

A P P E N D I X

B

THE LAPLACE TRANSFORM

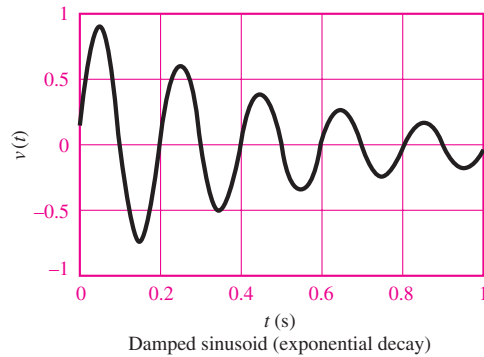
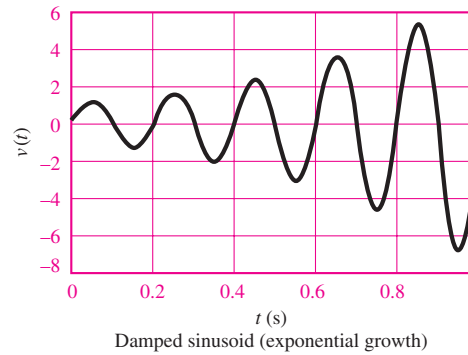
B.1 COMPLEX FREQUENCY AND THE LAPLACE TRANSFORM

The transient analysis methods illustrated in Chapter 5 for first- and second-order circuits can become rather cumbersome when applied to higher-order circuits. Moreover, solving the differential equations directly does not reveal the strong connection that exists between the transient response and the frequency response of a circuit. The aim of this section is to introduce an alternate solution method based on the notions of complex frequency and of the **Laplace transform**. The concepts presented will demonstrate that the frequency response of linear circuits is but a special case of the general transient response of the circuit, when analyzed by means of Laplace methods. In addition, the use of the Laplace transform method allows the introduction of *systems* concepts, such as poles, zeros, and transfer functions, that cannot be otherwise recognized.

Complex Frequency

In Chapter 4, we considered circuits with sinusoidal excitations such as

$$v(t) = A \cos(\omega t + \phi) \quad (\text{B.1})$$


Figure B.1(a) Damped sinusoid: negative σ

Figure B.1(b) Damped sinusoid: positive σ

which we also wrote in the equivalent phasor form

$$\mathbf{V}(j\omega) = Ae^{j\phi} = A\angle\phi \quad (\text{B.2})$$

The two expressions just given are related by

$$v(t) = \text{Re}(\mathbf{V}e^{j\omega t}) \quad (\text{B.3})$$

As was shown in Chapter 4, phasor notation is extremely useful in solving AC steady-state circuits, in which the voltages and currents are *steady-state sinusoids*. We now consider a different class of waveforms, useful in the transient analysis of circuits, namely, *damped sinusoids*. The most general form of a damped sinusoid is

$$v(t) = Ae^{\sigma t} \cos(\omega t + \phi) \quad (\text{B.4})$$

As one can see, a damped sinusoid is a sinusoid multiplied by a real exponential $e^{\sigma t}$. The constant σ is real and is usually zero or negative in most practical circuits. Figure B.1(a) and (b) depicts the case of a damped sinusoid with negative σ and with positive σ , respectively. Note that the case of $\sigma = 0$ corresponds exactly to a sinusoidal waveform. The definition of phasor voltages and currents given in Chapter 4 can easily be extended to account for the case of damped sinusoidal waveforms by defining a new variable s , called the *complex frequency*:

$$s = \sigma + j\omega \quad (\text{B.5})$$

Note that the special case of $\sigma = 0$ corresponds to $s = j\omega$, that is, the familiar steady-state sinusoidal (phasor) case. We shall now refer to the complex variable $\mathbf{V}(s)$ as the **complex frequency domain** representation of $v(t)$. It should be observed that from the viewpoint of circuit analysis, the use of the Laplace transform is analogous to phasor analysis; that is, substituting the variable s wherever $j\omega$ was used is the only step required to describe a circuit using the new notation.

CHECK YOUR UNDERSTANDING

B.1 Find the complex frequencies that are associated with

- a. $5e^{-4t}$ b. $\cos 2\omega t$ c. $\sin(\omega t + 2\theta)$ d. $4e^{-2t} \sin(3t - 50^\circ)$ e. $e^{-3t}(2 + \cos 4t)$

B.2 Find s and $\mathbf{V}(s)$ if $v(t)$ is given by

- a. $5e^{-2t}$ b. $5e^{-2t} \cos(4t + 10^\circ)$ c. $4 \cos(2t - 20^\circ)$

B.3 Find $v(t)$ if

- a. $s = -2, \mathbf{V} = 2\angle 0^\circ$ b. $s = j2, \mathbf{V} = 12\angle -30^\circ$ c. $s = -4 + j3, \mathbf{V} = 6\angle 10^\circ$

Answers: **B.1:** a. -4 ; b. $\pm j2\omega$; c. $\pm j\omega$; d. $-2 \pm j3$; e. -3 and $-3 \pm j4$. **B.2:** a. -2 , $5\angle 0^\circ$; b. $-2 + j4$, $5\angle 10^\circ$; c. $j2$, $4\angle -20^\circ$. **B.3:** a. $2e^{-2t}$; b. $12\cos(2t - 30^\circ)$; c. $6e^{-4t}\cos(3t + 10^\circ)$

All the concepts and rules used in AC network analysis (see Chapter 4), such as impedance, admittance, KVL, KCL, and Thévenin’s and Norton’s theorems, carry over to the damped sinusoid case exactly. In the complex frequency domain, the current $\mathbf{I}(s)$ and voltage $\mathbf{V}(s)$ are related by the expression

$$\mathbf{V}(s) = Z(s)\mathbf{I}(s) \tag{B.6}$$

where $Z(s)$ is the familiar impedance, with s replacing $j\omega$. We may obtain $Z(s)$ from $Z(j\omega)$ by simply replacing $j\omega$ by s . For a resistance R , the impedance is

$$Z_R(s) = R \tag{B.7}$$

For an inductance L , the impedance is

$$Z_L(s) = sL \tag{B.8}$$

For a capacitance C , it is

$$Z_C(s) = \frac{1}{sC} \tag{B.9}$$

Impedances in series or parallel are combined in exactly the same way as in the AC steady-state case, since we only replace $j\omega$ by s .

EXAMPLE B.1 Complex Frequency Notation

Problem:

Use complex impedance ideas to determine the response of a series RL circuit to a damped exponential voltage.

Solution:

Known Quantities: Source voltage, resistor, inductor values.

Find: The time-domain expression for the series current $i_L(t)$.

Schematics, Diagrams, Circuits, and Given Data: $v_s(t) = 10e^{-2t} \cos(5t)$ V; $R = 4 \Omega$; $L = 2$ H.

Assumptions: None.

Analysis: The input voltage phasor can be represented by the expression

$$\mathbf{V}(s) = 10\angle 0^\circ \text{ V}$$

The impedance seen by the voltage source is

$$Z(s) = R + sL = 4 + 2s$$

Thus, the series current is

$$\mathbf{I}(s) = \frac{\mathbf{V}(s)}{Z(s)} = \frac{10}{4 + 2s} = \frac{10}{4 + 2(-2 + j5)} = \frac{10}{j10} = j1 = 1\angle\left(-\frac{\pi}{2}\right)$$

Finally, the time-domain expression for the current is

$$i_L(t) = e^{-2t} \cos(5t - \pi/2) \quad \text{A}$$

Comments: The phasor analysis method illustrated here is completely analogous to the method introduced in Chapter 4, with the complex frequency $j\omega$ (steady-state sinusoidal frequency) related by s (damped sinusoidal frequency).

Just as frequency response functions $H(j\omega)$ were defined in this appendix, it is possible to define a **transfer function** $H(s)$. This can be a ratio of a voltage to a current, a ratio of a voltage to a voltage, a ratio of a current to a current, or a ratio of a current to a voltage. The transfer function $H(s)$ is a function of network elements and their interconnections. Using the transfer function and knowing the input (voltage or current) to a circuit, we can find an expression for the output either in the complex frequency domain or in the time domain. As an example, suppose $\mathbf{V}_i(s)$ and $\mathbf{V}_o(s)$ are the input and output voltages to a circuit, respectively, in complex frequency notation. Then

$$H(s) = \frac{\mathbf{V}_o(s)}{\mathbf{V}_i(s)} \tag{B.10}$$

from which we can obtain the output in the complex frequency domain by computing

$$\mathbf{V}_o(s) = H(s)\mathbf{V}_i(s) \tag{B.11}$$

If $\mathbf{V}_i(s)$ is a known damped sinusoid, we can then proceed to determine $v_o(t)$ by means of the method illustrated earlier in this section.

CHECK YOUR UNDERSTANDING

B.4 Given the transfer function $H(s) = 3(s+2)/(s^2+2s+3)$ and the input $\mathbf{V}_i(s) = 4\angle 0^\circ$, find the forced response $v_o(t)$ if

- a. $s = -1$ b. $s = -1 + j1$ c. $s = -2 + j1$

B.5 Given the transfer function $H(s) = 2(s+4)/(s^2+4s+5)$ and the input $\mathbf{V}_i(s) = 6\angle 30^\circ$, find the forced response $v_o(t)$ if

- a. $s = -4 + j1$ b. $s = -2 + j2$

Answers: **B.4:** a. $6e^{-t}$; b. $12\sqrt{2}e^{-t}\cos(t+45^\circ)$; c. $6e^{-2t}\cos(t+135^\circ)$.
B.5: a. $3e^{-4t}\cos(t+165^\circ)$; b. $8\sqrt{2}e^{-2t}\cos(2t-105^\circ)$.

The Laplace Transform

The Laplace transform, named after the French mathematician and astronomer Pierre Simon de Laplace, is defined by

$$\mathcal{L}[f(t)] = F(s) = \int_0^{\infty} f(t)e^{-st} dt \quad (\text{B.12})$$

The function $F(s)$ is the Laplace transform of $f(t)$ and is a function of the complex frequency $s = \sigma + j\omega$, considered earlier in this section. Note that the function $f(t)$ is defined only for $t \geq 0$. This definition of the Laplace transform applies to what is known as the **one-sided** or **unilateral Laplace transform**, since $f(t)$ is evaluated only for positive t . To conveniently express arbitrary functions only for positive time, we introduce a special function called the **unit step function** $u(t)$, defined by the expression

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases} \quad (\text{B.13})$$

EXAMPLE B.2 Computing a Laplace Transform

Problem:

Find the Laplace transform of $f(t) = e^{-at}u(t)$.

Solution:

Known Quantities: Function to be Laplace-transformed.

Find: $F(s) = \mathcal{L}[f(t)]$.

Schematics, Diagrams, Circuits, and Given Data: $f(t) = e^{-at}u(t)$.

Assumptions: None.

Analysis: From equation B.12,

$$F(s) = \int_0^{\infty} e^{-at}e^{-st} dt = \int_0^{\infty} e^{-(s+a)t} dt = \frac{1}{s+a} e^{-(s+a)t} \Big|_0^{\infty} = \frac{1}{s+a}$$

Comments: Table B.1 contains a list of common Laplace transform pairs.

EXAMPLE B.3 Computing a Laplace Transform

Problem:

Find the Laplace transform of $f(t) = \cos(\omega t)u(t)$.

Solution:

Known Quantities: Function to be Laplace-transformed.

Find: $F(s) = \mathcal{L}[f(t)]$.

Schematics, Diagrams, Circuits, and Given Data: $f(t) = \cos(\omega t)u(t)$.

Assumptions: None.

Analysis: Using equation B.12 and applying Euler's identity to $\cos(\omega t)$ give:

$$\begin{aligned} F(s) &= \int_0^\infty \frac{1}{2}(e^{j\omega t} + e^{-j\omega t})e^{-st} dt = \frac{1}{2} \int_0^\infty (e^{(-s+j\omega)t} + e^{(-s-j\omega)t}) dt \\ &= \frac{1}{-s+j\omega} e^{(-s+j\omega)t} \Big|_0^\infty + \frac{1}{-s-j\omega} e^{(-s-j\omega)t} \Big|_0^\infty \\ &= \frac{1}{-s+j\omega} + \frac{1}{-s-j\omega} = \frac{s}{s^2 + \omega^2} \end{aligned}$$

Comments: Table B.1 contains a list of common Laplace transform pairs.

Table B.1 Laplace transform pairs

$f(t)$	$F(s)$
$\delta(t)$ (unit impulse)	1
$u(t)$ (unit step)	$\frac{1}{s}$
$e^{-at}u(t)$	$\frac{1}{s+a}$
$\sin \omega t u(t)$	$\frac{\omega}{s^2 + \omega^2}$
$\cos \omega t u(t)$	$\frac{s}{s^2 + \omega^2}$
$e^{-at} \sin \omega t u(t)$	$\frac{\omega}{(s+a)^2 + \omega^2}$
$e^{-at} \cos \omega t u(t)$	$\frac{s+a}{(s+a)^2 + \omega^2}$
$tu(t)$	$\frac{1}{s^2}$

CHECK YOUR UNDERSTANDING

B.6 Find the Laplace transform of the following functions:

- a. $u(t)$ b. $\sin(\omega t) u(t)$ c. $tu(t)$

B.7 Find the Laplace transform of the following functions:

- a. $e^{-at} \sin \omega t u(t)$ b. $e^{-at} \cos \omega t u(t)$

Answers: **B.6:** a. $\frac{1}{s}$; b. $\frac{\omega}{s^2 + \omega^2}$; c. $\frac{1}{s^2}$. **B.7:** a. $\frac{\omega}{(s+a)^2 + \omega^2}$; b. $\frac{s+a}{(s+a)^2 + \omega^2}$.

From what has been said so far about the Laplace transform, it is obvious that we may compile a lengthy table of functions and their Laplace transforms by repeated application of equation B.12 for various functions of time $f(t)$. Then we could obtain a wide variety of inverse transforms by matching entries in the table. Table B.1 lists some of the more common **Laplace transform pairs**. The computation of the **inverse Laplace transform** is in general rather complex if one wishes to consider arbitrary functions of s . In many practical cases, however, it is possible to use combinations of known transform pairs to obtain the desired result.

EXAMPLE B.4 Computing an Inverse Laplace Transform

Problem:

Find the inverse Laplace transform of

$$F(s) = \frac{2}{s+3} + \frac{4}{s^2+4} + \frac{4}{s}$$

Solution:

Known Quantities: Function to be inverse Laplace-transformed.

Find: $f(t) = \mathcal{L}^{-1}[F(s)]$.

Schematics, Diagrams, Circuits, and Given Data:

$$F(s) = \frac{2}{s+3} + \frac{4}{s^2+4} + \frac{4}{s} = F_1(s) + F_2(s) + F_3(s)$$

Assumptions: None.

Analysis: Using Table B.1, we can individually inverse-transform each of the elements of $F(s)$:

$$f_1(t) = 2\mathcal{L}^{-1}\left(\frac{1}{s+3}\right) = 2e^{-3t}u(t)$$

$$f_2(t) = 2\mathcal{L}^{-1}\left(\frac{2}{s^2+2^2}\right) = 2\sin(2t)u(t)$$

$$f_3(t) = 4\mathcal{L}^{-1}\left(\frac{1}{s}\right) = 4u(t)$$

Thus

$$f(t) = f_1(t) + f_2(t) + f_3(t) = (2e^{-3t} + 2\sin 2t + 4)u(t).$$

EXAMPLE B.5 Computing an Inverse Laplace Transform

Problem:

Find the inverse Laplace transform of

$$F(s) = \frac{2s+5}{s^2+5s+6}$$

Solution

Known Quantities: Function to be inverse Laplace-transformed.

Find: $f(t) = \mathcal{L}^{-1}[F(s)]$.

Assumptions: None.

Analysis: A direct entry for the function cannot be found in Table B.1. In such cases, one must compute a *partial fraction expansion* of the function $F(s)$ and then individually transform each term in the expansion. A partial fraction expansion is the inverse operation of obtaining a common denominator and is illustrated below.

$$F(s) = \frac{2s+5}{s^2+5s+6} = \frac{A}{s+2} + \frac{B}{s+3}$$

To obtain the constants A and B , we multiply the above expression by each of the denominator terms:

$$(s+2)F(s) = A + \frac{(s+2)B}{s+3}$$

$$(s+3)F(s) = \frac{(s+3)A}{s+2} + B$$

From the above two expressions, we can compute A and B as follows:

$$A = (s + 2)F(s)|_{s=-2} = \frac{2s + 5}{s + 3} \Big|_{s=-2} = 1$$

$$B = (s + 3)F(s)|_{s=-3} = \frac{2s + 5}{s + 2} \Big|_{s=-3} = 1$$

Finally,

$$F(s) = \frac{2s + 5}{s^2 + 5s + 6} = \frac{1}{s + 2} + \frac{1}{s + 3}$$

and using Table B.1, we compute

$$f(t) = (e^{-2t} + e^{-3t}) u(t)$$

CHECK YOUR UNDERSTANDING

B.8 Find the inverse Laplace transform of each of the following functions:

- a. $F(s) = \frac{1}{s^2 + 5s + 6}$
- b. $F(s) = \frac{s - 1}{s(s + 2)}$
- c. $F(s) = \frac{3s}{(s^2 + 1)(s^2 + 4)}$
- d. $F(s) = \frac{1}{(s + 2)(s + 1)^2}$

Answers: a. $f(t) = e^{-2t} - e^{-3t}$; b. $f(t) = \frac{1}{2}e^{-2t} - \frac{1}{2}e^{-t}$; c. $f(t) = \frac{1}{2}(\cos 2t - \cos t)$; d. $f(t) = e^{-t} - e^{-2t} + te^{-2t}$

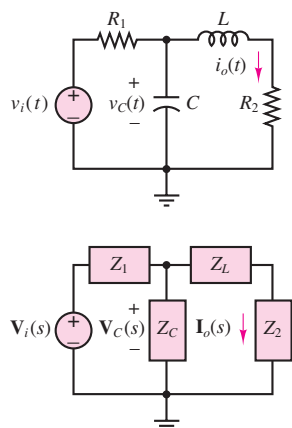


Figure B.2 A circuit and its Laplace transform domain equivalent

Transfer Functions, Poles, and Zeros

It should be clear that the Laplace transform can be quite a convenient tool for analyzing the transient response of a circuit. The Laplace variable s is an extension of the steady-state frequency response variable $j\omega$ already encountered in this appendix. Thus, it is possible to describe the input-output behavior of a circuit by using Laplace transform ideas in the same way in which we used frequency response ideas earlier. Now we can define voltages and currents in the complex frequency domain as $\mathbf{V}(s)$ and $\mathbf{I}(s)$, and we denote impedances by the notation $Z(s)$, where s replaces the familiar $j\omega$. We define an extension of the frequency response of a circuit, called the *transfer function*, as the ratio of any input variable to any output variable, that is,

$$H_1(s) = \frac{\mathbf{V}_o(s)}{\mathbf{V}_i(s)} \quad \text{or} \quad H_2(s) = \frac{\mathbf{I}_o(s)}{\mathbf{V}_i(s)} \quad \text{etc.} \tag{B.14}$$

As an example, consider the circuit of Figure B.2. We can analyze it by using a method analogous to phasor analysis by defining impedances

$$Z_1 = R_1 \quad Z_C = \frac{1}{sC} \quad Z_L = sL \quad Z_2 = R_2 \tag{B.15}$$

Then we can use mesh analysis methods to determine that

$$\mathbf{I}_o(s) = \mathbf{V}_i(s) \frac{Z_C}{(Z_L + Z_2)Z_C + (Z_L + Z_2)Z_1 + Z_1Z_C} \quad (\text{B.16})$$

or, upon simplifying and substituting the relationships of equation B.15,

$$H_2(s) = \frac{\mathbf{I}_o(s)}{\mathbf{V}_i(s)} = \frac{1}{R_1LCs^2 + (R_1R_2C + L)s + R_1 + R_2} \quad (\text{B.17})$$

If we were interested in the relationship between the input voltages and, say, the capacitor voltage, we could similarly calculate

$$H_1(s) = \frac{\mathbf{V}_C(s)}{\mathbf{V}_i(s)} = \frac{sL + R_2}{R_1LCs^2 + (R_1R_2C + L)s + R_1 + R_2} \quad (\text{B.18})$$

Note that a transfer function consists of a *ratio of polynomials*; this ratio can also be expressed in factored form, leading to the discovery of additional important properties of the circuit. Let us, for the sake of simplicity, choose numerical values for the components of the circuit of Figure B.2. For example, let $R_1 = 0.5 \Omega$, $C = \frac{1}{4} \text{ F}$, $L = 0.5 \text{ H}$, and $R_2 = 2 \Omega$. Then we can substitute these values into equation B.18 to obtain

$$H_1(s) = \frac{0.5s + 2}{0.0625s^2 + 0.375s + 2.5} = 8 \left(\frac{s + 4}{s^2 + 6s + 40} \right) \quad (\text{B.19})$$

Equation B.19 can be factored into products of first-order terms as follows:

$$H_1(s) = 8 \left[\frac{s + 4}{(s - 3.0000 + j5.5678)(s - 3.0000 - j5.5678)} \right] \quad (\text{B.20})$$

where it is apparent that the response of the circuit has very special characteristics for three values of s : $s = -4$; $s = +3.0000 + j5.5678$; and $s = +3.0000 - j5.5678$. In the first case, at the complex frequency $s = -4$, the numerator of the transfer function becomes zero, and the response of the circuit is zero, regardless of how large the input voltage is. We call this particular value of s a **zero** of the transfer function. In the latter two cases, for $s = +3.0000 \pm j5.5678$, the response of the circuit becomes infinite, and we refer to these values of s as **poles** of the transfer function.

It is customary to represent the response of electric circuits in terms of poles and zeros, since knowledge of the location of these poles and zeros is equivalent to knowing the transfer function and provides complete information regarding the response of the circuit. Further, if the poles and zeros of the transfer function of a circuit are plotted in the complex plane, it is possible to visualize the response of the circuit very effectively. Figure B.3 depicts the pole-zero plot of the circuit of Figure B.2; in plots of this type it is customary to denote zeros by a small circle and poles by an “×.”

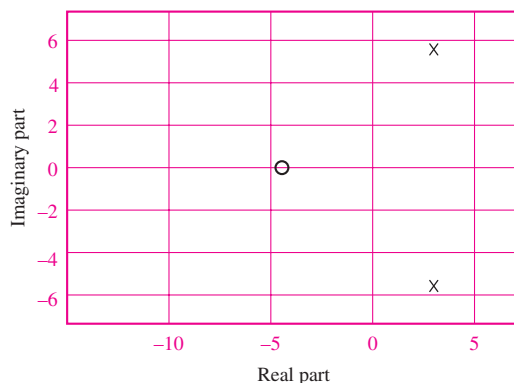


Figure B.3 Zero-pole plot for the circuit of Figure B.2

The poles of a transfer function have a special significance, in that they are equal to the roots of the natural response of the system. They are also called the **natural frequencies** of the circuit. Example B.6 illustrates this point.

EXAMPLE B.6 Poles of a Second-Order Circuit

Problem:

Determine the poles of a parallel *RLC* circuit. Express the homogeneous equation using i_L as the independent variable.

Solution:

Known Quantities: Values of resistor, inductor, and capacitor.

Find: Poles of the circuit.

Assumptions: None.

Analysis: The differential equation describing the natural response of the parallel *RLC* circuit is

$$\frac{d^2i}{dt^2} + \frac{R}{L} \frac{di}{dt} + \frac{1}{LC}i = 0$$

with characteristic equation given by

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

Now, let us determine the transfer function of the circuit, say, $\mathbf{V}_L(s)/\mathbf{V}_S(s)$. Applying the voltage divider rule, we can write

$$\begin{aligned} \frac{\mathbf{V}_L(s)}{\mathbf{V}_S(s)} &= \frac{sL}{1/sC + R + sL} \\ &= \frac{s^2}{s^2 + (R/L)s + 1/LC} \end{aligned}$$

The denominator of this function, which determines the poles of the circuit, is identical to the characteristic equation of the circuit: The poles of the transfer function are identical to the roots of the characteristic equation!

$$s_{1,2} = -\frac{R}{2L} \pm \frac{1}{2} \sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}$$

Comments: Describing a circuit by means of its transfer function is completely equivalent to representing it by means of its differential equation. However, it is often much easier to derive a transfer function by basic circuit analysis than it is to obtain the differential equation of a circuit.
