TWP.

Table 2.4 c_1 **and** c_2 **Parameters for Popular TWP Cables**

Type	c_1	c_2
Cat 3	4.31×10^{-3}	4.26×10^{-7}
Cat 4	3.89×10^{-3}	4.82×10^{-7}
Cat 5	3.83×10^{-3}	2.41×10^{-8}
AWG 26	4.8×10^{-3}	-1.71×10^{-8}
AWG 24	3.8×10^{-3}	-0.54×10^{-8}
AWG 22	3.0×10^{-3}	0.035×10^{-8}

Example 2.38

The element values in the lumped-circuit model of 24-AWG TWP are given as

$C = 0.05\ \mu$F/km
$L = 0.673$ mH/km
$R = 180$ ohms/km
$G = 0$

In telephone network, subscriber loops are limited to 18,000 feet (~ 5.45 km). Determine the output of 6-km TWP for an input sinusoidal signal $x(t) = 5\cos(6800\pi t)$.

Solution

Substituting these element values in (2.186) yields

$$\begin{aligned}\gamma(f)\big|_{f=3.4\text{ kHz}} &= \sqrt{(180 + j6.8\pi \times 0.673)(0 + j6.8\pi \times 0.05 \times 10^{-3})}\\ &= 0.2979 + j0.3227 = 0.4392\measuredangle 47.3^\circ\end{aligned}$$

Therefore, α at 3.4 kHz = 0.2979 km^{-1}. The attenuation at f = 3.4 kHz is obtained by substituting α into (2.191) as $8.686 \times 0.2979 \approx 2.6$ dB/km.

The transfer function value for 6-km TWP loop at 3.4 kHz is given from (2.189) as

$$\begin{aligned}H_{TWP}(f)\big|_{f=3.4\text{ kHz}} &= e^{-6\times\gamma(f)|_{f=3.4\times10^3}} = e^{-(0.2979+j0.3227)\times 6}\\ &= -0.0598 - j0.1563 = 0.1674\measuredangle -110.9^\circ\end{aligned}$$

Substituting into (2.116), the output of 6-km TWP loop for a sinusoidal input $x(t) = 5\cos(6800\pi t)$ can be written as

$$\begin{aligned}y(t) &= |H_{TWP}(f)|_{f=3.4\text{ kHz}} \times 5\cos\left[6800\pi t + \measuredangle H_{TWP}(f)|_{f=3.4\text{ kHz}}\right]\\ &= 5 \times 0.1674\cos(6800\pi t - 110.9^\circ) = 0.837\cos(6800\pi t - 110.9^\circ)\end{aligned}$$

Example 2.39

If the maximum run length using Category (Cat) 5 TWP from the desktop to the nearest wiring closet is restricted to 100 feet, what is the expected power level at the closet assuming the desktop network interface launches 250 mW at 100 MHz?

Solution

The attenuation of a Cat 5 TWP is given by

$$\text{Insertion Loss} = 8.686 \times 3.83 \times 10^{-3} \times \sqrt{10^8}\ \text{dB/mile} = 8.686 \times 38.3 = 332.67\ \text{dB/mile}$$

Therefore, loss for $\ell = 100$ foot drop of Cat 5 TWP cable = 332.67 × 100/5000 = 6.65 dB.

Power launched by the desktop = 250 mW = $10\log_{10}(250) = 24$ dBm

Power level at the wiring closet = Power launched by the desktop – loss = 24 − 6.65 = 17.35 dBm = $10^{17.35/10}$ = 54 mW

2.10.2 Coaxial Cable

The insertion loss of coaxial cables can be modeled by the following:

$$\text{Insertion Loss} = 20\log_{10}|H_{coax}(f,\ell)| = (k_1\sqrt{f} + k_2 f)\ell \ \text{dB} \qquad (2.194)$$

where

f = frequency in MHz
ℓ = cable length in kft

For high frequencies, $k_1\sqrt{f}$ term in (2.194) dominates. This allows us to write the following simplified expression for the attenuation of a coaxial cable:

$$\text{Insertion Loss} = k_1\sqrt{f}\,\ell \ \text{dB}, f \geq 300 \text{ kHz} \qquad (2.195)$$

where f and ℓ are specified in MHz and kft, respectively. Attenuation of the coaxial cable also increases both with frequency and the cable length. The increase in attenuation is linear with length and proportional to $\sqrt{f}$ at high frequencies because of the skin effect. The parameters k_1 and k_2 characterize the coaxial cable type; k_1 basically indicates the amount of conductor loss while k_2 indicates the amount of dielectric loss.

Table 2.5 k_1 and k_2 Parameters for Popular Coaxial Cables

	500-F	625-F	RG-6	RG-59
k_1	0.69	0.6058	2.1144	2.7155
k_2	3.7×10^{-3}	1.6×10^{-3}	2.1×10^{-3}	1.5×10^{-3}

The parameters for different types of cables are listed in Table 2.5.

RG-59 and RG-6 cables are used in the distribution segment of a CATV network for subscriber drops. 500-F and 625-F are examples of cables that originate from a fiber distribution node to form the trunk and feeder portion of the CATV network. Figure 2.49 displays the attenuation characteristics of both cable types. By comparing the results in Figure 2.49 with those in Figure 2.48, it can be seen that coaxial cables can provide much larger frequency range of operation (up to 1 GHz) than twisted wire pair (a few MHz). Today's **cable television (CATV)** systems use a frequency range of 1 GHz.

2.11 FOURIER TRANSFORMS FOR DISCRETE-TIME SIGNALS

For discrete-time signals $x[n]$, two alternative frequency domain representations are extremely useful.

- Discrete-time Fourier transform (DTFT)
- Discrete Fourier transform (DFT)

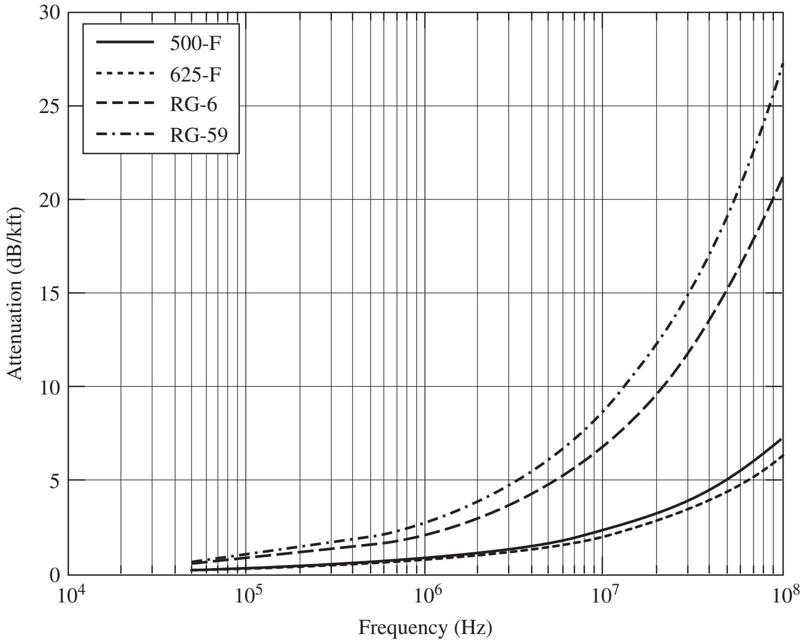


Figure 2.49 Attenuation characteristics of coaxial cables.

The DTFT $X(e^{j\hat{\omega}})$ of a sequence $x[n]$, obtained by sampling an analog signal $x(t)$ at the rate $f_s = 1/T_s$ samples/sec, is defined by

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\hat{\omega}n} \tag{2.196}$$

where $\hat{\omega}$ is normalized angular frequency $\hat{\omega} = 2\pi f T_s$ in radians/sample. $X(e^{j\hat{\omega}})$ is, in general, a complex and *continuous* function of the real variable $\hat{\omega}$ and can be written as

$$X(e^{j\hat{\omega}}) = |X(e^{j\hat{\omega}})|e^{j\measuredangle X(e^{j\hat{\omega}})} \tag{2.197}$$

where both magnitude $|X(e^{j\hat{\omega}})|$ and phase $\measuredangle X(e^{j\hat{\omega}})$ are real functions of $\hat{\omega}$. The DTFT $X(e^{j\hat{\omega}})$ of a sequence $x[n]$ is a periodic function of $\hat{\omega}$ with period 2π.

$$X[e^{j(\hat{\omega}+2\pi k)}] = \sum_{n=-\infty}^{\infty} x[n]e^{-j(\hat{\omega}+2\pi k)n} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\hat{\omega}n} = X(e^{j\hat{\omega}}) \tag{2.198}$$

Figure 2.50 displays the relationship between the FT of a continuous signal $x(t)$ and its sampled version $x[n]$. The values of $\hat{\omega} = \pm\pi$ correspond to half the sampling rate, that is, $f_s/2$.

In the case of a finite-length sequence $x[n]$, $0 \leq n \leq N-1$, there is a simpler frequency domain representation in terms of its DFT. The DFT $X[k]$ of a sequence $x[n]$ is defined by

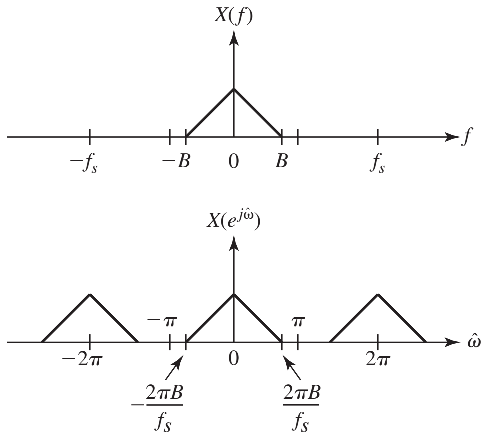


Figure 2.50 Relationship between FT and DTFT of a sampled signal.

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}, \quad k = 0, 1, \ldots, N-1 \tag{2.199}$$

Note that DFT is applicable *only* to a finite-length sequence. The length of the DFT sequence $X[k]$ is also N. $X[k]$ is, therefore, referred to as ***N*-point** DFT of $x[n]$. Because

$$X[k] = X(e^{j\hat{\omega}})|_{\hat{\omega}=2\pi k/N}, \tag{2.200}$$

the DFT $X[k]$ can be viewed as *uniformly spaced* samples of the corresponding DTFT $X(e^{j\hat{\omega}})$ over $[0, 2\pi]$ at frequencies $\hat{\omega}_k = \frac{2\pi k}{N}$, $k = 0, 1, \ldots, N-1$. It is easy to prove that $X[k]$ is periodic with period N.

$$X[k] = X[k + mN], \quad m \text{ any integer}$$

Because there are N frequency samples in the interval f_s (corresponding to $[0, 2\pi]$ on $\hat{\omega}$ axis), the frequency resolution of DFT is given by

$$\Delta f = \frac{f_s}{N} = \frac{1}{NT_s} = \frac{1}{T} \tag{2.201}$$

where T is the total duration of the signal. Equation (2.201) states that the frequency resolution of DFT is determined by the signal record length. To obtain the original sequence $x[n]$ from the DFT sequence, we use the **inverse DFT (IDFT)** defined by

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{+j2\pi kn/N}, \quad n = 0, 1, \ldots, N-1 \tag{2.202}$$

Equations (2.199) and (2.202) form a DFT pair.

The **Fast Fourier transform (FFT)** is an extremely efficient algorithm for computing DFT. The FFT requires that the sequence length N is an integer power of 2. This is usually accomplished by appending zeros on either side of discrete-time sequence $x[n]$.

Zero-padding increases the number of points in the DFT thereby improving the DFT's approximation of the DTFT $X(e^{j\hat{\omega}})$. In MATLAB, the m-file `fft(x)` is used for computing the N-point FFT of a length-N sequence $x[n]$. The m-file `ifft(X)` computes the inverse FFT of length-N DFT sequence $X[k]$.

Example 2.40

Consider a rectangular pulse of unit amplitude and duration $T = 1$ sec.

a. Sample the pulse at 20 Hz and append zeros on either side to generate a discrete-time sequence $x[n]$ of length 512.
b. Obtain the DFT X[k] of $x[n]$ by using m-file `fft(x)`. Plot `fftshift(X)`.

Solution

The m-file in Figure 2.51(b) computes the FFT as a rectangular pulse of unit amplitude and duration $T = 1$ sec. The sampling rate is chosen so that 20 samples are obtained within the pulse interval to account for the wide bandwidth of the pulse due to its sharp edges. Figure 2.54(a) displays the sampled sequence $x[n]$ and magnitude of the 512-point FFT.

```
% Example_2.41.m
% Matlab script to illustrate the FFT of a rectangular pulse
%

clear all;

% Time axis: Sampling period is 50 milliseconds
delt=1/20
t = −12.5:delt:12.5;
fs=1/delt

% A rectangular pulse of duration 1 second

x = rectpuls (t,1);
subplot (2,1,1);
stem(t,x)
axis([−5 5 0 1.1])
title ('x[n]')
xlabel('Time')

% Fast Fourier Transform of x[n]

X = fft(x,512);

% Compute the magnitude of FFT and center it

XX = abs(fftshift(X(1:512)));
f = fs*(−256:255)/512;
subplot (2,1,2);
stem(f,XX)
axis([−0.5*fs 0.5*fs 0 25])
title('DFT X[k]')
xlabel('Frequency (Hz)')
```

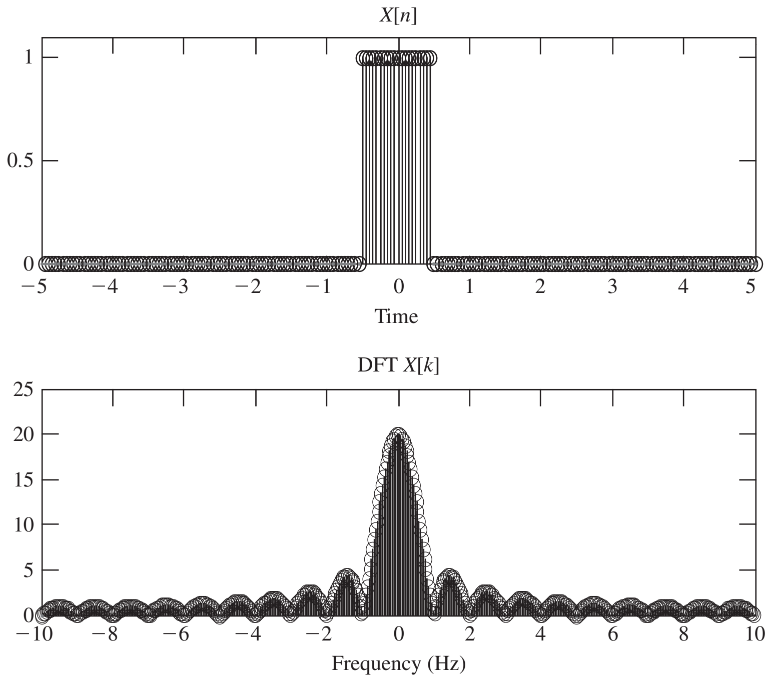


Figure 2.51 (a) Sampled sequence $x[n]$ and magnitude of the 512-point FFT; (b) m-file for computing the FFT of a rectangular pulse.

FINAL REMARKS

In this chapter we reviewed fundamental concepts about signals and their processing by linear systems. Although signals are usually described as functions of time, the frequency domain description was introduced to analyze the signals and linear systems. The Fourier transform serves as a fundamental tool in this context for relating the time-domain and frequency-domain descriptions.

An inverse relationship exists between the time-domain and frequency-domain parameters that characterize signals and systems. An important consequence of this inverse relationship is that the duration–bandwidth product of a signal is constant. Thus, a signal cannot arbitrarily be both duration and bandwidth limited.

The response of linear time-invariant systems to input signals was considered in both time- and frequency-domains. The output signal in general is a distorted version of the input as a result of nonideal magnitude and phase response characteristics of the system. An important signal processing operation in communication systems is that of linear filtering. We studied various ideal filter types and investigated realizable designs using MATLAB.

The transmission characteristics of different transmission media were then studied in the frequency domain. Signal transmission and distortion properties of wired media such as twisted wire pair and coaxial cable were reviewed.

FURTHER READINGS

Signals and systems are covered in the undergraduate texts on the subject [1–4]. References [5] and [6] review the material from the perspective of its relevance in the study of communication systems.

1. Kamen, E., and B. Heck. *Fundamentals of Signals and Systems,* 3rd ed. Upper Saddle River, NJ: Prentice Hall, 2006.
2. McClellan J., R. Schafer, and M. Yoder. *Signal Processing First.* Upper Saddle River, NJ: Prentice Hall, 2003.
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4. Oppenheim, A., A. Willsky, and S. Nawab. *Signals and Systems,* 2nd ed. Upper Saddle River, NJ: Prentice Hall, 1996.
5. Ziemer, R., and W. Tranter. *Principles of Communications: Systems, Modulation, and Noise,* 5th ed. New York: John Wiley, 2001.
6. Carlson, B., P. Crilly, and J. Rutledge. *Communication Systems,* 4th ed. New York: McGraw-Hill, 2002.
7. Mitra, S. *Digital Signal Processing: A Computer-Based Approach,* 3rd ed. New York: McGraw-Hill, 2006.
8. MATLAB + Signal Processing Toolbox, Student Version Release 14, available at www.mathworks.com/student.

PROBLEMS

2.1. Consider the signals displayed in Figure P2.1. Show that each of these signals can be expressed as the sum of rectangular $\Pi(t)$ or triangular $\Lambda(t)$ pulses.

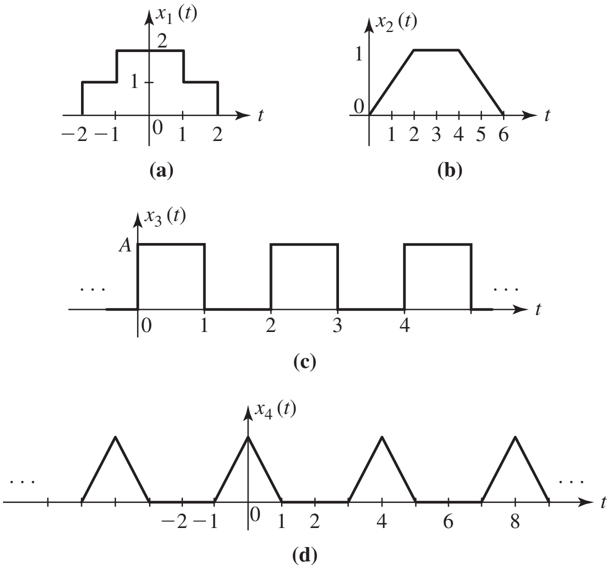

Figure P2.1

2.2. For the signal $x_2(t)$ in Figure P2.1(b) plot the following signals:

a. $x_2(t-3)$

b. $x_2(-t)$

c. $x_2(2t)$

d. $x_2(3-2t)$

2.3. Plot the following signals:

a. $x_1(t) = 2\Pi(t/2)\cos(6\pi t)$

b. $x_2(t) = 2\left[\frac{1}{2} + \frac{1}{2}\,\mathrm{sgn}(t)\right]$

c. $x_3(t) = x_2(-t+2)$

d. $x_4(t) = \mathrm{sinc}(2t)\Pi(t/2)$

2.4. Determine whether the following signals are periodic. For periodic signals, determine the fundamental period.

a. $x_1(t) = \sin(\pi t) + 5\cos(4\pi t/5)$

b. $x_2(t) = e^{j3t} + e^{j9t} + \cos(12t)$

c. $x_3(t) = \sin(2\pi t) + \cos(10t)$

d. $x_4(t) = \cos\left(2\pi t - \frac{\pi}{4}\right) + \sin(5\pi t)$

2.5. Classify the following signals as odd or even or neither.

a. $x_1(t) = -4t$

b. $x_2(t) = e^{-|t|}$

c. $x_3(t) = 5\cos(3t)$

d. $x_4(t) = \sin\left(3t - \frac{\pi}{2}\right)$

e. $x_5(t) = u(t)$

f. $x_6(t) = \sin(2t) + \cos(2t)$

2.6. Determine whether the following signals are energy or power, or neither and calculate the corresponding energy or power in the signal.

a. $x_1(t) = u(t)$

b. $x_2(t) = 4\cos(2\pi t) + 3\cos(4\pi t)$

c. $x_3(t) = \dfrac{1}{t}$

d $x_4(t) = e^{-\alpha t}\, u(t)$

e. $x_5(t) = \Pi(t/3) + \Pi\,(t)$

f. $x_6(t) = 5e^{(-2t + j10\pi t)}u(t)$

g. $x_7(t) = \sum_{n=-\infty}^{\infty} \Lambda[(t - 4n)/2]$

2.7. Evaluate the following expressions by using the properties of the delta function:

a. $x_1(t) = \delta(4t)\sin(2t)$

b. $x_2(t) = \delta(t)\cos\left(30\pi t + \dfrac{\pi}{4}\right)$

c. $x_3(t) = \delta(t)\mathrm{sinc}(t + 1)$

d. $x_4(t) = \delta(t - 2)e^{-t}\sin(2.5\pi t)$

e. $x_5(t) = \int_{-\infty}^{\infty} \delta(2t)\mathrm{sinc}(t)dt$

f. $x_6(t) = \int_{-\infty}^{\infty} \delta(t - 3)\cos(t)dt$

g. $x_7(t) = \int_{-\infty}^{\infty} \delta(2 - t)\dfrac{1}{1 - t^3}dt$

h. $x_8(t) = \int_{-\infty}^{\infty} \delta(3t - 4)e^{-3t}dt$

i. $x_9\,(t) = \delta'(t) \otimes \Pi\,(t)$

2.8. For each of the following continuous-time systems, determine whether or not the system is (1) linear, (2) time-invariant, (3) memoryless, and (4) causal.

a. $y(t) = x(t - 1)$

b. $y(t) = 3x(t) - 2$

c. $y(t) = |x(t)|$

d. $y(t) = [\cos(2t)]x(t)$

e. $y(t) = e^{x(t)}$

f. $y(t) = tx(t)$

g. $y(t) = \int_{-\infty}^{t} e^{-3(t-\tau)}x(\tau - 1)d\tau$

2.9. Calculate the output $y(t)$ of the LTI system for the following cases:

a. $x(t) = e^{-2t}\, u(t)$ and $h(t) = u(t - 2) - u(t - 4)$

b. $x(t) = e^{-t}\, u(t)$ and $h(t) = e^{-2t}u(t)$

c. $x(t) = u(-t)$ and $h(t) = \delta(t) - 3e^{-2t}u(t)$

d. $x(t) = \delta(t - 2) + 3e^{3t}u(-t)$ and $h(t) = u(t) - u(t - 1)$

2.10. The impulse response of a continuous-time LTI system is displayed in Figure P2.2(b). Assuming the input $x(t)$ to the system is waveform illustrated in Figure P2.2(a), determine the system output waveform $y(t)$ and sketch it.

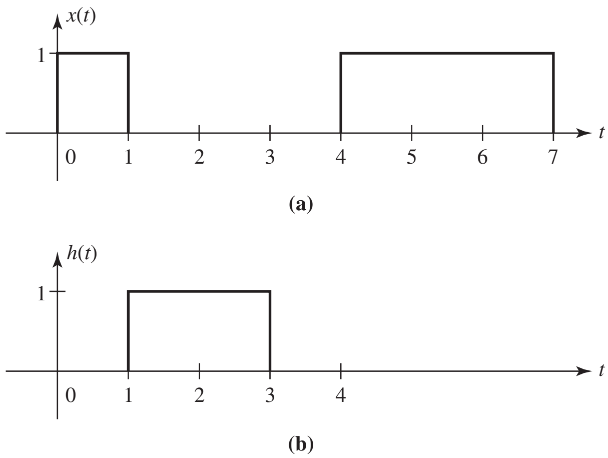

Figure P2.2

2.11. An LTI system has the impulse response

$$h(t) = e^{-0.5(t-2)}u(t - 2).$$

a. Is the system causal?

b. Is the system stable?

c. Repeat parts (a) and (b) for $h(t) = e^{-0.5(t+2)}u(t + 2)$.

2.12. a. Write down the exponential Fourier series coefficients of the signal

$x(t) = 5\sin(40\pi t) + 7\cos(80\pi t - \pi/2) - \cos(160\pi t + \pi/4).$

b. Is $x(t)$ periodic? If so, what is its period?

2.13. A signal has the two-sided spectrum shown in Figure P2.3.

a. Write the equation for $x(t)$.

b. Is the signal periodic? If so, what is its period?

c. Does the signal have energy at DC?

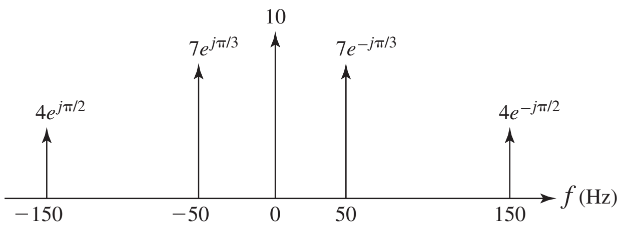

Figure P2.3

2.14. Write down the complex exponential Fourier series for each of the periodic signals shown in Figure P2.4. Plot the magnitude and phase spectra.

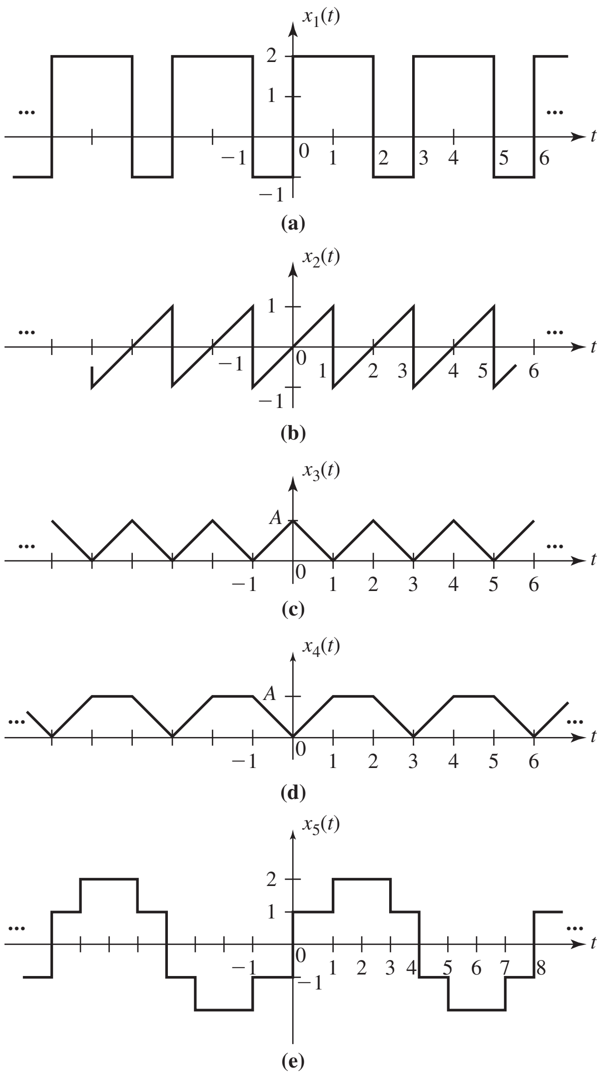

Figure P2.4

2.15. For the rectangular pulse train in Figure 2.24, compute the Fourier coefficients of the new periodic signal $y(t)$ given by

a. $y(t) = x(t - 0.5T_o)$

b. $y(t) = x(t)e^{j2\pi t/T_o}$

c. $y(t) = x(\alpha t)$

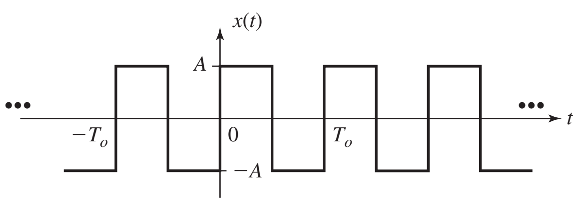

Figure P2.5

2.16. Draw the one-sided magnitude power spectrum for the square wave in Figure P2.5 with duty cycle 50%.

a. Calculate the normalized average power.

b. Determine the 99% power bandwidth of the pulse train.

2.17. Determine the Fourier transforms of the signals shown in Figure P2.6.

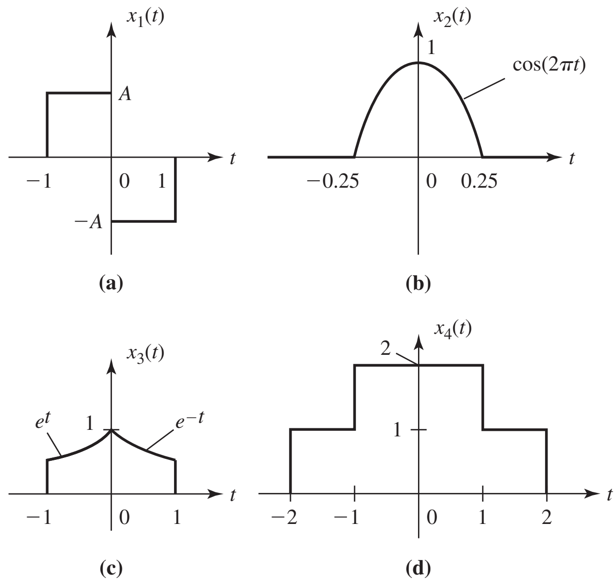

Figure P2.6

2.18. Use properties of the Fourier transform to compute the Fourier transform of following signals.

a. $\mathrm{sinc}^2(Wt)$

b. $\Pi\,(t/T)\cos(2\pi f_c t)$

c. $(e^{-t}\cos 10\pi t)u(t)$

d. $te^{-t}u(t)$

e. $e^{-\pi t^2}$

f. $4\mathrm{sinc}^2(t)\cos(100\pi t)$

2.19. Find the following convolutions:

a. $\mathrm{sinc}(Wt) \otimes \mathrm{sinc}(2Wt)$

b. $\mathrm{sinc}^2(Wt) \otimes \mathrm{sinc}(2Wt)$

2.20. The FT of a signal $x(t)$ is described by

$$X(f) = \frac{1}{5 + j2\pi f}$$

Determine the FT $V(f)$ of the following signals:

a. $v(t) = x(5t - 1)$ What is the impulse response $h(t)$?

b. $v(t) = x(t)\cos(100\pi t)$

c. $v(t) = x(t)e^{j10t}$

d. $v(t) = \dfrac{dx(t)}{dt}$

e. $v(t) = x(t) \otimes u(t)$

2.21. Consider the delay element $y(t) = x(t-3)$.

a. What is the impulse response $h(t)$?

b. What is the magnitude and phase response function of the system?

2.22. The periodic input $x(t)$ to an LTI system is displayed in Figure P2.7. The frequency response function of the system is given by

$$H(f) = \frac{2}{2 + j2\pi f}$$

a. Write the complex exponential FS of input $x(t)$.

b. Plot the magnitude and phase response functions for $H(f)$.

c. Compute the complex exponential FS of the output $y(t)$.

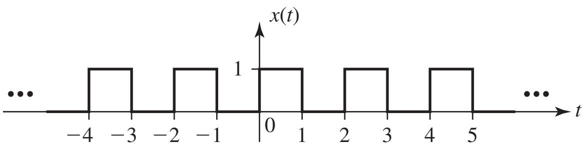

Figure P2.7

2.23. The frequency response of an ideal LP filter is given by

$$H(f) = \begin{cases} 5e^{-j0.0025\pi f}, & |f| < 1000 \text{ Hz} \\ 0, & |f| > 1000 \text{ Hz} \end{cases}$$

Determine the output signal in each of the following cases:

a. $x(t) = 5\sin(400\pi t) + 2\cos(1200\pi t - \pi/2) - \cos(2200\pi t + \pi/4)$

b. $x(t) = 2\sin(400\pi t) + \dfrac{\sin(2200\pi t)}{\pi t}$

c. $x(t) = \cos(400\pi t) + \dfrac{\sin(1000\pi t)}{\pi t}$

d $x(t) = 5\cos(800\pi t) + 2\delta(t)$

2.24. The frequency response of an ideal HP filter is given by

$$H(f) = \begin{cases} 4, & |f| > 20 \text{ Hz}, \\ 0, & |f| < 20 \text{ Hz} \end{cases}$$

Determine the output signal $y(t)$ for the input

a. $x(t) = 5 + 2\cos(50\pi t - \pi/2) - \cos(75\pi t + \pi/4)$

b. $x(t) = \cos(20\pi t - 3\pi/4) + 3\cos(100\pi t + \pi/4)$

2.25. The frequency response of an ideal BP filter is given by

$$H(f) = \begin{cases} 2e^{-j0.0005\pi f}, & 900 < |f| < 1000 \text{ Hz}, \\ 0, & \text{otherwise} \end{cases}$$

Determine the output signal $y(t)$ for the input

a. $x(t) = 2\cos(1850\pi t - \pi/2) - \cos(1900\pi t + \pi/4)$

b. $x(t) = \text{sinc}(60t)\cos(1900\pi t)$

c. $x(t) = \text{sinc}^2(30t)\cos(1900\pi t)$

2.26. The signal $2e^{-2t}u(t)$ is input to an ideal LP filter with passband edge frequency equal to 5 Hz. Find the energy density spectrum of the output of the filter. Calculate the energy of the input signal and the output signal.

2.27. Calculate and sketch the power spectral density of the following signals:

a. $x(t) = 2\cos(1000\pi t - \pi/2) - \cos(1850\pi t + \pi/4)$

b. $x(t) = [1 + \sin(200\pi t)]\cos(2000\pi t)$

c. $x(t) = \cos^2(200\pi t)\sin(1800\pi t)$

Calculate the normalized average power of the signal in each case.

MATLAB PROBLEMS

2.28. Consider the square wave $x(t)$ with $T_o = 1$ in Figure P2.5. It is applied to a filter with frequency response

$$H(f) = \frac{1}{1 + j0.2\pi f}$$

a. Verify using Symbolic MATLAB that FS coefficients of $x(t)$ are given by

$$C_n = \begin{cases} -j\dfrac{2A}{\pi n}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

b. Use MATLAB `stem` command to plot the magnitude spectrum $|C_n|$, $0 \le |n| \le 10$.

c. Calculate and plot the filter output magnitude spectrum $|D_n|$, $0 \le |n| \le 10$.

Plot the FS approximation of the filter output for $n = 10$.

d. Repeat parts (b) and (c) for $T_o = 0.1$. Comment on the differences in the filter output waveform.

2.29. Consider the periodic signal

$$x(t) = 1.5\sin(400\pi t) + 0.75\cos(800\pi t) + 2\sin(1200\pi t).$$

a. Generate a discrete-time sequence $x[n]$ of length 2048 by sampling the signal at 2.4 kHz. Plot $x[n]$.

b. Obtain the DFT `X[k]` of $x[n]$ by using function `fft(x)`. Plot `fftshift(X)`.

2.30. Consider the signal $x(t)$ described by

$$x(t) = \begin{cases} t + 4, & -4 \le t \le -1 \\ 1, & -1 < t \le 1 \\ t - 4, & 1 < t \le 4 \\ 0, & \text{otherwise} \end{cases}$$

a. Sample the pulse at 16 Hz and append zeros on either side to generate a discrete-time sequence $x[n]$ of length 256.

b. Obtain the DFT `X[k]` of $x[n]$ by using function `fft(x)`. Plot `fftshift(X)`.

2.31. A rectangular pulse of unit amplitude and duration $T = 50$ msec is applied to an Elliptic filter with following specifications:

Passband frequency $f_p = B$ Hz

Stopband frequency $f_s = 1.25B$ Hz

Maximum passband ripple $R_p = 1$ dB

Minimum stopband ripple $R_s = 40$ dB

a. Generate a discrete-time sequence `x[n]` of length 1024 by sampling the signal at 1 kHz over the interval $[-512,512)$ msec This has the effect of appending zeros prior to the beginning and at the end of the pulse.

b. Design an Elliptic filter $BT = 0.5$ as illustrated in Example 2.34. Now use function `filter(num,den,x)` to calculate the output `y[n]` of the filter. Calculate the 10–90% rise-time t_r of the output pulse.

c. Repeat (b) for $BT = 1, 2, 5, 10$. Derive an approximate relationship between B and t_r.

2.32. Consider the square wave $x(t)$ depicted in Figure P2.5. Assume $T_o = 20$ msec and sampling rate = 3.2 kHz.

a. Generate the 2,048-point sequence `x[n]` of 50% duty cycle square wave using m-file `square`

b. Obtain the DFT `X[k]` of `x[n]` after appending zeros on both sides. Plot using `stem` command the output of `fftshift(X)`. What is the frequency resolution achieved by FFT here?

c. Compute the FS coefficients of the square wave and plot them using `stem` command. Compare with plot obtained in (b) and comment.

2.33. Consider the triangular wave $x(t)$ in Figure P2.8. Assume $T_o = 50$ msec.

a. Calculate and plot sketch the power spectral density of $x(t)$ using MATLAB.

b. The signal is passed through an ideal LP filter with frequency response $H(f) = \Pi\,(f/B)$, where the filter bandwidth B is chosen so that $BT_o = 5$. Plot the power spectral density of the output.

c. Repeat (b) for the Elliptic filter designed in Problem 2.31 with $BT_o = 5$.

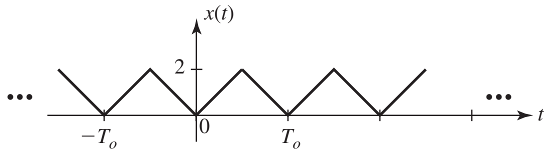

Figure P2.8

An Interview with Herwig Kogelnik

Courtesy of Herwig Kogelnik

Why did you choose a career in the optical communication field?
You know that many of the early laser pioneers thought of optical communications as a primary laser application and anticipated many communications concepts. I was doing research at Oxford University in plasma physics applied to energy production via nuclear fusion. I was just finishing up and writing my PhD thesis in 1960, just after the first laser had been demonstrated by Maiman. There was a visitor from the United States. He was Rudi Kompfner from Bell Labs, and already in 1960 he went around the world looking for people he was interested in, persuaded them to change fields, come to Bell Labs, and switch into the new field of lasers and optical communications. That's indeed what he did with me also. I still remember his key words that convinced me. He talked about lasers and reminded me of their extremely high carrier frequency compared to microwave sources. Usually you get 5–10% of this as signal bandwidth, so with lasers you get an enormous bandwidth to transmit information. He said "think of all this bandwidth." I still remember it clearly, and it is still an important guideline today. And in many ways, this was the key line that persuaded me to quit the field of plasma physics, go to America, and join Bell Labs and their laser and optical communications effort.

In your opinion what are the major innovations that have contributed to the information age we live in? What has been the impact of semiconductor revolution? Optical fiber revolution?
Clearly, transistors and integrated circuits are a highly important part of the revolution in information technology during the past three decades. There has been fine progress in the speed of electronics, but its growth rate is only a factor of 10 every 10 years. On the other hand, computing power is growing × 100 every 10 years, making our

(*continued*)

computers obsolete very quickly. Computers achieve their growth in processing speed by using parallel processing in addition to faster ICs. Now, there is Amdahl's law that says that networking speed has to match computing power: 1 MIPS networked computing power requires 1 Mbps I/O bandwidth. So optical fiber transmission capacity has to grow × 100 every 10 years to keep up. In recent years it has done this by using wavelength division multiplexing. It is hard to find such revolutionary growth rates in other modern technologies, but there is yet another one in information technology. This is the information storage density in magnetic storage.

Tell us about the invention of the distributed-feedback (DFB) laser which is widely used in optical fiber communications. What were the challenges in developing this single-mode laser design necessary to exploit the huge bandwidth offered by single-mode fibers?

Up to then feedback in lasers was provided by mirrors, not easily integrated. Integrated optics had just been proposed. So Chuck Shank and I were wondering whether we could make a compact laser, integrable on a photonic integrated circuit (PIC), and possibly make it to work in a single frequency. I had early contact with periodic structures during my PhD thesis in Vienna. So we came up with the idea of using a periodic structure for feedback and add gain to make an oscillator. Chuck was working on dye lasers. So, as a first test of the DFB principle, we thought of using a film of dichromated gelatin as in holography into which we dissolved a dye to provide the gain. This was pumped optically, and we were glad to see that DFB laser work. Of course we wanted the same principle applied to the semiconductor lasers envisaged for optical communications, and we patented several ideas right away. But to translate that idea to semiconductor lasers was difficult at first. The first room-temperature continuously operating double heterostructure lasers had just barely been demonstrated at the time. They required very delicate chemistry for their preparation. So it took time and effort by researchers worldwide to develop a practical DFB junction laser.

Multi-wavelength optical systems that can achieve 1 Tbps capacity are now widely available. What is the status of key component technologies, such as wide-range tunable lasers, cost-effective WDM devices, etc.? To build optical fiber rings, we need wavelength division multiplex add/drop devices which are remotely configurable. Are these devices commercially available for deployment in carrier networks?

Advances in optical fiber transmission are due to innovations such as wavelength-division-multiplexing and advanced modulation formats. The most recent innovation is coherent detection which can handle advanced modulation formats, correct linear impairments, and provide ideal optical filtering for high spectral efficiencies. The associated system complexity must be handled with ever higher levels of integration in PICs. Several of these PICs are already deployed in commercial systems. Examples are tunable optical dispersion compensator modules, tunable laser PICs, and the integrated coherent receivers of the recent transmission systems using 100 Gbps channels. The research community has already demonstrated highly sophisticated PICs, such as a monolithic four-channel dual polarization dual quadrature coherent receiver and the InP 16 QAM modulator PIC. However it will take time for the more sophisticated technologies to mature until they are ready for commercial introduction.

Although optical fiber provides almost infinite bandwidth, service providers continue to invest in extending the life of TWP plant (using technologies like DSL) instead of fiber infrastructure for the last mile. Do we need some major innovation in this area to make fiber a more attractive option?

Fiber-to-the-home (FTTH) technology, which provides the broadband communications infrastructure for the last mile, is a complex issue and the answers throughout the world depend strongly on local cost regulations and policies. Recent OECD statistics show that the U.S. is not among the top 10 countries with the most broadband subscribers where there are up to 35 subscribers per 100 inhabitants. While there is still quite a bit of DSL deployed, FTTH deployment is already larger than DSL deployment in advanced countries such as South Korea and Japan. As demand for bandwidth continues to increase worldwide, FTTH is regarded as a future-proof installation. Of course, reduction of costs will make it an even more attractive option. And it appears that fiber is already getting cheaper than copper. In the U.S., FTTH service is now available to about 18 million homes with about 6 million subscribing to the service. About 60% of this is provided by Verizon. Google has announced recently that it is planning to provide fiber-optic connections for up to 500,000 users with a capacity of 1 Gbps per user. They are starting with a small 1 Gbps pilot project at Stanford.

What are the new frontiers of innovation in optical fiber communications? High-spectral efficiency coherent communications?

In the past three decades, the capacity of long-haul optical fiber transmission has increased by a factor of 100 every 10 years. The latest advances have used wavelength division multiplexing and erbium doped fiber amplifiers. Recent research demonstrations have achieved capacities of up to 70 Tbps per fiber. This is approaching the Shannon capacity limits of a fiber, as modified for impairments due to fiber nonlinearities. New approaches are needed to continue this trend in the future. Among these are advanced modulation formats and the use of coherent detection allowing higher spectral efficiencies, as you mention. In addition, researchers are exploring amplifiers with larger bandwidth as well as the use of multimode and multicore fibers to increase the degrees of freedom available for higher capacity transmission via modal multiplexing.

Who inspired you professionally the most?
This was clearly Rudi Kompfner, the man who had recruited and hired me to Bell Labs. Rudi was a director in the Bell Labs Research area at the time, and he was the primary champion of optical communications at Bell Labs. He was building up a strong and broad research effort in all the relevant enabling technologies, and this was more than 20 years before fiber communications was a success in the market. I learned from him another line: "You have to have a hundred new ideas before you have a really good one." He practiced that one diligently, and asked us to help him throw out the bad ideas among the many he created.

Do you have any advice for the new generation of students and researchers entering the optical fiber communications field?
Well, in information technology we live in a time of tremendous innovation and constant change. Learn to use and enjoy that change, even though it may appear painful at first. Think of the line "Change is inevitable, suffering is optional." Switching fields can bring a tremendous benefit, namely cross-fertilization by transporting the ideas and experiences of one field to the other.

Herwig Kogelnik was born in Graz, Austria, in 1932. He received the Dipl. Ing. and Doctor of Technology Degrees, both from the Technische Hochschule Wien, Vienna, Austria, in 1955 and 1958, respectively, and the Ph.D. Degree from Oxford University, Oxford, England, in 1960.

He joined Bell Labs (currently Alcatel-Lucent), Holmdel, New Jersey, in 1961, where he has been concerned with research in optics, electronics, and communications, including work on lasers, holography, optical guided-wave devices, and integrated optics. He is presently Adjunct Photonics Research Vice President.

Dr. Kogelnik is a Fellow of the IEEE and the OSA. He was elected to the National Academy of Engineering in 1978 and to the National Academy of Sciences in 1994. He is the recipient of the 1984 Frederic Ives Medal of OSA, the 1989 David Sarnoff Award, and the 1991 Quantum Electronics Award from the IEEE. He was elected Honorary Fellow of St. Peter's College at Oxford University in 1992. He is the recipient of the 2001 IEEE Medal of Honor and received the 2001 Marconi International Fellowship. In 2006 he was named Honorary Member of OSA, was awarded the Okawa Prize from the Okawa Foundation for Information and Telecommunications, and received the 2006 National Medal of Technology.