CHAPTER 8 - SECOND-ORDER CIRCUITS

List of topics for this chapter :

FINDING INITIAL VALUES

Problem 8.1 Given the circuit shown in Figure 8.1, which has existed for a long time, find $v_{C1}(0)$, $v_{C2}(0)$, $i_{L1}(0)$, and $i_{L2}(0)$.

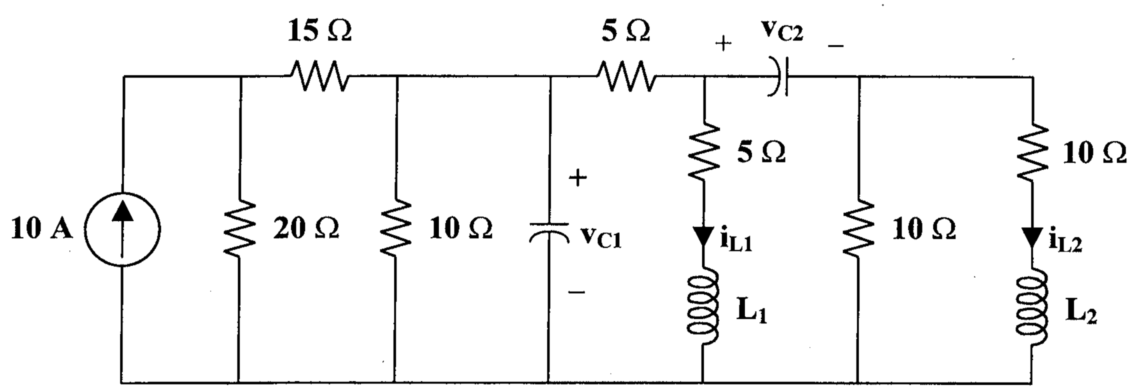
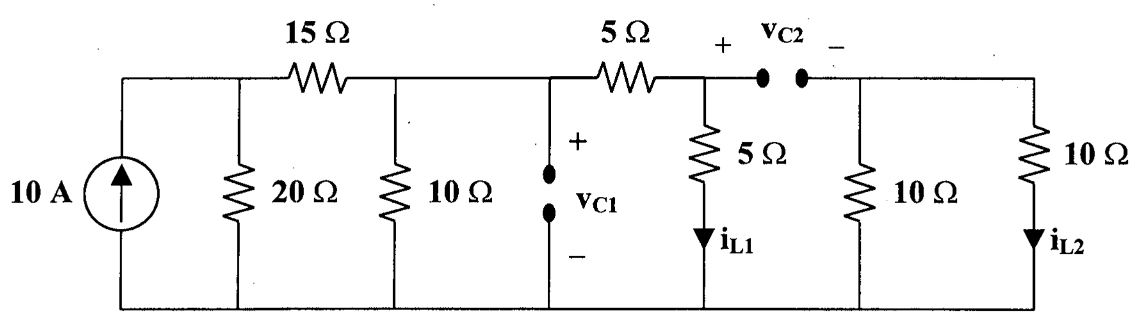
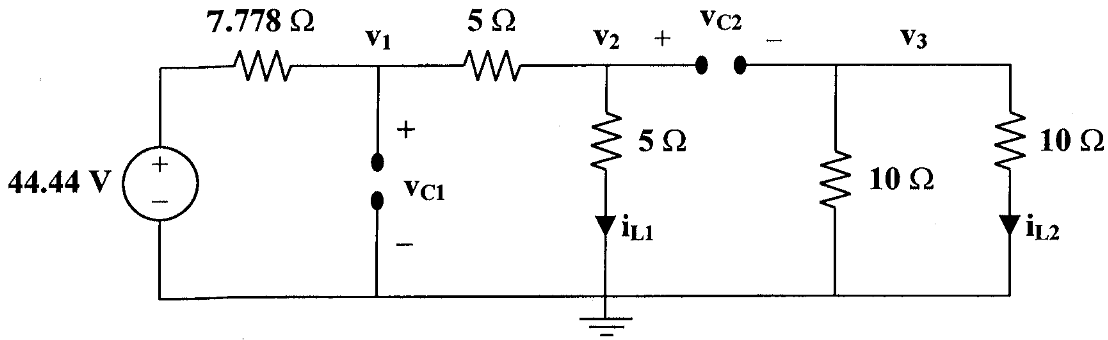


Figure 8.1

When a circuit reaches steady state, an inductor looks like a short circuit and a capacitor looks like an open circuit. So, use the following circuit to find the initial values.



Use source transformations to simplify the circuit.



Now, it is evident that

$$v_{C1}(0) = v_1 \qquad i_{L1}(0) = \frac{v_2}{5}$$

$$v_{C2}(0) = v_2 - v_3 \qquad i_{L2}(0) = \frac{v_3}{10}$$

Use nodal analysis to find v_1, v_2, and v_3.

At node 1 :
$$\frac{v_1 - 44.44}{7.778} + \frac{v_1 - v_2}{5} = 0$$
$$5(v_1 - 44.44) + 7.778(v_1 - v_2) = 0$$
$$12.778 v_1 - 7.778 v_2 = 222.2$$

At node 2 :
$$\frac{v_2 - v_1}{5} + \frac{v_2}{5} = 0$$
$$-v_1 + 2v_2 = 0$$
$$v_2 = \frac{v_1}{2}$$

At node 3 :
$$\frac{v_3}{10} + \frac{v_3}{10} = 0$$
$$2v_3 = 0$$
$$v_3 = 0 \text{ volts}$$

Substitute the equation from node 2 into the equation for node 1.

$$12.778 v_1 - 7.778 \frac{v_1}{2} = 222.2$$
$$8.889 v_1 = 222.2$$
$$v_1 = 25 \text{ volts}$$

Then, $$v_2 = \frac{v_1}{2} = \frac{25}{2} = 12.5 \text{ volts}$$

Therefore,

$v_{C1}(0) = \mathbf{25\ volts}$ $\qquad$ $i_{L1}(0) = \mathbf{2.5\ amps}$

$v_{C2}(0) = \mathbf{12.5\ volts}$ $\qquad$ $i_{L2}(0) = \mathbf{0\ amps}$

Problem 8.2 Given the circuit shown in Figure 8.2, which has existed for a long time, find $i_L(0)$ and $v_C(0)$.

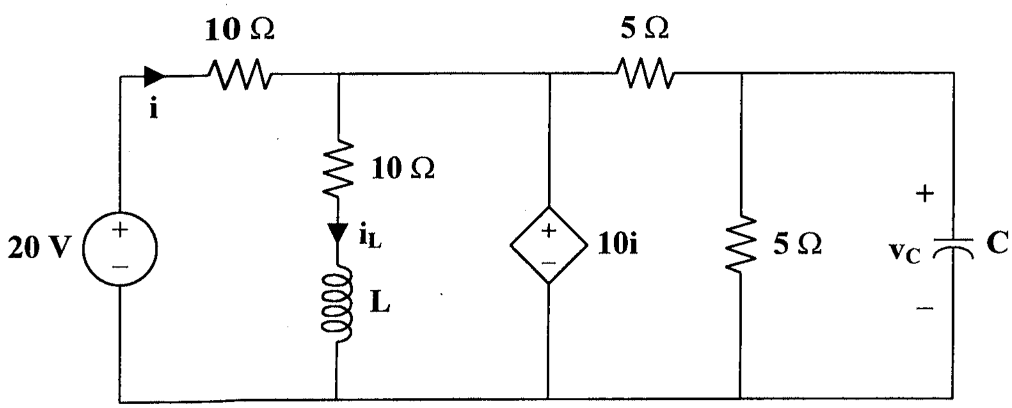


Figure 8.2

$i_L(0) = \mathbf{1\ amp}$ $\qquad$ $v_C(0) = \mathbf{5\ volts}$

THE SOURCE-FREE SERIES RLC CIRCUIT

Problem 8.3 Given the circuit in Figure 8.3, which has reached steady state before the switch closes, find $i(t)$ for all $t > 0$.

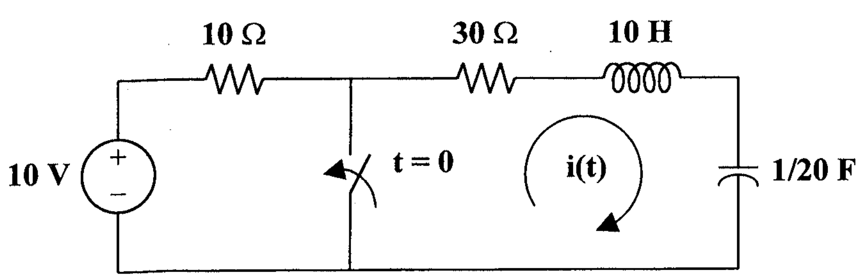


Figure 8.3

Use KVL to write a loop equation for $t > 0$.

$$Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)\,dt = 0$$

Multiply by $1/L$ and differentiate with respect to time.

$$\frac{R}{L}\frac{di(t)}{dt} + \frac{d^2i(t)}{dt^2} + \frac{1}{LC}i(t) = 0$$

Rearranging the terms and inserting the values for R, L, and C,

$$\frac{d^2i(t)}{dt^2} + 3\frac{di(t)}{dt} + 2i(t) = 0$$

Assume a solution of Ae^{st}.

$$s^2Ae^{st} + 3sAe^{st} + 2Ae^{st} = 0$$

$$(s^2 + 3s + 2)Ae^{st} = 0$$

Thus,

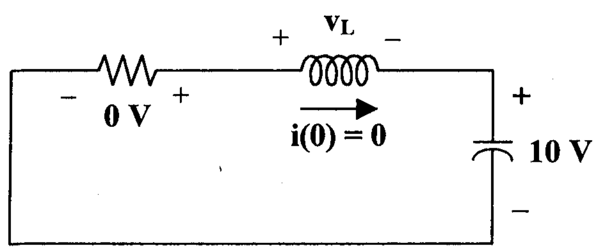
$$(s+1)(s+2) = 0$$

which gives real and unequal roots at $s_1 = -1$ and $s_2 = -2$.

Hence,

$$i(t) = A_1e^{-t} + A_2e^{-2t}$$

At $t = 0^+$, the circuit is



So,

$$i(0) = 0 = A_1 + A_2 \qquad \text{or} \qquad A_2 = -A_1$$

Also,

$$v_L(0^+) = 10\frac{di(0)}{dt} = -10 \text{ volts} \qquad \text{or} \qquad \frac{di(0)}{dt} = -1$$

and
$$\frac{di(0)}{dt} = -A_1e^0 - 2A_2e^0 = -A_1 - 2A_2$$

So,

$$-1 = -A_1 - 2A_2 = -A_1 + 2A_1 = A_1$$

Hence,

$$A_1 = -1 \qquad \text{and} \qquad A_2 = 1$$

Therefore,

$$\underline{i(t) = \left(-e^{-t} + e^{-2t}\right) \textbf{ amps } \forall\ t > 0}$$

Problem 8.4 Given the circuit in Figure 8.4, which has reached steady state before the switch closes, find $i(t)$ for all $t > 0$.

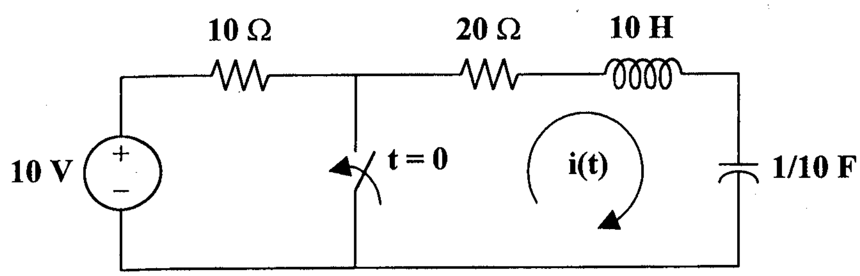


Figure 8.4

At $t = 0^-$,

$$i(0^-) = i(0^+) = 0 \text{ amps} \qquad \text{and} \qquad v_C(0^-) = v_C(0^+) = 10 \text{ volts}$$

For $t > 0$,

$$20\,i(t) + 10\frac{di(t)}{dt} + \frac{1}{1/10}\int i(t)\,dt = 0$$

Multiply by $1/10$, differentiate with respect to time, and rearrange the terms.

$$\frac{d^2i(t)}{dt^2} + 2\frac{di(t)}{dt} + i(t) = 0$$

Again, using a solution of Ae^{st},

$$s^2Ae^{st} + 2sAe^{st} + Ae^{st} = 0$$
$$(s^2 + 2s + 1)\,Ae^{st} = 0$$

Thus,

$$(s+1)^2 = 0$$

which gives a real and repeated root at $s_{1,2} = -1$.

A repeated root gives the following solution,

$$i(t) = A_1e^{-t} + A_2te^{-t}$$

At $t = 0$,

$$i(0) = 0 = A_1e^0 + A_2(0)e^0 = A_1 \qquad \text{or} \qquad A_1 = 0$$

Also,

$$v_L(0) = 10\frac{di(0)}{dt} = -10 \qquad \text{or} \qquad \frac{di(0)}{dt} = -1$$

and

$$\frac{di(0)}{dt} = 0 + A_2 e^0 - A_2(0)e^0 = A_2$$

Hence,

$$A_2 = -1$$

Therefore,

$$i(t) = \underline{(-te^{-t})\ \textbf{amps}\ \forall\ t > 0}$$

Problem 8.5 Given the circuit in Figure 8.5, which has reached steady state before the switch closes, find $i(t)$ for all $t > 0$.

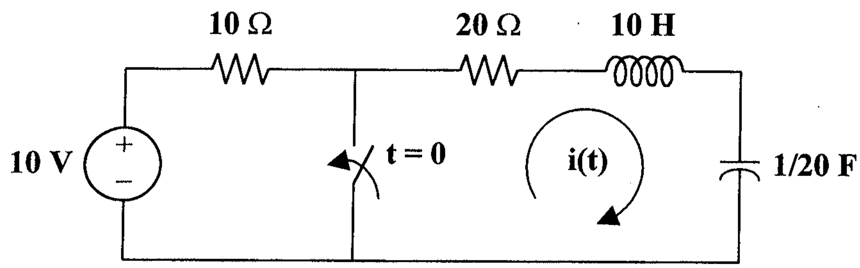


Figure 8.5

Writing a loop equation for $t > 0$ gives,

$$20i(t) + 10\frac{di(t)}{dt} + \frac{1}{1/20}\int i(t)\, dt = 0$$

Multiply by $1/10$, differentiate with respect to time, and rearrange the terms.

$$\frac{d^2 i(t)}{dt^2} + 2\frac{di(t)}{dt} + 2i(t) = 0$$

Again, using a solution of Ae^{st},

$$s^2 Ae^{st} + 2sAe^{st} + 2Ae^{st} = 0$$
$$(s^2 + 2s + 2)Ae^{st} = 0$$

Thus,

$$(s + 1 + j)(s + 1 - j) = 0$$

which gives complex roots at $s_{1,2} = -1 \mp j$.

Hence, we have a solution

$$i(t) = A_1 e^{(-1-j)t} + A_2 e^{(-1+j)t}$$

At $t = 0$,

$$i(0) = 0 = A_1 e^0 + A_2 e^0 = A_1 + A_2 \quad \text{or} \quad A_2 = -A_1$$

Also,

$$\frac{di(0)}{dt} = -1 = (-1 - j)A_1 e^0 + (-1 + j)A_2 e^0$$

Using $A_2 = -A_1$, we get

$$-1 = (-1 - j)\, A_1 + (-1 + j)(-A_1)$$

$$-1 = (-1 - j + 1 - j)A_1 = -2jA_1$$

or $\quad A_1 = \frac{1}{2j} \quad$ and $\quad A_2 = \frac{-1}{2j}$

Therefore,

$$i(t) = \frac{1}{2j} e^{(-1-j)\,t} + \frac{-1}{2j} e^{(-1+j)\,t}$$

$$i(t) = -e^{-t} \left\{ \frac{e^{jt} - e^{-jt}}{2j} \right\}$$

$$i(t) = \underline{\left(-\mathbf{e^{-t}\,\sin(t)} \right)\ \textbf{amps}\ \forall\ \mathbf{t > 0}}$$

THE SOURCE-FREE PARALLEL RLC CIRCUIT

Problem 8.6 Given the circuit in Figure 8.6, find $v_C(t)$ for all $t > 0$.

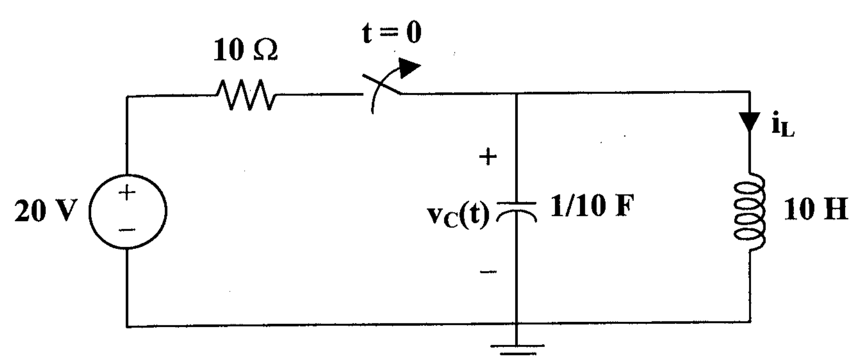


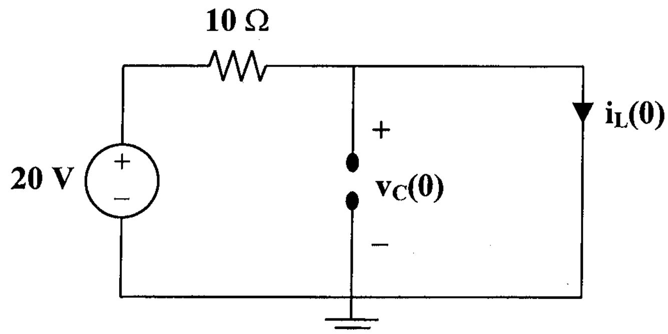
Figure 8.6

➢ **Carefully DEFINE the problem.**
Each component is labeled, indicating the value and polarity. The problem is clear.

➢ **PRESENT everything you know about the problem.**

The goal of the problem is to find $v_C(t)$ for all $t > 0$.

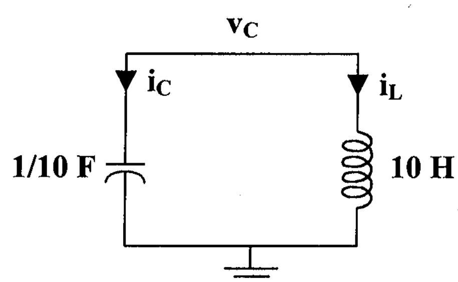
There is a switch which opens at $t = 0$. So, there are two circuits. The first circuit, when the switch is closed, is used to find the initial values of the capacitor and inductor. Note that there is a dc source. At dc, a capacitor is an open circuit and an inductor is a short circuit. Thus, we have the following circuit.



Recall that the voltage of a capacitor cannot change instantaneously and the current through an inductor cannot change instantaneously.

$$v_C(0) = v_C(0^-) = v_C(0^+) \qquad \text{and} \qquad i_L(0) = i_L(0^-) = i_L(0^+)$$

The second circuit, after the switch opens, is used to find the final solution.



➢ **Establish a set of ALTERNATIVE solutions and determine the one that promises the greatest likelihood of success.**

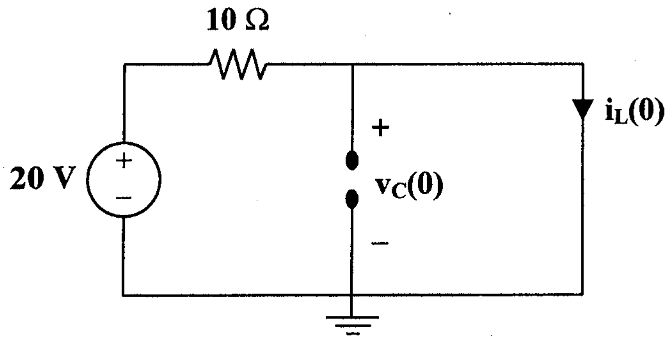
The first circuit was simplified by applying the characteristics of capacitors and inductors with a dc source. Ohm's law should provide the answer you need when the circuit consists of a dc voltage source and a resistor.

For the second circuit, there is only one node or one loop. In this case, the use of KCL or KVL should provide the desired equation to find the solution to the problem. Because the components are in parallel, the voltage across each component is the same. So, use KCL to find the currents in terms of the voltage v_C.

➢ **ATTEMPT a problem solution.**

Begin by finding the initial values of the capacitor and inductor.

At $t = 0$,



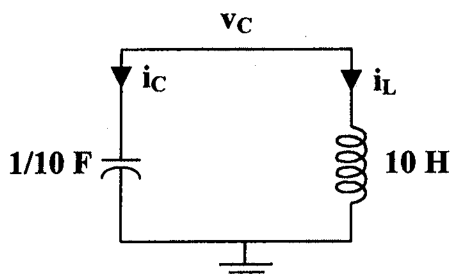
It is evident from the circuit that

$$v_C(0) = 0 \text{ volts.}$$

Using Ohm's law,

$$i_L(0) = 20/10 = 2 \text{ amps}$$

After the switch opens, the circuit becomes



Using KCL,

$$i_C + i_L = 0$$

$$\frac{1}{10}\frac{d(v_C - 0)}{dt} + \frac{1}{10}\int (v_C - 0)\, dt = 0$$

Multiply both sides of the equation by 10 and differentiate both sides with respect to time.

$$\frac{d^2 v_C}{dt^2} + v_C = 0$$

but Ae^{st} must be a solution.

Substituting,

$$\frac{d^2(Ae^{st})}{dt^2} + Ae^{st} = 0$$

$$s^2 Ae^{st} + Ae^{st} = 0$$

$$(s^2 + 1)Ae^{st} = 0$$

Thus,

$$(s^2+1) = (s-j)(s+j) = 0$$

which gives complex roots at $s_1, s_2 = \pm j$

So, we have a solution

$$v_C(t) = A_1 e^{jt} + A_2 e^{-jt}$$

Now, to solve for A_1 and A_2.

$$v_C(0) = A_1 + A_2 = 0 \qquad \text{or} \qquad A_2 = -A_1$$

Now,

$$v_C(t) = A_1 e^{jt} - A_1 e^{-jt}$$

$$i_C(0) = -i_L(0) = -2 \text{ amps}$$

$$i_C(0) = \frac{1}{10}\frac{dv_C(0)}{dt} = \frac{1}{10}(jA_1e^0 + jA_1e^0)$$

or

$$\frac{j2A_1}{10} = -2$$

Hence,

$$A_1 = \frac{-20}{j2} = \frac{-10}{j} = j10$$

Therefore,

$$v_C(t) = j10e^{jt} - j10e^{-jt}$$

$$v_C(t) = 10\left[e^{j90^\circ}e^{jt} + e^{-j90^\circ}e^{-jt}\right]$$

$$v_C(t) = 10\left[e^{j(t+90^\circ)} + e^{-j(t+90^\circ)}\right]$$

$$v_C(t) = 10\left[2\cos(t+90^\circ)\right]$$

$$v_C(t) = 20\cos(t+90^\circ) \text{ volts } \forall\ t > 0$$

- **EVALUATE the solution and check for accuracy.**
  The circuit must satisfy the conservation of energy. In this case, the energy supplied from one component is absorbed by the other component.

  Recall, from Chapter 6 – Capacitors and Inductors, that the energy stored in a capacitor is

$$w = \frac{1}{2}Cv^2$$

and the energy stored in an inductor is

$$w = \frac{1}{2}Li^2$$

Thus,

$$w = \frac{1}{2}Cv^2 = \left(\frac{1}{2}\right)\left(\frac{1}{10}\right)(20)^2 = 20 \text{ joules}$$

and

$$w = \frac{1}{2}Li^2 = \left(\frac{1}{2}\right)(10)(2)^2 = 20 \text{ joules}$$

These two answers match. This circuit satisfies conservation of energy.

- **Has the problem been solved SATISFACTORILY? If so, present the solution; if not, then return to "ALTERNATIVE solutions" and continue through the process again.**
This problem has been solved satisfactorily.

$$v_C(t) = \underline{\mathbf{20\cos(t + 90°)\ volts\ \forall\ t > 0}}$$

Problem 8.7 *[8.23]* In the circuit in Figure 8.7, calculate $i_o(t)$ and $v_o(t)$ for $t > 0$.

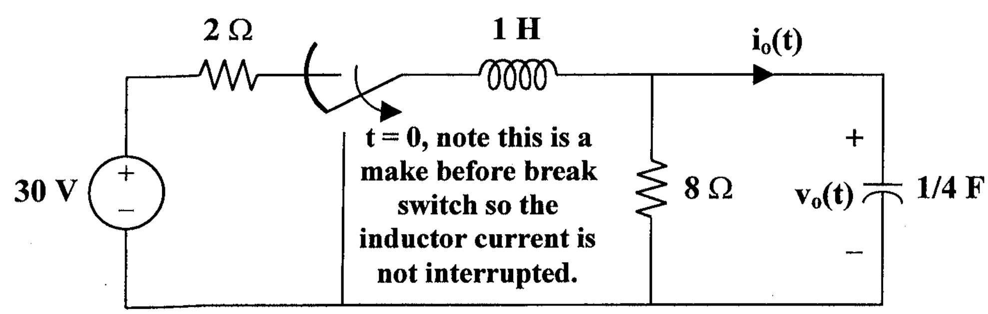
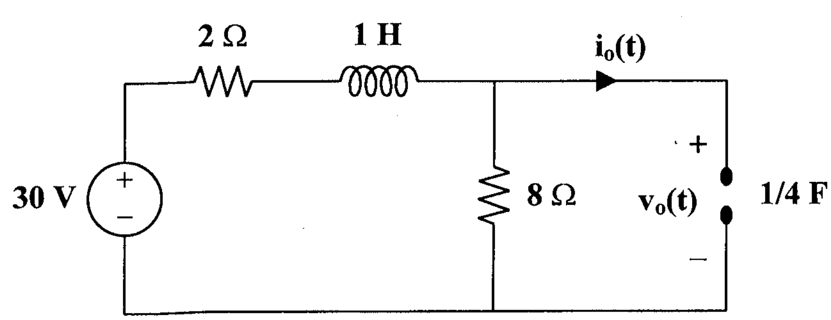


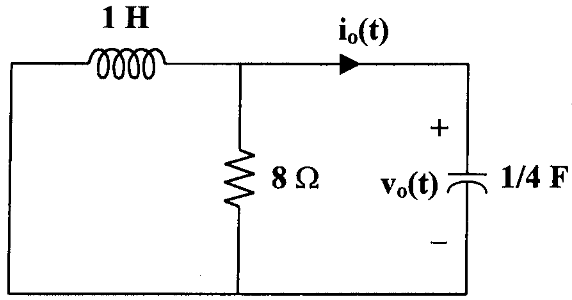
Figure 8.7

At $t = 0$,



$$v_o(0) = \frac{8}{2+8}(30) = 24 \qquad \text{and} \qquad i_o(0) = 0$$

For $t > 0$, we have a source-free parallel RLC circuit.



$$\alpha = \frac{1}{2RC} = \frac{1}{(2)(8)(1/4)} = \frac{1}{4}$$

$$\omega_o = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(1)(1/4)}} = 2$$

Since α is less than ω_o, we have an underdamped response with a damping frequency of

$$\omega_d = \sqrt{\omega_o^2 - \alpha^2} = \sqrt{4-(1/16)} = 1.9843$$

and the natural response is

$$v_o(t) = (A_1\cos(\omega_d t) + A_2\sin(\omega_d t))e^{-\alpha t}$$

Use the initial values to find the unknowns.

$$v_o(0) = 24 = A_1$$

Also,

$$i_o(0) = C\frac{dv_o(0)}{dt} = 0$$

where $\dfrac{dv_o}{dt} = -\alpha(A_1\cos(\omega_d t) + A_2\sin(\omega_d t))e^{-\alpha t} + (-\omega_d A_1\sin(\omega_d t) + \omega_d A_2\cos(\omega_d t))e^{-\alpha t}$

So,

$$\frac{dv_o(0)}{dt} = 0 = -\alpha A_1 + \omega_d A_2$$

Thus,

$$A_2 = \frac{\alpha}{\omega_d}A_1 = \frac{(1/4)(24)}{1.9843} = 3.024$$

Therefore,

$$v_o(t) = \underline{\left[(\mathbf{24\cos(\omega_d t) + 3.024\sin(\omega_d t)})\mathbf{e^{-t/4}}\right]\ \textbf{volts}\ \forall\ \mathbf{t > 0}}$$

Problem 8.8 Given the circuit in Figure 8.8, which has reached steady state before the switch closes, find $v(t)$ for all $t > 0$.

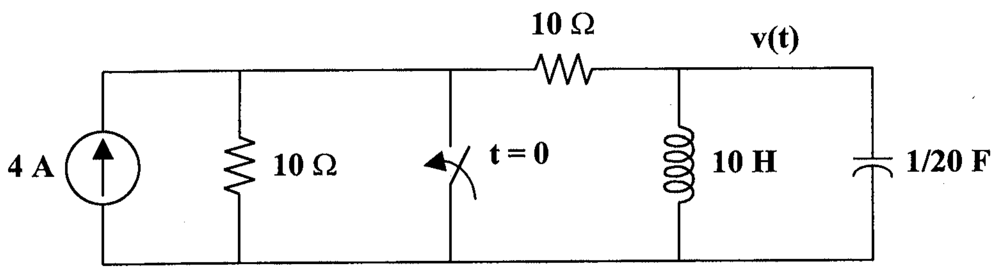


Figure 8.8

$$v(t) = \underline{\mathbf{40\,e^{-t}\cos(t + 90^\circ)\ volts\ \forall\ t > 0}}$$

STEP RESPONSE OF A SERIES RLC CIRCUIT

Problem 8.9 *[8.25]* A branch voltage in a series RLC circuit is described by

$$\frac{d^2v}{dt^2} + 4\frac{dv}{dt} + 8v = 24$$

If the initial conditions are $v(0) = 0 = \dfrac{dv(0)}{dt}$, find $v(t)$.

Recall the general loop equation for a series RLC circuit.

$$L\frac{di}{dt} + Ri + v = V_S$$

where $i = C\,dv/dt$. Then,

$$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{v}{LC} = \frac{V_S}{LC}$$

The complete response of this circuit has a forced response and a natural response. To find the forced response, realize that $8V_S = 24$ which means that $V_S = 3$. This is the forced response. Now, to find the natural response, let $V_S = 0$ and

$$\frac{d^2v}{dt^2} + 4\frac{dv}{dt} + 8v = 0$$

$$(s^2 + 4s + 8)v = 0$$

Thus,

$$s^2 + 4s + 8 = 0$$

which gives complex roots at $s_{1,2} = \dfrac{-4 \pm \sqrt{16-32}}{2} = -2 \pm j2$.

So, we have the solution

$$v(t) = V_S + (A_1 \cos(2t) + A_2 \sin(2t))e^{-2t}$$

Solve for the unknowns.

$$v(0) = 0 = V_s + A_1 = 3 + A_1 \qquad \text{or} \qquad A_1 = -3$$

$$\frac{dv}{dt} = -2(A_1 \cos(2t) + A_2 \sin(2t))e^{-2t} + (-2A_1 \sin(2t) + 2A_2 \cos(2t))e^{-2t}$$

$$\frac{dv(0)}{dt} = 0 = -2A_1 + 2A_2 \qquad \text{or} \qquad A_2 = A_1 = -3$$

Therefore,

$$v(t) = \underline{\mathbf{[3 - 3(\cos(2t) + \sin(2t))e^{-2t}]\ volts}}$$

Problem 8.10 *[8.31]* Calculate $i(t)$ for $t > 0$ using the circuit in Figure 8.9.

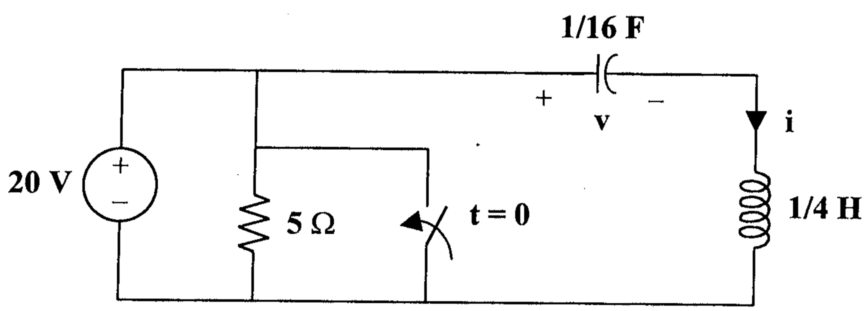
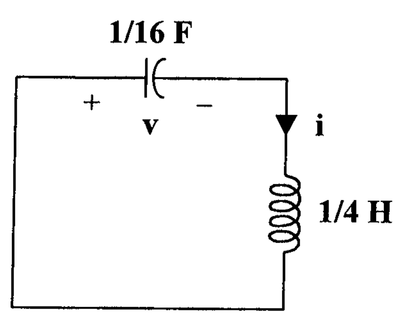


Figure 8.9

Before $t = 0$, the capacitor acts like an open circuit while the inductor acts like a short circuit.

$i(0) = 0$ amps and $v(0) = 20$ volts

For $t > 0$, the LC circuit is disconnected from the voltage source as shown below.



This is a lossless, source-free series RLC circuit.

$$\alpha = \frac{R}{2L} = 0 \qquad \omega_o = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(1/16)+(1/4)}} = 8$$

$$s = -\alpha \pm j\omega_d = -\alpha \pm j\sqrt{\omega_o^2 - \alpha^2} = \pm j\omega_o = \pm j8$$

Since α is equal to zero, we have an undamped response. Hence,

$$i(t) = A_1 \cos(8t) + A_2 \sin(8t)$$

where $i(0) = 0 = A_1$.

So,

$$i(t) = A_2 \sin(8t)$$

To solve for A_2, we know that

$$\frac{di(0)}{dt} = \frac{1}{L} v_L(0) = \frac{-1}{L} v(0) = (-4)(20) = -80$$

However,

$$\frac{di}{dt} = 8A_2 \cos(8t)$$

and $$\frac{di(0)}{dt} = -80 = 8A_2$$

which leads to $A_2 = -10$.

Therefore,

$$i(t) = \underline{\mathbf{-10\sin(8t)\ amps\ \forall\ t > 0}}$$

STEP RESPONSE OF A PARALLEL RLC CIRCUIT

Problem 8.11 Given the circuit in Figure 8.10, find $v(t)$ for all $t > 0$.

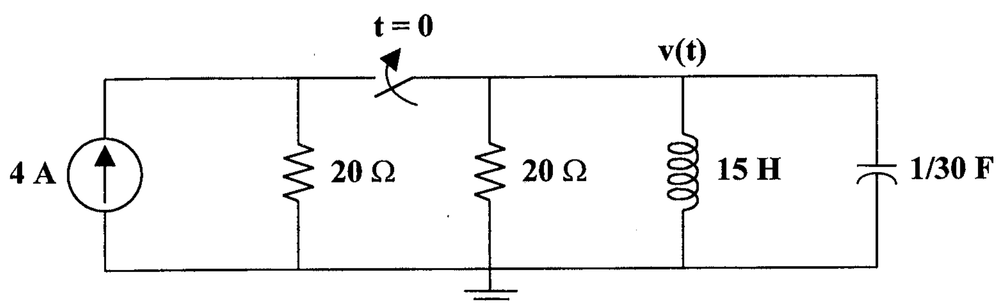


Figure 8.10

$$v(t) = \underline{\left(120\,e^{-t} - 120\,e^{-2t}\right) \text{ volts } \forall\ t > 0}$$

Problem 8.12 Given the circuit in Figure 8.11, find $v(t)$ for all $t > 0$.

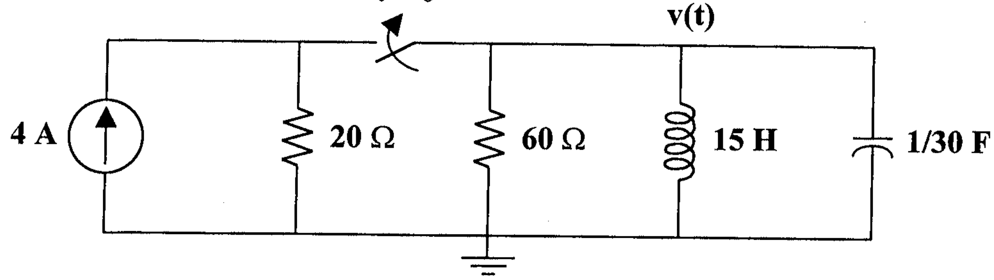


Figure 8.11

At $t = 0^-$,

$v_C(0) = 0$ volts and $i_L(0) = 0$ amps.

At $t = 0^+$,

$v_C = v_L = v_R = 0$ volts

So, the 4 amp current source must all go through the capacitor or

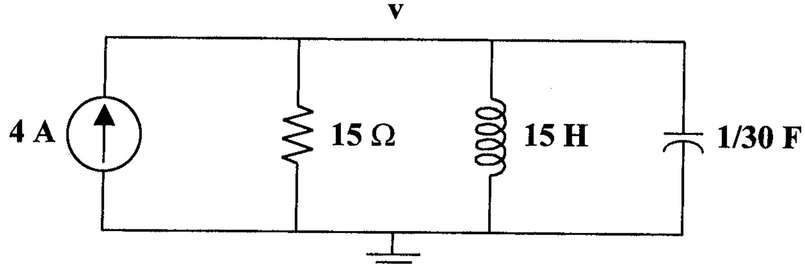
$i_C(0^+) = 4$ amps.

Hence,

$$C\frac{dv(0^+)}{dt} = i_C(0^+) = 4$$

or
$$\frac{dv(0^+)}{dt} = \frac{4}{C} = \frac{4}{1/30} = 120 \text{ volts/sec}$$

For $t > 0$, the circuit becomes



where the 20 ohm resistor in parallel with the 60 ohm resistor is equivalent to a 15 ohm resistor.

Using nodal analysis,

$$-4+\frac{v-0}{15}+\frac{1}{15}\int(v-0)\,dt+\frac{1}{30}\frac{d(v-0)}{dt}=0$$

Multiplying by 30 and differentiating with respect to time yields

$$\frac{d^2v}{dt^2}+2\frac{dv}{dt}+2v=0$$

Substituting the solution of Ae^{st} produces

$$s^2Ae^{st}+2sAe^{st}+2Ae^{st}=0$$
$$(s^2+2s+2)Ae^{st}=0$$

Hence,

$$s^2+2s+2=(s+1+j)(s+1-j)=0$$

which gives complex roots at $s_{1,2}=-1\mp j$.

So, we have the solution

$$v(t)=A_1e^{-(1+j)t}+A_2e^{-(1-j)t}$$

At $t=0$,

$$v(0)=A_1\,e^0+A_2\,e^0=A_1+A_2=0 \qquad \text{or} \qquad A_2=-A_1$$

Also,

$$\frac{dv(0)}{dt}=\left[-(1+j)\right]A_1-\left[-(1-j)\right]A_1=120$$
$$-A_1-jA_1+A_1-jA_1=-2jA_1=120$$
$$A_1=\frac{120}{-2j}=60j=60e^{j} \qquad \text{and} \qquad A_2=-A_1=60e^{-j}$$

Therefore,

$$\begin{aligned} v(t)&=60e^{j}e^{-(1+j)t}+60e^{-j}e^{-(1-j)t}\\ &=60e^{-t}\left\{e^{-jt+j}+e^{jt-j}\right.\\ &=60e^{-t}\,[2\cos(t-90°)]\end{aligned}$$

$$v(t)=\underline{\mathbf{120e^{-t}\cos(t-90°)\ volts}\ \forall\ \mathbf{t>0}}$$

The answer can also be written as

$$v(t)=\underline{\mathbf{120e^{-t}\sin(t)\ volts}\ \forall\ \mathbf{t>0}}$$

Problem 8.13 Given the circuit in Figure 8.12, find $v(t)$ for all $t > 0$.

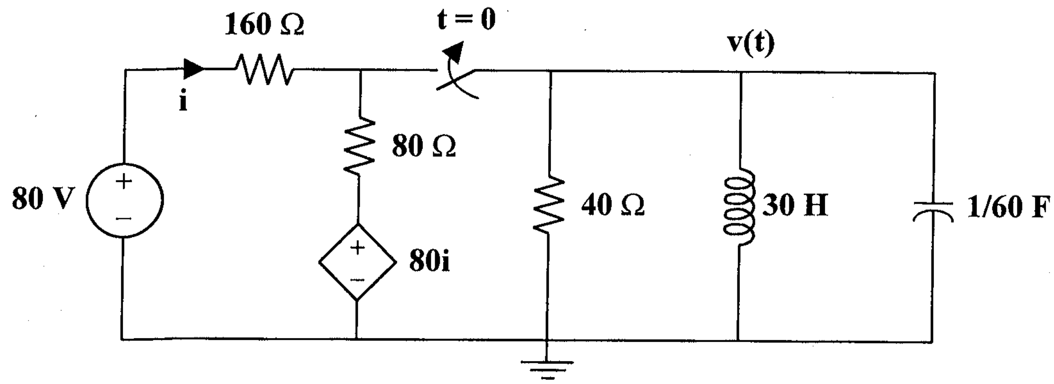


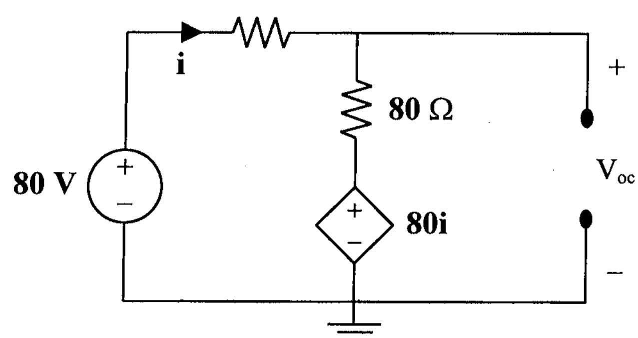
Figure 8.12

At $t = 0$,

$$i_L(0) = 0 \text{ amps} \qquad \text{and} \qquad v_C(0) = 0 \text{ volts}$$

For complicated circuits that can be simplified using either a Thevenin or Norton equivalent, do so immediately!

Solving for V_{oc},



Using nodal analysis,

$$\frac{V_{oc} - 80}{160} + \frac{V_{oc} - 80i}{80} = 0$$

$$(V_{oc} - 80) + 2(V_{oc} - 80i) = 0$$

$$3V_{oc} - 80 - 160i = 0$$

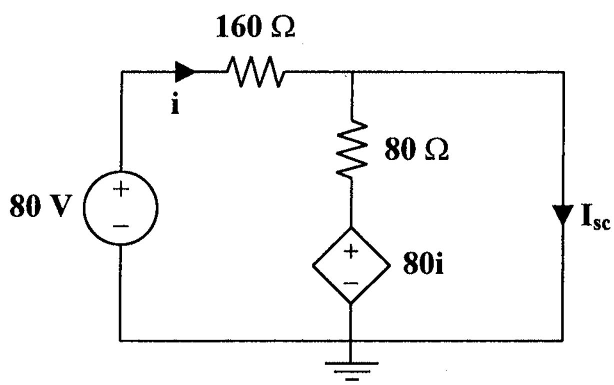
where $i = \dfrac{80 - V_{oc}}{160}$.

Hence,

$$3V_{oc} = 80 + 160\left(\frac{80 - V_{oc}}{160}\right)$$

$$3V_{oc} = 80 + (80 - V_{oc})$$
$$4V_{oc} = 160$$
$$V_{oc} = 40 \text{ volts}$$

Solving for I_{sc},



Using KCL,

$$I_{sc} = \frac{80-0}{160} + \frac{80i-0}{80}$$

where $i = \frac{80-0}{160} = \frac{1}{2}$ amp

Hence,

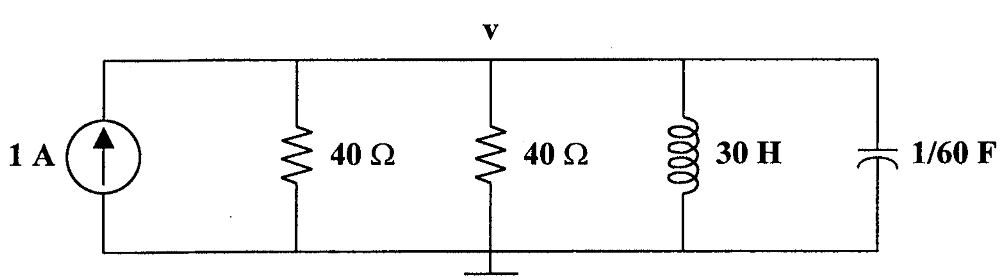
$$I_{sc} = \frac{80}{160} + \frac{80i}{80} = \frac{1}{2} + i = \frac{1}{2} + \frac{1}{2} = 1 \text{ amp}$$

Finding a Norton equivalent circuit,

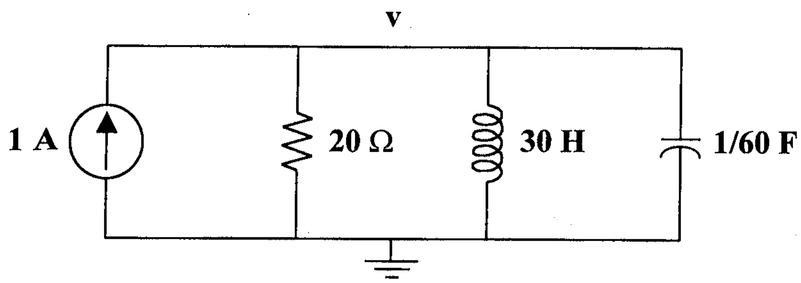
$$I_N = I_{sc} = 1\,\text{amp}$$

and $$R_{eq} = R_N = \frac{V_{oc}}{I_{sc}} = \frac{40}{1} = 40 \text{ ohms}$$

Now, we can work with the following simplified circuit,



The two 40 ohm resistors in parallel convert to a 20 ohm resistor, so we can simplify the circuit further to become



Writing a single node equation results in

$$-1 + \frac{v-0}{20} + \frac{1}{30}\int (v-0)\,dt + \frac{1}{60}\frac{d(v-0)}{dt} = 0$$

Multiplying by 60 and differentiating with respect to time leads to

$$\frac{d^2v}{dt^2} + 3\frac{dv}{dt} + 2v = 0$$

Substituting the solution Ae^{st}, we get

$$s^2Ae^{st} + 3sAe^{st} + 2Ae^{st} = 0$$
$$(s^2 + 3s + 2)Ae^{st} = 0$$

Hence,

$$s^2 + 3s + 2 = (s+1)(s+2) = 0$$

which gives real and unequal roots at $s_1 = -1$ and $s_2 = -2$.

So, we have the solution

$$v(t) = A_1e^{-t} + A_2e^{-2t}$$

At $t = 0$,

$v(0) = 0$ volts and $i_L(0) = 0$ amps.

Since $v(0) = 0$ volts,

$v_R = 0$ volts and $i_R = 0$ amps.

Hence, the entire 1 amp of current flows through the capacitor. So, we have

$i_C(0) = 1$ amp and $i_C(0) = C\frac{dv(0)}{dt}$

Thus,

$$\frac{dv(0)}{dt} = \frac{1}{C}i_C(0) = \frac{1}{1/60}(1) = 60 \text{ volts/sec}$$

but $\frac{dv(0)}{dt} = -A_1e^0 - 2A_2e^0 = -A_1 - 2A_2 = 60$

Also,

$$v(0) = A_1 e^0 + A_2 e^0 = A_1 + A_2 = 0$$

implies that $A_2 = -A_1$.

So,

$$-A_1 - 2A_2 = 60$$

or

$$-A_1 + 2A_1 = A_1 = 60$$

Hence,

$$A_1 = 60 \quad \text{and} \quad A_2 = -60.$$

Therefore,

$$\underline{v(t) = \left(60e^{-t} - 60e^{-2t}\right) \text{volts } \forall\ t > 0}$$

The power of using a Norton equivalent circuit is clearly demonstrated by this problem.

GENERAL SECOND-ORDER CIRCUITS

Problem 8.14 Given the circuit in Figure 8.13, find $v_C(t)$ for all $t > 0$.

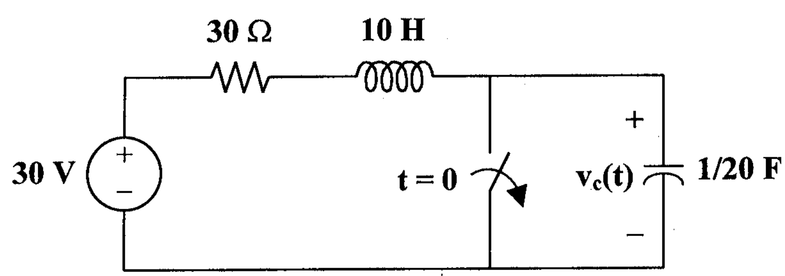


Figure 8.13

This is a second-order circuit with a forced response and a natural response.

Solve for initial conditions.
At $t = 0$,

$$v_C(0^-) = v_C(0^+) = 0 \quad \text{and} \quad i_L(0^-) = i_L(0^+) = \frac{30}{30} = 1 \text{ amp.}$$

Hence,

$$i_C(0^+) = C\frac{dv_C(0^+)}{dt} = 1 \text{ amp}$$

or $$\frac{dv_C(0^+)}{dt} = \frac{1}{C} i_C(0^+) = \frac{1}{1/20}(1) = 20 \text{ volts/sec}$$

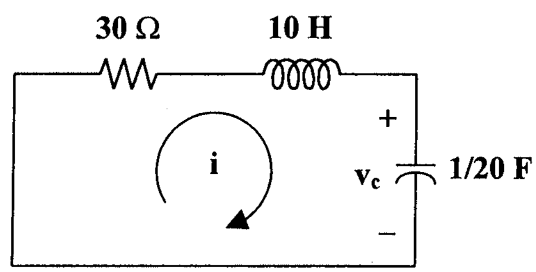
Solving for final values,

$$i_C(\infty) = 0 \text{ amps} \quad \text{and} \quad v_C(\infty) = 30 \text{ volts}$$

which is also the forced response,

$$v_{C_f} = 30 \text{ volts.}$$

Solving for the natural response,



The loop equation is

$$Ri + L\frac{di}{dt} + v_C = 0$$

where $i = C\frac{dv_C}{dt}$.

So, we now have,

$$RC\frac{dv_C}{dt} + LC\frac{d^2v_C}{dt^2} + v_C = 0$$

Substituting for the values of R, L, and C

$$\frac{30}{20}\frac{dv_C}{dt} + \frac{10}{20}\frac{d^2v_C}{dt^2} + v_C = 0$$

Simplifying and rearranging the terms,

$$\frac{d^2v_C}{dt^2} + 3\frac{dv_C}{dt} + 2v_C = 0$$

Substituting a solution $v_C = Ae^{st}$ yields

$$s^2Ae^{st} + 3sAe^{st} + 2Ae^{st} = 0$$
$$(s^2 + 3s + 2)Ae^{st} = 0$$

Thus,

$$s^2 + 3s + 2 = (s+1)(s+2) = 0$$

which gives real and unequal roots at $s_1 = -1$ and $s_2 = -2$.

Hence, the natural response is

$$v_{C_n} = Ae^{-t} + Be^{-2t}$$

and the complete response is

$$v_C = v_{C_f} + v_{C_n} = 30 + Ae^{-t} + Be^{-2t}$$

Now, we need to solve for the unknowns, A and B.

$$v_C(0) = 30 + Ae^0 + Be^0 = 0$$

$$A + B = -30 \qquad \text{or} \qquad B = -A - 30$$

Also,

$$\frac{dv_C(0)}{dt} = 0 + (-A)e^0 + (-2B)e^0 = -A - 2B = 20$$

Now,

$$A + 2B = -20$$

which leads to

$$A + (2)(-A - 30) = -20$$

Hence,

$$-A - 60 = -20 \qquad \text{or} \qquad A = -40$$

Then,

$$B = -(-40) - 30 = 10$$

Therefore,

$$v_C(t) = \underline{\left(\mathbf{30 - 40\,e^{-t} + 10\,e^{-2t}}\right)\ \mathbf{volts}\ \forall\ \mathbf{t > 0}}$$

Problem 8.15 Given the circuit shown in Figure 8.14, find $i_L(t)$ for all $t > 0$.

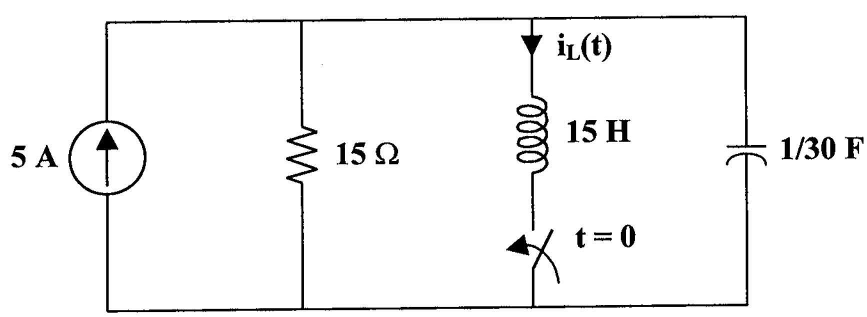


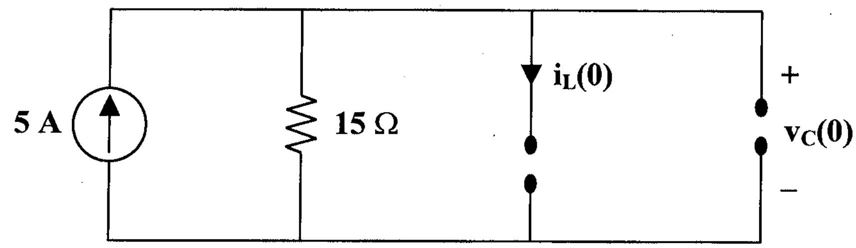
Figure 8.14

- **Carefully DEFINE the problem.**
  Each component is labeled, indicating the value and polarity. The problem is clear.

➢ **PRESENT everything you know about the problem.**

The goal of the problem is to find $i_L(t)$ for all $t > 0$.

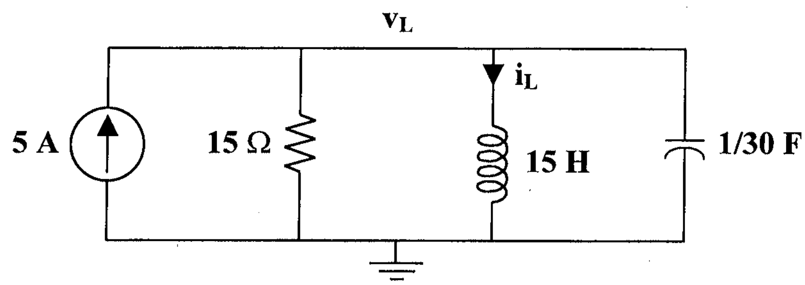
There is a switch in the circuit. So, we will need to look at two different circuits. The first circuit, when the switch is open, is used to find the initial values of the capacitor and inductor. Note that there is a dc source. At dc, a capacitor is an open circuit and an inductor is a short circuit. Thus, we have the following circuit.



Recall that the voltage of a capacitor cannot change instantaneously and the current through an inductor cannot change instantaneously.

$$v_C(0) = v_C(0^-) = v_C(0^+) \qquad \text{and} \qquad i_L(0) = i_L(0^-) = i_L(0^+)$$

The second circuit, after the switch opens is used to find the final solution.



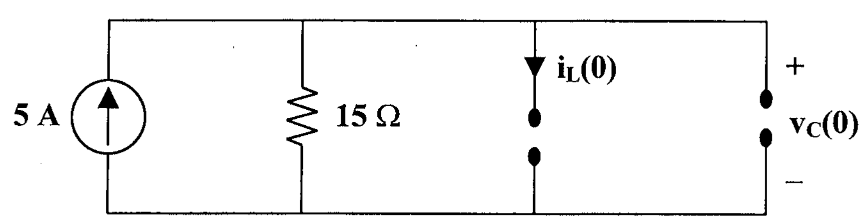
➢ **Establish a set of ALTERNATIVE solutions and determine the one that promises the greatest likelihood of success.**

This is a second-order circuit with a forced response and a natural response. The forced response is the response due to the current source at steady state. The natural response is the response of the parallel RLC circuit without the current source.

For simple resistive circuits such as the one used to find the initial values and final values, we will use observation and Ohm's law. For the parallel RLC circuit, use nodal analysis. There will be only one equation with this technique versus three equations with mesh analysis.

➢ **ATTEMPT a problem solution.**

Solve for initial conditions using the circuit below.



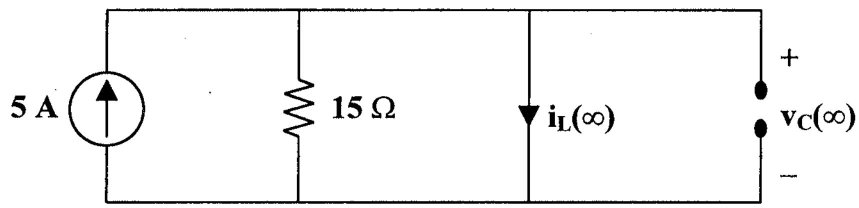
At $t = 0$,

$$v_C(0^-) = v_C(0^+) = (5)(15) = 75 \text{ volts}$$
$$i_L(0^-) = i_L(0^+) = 0$$
$$v_L(0^+) = v_C(0^+) = 75 \text{ volts}$$

Thus,

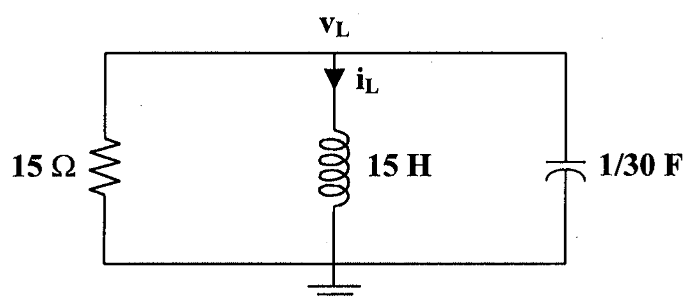
$$\frac{di_L(0^+)}{dt} = \frac{v_L(0^+)}{L} = \frac{75}{15} = 5 \text{ amp/sec}$$

Solving for final values, the forced response,



$$i_L(\infty) = i_{L_f} = 5 \text{ amps}$$

Now, solve for the natural response.



Writing the node equation,

$$\frac{v_L - 0}{15} + i_L + \frac{1}{30}\frac{d(v_L - 0)}{dt} = 0$$

where $v_L = 15\frac{di_L}{dt}$

So,

$$\frac{15}{15}\frac{di_L}{dt} + i_L + \frac{15}{30}\frac{d^2 i_L}{dt^2} = 0$$

Simplifying and rearranging the terms,

$$\frac{d^2 i_L}{dt^2} + 2\frac{di_L}{dt} + 2i_L = 0$$

Substituting the solution $i_L = Ae^{st}$ yields

$$s^2Ae^{st} + 2sAe^{st} + 2Ae^{st} = 0$$
$$(s^2 + 2s + 2)Ae^{st} = 0$$

Thus,

$$(s^2 + 2s + 2) = (s + 1 + j)(s + 1 - j) = 0$$

which gives complex roots at $s_{1,2} = -1 \mp j$.

Hence,

$$i_{L_n} = Ae^{-(1+j)t} + Be^{-(1-j)t}$$

This means that the complete response is

$$i_L = i_{L_f} + i_{L_n} = 5 + Ae^{-(1+j)t} + Be^{-(1-j)t}$$

At $t = 0$,

$$i_L(0) = 5 + Ae^0 + Be^0 = 5 + A + B = 0$$
$$A + B = -5 \qquad \text{or} \qquad B = -A - 5$$

Also,

$$\frac{di_L(0)}{dt} = 0 - (1 + j)\,A - (1 - j)\,B = 5$$
$$-A - jA - B + jB = 5$$
$$-A - jA - (-A - 5) + j(-A - 5) = 5$$
$$-2jA + 5 - j5 = 5$$
$$A = \frac{5 - 5 + j5}{-2j} = \frac{-5}{2}$$

Then,

$$B = -A - 5 = \frac{5}{2} - 5 = \frac{-5}{2}$$

Therefore,

$$i_L = 5 + (-5/2)e^{-(1+j)t} + (-5/2)e^{-(1-j)t}$$
$$i_L(t) = 5 - 5e^{-t}\left[\frac{e^{-jt} + e^{jt}}{2}\right]$$
$$i_L(t) = \left[5 - 5e^{-t}\cos(t)\right] \text{amps} \ \forall\ t > 0$$

- **EVALUATE the solution and check for accuracy.**
  Check to see if the answer satisfies the initial conditions.

$$i_L(0) = 5 - 5e^{-0}\cos(0) = 5 - 5 = 0 \text{ amps}$$

This matches the initial condition that

$$i_L(0^-) = i_L(0^+) = 0 \text{ amps}$$

$$\frac{di_L(t)}{dt} = 0 + 5e^{-t}\sin(t) + 5e^{-t}\cos(t)$$

$$\frac{di_L(0)}{dt} = 5e^{-0}\sin(0) + 5e^{-0}\cos(0) = 0 + 5 = 5 \text{ amps/sec}$$

This matches the initial condition that

$$\frac{di_L(0^+)}{dt} = \frac{v_L(0^+)}{L} = \frac{75}{15} = 5 \text{ amp/sec}$$

- **Has the problem been solved SATISFACTORILY? If so, present the solution; if not, then return to "ALTERNATIVE solutions" and continue through the process again.**
  This problem has been solved satisfactorily.

$$i_L(t) = \underline{[5 - 5e^{-t}\cos(t)]\,\text{amps} \ \forall\ t > 0}$$

SECOND-ORDER OP AMP CIRCUITS

Problem 8.16 Given the circuit shown in Figure 8.15, find v_o in terms of v_i.

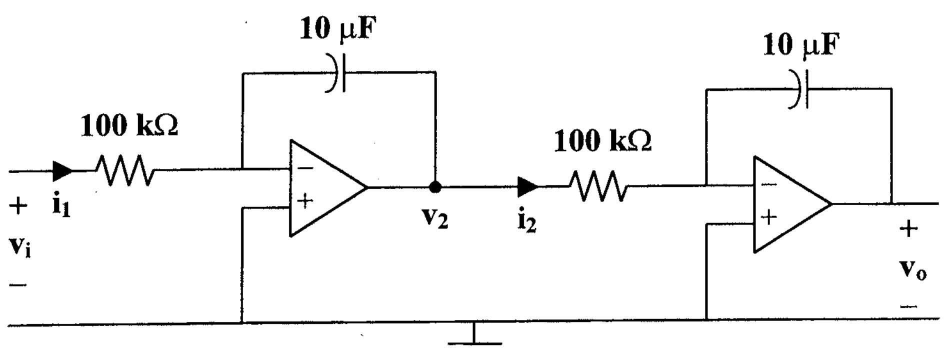


Figure 8.15

Clearly,

$$i_1 = \frac{v_i}{R_1}$$

and

$$v_2 = \frac{-1}{C_1}\int\frac{v_i}{R_1}\,dt = \frac{-1}{R_1C_1}\int v_i\,dt$$

where $R_1C_1 = (10^5)(10^{-5}) = 1$.
Hence,

$$v_2 = -\int v_i\,dt$$

Also,

$$i_2 = \frac{v_2}{R_2}$$

and

$$v_o = \frac{-1}{C_2}\int\frac{v_2}{R_2}\,dt = \frac{-1}{R_2C_2}\int v_2\,dt$$

where $R_2C_2 = (10^5)(10^{-5}) = 1$.

So,

$$v_o = -\int v_2\,dt$$

Therefore,

$$v_o = \underline{\int\int \mathbf{v_i\,dt^2}}$$

Circuits like these have many applications, especially in analog computers.