

CHAPTER 7 - FIRST-ORDER CIRCUITS

List of topics for this chapter :

Source-Free RC Circuit
Source-Free RL Circuit
Singularity Functions
Step Response of an RC Circuit
Step Response of an RL Circuit
First-Order Op Amp Circuits
Transient Analysis with PSpice
Applications

SOURCE-FREE RC CIRCUIT

Problem 7.1 For the circuit in Figure 7.1, find $v_C(t)$ and $i_C(t)$ given $v_C(0) = 10 \text{ V}$.

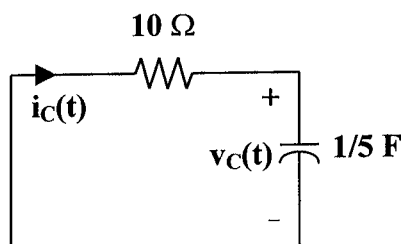


Figure 7.1

- **Carefully DEFINE the problem.**
Each component is labeled completely. The problem is clear.
- **PRESENT everything you know about the problem.**
This is a source-free RC circuit. The natural response of this source-free RC circuit is

$$v_C(t) = V_0 e^{-t/\tau}, \quad \text{where } V_0 = v_C(0) \quad \text{and} \quad \tau = RC$$

We know the initial voltage across the capacitor. To find the capacitor voltage for any time greater than zero, we need to calculate the time constant of the circuit.

- **Establish a set of ALTERNATIVE solutions and determine the one that promises the greatest likelihood of success.**
The three solution techniques that can be used are nodal analysis, mesh analysis, and basic circuit analysis. Basic circuit analysis can be used to solve this problem.

➤ **ATTEMPT a problem solution.**

$$V_0 = v_C(0) = 10 \text{ V} \quad \text{and} \quad \tau = RC = (10)(1/5) = 2 \text{ s}$$

$$v_C(t) = 10e^{-t/2} \text{ V}$$

$$i_C(t) = \frac{-v_C(t)}{R} \quad \text{or} \quad i_C(t) = C \frac{dv_C(t)}{dt}$$

In either case, $i_C(t) = -e^{-t/2} \text{ A}$

➤ **EVALUATE the solution and check for accuracy.**

Using KVL,

$$10i_C(t) + v_C(t) = (10)(-e^{-t/2}) + 10e^{-t/2} = 0$$

Our check for accuracy was successful.

➤ **Has the problem been solved SATISFACTORILY? If so, present the solution; if not, then return to “ALTERNATIVE solutions” and continue through the process again.**
This problem has been solved satisfactorily.

$$i_C(t) = \underline{-e^{-t/2} \text{ amps}} \text{ for all } t > 0.$$

Problem 7.2 [7.23] Express the signals in Figure 7.2 in terms of singularity functions.

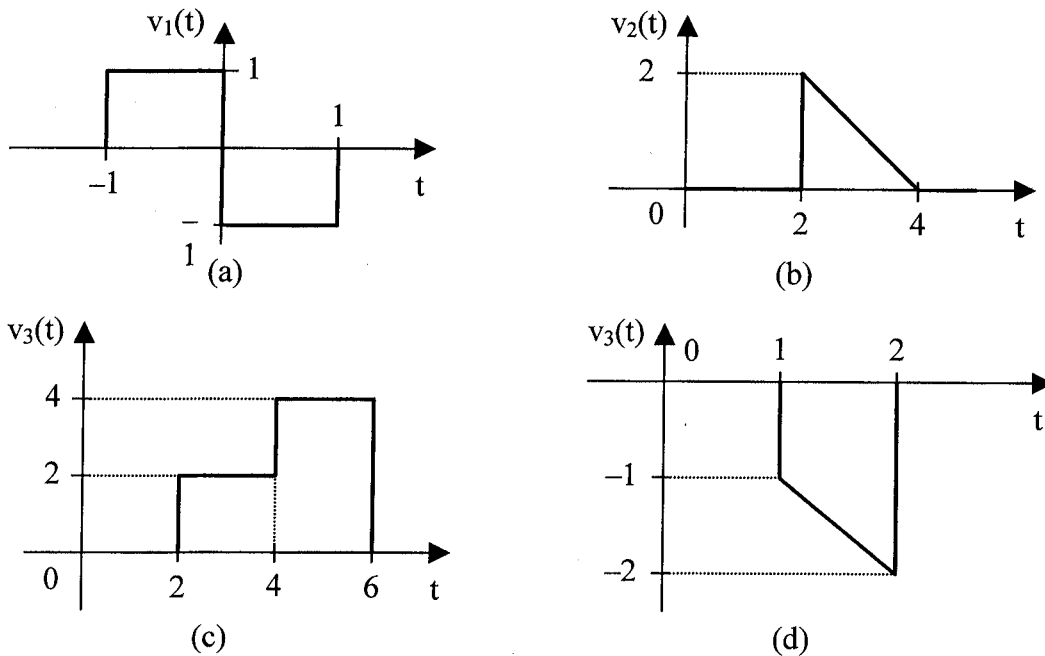


Figure 7.2

- (a) $v_1(t) = u(t+1) - u(t) + [u(t-1) - u(t)]$
 $v_1(t) = \underline{u(t+1) - 2u(t) + u(t-1)}$
- (b) $v_2(t) = (4-t)[u(t-2) - u(t-4)]$
 $v_2(t) = -(t-4)u(t-2) + (t-4)u(t-4)$
 $v_2(t) = \underline{2u(t-2) - r(t-2) + r(t-4)}$
- (c) $v_3(t) = 2[u(t-2) - u(t-4)] + 4[u(t-4) - u(t-6)]$
 $v_3(t) = \underline{2u(t-2) + 2u(t-4) - 4u(t-6)}$
- (d) $v_4(t) = -t[u(t-1) - u(t-2)] = -tu(t-1) + tu(t-2)$
 $v_4(t) = (-t+1-1)u(t-1) + (t-2+2)u(t-2)$
 $v_4(t) = \underline{-r(t-1) - u(t-1) + r(t-2) + 2u(t-2)}$

Problem 7.3 Given $i(t) = 3e^{-t/2}$ A, find $v_C(t)$ for the circuit shown in Figure 7.3.

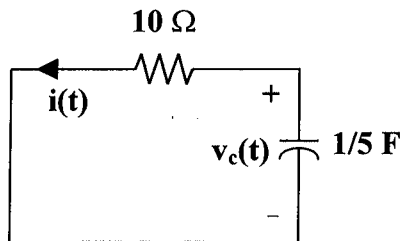


Figure 7.3

$$v_C(t) = \underline{30e^{-t/2} \text{ V}}$$

Problem 7.4 Given $v_C(1) = 10$ V, find $v_C(t)$ for all $t > 0$ in Figure 7.4.

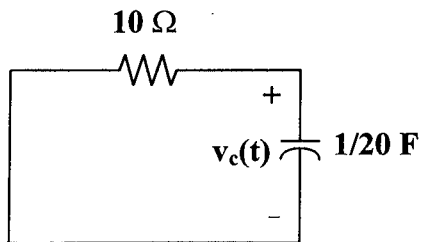


Figure 7.4

$$v_C(t) = \underline{\left(\frac{10}{e^{-2}}\right)e^{-2t} \text{ V}}$$

SOURCE-FREE RL CIRCUIT

Problem 7.5 For the circuit in Figure 7.5, find $i(t)$ and $v_L(t)$ given $i(0) = 4 \text{ A}$.

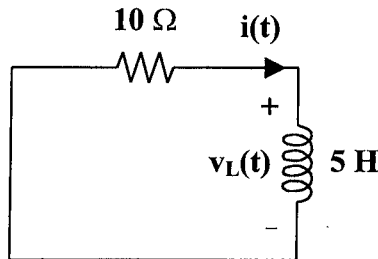


Figure 7.5

- **Carefully DEFINE the problem.**
Each component is labeled completely. The problem is clear.
- **PRESENT everything you know about the problem.**
This is a source-free RL circuit. The natural response of this source-free RL circuit is

$$i_L(t) = I_0 e^{-t/\tau}, \quad \text{where } I_0 = i_L(0) \quad \text{and} \quad \tau = L/R$$

We know the initial current through the inductor. To find the current through the inductor for any value of time greater than zero, we need to calculate the time constant of the circuit.

- **Establish a set of ALTERNATIVE solutions and determine the one that promises the greatest likelihood of success.**
The three solution techniques that can be used are nodal analysis, mesh analysis, and basic circuit analysis. Basic circuit analysis can be used to solve this problem.
- **ATTEMPT a problem solution.**

$$I_0 = i_L(0) = 4 \text{ A} \quad \text{and} \quad \tau = L/R = 5/10 = 0.5 \text{ s}$$

$$i(t) = 4e^{-2t} \text{ A}$$

$$v_L(t) = -10i(t) \quad \text{or} \quad v_L(t) = L \frac{di(t)}{dt}$$

In either case,

$$v_L(t) = -40e^{-2t} \text{ V}$$

- **EVALUATE the solution and check for accuracy.**

Using KVL,

$$10i(t) + v_L(t) = (10)(4e^{-2t}) - 40e^{-2t} = 0$$

Our check for accuracy was successful.

- **Has the problem been solved SATISFACTORILY? If so, present the solution; if not, then return to “ALTERNATIVE solutions” and continue through the process again.**
This problem has been solved satisfactorily.

$$i(t) = \underline{4e^{-2t}u(t)} \text{ A} \quad \text{and} \quad v_L(t) = \underline{-40e^{-2t}u(t)} \text{ V}$$

Problem 7.6 For the circuit in Figure 7.6, find $i(t)$ given $v_L(t) = 20e^{-2t}$ V.

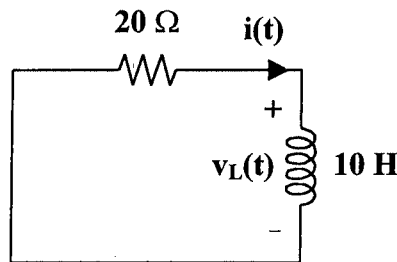


Figure 7.6

$$L/R = 10/20 = 0.5 \text{ s}$$

$$i(t) = \frac{1}{L} \int v_L(\tau) d\tau$$

but it is also

$$i(t) = \frac{-v_L(t)}{R} = \underline{-e^{-2t}u(t)} \text{ A}$$

Problem 7.7 Given $i(0) = 2$ A, find $i(t)$, $p_{10\Omega}$ (power absorbed by the 10 ohm resistor), and $w_{10\Omega}$ (total energy dissipated by the 10 ohm resistor) for the circuit in Figure 7.7.

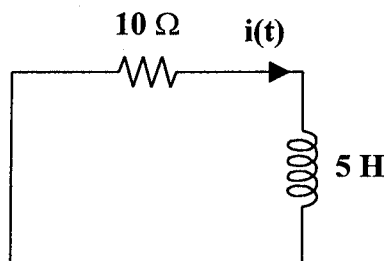


Figure 7.7

$$i(t) = \underline{2e^{-2t} \text{ A}}$$

$$p_{10\Omega} = \underline{40e^{-4t} \text{ W}}$$

$$w_{10\Omega} = \underline{10 \text{ J}}$$

SINGULARITY FUNCTIONS

Problem 7.8 Solve for

(a) $\frac{du(t)}{dt}$

(b) $\frac{dr(t)}{dt}$

(a) $u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$

$$\frac{d}{dt}u(t) = \begin{cases} 0 & t < 0 \\ \text{undefined} & t = 0 \\ 0 & t > 0 \end{cases} \longrightarrow \frac{d}{dt}u(t) = \underline{\delta(t)}$$

(b) $r(t) = \begin{cases} 0 & t \leq 0 \\ t & t > 0 \end{cases}$

$$\frac{d}{dt}r(t) = \begin{cases} 0 & t \leq 0 \\ 1 & t > 0 \end{cases} \longrightarrow \frac{d}{dt}r(t) = \underline{u(t)}$$

Problem 7.9 Given $v_C(t) = [5u(t) + 6r(t)] \text{ V}$, find $i_C(t)$ for the circuit in Figure 7.8.

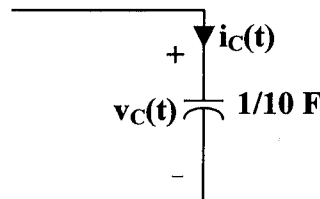


Figure 7.8

$$i_C(t) = C \frac{d}{dt}v_C(t)$$

$$i_c(t) = \frac{1}{10} \cdot \frac{d}{dt} [5u(t) + 6r(t)] = \frac{1}{10} \left[5 \frac{d}{dt} u(t) + 6 \frac{d}{dt} r(t) \right]$$

Using Problem 7.8, it is clear that

$$i_c(t) = \underline{\underline{\frac{1}{10} [5\delta(t) + 6u(t)] \text{ A}}}$$

Problem 7.10 Solve for

(a) $\int \delta(t) dt$

(b) $\int u(t) dt$

(a) $u(t)$

(b) $r(t)$

STEP RESPONSE OF AN RC CIRCUIT

Problem 7.11 Given $v(t) = 20u(t)$ V, find $v_c(t)$ and $i_c(t)$ in Figure 7.9.

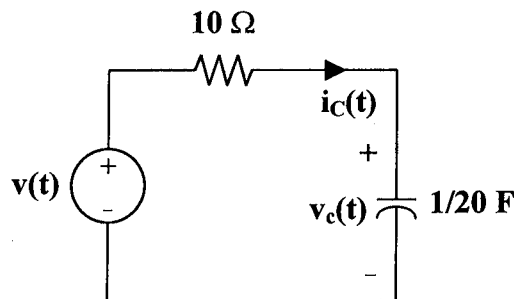


Figure 7.9

$$\tau = RC = (10) \left(\frac{1}{20} \right) = \frac{1}{2} \text{ s}$$

$$v_c(0) = 0 \text{ V} \qquad v_c(\infty) = 20 \text{ V}$$

$$v_c(t) = \underline{\underline{(20)(1 - e^{-2t})u(t) \text{ V}}}$$

$$i_c(t) = C \frac{dv_c(t)}{dt} = \left(\frac{1}{20} \right) (-20)(-2e^{-2t}) = \underline{\underline{2e^{-2t}u(t) \text{ A}}}$$

Problem 7.12 [7.37] Find the step responses $v(t)$ and $i(t)$ to $v_s = 5u(t)$ V in the circuit of Figure 7.10

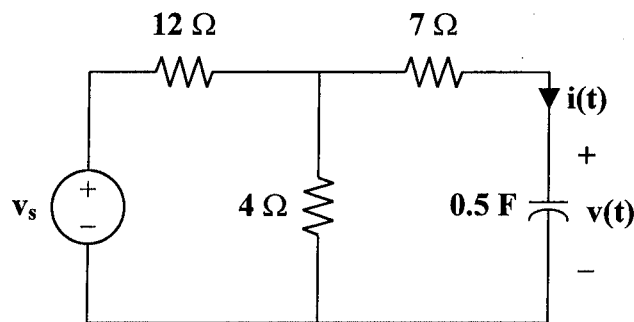


Figure 7.10

For $t < 0$, $v_s = 5u(t) = 0 \longrightarrow v(0) = 0$ V

For $t > 0$, $v_s = 5$ V, $v(\infty) = \frac{4}{4+12} (5) = \frac{5}{4}$ V

$R_{eq} = 7 + 4 \parallel 12 = 10 \Omega$, $\tau = R_{eq}C = (10)(1/2) = 5$ s

$v(t) = v(\infty) + [v(0) - v(\infty)] e^{-t/\tau}$

$v(t) = \underline{1.25(1 - e^{-t/5})}$ V

$i(t) = C \frac{dv}{dt} = \left(\frac{1}{2}\right)\left(\frac{-5}{4}\right)\left(\frac{-1}{5}\right) e^{-t/5}$

$i(t) = \underline{0.125e^{-t/5}}$ A

Problem 7.13 Given $v(t) = 10[u(t) - u(t - 2)]$ V, find $v_c(t)$ in Figure 7.11.

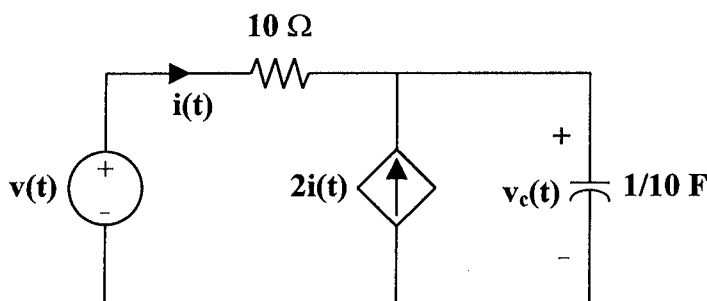
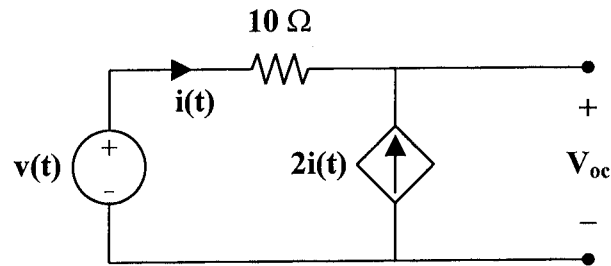


Figure 7.11

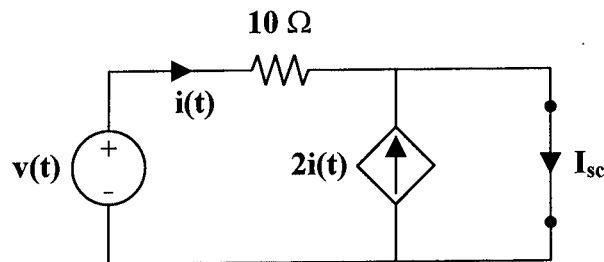
Find the Thevenin equivalent of the circuit at the terminals of the capacitor. This will simplify the circuit, forming an RC circuit with a voltage source.

Use the following circuit to find the open circuit voltage.



V_{oc} must be equal to $v(t)$, since $i(t) + 2i(t) = 0 \longrightarrow i(t) = 0 \text{ A}$.

To find the short circuit current,



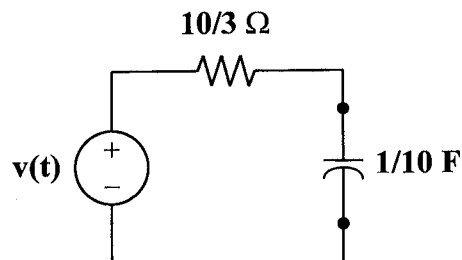
$$I_{sc} = i(t) + 2i(t) = 3i(t) \quad \text{where } i(t) = \frac{v(t)}{10}$$

$$I_{sc} = \frac{3}{10}v(t)$$

Thus,

$$R_{Th} = \frac{V_{oc}}{I_{sc}} = \frac{10}{3} \Omega$$

which leads to the following Thevenin equivalent circuit.



Using the Thevenin equivalent circuit with the capacitor as the load, we can see that

$$\tau = R_{th}C = \left(\frac{10}{3}\right)\left(\frac{1}{10}\right) = \frac{1}{3} \text{ s}$$

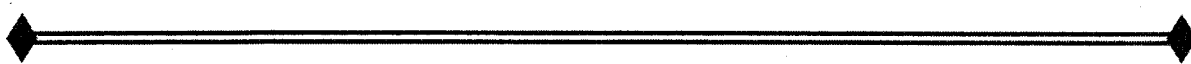
For $t < 0$, $v(t) = 0 \text{ V}$, $v_C(0) = 0 \text{ V}$, $v_C(\infty) = 0 \text{ V}$
 $v_C(t) = 0 \text{ V}$

For $0 < t < 2$, $v(t) = 10 \text{ V}$, $v_C(0) = 0 \text{ V}$, $v_C(\infty) = 10 \text{ V}$
 $v_C(t) = (10)(1 - e^{-3t}) \text{ V}$

For $2 < t$, $v(t) = 0 \text{ V}$, $v_C(2) = (10)(1 - e^{-6}) \text{ V}$, $v_C(\infty) = 0 \text{ V}$
 $v_C(t) = (10)(1 - e^{-6})e^{-3(t-2)} \text{ V}$

Combining these cases,

$$v_C(t) = \left\{ [(10)(1 - e^{-3t})][u(t) - u(t - 2)] + [9.975e^{-3(t-2)}u(t - 2)] \right\} \text{ V}$$



STEP RESPONSE OF AN RL CIRCUIT

Problem 7.14 Given $v(t) = 40u(t) \text{ V}$, find $i_L(t)$ and $v_L(t)$ in Figure 7.12.

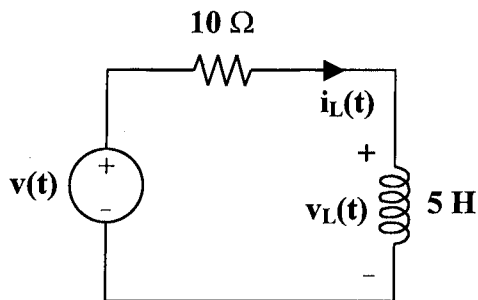


Figure 7.12

$$\tau = \frac{L}{R} = \frac{5}{10} = \frac{1}{2} \text{ s}$$

$$i_L(0) = 0 \text{ A}$$

$$i_L(t) = \underline{(4)(1 - e^{-2t})u(t) \text{ A}}$$

$$v_L(t) = L \frac{di_L(t)}{dt} = (5)(4)(2)e^{-2t}u(t) = \underline{40e^{-2t}u(t) \text{ V}}$$

Problem 7.15 [7.55] Find $v_o(t)$ for $t > 0$ in the circuit of Figure 7.13.

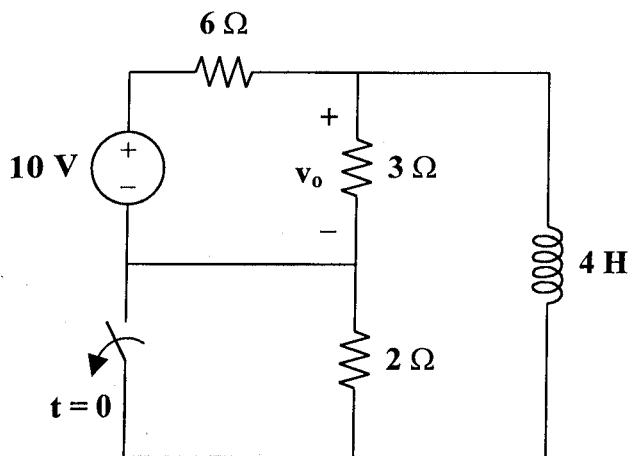
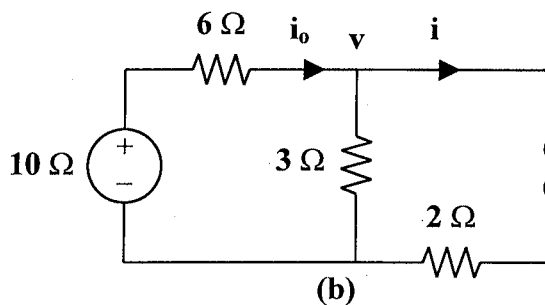
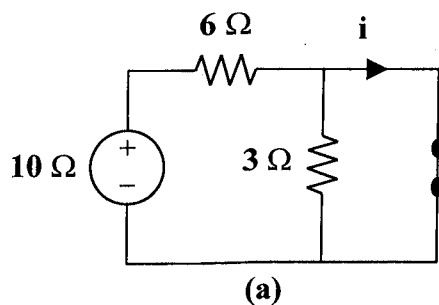


Figure 7.13

Let i be the inductor current. For $t < 0$, the inductor acts like a short circuit and the 2Ω resistor is short-circuited so that the equivalent circuit is shown in Fig. (a).



$$i = i(0) = \frac{10}{6} = 1.6667 \text{ A}$$

$$\text{For } t > 0, \quad R_{th} = 2 + 3 \parallel 6 = 4 \Omega, \quad \tau = \frac{L}{R_{th}} = \frac{4}{4} = 1 \text{ s}$$

To find $i(\infty)$, consider the circuit in Fig. (b).

$$\frac{10 - v}{6} = \frac{v}{3} + \frac{v}{2} \longrightarrow v = \frac{10}{6} \text{ V}$$

$$i = i(\infty) = \frac{v}{2} = \frac{5}{6} \text{ A}$$

$$i(t) = i(\infty) + [i(0) - i(\infty)] e^{-t/\tau}$$

$$i(t) = \frac{5}{6} + \left(\frac{10}{6} - \frac{5}{6} \right) e^{-t} = \frac{5}{6} (1 + e^{-t}) \text{ A}$$

v_o is the voltage across the 4 H inductor and the 2 Ω resistor

$$v_o(t) = 2i + L \frac{di}{dt} = \frac{10}{6} + \frac{10}{6} e^{-t} + (4) \left(\frac{5}{6} \right) (-1) e^{-t} = \frac{10}{6} - \frac{10}{6} e^{-t}$$

$$v_o(t) = \underline{1.6667(1 - e^{-t}) \text{ V}}$$

Problem 7.16 Find $i_L(t)$ and $v_L(t)$ in Figure 7.14 for $v(t) = [20u(t) - 40u(t-1)] \text{ V}$.

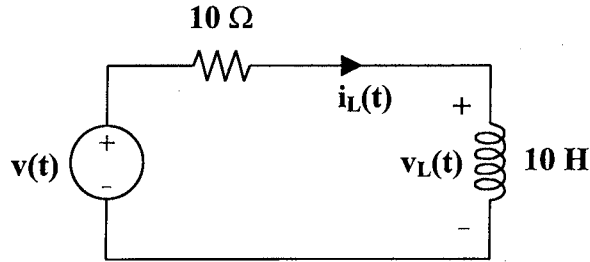


Figure 7.14

$$i_L(t) = \underline{(2)(1 - e^{-t})[u(t) - u(t-1)] + [-2 + (4 - 2e^{-1})e^{-(t-1)}]u(t-1) \text{ A}}$$

$$v_L(t) = \underline{20e^{-t}u(t) - [20e^{-t} + (20)(2 - e^{-1})e^{-(t-1)}]u(t-1) \text{ V}}$$

FIRST-ORDER OP AMP CIRCUITS

Problem 7.17 Given $v(t) = 10u(t) \text{ V}$, find $i_o(t)$ for the circuit in Figure 7.15.

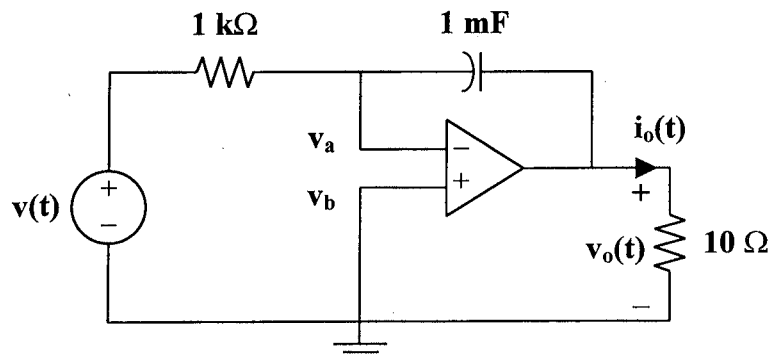


Figure 7.15

$$\frac{v_a - v(t)}{1k} + 1m \frac{d}{dt} [v_a - v_o(t)] = 0,$$

$$\text{where } v_a = v_b = 0 \text{ V.}$$

$$-v(t) = \frac{dv_o(t)}{dt}$$

$$v_o(t) = -\int v(\tau) d\tau = -\int 10 u(\tau) d\tau = -10t \text{ V}$$

$$i_o(t) = \frac{v_o(t)}{10} = \underline{\underline{-t u(t) \text{ A}}}$$

Problem 7.18 [7.59] Obtain v_o for $t > 0$ in the circuit of Figure 7.16.

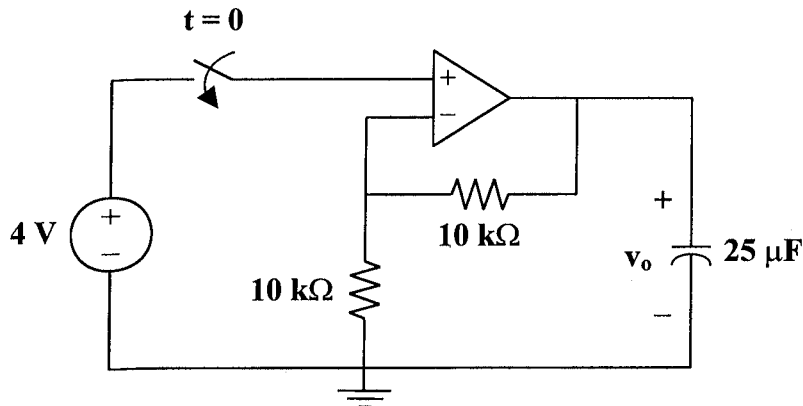


Figure 7.16

This is a very interesting problem and has both an important ideal solution as well as an important practical solution. Let us look at the ideal solution first. Just before the switch closes, the value of the voltage across the capacitor is zero, which means that the voltage at both input terminals of the op amp are zero. As soon as the switch closes, the output tries to go to a voltage such that the inputs to the op amp both go to 4 volts. The ideal op amp puts out whatever current is necessary to reach this condition. An infinite (impulse) current is necessary if the voltage across the capacitor is to go to 8 volts in zero time (8 volts across the capacitor will result in 4 volts appearing at the negative terminal of the op amp). So v_o will be equal to 8 volts for all $t > 0$.

What happens in a real circuit? Essentially, the output of the amplifier portion of the op amp goes to whatever its maximum value can be. Then, this maximum voltage appears across the output resistance of the op amp and the capacitor that is in series with it. This then results in an exponential rise in the capacitor voltage to the steady-state value of 8 volts.

For all values of $v_c(t)$ less than 8 V,

$$v_c(t) = \underline{\underline{V_{\text{op-amp-max}} (1 - e^{-t/(R_{\text{out}}C)}) \text{ V}}}$$

where $V_{\text{op-amp-max}}$ is the maximum value of the op amp and R_{out} is the real output resistance of the practical op amp.

When t is large enough so that the 8 V is reached,

$$v_c(t) = \underline{\underline{8 \text{ V}}}$$

TRANSIENT ANALYSIS WITH PSPICE

Problem 7.19 [7.69] The switch in Figure 7.17 moves from position a to b at $t = 0$. Use PSpice to find $i(t)$ for $t > 0$.

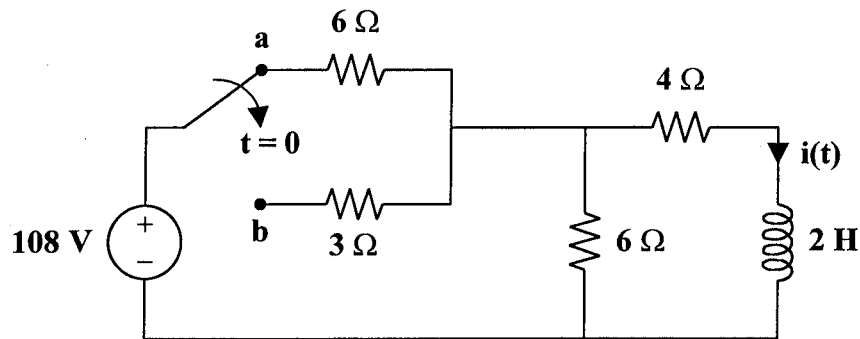
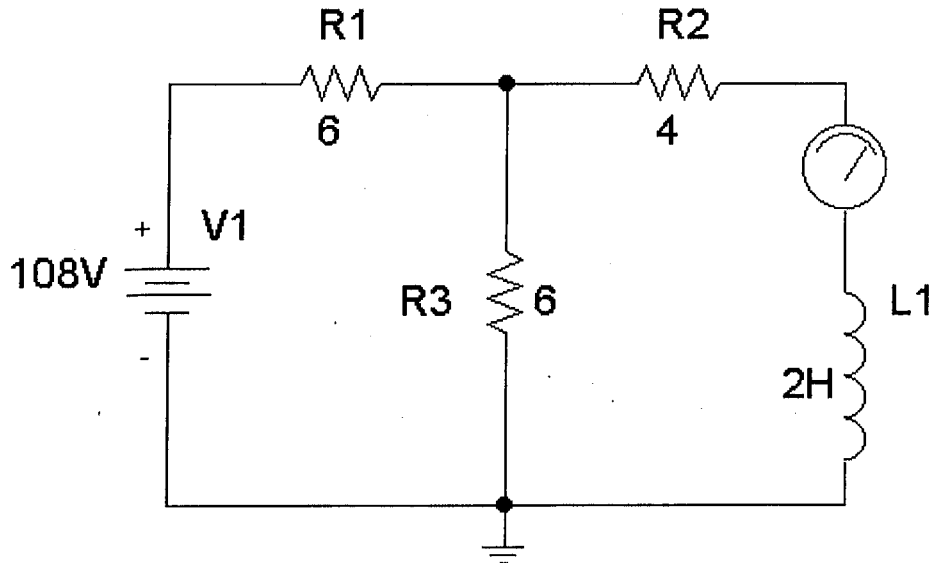


Figure 7.17

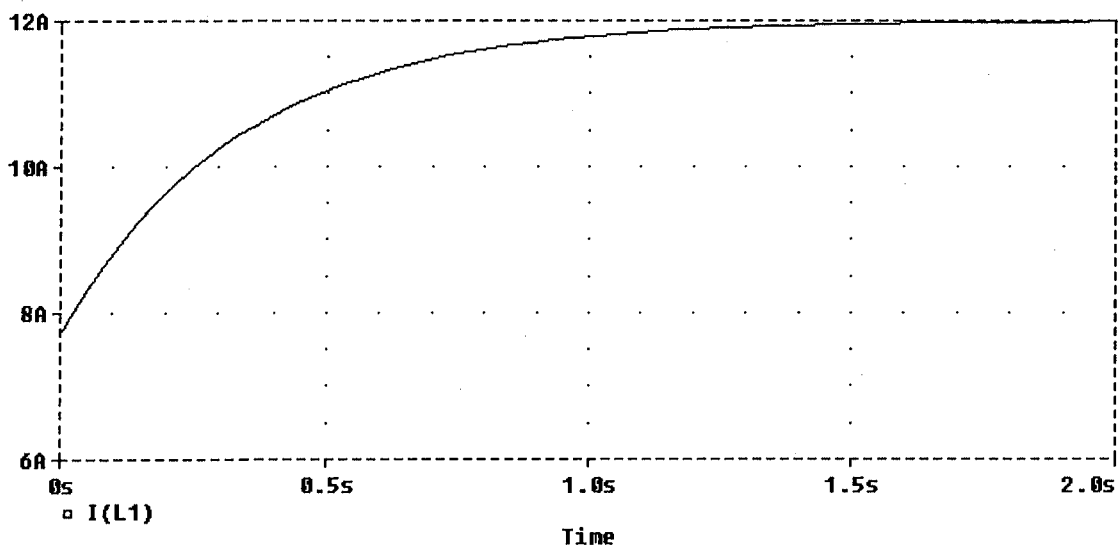
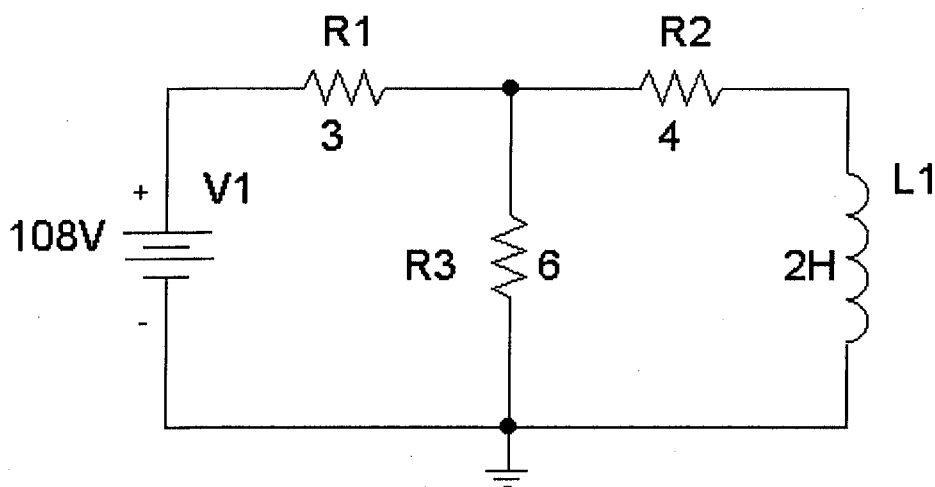
- (a) When the switch is in position a, the schematic is shown below. We insert IPROBE to display i . After simulation, we obtain,

$$i(0) = 7.714 \text{ A}$$

from the display of IPROBE.



- (b) When the switch is in position b, the schematic is as shown below. For inductor L1, we let $I_C = 7.714 \text{ A}$. By clicking Analysis/Setup/Transient, we let Print Step = 25 ms and Final Step = 2 s. After Simulation, we click Trace/Add in the probe menu and display $I(L1)$ as shown below. Note that $i(\infty) = 12 \text{ A}$, which is correct.



We now know the initial and final values of the current through the inductor.

$$i(0) = 7.714 \text{ A} \qquad i(\infty) = 12 \text{ A}$$

To find the current through the inductor for any value of time, we need to know the time constant of the circuit. Using the circuit from part (b),

$$R_{eq} = 3 \parallel 6 + 4 = 2 + 4 = 6 \Omega$$

$$\tau = L/R_{eq} = 2/6 = 1/3 \text{ s}$$

Therefore, $i(t) = i(\infty) + [i(0) - i(\infty)] e^{-t/\tau}$

$$i(t) = 12 + [7.714 - 12]e^{-3t} = \underline{12 - 4.286e^{-3t} \text{ A}}$$

APPLICATIONS

Problem 7.20 [7.73] Figure 7.18 shows a circuit for setting the length of time voltage is applied to the electrodes of a welding machine. The time is taken as how long it takes the capacitor to charge from 0 to 8 V. What is the time range covered by the variable resistor?

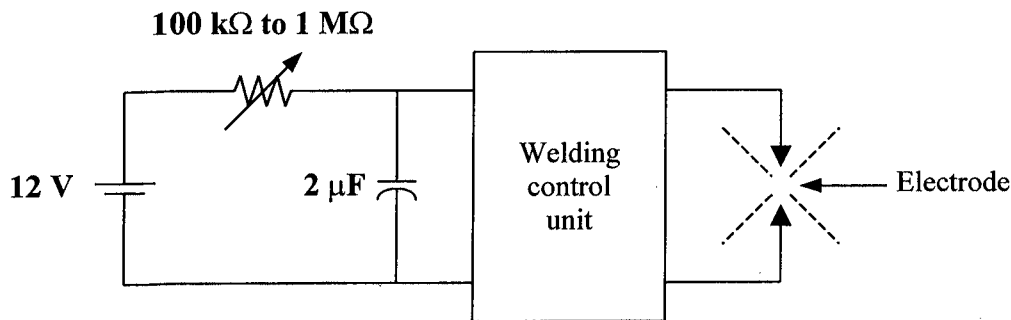


Figure 7.18

- **Carefully DEFINE the problem.**
Each component is labeled completely. The problem is clear.
- **PRESENT everything you know about the problem.**
This is an RC circuit with a dc voltage source. When the welding machine is activated, the dc source supplies power to the RC circuit. To find the time in which it takes the capacitor to charge from 0 to 8 V, we need to find the response of the RC circuit, written as

$$v(t) = v(\infty) + [v(0) - v(\infty)] e^{-t/\tau}$$

where $v(0)$ is the initial voltage across the capacitor, $v(\infty)$ is the steady-state value of the voltage across the capacitor, and τ is the time constant of the RC circuit.

- **Establish a set of ALTERNATIVE solutions and determine the one that promises the greatest likelihood of success.**
The three solution techniques that can be used are nodal analysis, mesh analysis, and basic circuit analysis. Basic circuit analysis can be used to solve this problem.
- **ATTEMPT a problem solution.**

$$v(t) = v(\infty) + [v(0) - v(\infty)] e^{-t/\tau}$$

$$v(0) = 0 \text{ V} \quad \text{and} \quad v(\infty) = 12 \text{ V}$$

$$v(t) = (12)(1 - e^{-t/\tau})$$

Let the voltage at an unknown time, t_0 , be equal to 8 V.

$$v(t_0) = 8 = (12)(1 - e^{-t_0/\tau})$$

$$\frac{8}{12} = 1 - e^{-t_0/\tau} \longrightarrow e^{-t_0/\tau} = \frac{1}{3}$$

$$t_0 = \tau \ln(3)$$

For $R = 100 \text{ k}\Omega$,

$$\tau = RC = (100 \times 10^3)(2 \times 10^{-6}) = 0.2 \text{ s}$$

$$t_0 = 0.2 \ln(3) = 0.2197 \text{ s}$$

For $R = 1 \text{ M}\Omega$,

$$\tau = RC = (1 \times 10^6)(2 \times 10^{-6}) = 2 \text{ s}$$

$$t_0 = 2 \ln(3) = 2.197 \text{ s}$$

Thus,

$$0.2197 \text{ s} < t_0 < 2.197 \text{ s}$$

➤ **EVALUATE the solution and check for accuracy.**

For $R = 100 \text{ k}\Omega$,

$$\tau = RC = (100 \times 10^3)(2 \times 10^{-6}) = 0.2 \text{ s}$$

$$v(0.2197) = (12)(1 - e^{-t/\tau}) = (12)(1 - e^{-0.2197/0.2}) = 8 \text{ V}$$

For $R = 1 \text{ M}\Omega$,

$$\tau = RC = (1 \times 10^6)(2 \times 10^{-6}) = 2 \text{ s}$$

$$v(2.197) = (12)(1 - e^{-t/\tau}) = (12)(1 - e^{-2.197/2}) = 8 \text{ V}$$

Our check for accuracy was successful.

➤ **Has the problem been solved SATISFACTORILY? If so, present the solution; if not, then return to “ALTERNATIVE solutions” and continue through the process again.**
This problem has been solved satisfactorily.

$$\underline{0.2197 \text{ s} < t_0 < 2.197 \text{ s}}$$