CHAPTER 15

P.P.15.1 $\mathcal{L}\left[\, t\, u(t)\right] = \int_0^\infty t\, e^{-st}\, dt$

Using integration by parts,

$$\int u\, dv = uv - \int v\, du$$

Let $u = t \longrightarrow du = dt$.

$$e^{-st}\, dt = dv \longrightarrow v = \frac{-1}{s} e^{-st}$$

$$\mathcal{L}\left[\, t\, u(t)\right] = \frac{-t}{s} e^{-st}\Big|_0^\infty + \int_0^\infty \frac{1}{s} e^{-st}\, dt = 0 + \frac{e^{-st}}{s^2}\Big|_0^\infty = \mathbf{\frac{1}{s^2}}$$

Next,

$$\mathcal{L}\left[\, Ae^{at}\, u(t)\right] = A\int_0^\infty e^{-at} e^{-st}\, dt = \frac{-A}{s+a} e^{-(s+a)t}\Big|_0^\infty$$

$$= \mathbf{A/(s+a)}$$

From this, all we need to do is to solve for the Laplace transform of $e^{-j\omega t}$ is to let a = jω in the above and B = A, we get,

$$\mathcal{L}[Be^{-j\omega t}u(t)] = \mathbf{B/(s+j\omega)}$$

P.P.15.2

$$\mathcal{L}\left[50\cos(\omega t)\right] = \int_0^\infty \frac{50}{2}\left(e^{j\omega t} + e^{-j\omega t}\right)e^{-st}\, dt$$

$$\mathcal{L}\left[50\cos(\omega t)\right] = 25\int_0^\infty e^{-(s-j\omega)t}\, dt + 25\int_0^\infty e^{-(s+j\omega)t}\, dt$$

$$\mathcal{L}\left[10\cos(\omega t)\right] = 25\left(\frac{1}{s-j\omega} + \frac{1}{s+j\omega}\right) = \mathbf{\frac{50s}{s^2+\omega^2}}$$

P.P.15.3 If $f(t) = \cos(2t) + e^{-4t}$,

$$F(s) = \frac{s}{s^2+4} + \frac{1}{s+4} = \frac{s^2+4s+s^2+4}{(s^2+4)(s+4)}$$

$$F(s) = \mathbf{\frac{2s^2+4s+4}{(s+4)(s^2+4)}}$$

P.P.15.4 Given $f(t) = t^2 \cos(3t)$

From P.P.15.2, $\mathcal{L}\left[\cos(3t)\right] = \frac{s}{s^2 + 9}$

Using Eq. 15.34, $F(s) = \mathcal{L}\left[t^2 \cos(3t)\right] = (-1)^2 \frac{d^2}{ds^2}\left(\frac{s}{s^2 + 9}\right)$

$$F(s) = \frac{d^2}{ds^2}\left[s\left(s^2 + 9\right)^{-1}\right] = \frac{d^2}{ds^2}\left[(1)\left(s^2 + 9\right)^{-1} - (s)(2s)\left(s^2 + 9\right)^{-2}\right]$$

$$F(s) = (-2s)\left(s^2 + 9\right)^{-2} - (4s)\left(s^2 + 9\right)^{-2} + \left(4s^2\right)(2s)\left(s^2 + 9\right)^{-3}$$

$$F(s) = (-6s)\left(s^2 + 9\right)^{-2} + \left(8s^3\right)\left(s^2 + 9\right)^{-3} = \frac{2s^3 - 54s}{\left(s^2 + 9\right)^3}$$

$$F(s) = \boldsymbol{\frac{2s\left(s^2 - 27\right)}{\left(s^2 + 9\right)^3}}$$

P.P.15.5 $h(t) = 20\left[u(t) - u(t-4)\right] + 10\left[u(t-4) - u(t-8)\right]$

$$H(s) = 20\left(\frac{1}{s} - \frac{e^{-4s}}{s}\right) + 10\left(\frac{e^{-4s}}{s} - \frac{e^{-8s}}{s}\right)$$

$$H(s) = \boldsymbol{\frac{10}{s}\left(2 - e^{-4s} - e^{-8s}\right)}$$

P.P.15.6 $T = 5$

$$f_1(t) = u(t) - u(t-2)$$

$$F_1(s) = \frac{1}{s}\left(1 - e^{-2s}\right)$$

$$F(s) = \frac{F_1(s)}{1 - e^{-Ts}} = \boldsymbol{\frac{1 - e^{-2s}}{s\,(1 - e^{-5s})}}$$

P.P.15.7 $g(0) = \lim_{s \to \infty} sF(s) = \lim_{s \to \infty} \frac{6s^3 + 2s + 5}{(s^2 + 4s + 4)(s + 3)}$

$$g(0) = \lim_{s \to \infty} \frac{6 + \frac{2}{s^2} + \frac{5}{s^3}}{\left(1 + \frac{4}{s} + \frac{4}{s^2}\right)\left(1 + \frac{3}{s}\right)} = \mathbf{6}$$

Since all poles s = 0, –2, –2, –3 lie in the left-hand s-plane, we can apply the final-value theorem.

$$g(\infty) = \lim_{s \to 0} sF(s) = \lim_{s \to 0} \frac{6s^3 + 2s + 5}{(s + 2)^2 (s + 3)}$$

$$g(\infty) = \lim_{s\to 0} \frac{5}{(2)^2(3)} = \mathbf{0.4167}$$

P.P.15.8 $$F(s) = 5 + \frac{6}{s+4} - \frac{7s}{s^2+25}$$

$$f(t) = L^{-1}[5] + L^{-1}\left[\frac{6}{s+4}\right] - L^{-1}\left[\frac{7s}{s^2+25}\right]$$

$$f(t) = \mathbf{5\delta(t) + (6e^{-4t} - 7\cos(5t))u(t)}$$

P.P.15.9 $$F(s) = \frac{A}{s+1} + \frac{B}{s+3} + \frac{C}{s+4}$$

$$A = F(s)(s+1)\big|_{s=-1} = \frac{6(s+2)}{(s+3)(s+4)}\Big|_{s=-1} = \frac{6}{(2)(3)} = 1$$

$$B = F(s)(s+3)\big|_{s=-3} = \frac{6(s+2)}{(s+1)(s+4)}\Big|_{s=-3} = \frac{(6)(-1)}{(-2)(1)} = 3$$

$$C = F(s)(s+4)\big|_{s=-4} = \frac{6(s+2)}{(s+1)(s+3)}\Big|_{s=-4} = \frac{(6)(-2)}{(-3)(-1)} = -4$$

$$F(s) = \frac{1}{s+1} + \frac{3}{s+3} - \frac{4}{s+4}$$

$$f(t) = \mathbf{(e^{-t} + 3e^{-3t} - 4e^{-4t})u(t)}$$

P.P.15.10 $$G(s) = \frac{s^3 + 2s + 6}{s(s+1)^2(s+3)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2} + \frac{D}{s+3}$$

Multiplying both sides by $s(s+1)^2(s+3)$ gives

$$s^3 + 2s + 6 = A(s+3)(s^2+2s+1) + Bs(s+1)(s+3) + Cs(s+3) + Ds(s+1)^2$$
$$= A(s^3 + 5s^2 + 7s + 3) + B(s^3 + 4s^2 + 3s) + C(s^2 + 3s) + D(s^3 + 2s^2 + s)$$

Equating coefficients :

s^0: $6 = 3A \longrightarrow A = 2$ (1)

s^1: $2 = 7A + 3B + 3C + D \longrightarrow 3B + 3C + D = -12$ (2)

s^2: $0 = 5A + 4B + C + 2D \longrightarrow 4B + C + 2D = -10$ (3)

s^3: $1 = A + B + D \longrightarrow B + D = -1$ (4)

Solving (2), (3), and (4) gives

$A = 2, \qquad B = \frac{-13}{4}, \qquad C = \frac{-3}{2}, \qquad D = \frac{9}{4}$

$$G(s) = \frac{2}{s} - \frac{13/4}{s+1} - \frac{3/2}{(s+1)^2} + \frac{9/4}{s+3}$$

$$g(t) = \mathbf{(2 - 3.25\,e^{-t} - 1.5\,t\,e^{-t} + 2.25\,e^{-3t})u(t)}$$

P.P.15.11 $\quad G(s) = \dfrac{60}{(s+1)(s^2+4s+13)} = \dfrac{A}{s+1} + \dfrac{Bs+C}{s^2+4s+13}$

Multiplying both sides by $(s+1)(s^2+4s+13)$ gives

$$60 = A(s^2+4s+13) + B(s^2+s) + C(s+1)$$

Equating coefficients :

$s^2: \quad 0 = A + B \longrightarrow A = -B \qquad (1)$

$s^1: \quad 0 = 4A + B + C \longrightarrow C = -3A \qquad (2)$

$s^0: \quad 60 = 13A + C \longrightarrow 60 = 10A \qquad (3)$

Solving (1), (2), and (3) gives

$A = 6, \qquad B = -6, C = -18$

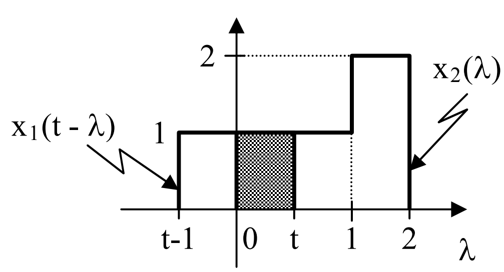
$$G(s) = \frac{6}{s+1} + \frac{-6s-18}{(s+2)^2+9} = \frac{6}{s+1} - 6\frac{s+2}{(s+2)^2+9} - \frac{(6/3)3}{(s+2)^2+9}$$

$$g(t) = \mathbf{(6e^{-t} - 6e^{-2t}\cos(3t) - 2\,e^{-2t}\sin(3t))u(t)}$$

P.P.15.12

For $0 < t < 1$, consider Fig. (a).

$$y(t) = \int_0^t (1)(1)\,d\lambda = t$$

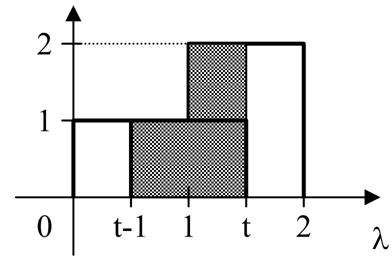


(a)

For $1 < t < 2$, consider Fig. (b).

$$y(t) = \int_{t-1}^{1} (1)(1)\, d\lambda + \int_{1}^{t} (1)(2)\, d\lambda = \lambda \Big|_{t-1}^{t} + 2\lambda \Big|_{1}^{t}$$

$$y(t) = 1 - t + 1 + 2(t-1) = t$$

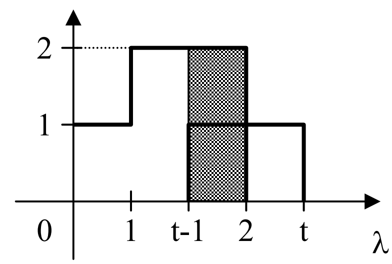


(b)

For $2 < t < 3$, consider Fig. (c).

$$y(t) = \int_{t-1}^{2} (1)(2)\, d\lambda = 2\lambda \Big|_{t-1}^{2}$$

$$y(t) = 2(2 - t + 1) = 6 - 2t$$

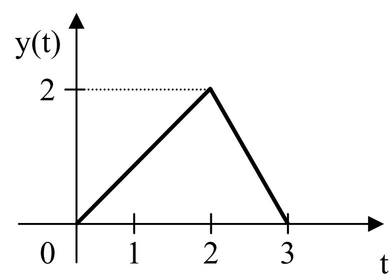


(c)

For $t > 3$, there is no overlap so $y(t) = 0$.

Thus,

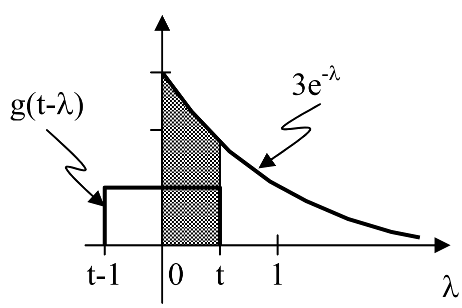
$$y(t) = \begin{cases} \mathbf{t} & \mathbf{0 < t < 2} \\ \mathbf{6 - 2t} & \mathbf{2 < t < 3} \\ \mathbf{0} & \mathbf{otherwise} \end{cases}$$



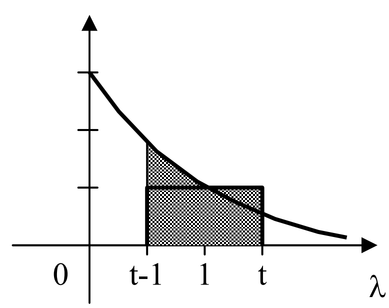
(d)

The result of the convolution is shown in Fig. (d).

P.P.15.13



(a)



(b)

For $0 < t < 1$, consider Fig. (a).

$$y(t) = \int_{0}^{t} (1)\, 3e^{-\lambda}\, d\lambda = -3e^{-\lambda} \Big|_{0}^{t} = 3(1 - e^{-t})$$

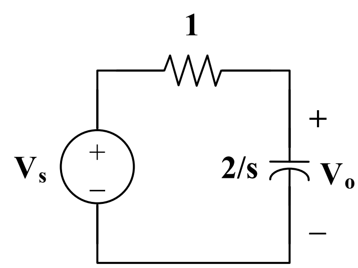
For $t > 1$, consider Fig. (b).

$$y(t) = \int_{t-1}^{t} (1)\, 3\, e^{-\lambda}\, d\lambda = -3\, e^{-\lambda}\Big|_{t-1}^{t} = 3\, e^{-t}\,(e-1)$$

Thus,

$$y(t) = \begin{cases} \mathbf{3\,(1 - e^{-t})} & \mathbf{0 \le t \le 1} \\ \mathbf{3\,e^{-t}\,(e-1)} & \mathbf{t \ge 1} \\ \mathbf{0} & \mathbf{elsewhere} \end{cases}$$

P.P.15.14 The circuit in the s-domain is shown below.



$$V_o = \frac{2/s}{1 + 2/s} V_s$$

$$H(s) = \frac{V_o}{V_s} = \frac{2}{s+2} \longrightarrow h(t) = 2\, e^{-2t}$$

$$\begin{aligned} v_o(t) = h(t) * v_s(t) &= \int_0^t h(\lambda)\, v_s(t-\lambda)\, d\lambda \\ &= \int_0^t 2\, e^{-2\lambda}\, 10 e^{-(t-\lambda)}\, d\lambda \\ &= 20 e^{-t} \int_0^t e^{-2\lambda}\, e^{\lambda}\, d\lambda = 20 e^{-t} (-e^{-\lambda})\Big|_0^t \\ &= \mathbf{20\,(e^{-t} - e^{-2t})u(t)\ V} \end{aligned}$$

P.P.15.15 Taking the Laplace transform of each term gives

$$\left[s^2 V(s) - s v(0) - v'(0) \right] + 4 \left[s V(s) - v(0) \right] + 4\, V(s) = \frac{2}{s+1}$$

$$(s^2 + 4s + 4)\, V(s) = 2s + 10 + \frac{2}{s+1} = \frac{2s^2 + 12s + 12}{s+1}$$

$$V(s) = \frac{2s^2 + 12s + 12}{(s+1)(s+2)^2} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$$

$$2s^2 + 12s + 12 = A\,(s^2 + 4s + 4) + B\,(s^2 + 3s + 2) + C\,(s+1)$$

Equating coefficients :

s^2: $2 = A + B \longrightarrow B = 2 - A$ or $A = 2 - B$

s^1: $12 = 4A + 3B + C \longrightarrow 12 = A + 6 + C$ or $C = 6 - A$

s^0: $12 = 4A + 2B + C \longrightarrow 12 = 12 - B$ or $B = 0$

Thus,

$$A = 2, \quad B = 0, \quad C = 4$$

and

$$V(s) = \frac{2}{s+1} + \frac{4}{(s+2)^2}$$

Therefore,

$v(t) = \mathbf{(2e^{-t} + 4\,t\,e^{-2t})\,u(t)}$ **Note, there were no units given for v(t).**

P.P.15.16 Taking the Laplace transform of each term gives

$$sY(s) - y(0) + 3Y(s) + \frac{2}{s}Y(s) = \frac{2}{s+3}$$

$$\left[s^2 + 3s + 2\right]Y(s) = \frac{2s}{s+3}$$

$$Y(s) = \frac{2s}{(s+1)(s+2)(s+3)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3}$$

$$A = Y(s)\,(s+1)\Big|_{s=-1} = -1$$

$$B = Y(s)\,(s+2)\Big|_{s=-2} = 4$$

$$C = Y(s)\,(s+3)\Big|_{s=-3} = -3$$

$$Y(s) = \frac{-1}{s+1} + \frac{4}{s+2} - \frac{3}{s+3}$$

$y(t) = \mathbf{(-e^{-t} + 4\,e^{-2t} - 3\,e^{-3t})\,u(t)}$