Percentage change approximation

Suppose we know that $Z = AB$. If A changes by ΔA and B changes by ΔB, what will be the change in Z? What about percentage changes?

We can start by noting that $Z + \Delta Z = (A + \Delta A)\cdot(B + \Delta B)$. Multiplying out the terms on the right, we have $Z + \Delta Z = AB + B\Delta A + A\Delta B + \Delta A\Delta B$. But since $Z = AB$, this simplifies to $\Delta Z = B\Delta A + A\Delta B + \Delta A\Delta B$. For small changes, $\Delta A\Delta B$ will be an order of magnitude smaller and so we have as an approximation, $\Delta Z \approx B\Delta A + A\Delta B$. To convert this to percentage changes, we can divide by Z on the left side and the equal term AB on the right to obtain $\frac{\Delta Z}{Z} \approx \frac{B\Delta A}{AB} + \frac{A\Delta B}{AB} = \frac{\Delta A}{A} + \frac{\Delta B}{B}$. In words, the percentage change in the product of two variables is approximately equal to the sum of their component percentage changes.

What if $Z = A/C$? Does a similar relationship hold? As before, let A change by ΔA and C change by ΔC to obtain $Z + \Delta Z = \frac{A + \Delta A}{C + \Delta C}$. To obtain percentage changes, divide the left side by Z and the right side by the equal amount A/C (equivalent to multiplying by C/A). Then $\frac{Z + \Delta Z}{Z} = 1 + \frac{\Delta Z}{Z} = \frac{A + \Delta A}{C + \Delta C} \cdot \frac{C}{A}$. Next we will subtract 1 from each side, but on the right side this 1 will take the form $\frac{A(C + \Delta C)}{A(C + \Delta C)}$ to get a common denominator. This leaves us with $\frac{\Delta Z}{Z} = \frac{AC + C\Delta A - AC - A\Delta C}{A(C + \Delta C)}$. The first and third terms in the numerator cancel, and if we multiply the last term by C/C = 1, we can then factor out the common term $\frac{C}{C + \Delta C}$ to obtain $\frac{\Delta Z}{Z} = \frac{C}{C + \Delta C}\left[\frac{\Delta A}{A} - \frac{\Delta C}{C}\right]$.

For small values of ΔC, $\frac{C}{C + \Delta C}$ is approximately equal to 1, which gives us our final result: if $Z = A/C$, then $\frac{\Delta Z}{Z} \approx \frac{\Delta A}{A} - \frac{\Delta C}{C}$.

In the text example, real income (Z) is equal to nominal income (A) divided by the price index (C). Consequently, for small changes in the price level, the approximation applies: the percentage change in real income ≈ the percentage change in nominal income minus the percentage change in the price level.

There is a shortcut to these relationships as well: Note that for any Z, the differential of the natural logarithm of Z is $d(\ln Z) = dZ/Z$. At the limit as dZ approaches zero, the term on the right is simply the point-estimate of the percentage change in Z—an approximation of $\frac{\Delta Z}{Z}$. As before suppose that $Z = AB$. Taking the natural log of both sides, we have $\ln Z = \ln A + \ln B$. Differentiating, we obtain $dZ/Z = dA/A + dB/B$ as before. Likewise, if $Z = A/C$, then $\ln Z = \ln A - \ln C$ so that $d(\ln Z) = dZ/Z = dA/A - dC/C$.