22

Project Management with PERT/CPM

One of the most challenging jobs that any manager can take on is the management of a large-scale project that requires coordinating numerous activities throughout the organization. A myriad of details must be considered in planning how to coordinate all these activities, in developing a realistic schedule, and then in monitoring the progress of the project.

Fortunately, two closely related operations research techniques, **PERT** (program evaluation and review technique) and **CPM** (critical path method), are available to assist the project manager in carrying out these responsibilities. These techniques make heavy use of *networks* (as introduced in the preceding chapter) to help plan and display the coordination of all the activities. They also normally use a *software package* to deal with all the data needed to develop schedule information and then to monitor the progress of the project. *Project management software* now is widely available for these purposes.

PERT and CPM have been used for a variety of projects, including the following types:

1. Construction of a new plant
2. Research and development of a new product
3. NASA space exploration projects
4. Movie productions
5. Building a ship
6. Government-sponsored projects for developing a new weapons system
7. Relocation of a major facility
8. Maintenance of a nuclear reactor
9. Installation of a management information system
10. Conducting an advertising campaign

PERT and CPM were independently developed in the late 1950s. Ever since, they have been among the most widely used OR techniques.

The original versions of PERT and CPM had some important differences, as we will point out later in the chapter. However, they also had a great deal in common, and the two techniques have gradually merged further over the years. In fact, today's software packages often include all the important options from both original versions.

Consequently, practitioners now commonly use the two names interchangeably, or combine them into the single acronym PERT/CPM, as we often will do. We will make the distinction between them only when we are describing an option that was unique to one of the original versions.

Section 10.8 has presented one of the key techniques of PERT/CPM, namely, a network model for optimizing a project's time-cost trade-off. For the sake of having a complete, self-contained chapter on project management with PERT/CPM, we will present this technique again in Sec. 22.5.

The next section introduces a prototype example that will carry through the chapter to illustrate the various options for analyzing projects provided by PERT/CPM.

■ 22.1   A PROTOTYPE EXAMPLE—THE RELIABLE CONSTRUCTION CO. PROJECT

The RELIABLE CONSTRUCTION COMPANY has just made the winning bid of $5.4 million to construct a new plant for a major manufacturer. The manufacturer needs the plant to go into operation within a year. Therefore, the contract incudes the following provisions:

- A penalty of $300,000 if Reliable has not completed construction by the deadline 47 weeks from now.
- To provide additional incentive for speedy construction, a *bonus* of $150,000 will be paid to Reliable if the plant is completed within 40 weeks.

Reliable is assigning its best construction manager, David Perty, to this project to help ensure that it stays on schedule. He looks forward to the challenge of bringing the project in on schedule, and perhaps even finishing early. However, since he is doubtful that it will be feasible to finish within 40 weeks without incurring excessive costs, he has decided to focus his initial planning on meeting the deadline of 47 weeks.

Mr. Perty will need to arrange for a number of crews to perform the various construction activities at different times. Table 22.1 shows his list of the various activities. The third column provides important additional information for coordinating the scheduling of the crews.

■ **TABLE 22.1** Activity list for the Reliable Construction Co. project

Activity	Activity Description	Immediate Predecessors	Estimated Duration
A	Excavate	—	2 weeks
B	Lay the foundation	A	4 weeks
C	Put up the rough wall	B	10 weeks
D	Put up the roof	C	6 weeks
E	Install the exterior plumbing	C	4 weeks
F	Install the interior plumbing	E	5 weeks
G	Put up the exterior siding	D	7 weeks
H	Do the exterior painting	E, G	9 weeks
I	Do the electrical work	C	7 weeks
J	Put up the wallboard	F, I	8 weeks
K	Install the flooring	J	4 weeks
L	Do the interior painting	J	5 weeks
M	Install the exterior fixtures	H	2 weeks
N	Install the interior fixtures	K, L	6 weeks

For any given activity, its **immediate predecessors** (as given in the third column of Table 22.1) are those activities that must be completed by no later than the starting time of the given activity. (Similarly, the given activity is called an **immediate successor** of each of its immediate predecessors.)

For example, the top entries in this column indicate that

1. Excavation does not need to wait for any other activities.
2. Excavation must be completed before starting to lay the foundation.
3. The foundation must be completely laid before starting to put up the rough wall, etc.

When a given activity has *more than one* immediate predecessor, all must be finished before the activity can begin.

In order to schedule the activities, Mr. Perty consults with each of the crew supervisors to develop an estimate of how long each activity should take when it is done in the normal way. These estimates are given in the rightmost column of Table 22.1.

Adding up these times gives a grand total of 79 weeks, which is far beyond the deadline for the project. Fortunately, some of the activities can be done in parallel, which substantially reduces the project completion time.

Given all the information in Table 22.1, Mr. Perty now wants to develop answers to the following questions.

1. How can the project be displayed graphically to better visualize the flow of the activities? (Section 22.2)
2. What is the total time required to complete the project if no delays occur? (Section 22.3)
3. When do the individual activities need to start and finish (at the latest) to meet this project completion time? (Section 22.3)
4. When can the individual activities start and finish (at the earliest) if no delays occur? (Section 22.3)
5. Which are the critical bottleneck activities where any delays must be avoided to prevent delaying project completion? (Section 22.3)
6. For the other activities, how much delay can be tolerated without delaying project completion? (Section 22.3)
7. Given the uncertainties in accurately estimating activity durations, what is the probability of completing the project by the deadline? (Section 22.4)
8. If extra money is spent to expedite the project, what is the least expensive way of attempting to meet the target completion time (40 weeks)? (Section 22.5)
9. How should ongoing costs be monitored to try to keep the project within budget? (Section 22.6)

Being a regular user of PERT/CPM, Mr. Perty knows that this technique will provide invaluable help in answering these questions (as you will see in the sections indicated in parentheses above).

■ 22.2   USING A NETWORK TO VISUALLY DISPLAY A PROJECT

Chapter 10 describes how valuable *networks* can be to represent and help analyze many kinds of problems. In much the same way, networks play a key role in dealing with projects. They enable showing the relationships between the activities and succinctly displaying the overall plan for the project. They then are used to help analyze the project and answer the kinds of questions raised at the end of the preceding section.

Project Networks

A network used to represent a project is called a **project network.** A project network consists of a number of *nodes* (typically shown as small circles or rectangles) and a number of *arcs* (shown as arrows) that connect two different nodes. (If you have not previously studied Chap. 10, where nodes and arcs are discussed extensively, just think of them as the names given to the small circles or rectangles and to the arrows in the network.)

As Table 22.1 indicates, three types of information are needed to describe a project:

1. Activity information: Break down the project into its individual *activities* (at the desired level of detail).
2. Precedence relationships: Identify the *immediate predecessor(s)* for each activity.
3. Time information: Estimate the *duration* of each activity.

The project network should convey all this information. Two alternative types of project networks are available for doing this.

One type is the **activity-on-arc (AOA)** project network, where each activity is represented by an *arc.* A node is used to separate an activity (an outgoing arc) from each of its immediate predecessors (an incoming arc). The sequencing of the arcs thereby shows the precedence relationships between the activities.

The second type is the **activity-on-node (AON)** project network, where each activity is represented by a *node.* Then the arcs are used just to show the precedence relationships that exist between the activities. In particular, the node for each activity with immediate predecessors has an arc coming in from each of these predecessors.

The original versions of PERT and CPM used AOA project networks, so this was the conventional type for some years. However, AON project networks have some important advantages over AOA project networks for conveying the same information.
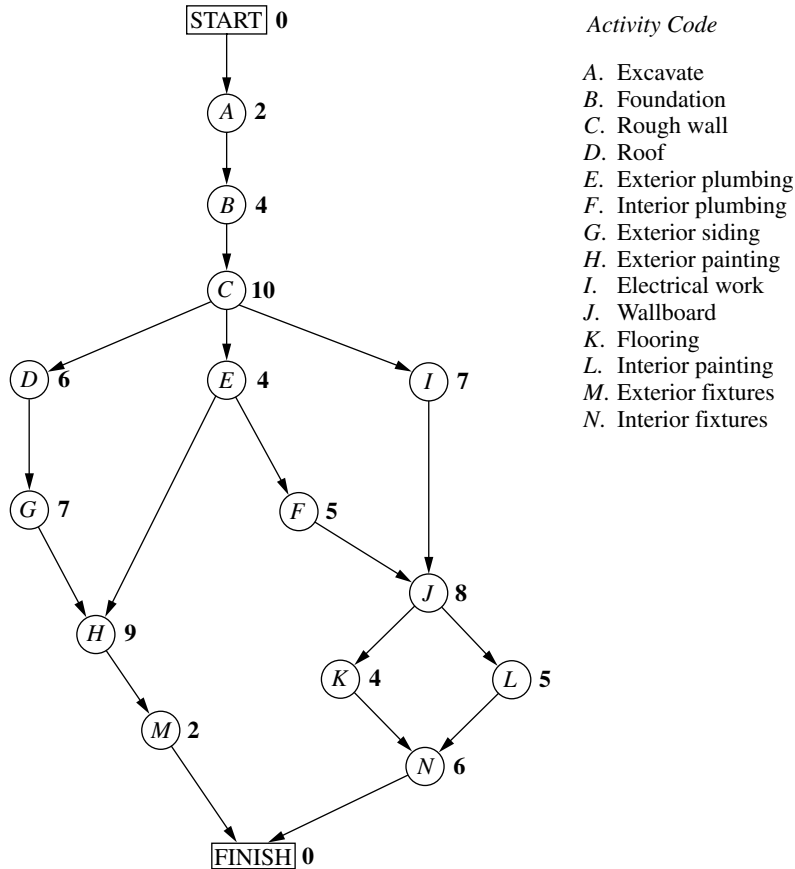
1. AON project networks are considerably easier to construct than AOA project networks.
2. AON project networks are easier to understand than AOA project networks for inexperienced users, including many managers.
3. AON project networks are easier to revise than AOA project networks when there are changes in the project.

For these reasons, AON project networks have become increasingly popular with practitioners. It appears that they may become the standard format for project networks. Therefore, we now will focus solely on AON project networks and will drop the adjective AON.

Figure 22.1 shows the project network for Reliable's project.[1] Referring also to the third column of Table 22.1, note how there is an arc leading to each activity from each of its immediate predecessors. Because activity *A* has no immediate predecessors, there is an arc leading from the start node to this activity. Similarly, since activities *M* and *N* have no immediate successors, arcs lead from these activities to the finish node. Therefore, the project network nicely displays at a glance all the precedence relationships between all the activities (plus the start and finish of the project). Based on the rightmost column of Table 22.1, the number next to the node for each activity then records the estimated duration (in weeks) of that activity.

In real applications, software commonly is used to construct the project network, etc. For example, Microsoft Project is widely used for this purpose. Several dozen other commercially available software packages also are available for dealing with the various aspects of project management.

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[1]Although project networks often are drawn from left to right, we go from top to bottom to better fit on the printed page.

Activity Code

A. Excavate
B. Foundation
C. Rough wall
D. Roof
E. Exterior plumbing
F. Interior plumbing
G. Exterior siding
H. Exterior painting
I. Electrical work
J. Wallboard
K. Flooring
L. Interior painting
M. Exterior fixtures
N. Interior fixtures

■ **FIGURE 22.1**
The project network for the
Reliable Construction Co.
project.

■ 22.3 SCHEDULING A PROJECT WITH PERT/CPM

At the end of Sec. 22.1, we mentioned that Mr. Perty, the project manager for the Reliable Construction Co. project, wants to use PERT/CPM to develop answers to a series of questions. His first question has been answered in the preceding section. Here are the five questions that will be answered in this section.

Question 2: What is the total time required to complete the project if no delays occur?
Question 3: When do the individual activities need to start and finish (at the latest) to meet this project completion time?
Question 4: When can the individual activities start and finish (at the earliest) if no delays occur?
Question 5: Which are the critical bottleneck activities where any delays must be avoided to prevent delaying project completion?
Question 6: For the other activities, how much delay can be tolerated without delaying project completion?

The project network in Fig. 22.1 enables answering all these questions by providing two crucial pieces of information, namely, the *order* in which certain activities must be performed and the (estimated) *duration* of each activity. We begin by focusing on Questions 2 and 5.

The Critical Path

How long should the project take? We noted earlier that summing the durations of all the activities gives a grand total of 79 weeks. However, this isn't the answer to the question because some of the activities can be performed (roughly) simultaneously.

What is relevant instead is the *length* of each *path* through the network.

> A **path** through a project network is one of the routes following the arcs from the START node to the FINISH node. The **length** of a path is the *sum* of the (estimated) *durations* of the activities on the path.

The six paths through the project network in Fig. 22.1 are given in Table 22.2, along with the calculations of the lengths of these paths. The path lengths range from 31 weeks up to 44 weeks for the longest path (the fourth one in the table).

So given these path lengths, what should be the (estimated) **project duration** (the total time required for the project)? Let us reason it out.

Since the activities on any given path must be done in sequence with no overlap, the project duration cannot be *shorter* than the path length. However, the project duration can be *longer* because some activity on the path with multiple immediate predecessors might have to wait longer for an immediate predecessor *not* on the path to finish than for the one on the path. For example, consider the second path in Table 22.2 and focus on activity *H*. This activity has two immediate predecessors, one (activity *G*) *not* on the path and one (activity *E*) that is. After activity *C* finishes, only 4 more weeks are required for activity *E* but 13 weeks will be needed for activity *D* and then activity *G* to finish. Therefore, the project duration must be considerably longer than the length of the second path in the table.

However, the project duration will not be longer than one particular path. This is the *longest path* through the project network. The activities on this path can be performed sequentially without interruption. (Otherwise, this would not be the longest path.) Therefore, the time required to reach the FINISH node equals the length of this path. Furthermore, all the shorter paths will reach the FINISH node no later than this.

Here is the key conclusion.

> The (estimated) *project duration* equals the *length of the longest path* through the project network. This longest path is called the **critical path.** (If more than one path tie for the longest, they all are critical paths.)

Thus, for the Reliable Construction Co. project, we have

Critical path: START →*A*→*B*→*C*→*E*→*F*→*J*→*L*→*N*→ FINISH
(Estimated) project duration = 44 weeks.

We now have answered Mr. Perty's Questions 2 and 5 given at the beginning of the section. If no delays occur, the total time required to complete the project should be about 44 weeks. Furthermore, the activities on this critical path are the critical bottleneck

■ **TABLE 22.2 The paths and path lengths through Reliable's project network**

Path	Length
START →*A*→*B*→*C*→*D*→*G*→*H*→*M*→ FINISH	2 + 4 + 10 + 6 + 7 + 9 + 2 = 40 weeks
START →*A*→*B*→*C*→*E*→*H*→*M*→ FINISH	2 + 4 + 10 + 4 + 9 + 2 = 31 weeks
START →*A*→*B*→*C*→*E*→*F*→*J*→*K*→*N*→ FINISH	2 + 4 + 10 + 4 + 5 + 8 + 4 + 6 = 43 weeks
START →*A*→*B*→*C*→*E*→*F*→*J*→*L*→*N*→ FINISH	2 + 4 + 10 + 4 + 5 + 8 + 5 + 6 = 44 weeks
START →*A*→*B*→*C*→*I*→*J*→*K*→*N*→ FINISH	2 + 4 + 10 + 7 + 8 + 4 + 6 = 41 weeks
START →*A*→*B*→*C*→*I*→*J*→*L*→*N*→ FINISH	2 + 4 + 10 + 7 + 8 + 5 + 6 = 42 weeks

activities where any delays in their completion must be avoided to prevent delaying project completion. This is valuable information for Mr. Perty, since he now knows that he should focus most of his attention on keeping these particular activities on schedule in striving to keep the overall project on schedule. Furthermore, if he decides to reduce the duration of the project (remember that bonus for completion within 40 weeks), these are the main activities where changes should be made to reduce their durations.

For small project networks like Fig. 22.1, finding all the paths and determining the longest path is a convenient way to identify the critical path. However, this is not an efficient procedure for larger projects. PERT/CPM uses a considerably more efficient procedure instead.

Not only is this PERT/CPM procedure very efficient for larger projects, it also provides much more information than is available from finding all the paths. In particular, it answers *all five* of Mr. Perty's questions listed at the beginning of the section rather than just two. These answers provide the key information needed to schedule all the activities and then to evaluate the consequences should any activities slip behind schedule.

The components of this procedure are described in the remainder of this section.

Scheduling Individual Activities

The PERT/CPM scheduling procedure begins by addressing Question 4: When can the individual activities start and finish (at the earliest) if no delays occur? Having no delays means that (1) the *actual* duration of each activity turns out to be the same as its *estimated* duration and (2) each activity begins as soon as all its immediate predecessors are finished. The starting and finishing times of each activity if no delays occur anywhere in the project are called the **earliest start time** and the **earliest finish time** of the activity. These times are represented by the symbols

ES = earliest start time for a particular activity,
EF = earliest finish time for a particular activity,

where

EF = ES + (estimated) duration of the activity.

Rather than assigning calendar dates to these times, it is conventional instead to count the number of time periods (weeks for Reliable's project) from when the project started. Thus,

Starting time for project = 0.

Since activity *A* starts Reliable's project, we have

Activity *A:*   ES = 0,
                 EF = 0 + duration (2 weeks)
                    = 2,

where the duration (in weeks) of activity *A* is given in Fig. 22.1 as the boldfaced number next to this activity. Activity *B* can start as soon as activity *A* finishes, so

Activity *B:*   ES = EF for activity *A*
                    = 2,
                 EF = 2 + duration (4 weeks)
                    = 6.

This calculation of ES for activity *B* illustrates our first rule for obtaining ES.

If an activity has only a *single* immediate predecessor, then

ES for the activity = EF for the immediate predecessor.

This rule (plus the calculation of each EF) immediately gives ES and EF for activity *C,* then for activities *D, E, I,* and then for activities *G, F* as well. Figure 22.2 shows ES and EF for each of these activities to the right of its node. For example,

Activity *G:*   ES = EF for activity *D*
                     = 22,
               EF = 22 + duration (7 weeks)
                     = 29,

which means that this activity (putting up the exterior siding) should start 22 weeks and finish 29 weeks after the start of the project.

Now consider activity *H,* which has *two* immediate predecessors, activities *G* and *E.* Activity *H* must wait to start until *both* activities *G* and *E* are finished, which gives the following calculation.

Immediate predecessors of activity *H:*

Activity *G* has EF = 29.
Activity *E* has EF = 20.
Larger EF = 29.

Therefore,

ES for activity *H* = larger EF above
                         = 29.

■ **FIGURE 22.2**
Earliest start time (ES) and earliest finish time (EF) values for the initial activities in Fig. 22.1 that have only a single immediate predecessor.
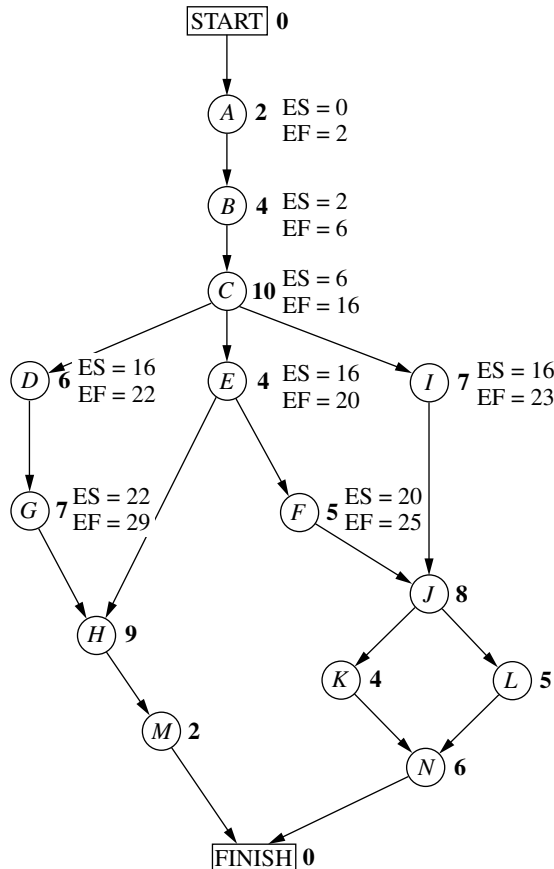
This calculation illustrates the general rule for obtaining the earliest start time for any activity.
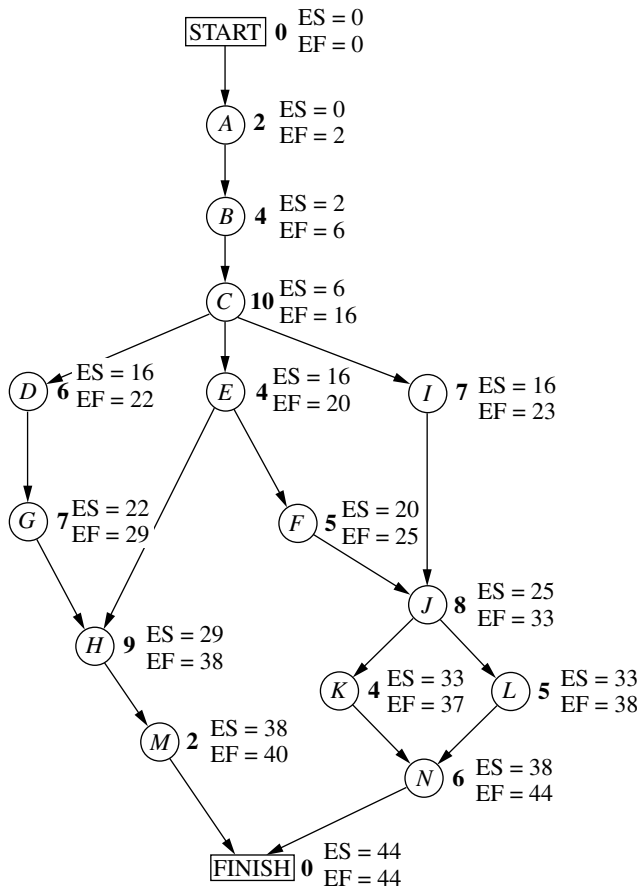
Earliest Start Time Rule

The earliest start time of an activity is equal to the *largest* of the earliest finish times of its immediate predecessors. In symbols,

ES = largest EF of the immediate predecessors.

When the activity has only a single immediate predecessor, this rule becomes the same as the first rule given earlier. However, it also allows any larger number of immediate predecessors as well. Applying this rule to the rest of the activities in Fig. 22.2 (and calculating each EF from ES) yields the complete set of ES and EF values given in Fig. 22.3.

Note that Fig. 22.3 also includes ES and EF values for the START and FINISH nodes. The reason is that these nodes are conventionally treated as *dummy activities* that require no time. For the START node, ES=0=EF automatically. For the FINISH

■ **FIGURE 22.3**
Earliest start time (ES) and earliest finish time (EF) values for all the activities (plus the START and FINISH nodes) of the Reliable Construction Co. project.

node, the earliest start time rule is used to calculate ES in the usual way, as illustrated below.

Immediate predecessors of the FINISH node:

$$\text{Activity } M \text{ has EF} = 40.$$
$$\text{Activity } N \text{ has EF} = 44.$$
$$\text{Larger EF} = 44.$$

Therefore,

ES for the FINISH node = larger EF above
$$= 44.$$
EF for the FINISH node $= 44 + 0 = 44.$

This last calculation indicates that the project should be completed in 44 weeks if everything stays on schedule according to the start and finish times for each activity given in Fig. 22.3. (This answers Question 2.) Mr. Perty now can use this schedule to inform the crew responsible for each activity as to when it should plan to start and finish its work.

This process of starting with the initial activities and working *forward* in time toward the final activities to calculate all the ES and EF values is referred to as making a **forward pass** through the network.

Keep in mind that the schedule obtained from this procedure assumes that the *actual* duration of each activity will turn out to be the same as its *estimated* duration. What happens if some activity takes longer than expected? Would this delay project completion? Perhaps, but not necessarily. It depends on which activity and the length of the delay.

The next part of the procedure focuses on determining how much later than indicated in Fig. 22.3 can an activity start or finish without delaying project completion.

> The **latest start time** for an activity is the latest possible time that it can start without delaying the completion of the project (so the FINISH node still is reached at its earliest finish time), assuming no subsequent delays in the project. The **latest finish time** has the corresponding definition with respect to finishing the activity.

In symbols,

LS = latest start time for a particular activity,
LF = latest finish time for a particular activity,

where

LS = LF − (estimated) duration of the activity.

To find LF, we have the following rule.

Latest Finish Time Rule

The latest finish time of an activity is equal to the *smallest* of the latest start times of its immediate successors. In symbols,

LF = smallest LS of the immediate successors.

Since an activity's immediate successors cannot start until the activity finishes, this rule is saying that the activity must finish in time to enable *all* its immediate successors to begin by their latest start times.

For example, consider activity *M* in Fig. 22.1. Its only immediate successor is the FINISH node. This node must be reached by time 44 in order to complete the project within 44 weeks, so we begin by assigning values to this node as follows.

FINISH node: LF = its EF = 44,
                        LS = 44 − 0 = 44.

Now we can apply the latest finish time rule to activity *M*.

Activity *M*:   LF = LS for the FINISH node
                        = 44,
                  LS = 44 − duration (2 weeks)
                        = 42.

(Since activity *M* is one of the activities that together complete the project, we also could have automatically set its LF equal to the earliest finish time of the FINISH node without applying the latest finish time rule.)

Since activity *M* is the only immediate successor of activity *H,* we now can apply the latest finish time rule to the latter activity.

Activity *H*:   LF = LS for activity *M*
                        = 42,
                  LS = 42 − duration (9 weeks)
                        = 33.

Note that the procedure being illustrated above is to start with the final activities and work *backward* in time toward the initial activities to calculate all the LF and LS values. Thus, in contrast to the *forward pass* used to find earliest start and finish times, we now are making a **backward pass** through the network.

Figure 22.4 shows the results of making a backward pass to its completion. For example, consider activity *C,* which has three immediate successors.

Immediate successors of activity *C:*

Activity *D* has LS = 20.
Activity *E* has LS = 16.
Activity *I* has LS = 18.
Smallest LS = 16.

Therefore,

LF for activity *C* = smallest LS above
                        = 16.

Mr. Perty now knows that the schedule given in Fig. 22.4 represents his "last chance schedule." Even if an activity starts and finishes as late as indicated in the figure, he still will be able to avoid delaying project completion beyond 44 weeks as long as there is no subsequent slippage in the schedule. However, to allow for unexpected delays, he would prefer to stick instead to the *earliest time schedule* given in Fig. 22.3 whenever possible in order to provide some slack in parts of the schedule.

If the start and finish times in Fig. 22.4 for a particular activity are later than the corresponding earliest times in Fig. 22.3, then this activity has some slack in the schedule. The last part of the PERT/CPM procedure for scheduling a project is to identify this slack, and then to use this information to find the *critical path.* (This will answer both Questions 5 and 6.)
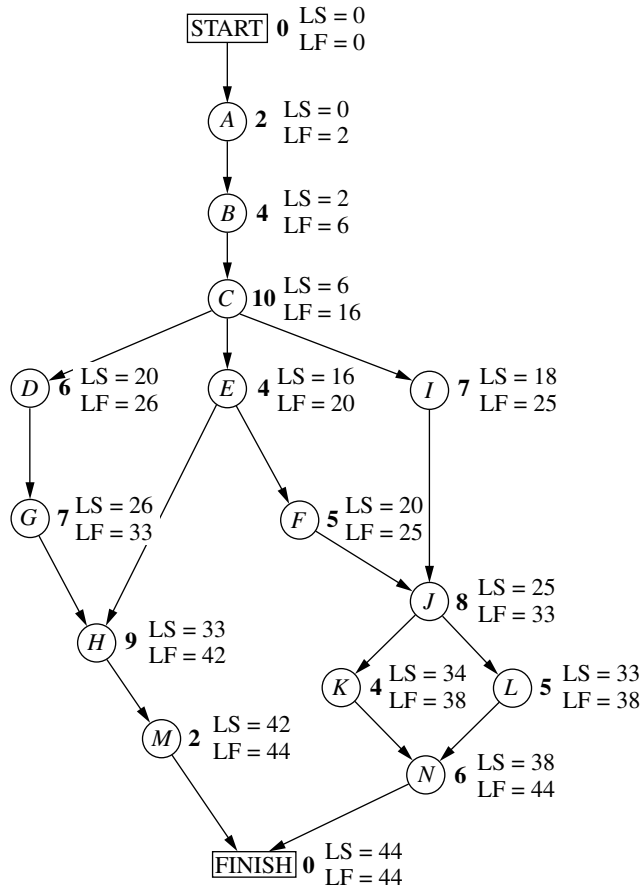
■ **FIGURE 22.4**
Latest start time (LS) and latest finish time (LF) for all the activities (plus the START and FINISH nodes) of the Reliable Construction Co. project.

Identifying Slack in the Schedule

To identify slack, it is convenient to combine the latest times in Fig. 22.4 and the earliest times in Fig. 22.3 into a single figure. Using activity *M* as an example, this is done by displaying the information for each activity as follows:
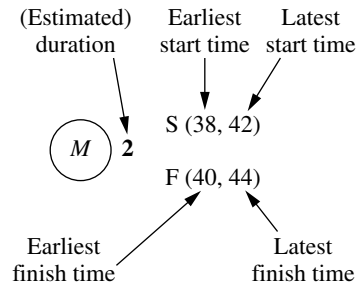
(Note that the S or F in front of each parentheses will remind you of whether these are Start times or Finish times.) Figure 22.5 displays this information for the entire project.
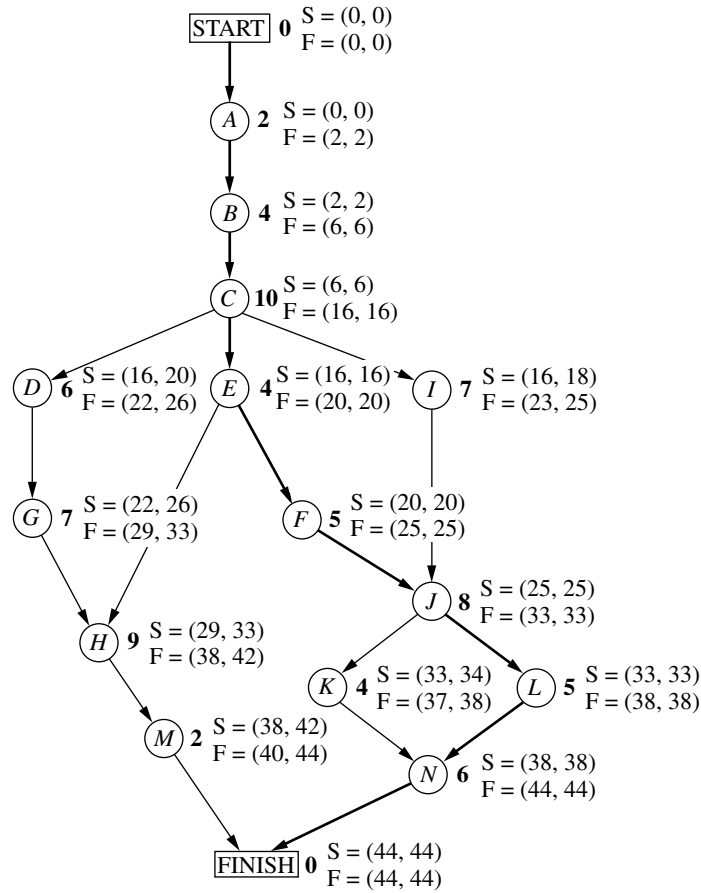
■ **FIGURE 22.5**
The complete project network showing ES and LS (in parentheses above the node) and EF and LF (in parentheses below the node) for each activity of the Reliable Construction Co. project. The darker arrows show the critical path through the project network.

This figure makes it easy to see how much slack each activity has.

The **slack** for an activity is the difference between its latest finish time and its earliest finish time. In symbols,

Slack = LF − EF.

(Since LF − EF = LS − ES, either difference actually can be used to calculate slack.)

For example,

Slack for activity $M = 44 − 40 = 4$.

This indicates that activity M can be delayed up to 4 weeks beyond the earliest time schedule without delaying the completion of the project at 44 weeks. This makes sense, since the project is finished as soon as both activities M and N are completed and the earliest finish time for activity N (44) is 4 weeks later than for activity M (40). As long as activity N stays on schedule, the project still will finish at 44 weeks if any delays in starting activity M (perhaps due to preceding activities taking longer than expected) and in performing activity M do not cumulate more than 4 weeks.

Table 22.3 shows the slack for each of the activities. Note that some of the activities have *zero slack,* indicating that any delays in these activities will delay project completion. This is how PERT/CPM identifies the critical path(s).

■ **TABLE 22.3** Slack for Reliable's activities

Activity	Slack (LF − EF)	On Critical Path?
A	0	Yes
B	0	Yes
C	0	Yes
D	4	No
E	0	Yes
F	0	Yes
G	4	No
H	4	No
I	2	No
J	0	Yes
K	1	No
L	0	Yes
M	4	No
N	0	Yes

> Each activity with *zero slack* is on a **critical path** through the project network such that any delay along this path will delay project completion.

Thus, the critical path is

START →A→B→C→E→F→J→L→N→ FINISH,

just as we found by a different method at the beginning of the section. This path is highlighted in Fig. 22.5 by the darker arrows. It is the activities on this path that Mr. Perty must monitor with special care to keep the project on schedule.

Review

Now let us review Mr. Perty's questions at the beginning of the section and see how all of them have been answered by the PERT/CPM scheduling procedure.

Question 2: What is the total time required to complete the project if no delays occur? This is the earliest finish time at the FINISH node (EF = 44 weeks), as given at the bottom of Figs. 22.3 and 22.5.

Question 3: When do the individual activities need to start and finish (at the latest) to meet this project completion time? These times are the latest start times (LS) and latest finish times (LF) given in Figs. 22.4 and 22.5. These times provide a "last chance schedule" to complete the project in 44 weeks if no further delays occur.

Question 4: When can the individual activities start and finish (at the earliest) if no delays occur? These times are the earliest start times (ES) and earliest finish times (EF) given in Figs. 22.3 and 22.5. These times usually are used to establish the initial schedule for the project. (Subsequent delays may force later adjustments in the schedule.)

Question 5: Which are the critical bottleneck activities where any delays must be avoided to prevent delaying project completion? These are the activities on the critical path shown by the darker arrows in Fig. 22.5. Mr. Perty needs to focus most of his attention on keeping these particular activities on schedule in striving to keep the overall project on schedule.

Question 6: For the other activities, how much delay can be tolerated without delaying project completion? These tolerable delays are the positive slacks given in the middle column of Table 22.3.

22.4   DEALING WITH UNCERTAIN ACTIVITY DURATIONS

Now we come to the next of Mr. Perty's questions posed at the end of Sec. 22.1.

Question 7: Given the uncertainties in accurately estimating activity durations, what is the probability of completing the project by the deadline (47 weeks)?

Recall that Reliable will incur a large penalty ($300,000) if this deadline is missed. Therefore, Mr. Perty needs to know the probability of meeting the deadline. If this probability is not very high, he will need to consider taking costly measures (using overtime, etc.) to shorten the duration of some of the activities.

It is somewhat reassuring that the PERT/CPM scheduling procedure in the preceding section obtained an estimate of 44 weeks for the project duration. However, Mr. Perty understands very well that this estimate is based on the assumption that the *actual* duration of each activity will turn out to be the same as its *estimated* duration for at least the activities on the critical path. Since the company does not have much prior experience with this kind of project, there is considerable uncertainty about how much time actually will be needed for each activity. In reality, the duration of each activity is a *random variable* having some probability distribution.

The original version of PERT took this uncertainty into account by using three different types of estimates of the duration of an activity to obtain basic information about its probability distribution, as described below.

The PERT Three-Estimate Approach

The three estimates to be obtained for each activity are

Most likely estimate (*m*) = estimate of the most likely value of the duration,

Optimistic estimate (*o*) = estimate of the duration under the most favorable conditions,

Pessimistic estimate (*p*) = estimate of the duration under the most unfavorable conditions.

The intended location of these three estimates with respect to the probability distribution is shown in Fig. 22.6.

Thus, the optimistic and pessimistic estimates are meant to lie at the extremes of what is possible, whereas the most likely estimate provides the highest point of the probability

■ **FIGURE 22.6**
Model of the probability distribution of the duration of an activity for the PERT three-estimate approach: *m* = most likely estimate, *o* = optimistic estimate, and *p* = pessimistic estimate.
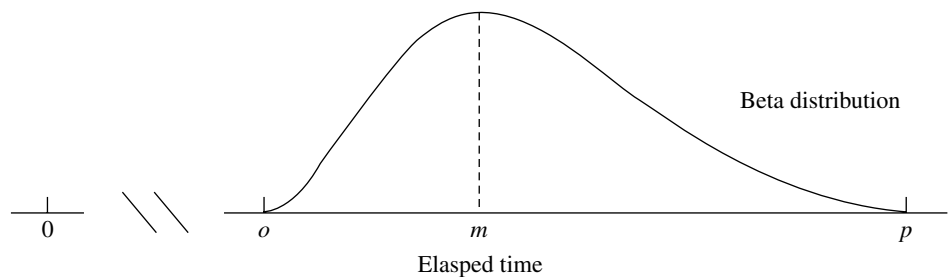
■ **TABLE 22.4** Expected value and variance of the duration of each activity for Reliable's project

Activity	Optimistic Estimate o	Most Likely Estimate m	Pessimistic Estimate p	Mean $\mu = \dfrac{o + 4m + p}{6}$	Variance $\sigma^2 = \left(\dfrac{p - o}{6}\right)^2$
A	1	2	3	2	$\frac{1}{9}$
B	2	$3\frac{1}{2}$	8	4	1
C	6	9	18	10	4
D	4	$5\frac{1}{2}$	10	6	1
E	1	$4\frac{1}{2}$	5	4	$\frac{4}{9}$
F	4	4	10	5	1
G	5	$6\frac{1}{2}$	11	7	1
H	5	8	17	9	4
I	3	$7\frac{1}{2}$	9	7	1
J	3	9	9	8	1
K	4	4	4	4	0
L	1	$5\frac{1}{2}$	7	5	1
M	1	2	3	2	$\frac{1}{9}$
N	5	$5\frac{1}{2}$	9	6	$\frac{4}{9}$

distribution. PERT also assumes that the *form* of the probability distribution is a *beta distribution* (which has a shape like that in the figure) in order to calculate the *mean* (μ) and *variance* (σ^2) of the probability distribution. For most probability distributions such as the beta distribution, essentially the entire distribution lies inside the interval between ($\mu - 3\sigma$) and ($\mu + 3\sigma$). (For example, for a normal distribution, 99.73 percent of the distribution lies inside this interval.) Thus, the spread between the smallest and largest elapsed times in Fig. 22.8 is roughly 6σ. Therefore, an approximate formula for σ^2 is

$$\sigma^2 = \left(\frac{p - o}{6}\right)^2.$$

Similarly, an approximate formula for μ is

$$\mu = \frac{o + 4m + p}{6}.$$

Intuitively, this formula is placing most of the weight on the *most likely estimate* and then small equal weights on the other two estimates.[1]

Mr. Perty now has contacted the supervisor of each crew that will be responsible for one of the activities to request that these three estimates be made of the duration of the activity. The responses are shown in the first four columns of Table 22.4.

---

[1]For a justification of this formula, see R. H. Pleguezuelo, J. G. Pérez, and S. C. Rambaud, "A Note on the Reasonableness of PERT Hypotheses," *Operations Research Letters,* **31:** 60–62, 2003.

The last two columns show the approximate mean and variance of the duration of each activity, as calculated from the formulas on p. 22–16. In this example, all the means happen to be the same as the estimated duration obtained in Table 22.1 of Sec. 22.1. Therefore, if all the activity durations were to equal their means, the duration of the project still would be 44 weeks, so 3 weeks before the deadline. (See Fig. 22.5 for the critical path requiring 44 weeks.)

However, this piece of information is not very reassuring to Mr. Perty. He knows that the durations fluctuate around their means. Consequently, it is inevitable that the duration of some activities will be larger than the mean, perhaps even nearly as large as the pessimistic estimate, which could greatly delay the project.

To check the *worst case scenario,* Mr. Perty reexamines the project network with the duration of each activity set equal to the *pessimistic estimate* (as given in the fourth column of Table 22.4). Table 22.5 shows the six paths through this network (as given previously in Table 22.2) and the length of each path using the pessimistic estimates. The fourth path, which was the critical path in Fig. 22.3, now has increased its length from 44 weeks to 69 weeks. However, the length of the first path, which originally was 40 weeks (as given in Table 22.2), now has increased all the way up to 70 weeks. Since this is the longest path, it is the critical path with pessimistic estimates, which would give a project duration of 70 weeks.

Given this dire (albeit unlikely) worst case scenario, Mr. Perty realizes that it is far from certain that the deadline of 47 weeks will be met. But what is the probability of doing so?

PERT/CPM makes three *simplifying approximations* to help calculate this probability.

Three Simplifying Approximations

To calculate the probability that *project duration* will be no more than 47 weeks, it is necessary to obtain the following information about the probability distribution of project duration.

Probability Distribution of Project Duration.

1. What is the *mean* (denoted by μ_p) of this distribution?
2. What is the *variance* (denoted by σ_p^2) of this distribution?
3. What is the *form* of this distribution?

Recall that project duration equals the *length* (total elapsed time) of the *longest path* through the project network. However, just about any of the six paths listed in Table 22.5 can turn out to be the longest path (and so the critical path), depending upon what the duration of each activity turns out to be between its optimistic and pessimistic estimates.

■ **TABLE 22.5** The paths and path lengths through Reliable's project network when the duration of each activity equals its pessimistic estimate

Path	Length
START→A→B→C→D→G→H→M→FINISH	3 + 8 + 18 + 10 + 11 + 17 + 3 = 70 weeks
START→A→B→C→E→H→M→FINISH	3 + 8 + 18 + 5 + 17 + 3 = 54 weeks
START→A→B→C→E→F→J→K→N→FINISH	3 + 8 + 18 + 5 + 10 + 9 + 4 + 9 = 66 weeks
START→A→B→C→E→F→J→L→N→FINISH	3 + 8 + 18 + 5 + 10 + 9 + 7 + 9 = 69 weeks
START→A→B→C→I→J→K→N→FINISH	3 + 8 + 18 + 9 + 9 + 4 + 9 = 60 weeks
START→A→B→C→I→J→L→N→FINISH	3 + 8 + 18 + 9 + 9 + 7 + 9 = 63 weeks

Since dealing with all these paths would be complicated, PERT/CPM focuses on just the following path.

> The **mean critical path** is the path through the project network that would be the critical path if the duration of each activity equals its *mean.*

Reliable's mean critical path is

> START→A→B→C→E→F→J→L→N→FINISH,

as highlighted in Fig. 22.5.

> **Simplifying Approximation 1:** Assume that the *mean critical path* will turn out to be the longest path through the project network. This is only a rough approximation, since the assumption occasionally does not hold in the usual case where some of the activity durations do not equal their means. Fortunately, when the assumption does not hold, the true longest path commonly is not much longer than the mean critical path (as illustrated in Table 22.5).

Although this approximation will enable us to calculate μ_p, we need one more approximation to obtain σ_p^2.

> **Simplifying Approximation 2:** Assume that the durations of the activities on the mean critical path are *statistically independent.* This assumption should hold if the activities are performed truly independently of each other. However, the assumption becomes only a rough approximation if the circumstances that cause the duration of one activity to deviate from its mean also tend to cause similar deviations for some other activities.

We now have a simple method for computing μ_p and σ_p^2.

> **Calculation of μ_p and σ_p^2:** Because of simplifying approximation 1, the *mean* of the probability distribution of project duration is approximately
>
> > μ_p = sum of the *means* of the durations for the activities on the mean critical path.
>
> Because of both simplifying approximations 1 and 2, the *variance* of the probability distribution of project duration is approximately
>
> > σ_p^2 = sum of the variances of the durations for the activities on the mean critical path.

Since the means and variances of the durations for all the activities of Reliable's project already are given in Table 22.4, we only need to record these values for the activities on the mean critical path as shown in Table 22.6. Summing the second column and then summing the third column give

> $\mu_p = 44,$ $\sigma_p^2 = 9.$

Now we just need an approximation for the *form* of the probability distribution of project duration.

> **Simplifying Approximation 3:** Assume that the form of the probability distribution of project duration is a *normal distribution,* as shown in Fig. 22.7. By using simplifying approximations 1 and 2, one version of the central limit theorem justifies this assumption as being a reasonable approximation if the number of activities on the mean critical path is not too small (say, at least 5). The approximation becomes better as this number of activities increases.

■ **FIGURE 22.7**
The three simplifying
approximations lead to the
probability distribution of the
duration of Reliable's project
being approximated by the
normal distribution shown
here. The shaded area is the
portion of the distribution
that meets the deadline of
47 weeks.

$$\sigma_p^2 = 9$$

$$\frac{d - \mu_p}{\sigma_p} = \frac{47 - 44}{3} = 1$$

Project duration
(in weeks)

44
(Mean)

47
(Deadline)
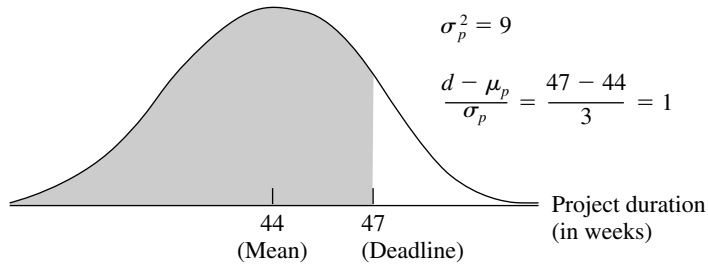
■ **TABLE 22.6** Calculation of μ_p and σ_p^2 for Reliable's project

Activities on Mean Critical Path	Mean	Variance
A	2	$\frac{1}{9}$
B	4	1
C	10	4
E	4	$\frac{4}{9}$
F	5	1
J	8	1
L	5	1
N	6	$\frac{4}{9}$
Project duration	$\mu_p = 44$	$\sigma_p^2 = 9$

Now we are ready to determine (approximately) the probability of completing Reliable's project within 47 weeks.

Approximating the Probability of Meeting the Deadline

Let

T = project duration (in weeks), which has (approximately) a normal distribution
with mean $\mu_p = 44$ and variance $\sigma_p^2 = 9$,
d = deadline for the project = 47 weeks.

Since the standard deviation of T is $\sigma_p = 3$, the number of standard deviations by which d exceeds μ_p is

$$K_\alpha = \frac{d - \mu_p}{\sigma_p} = \frac{47 - 44}{3} = 1.$$

Therefore, using Table A5.1 in Appendix 5 for a *standard* normal distribution (a normal distribution with mean 0 and variance 1), the probability of meeting the deadline (given the three simplifying approximations) is

$$P(T \leq d) = P(\text{standard normal} \leq K_\alpha)$$
$$= 1 - P(\text{standard normal} > K_\alpha) = 1 - 0.1587 \approx 0.84.$$

Warning: This $P(T \leq d)$ is only a rough approximation of the true probability of meeting the project deadline. Furthermore, because of simplifying approximation 1, it usually overstates the true probability somewhat. Therefore, the project manager should view $P(T \leq d)$ as only providing rough guidance on the best odds of meeting the deadline without taking new costly measures to try to reduce the duration of some activities. (Section 22.7 will discuss other alternatives, including the use of the technique of simulation described in Chap. 20, for obtaining a better approximation of the probability of meeting the project deadline.)

To assist you in carrying out this procedure for calculating $P(T \leq d)$, we have provided an Excel template (labeled PERT) in this chapter's Excel files in your OR Courseware. Figure 22.8 illustrates the use of this template for Reliable's project. The data for the problem is entered in the light sections of the spreadsheet. After entering data, the results immediately appear in the dark sections. In particular, by entering the three time estimates for each activity, the spreadsheet will automatically calculate the corresponding estimates for the mean and variance. Next, by specifying the mean critical path (by entering * in column G for each activity on the mean critical path) and the deadline (in cell L10), the spreadsheet automatically calculates the mean and variance of the length of the mean critical path along with the probability that the project will be completed by the deadline. (If you are not sure which path is the mean critical path, the mean length of *any* path can be checked by entering a * for each activity on that path in column G. The path with the longest mean length then is the mean critical path.)

■ **FIGURE 22.8**
This PERT template in your OR Courseware enables efficient application of the PERT three-estimate approach, as illustrated here for Reliable's project.

	A	B	C	D	E	F	G	H	I	J	K
1		**Template for PERT Three-Estimate Approach**									
2											
3				Time Estimates		On Mean					
4		Activity	o	m	p	Critical Path	μ	σ^2			
5		A	1	2	3	*	2	0.1111		Mean Critical	
6		B	2	3.5	8	*	4	1		Path	
7		C	6	9	18	*	10	4		$\mu =$	44
8		D	4	5.5	10		6	1		$\sigma^2 =$	9
9		E	1	4.5	5	*	4	0.4444			
10		F	4	4	10	*	5	1		$P(T<=d) =$	0.8413
11		G	5	6.5	11		7	1		where	
12		H	5	8	17		9	4		$d =$	47
13		I	3	7.5	9		7	1			
14		J	3	9	9	*	8	1			
15		K	4	4	4		4	0			
16		L	1	5.5	7	*	5	1			
17		M	1	2	3		2	0.1111			
18		N	5	5.5	9	*	6	0.4444			

Range Name	Cells
Activity	B5:B18
ActivityMean	G5:G18
ActivityVariance	H5:H18
CompletionProbability	K10
CriticalPathMean	K7
CriticalPathVariance	K8
d	K12
m	D5:D18
o	C5:C18
OnMeanCriticalPath	F5:F18
p	E5:E18

	G	H
4	μ	σ^2
5	=IF(o="","",(o+4*m+p)/6)	=IF(o="","",((p-o)/6)^2)
6	=IF(o="","",(o+4*m+p)/6)	=IF(o="","",((p-o)/6)^2)
7	=IF(o="","",(o+4*m+p)/6)	=IF(o="","",((p-o)/6)^2)
8	=IF(o="","",(o+4*m+p)/6)	=IF(o="","",((p-o)/6)^2)
9	:	:
10	:	:

	J	K
5		Mean Critical
6		Path
7	$\mu =$	=SUMIF(OnMeanCriticalPath,"*",ActivityMean)
8	$\sigma^2 =$	=SUMIF(OnMeanCriticalPath,"*",ActivityVariance)
9		
10	$P(T<=d) =$	=NORMDIST(d,CriticalPathMean,SQRT(CriticalPathVariance),1)
11	where	
12	$d =$	47

Realizing that $P(T \leq d) = 0.84$ is probably an optimistic approximation, Mr. Perty is somewhat concerned that he may have perhaps only a 70 to 80 percent chance of meeting the deadline with the current plan.[1] Therefore, rather than taking the significant chance of the company incurring the late penalty of $300,000, he decides to investigate what it would cost to reduce the project duration to about 40 weeks. If the *time-cost trade-off* for doing this is favorable, the company might then be able to earn the bonus of $150,000 for finishing within 40 weeks.

You will see this story unfold in the next section.

■ 22.5   CONSIDERING TIME-COST TRADE-OFFS[2]

Mr. Perty now wants to investigate how much extra it would cost to reduce the expected project duration down to 40 weeks (the deadline for the company earning a bonus of $150,000 for early completion). Therefore, he is ready to address the next of his questions posed at the end of Sec. 22.1.

Question 8:   If extra money is spent to expedite the project, what is the least expensive way of attempting to meet the target completion time (40 weeks)?

Mr. Perty remembers that CPM provides an excellent procedure for using *linear programming* to investigate such *time-cost trade-offs,* so he will use this approach again to address this question.

We begin with some background.

Time-Cost Trade-Offs for Individual Activities

The first key concept for this approach is that of *crashing.*

Crashing an activity refers to taking special costly measures to reduce the duration of an activity below its normal value. These special measures might include using overtime, hiring additional temporary help, using special time-saving materials, obtaining special equipment, etc. **Crashing the project** refers to crashing a number of activities in order to reduce the duration of the project below its normal value.

The **CPM method of time-cost trade-offs** is concerned with determining how much (if any) to crash each of the activities in order to reduce the anticipated duration of the project to a desired value.

The data necessary for determining how much to crash a particular activity are given by the *time-cost graph* for the activity. Figure 22.9 shows a typical time-cost graph. Note the two key points on this graph labeled *Normal* and *Crash.*

The **normal point** on the time-cost graph for an activity shows the time (duration) and cost of the activity when it is performed in the normal way. The **crash point** shows the time and cost when the activity is *fully crashed,* i.e., it is fully expedited with no cost spared to reduce its duration as much as possible. As an approximation, CPM assumes that these times and costs can be reliably predicted without significant uncertainty.

For most applications, it is assumed that *partially crashing* the activity at any level will give a combination of time and cost that will lie somewhere on the line segment between

---

[1]In fact, when simulation is applied in Sec. 28.2 to obtain a better estimate of the probability of meeting this deadline, the estimated probability is found to be only 0.577.

[2]This section also is included (with only slight differences) in Sec. 10.8, and so can be omitted if you have previously studied Sec. 10.8.
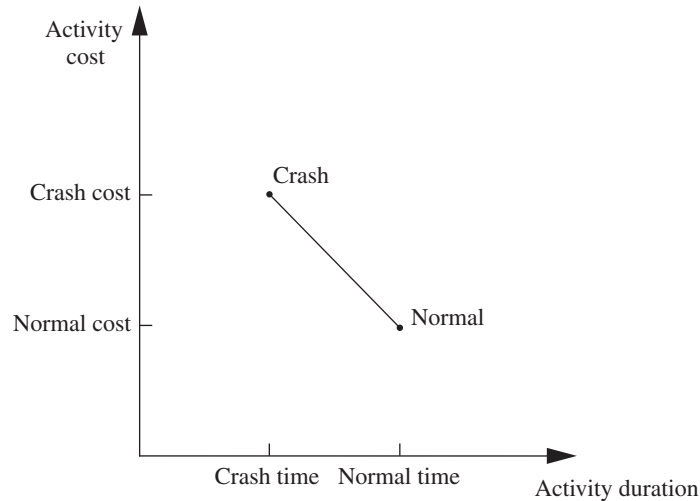
■ FIGURE 22.9
A typical time-cost graph for an activity.

these two points.[1] (For example, this assumption says that *half* of a full crash will give a point on this line segment that is midway between the normal and crash points.) This simplifying approximation reduces the necessary data gathering to estimating the time and cost for just two situations: *normal conditions* (to obtain the normal point) and a *full crash* (to obtain the crash point).

Using this approach, Mr. Perty has his staff and crew supervisors working on developing these data for each of the activities of Reliable's project. For example, the supervisor of the crew responsible for putting up the wallboard indicates that adding two temporary employees and using overtime would enable him to reduce the duration of this activity from 8 weeks to 6 weeks, which is the minimum possible. Mr. Perty's staff then estimates the cost of fully crashing the activity in this way as compared to following the normal 8-week schedule, as shown below:

Activity *J* (put up the wallboard):

Normal point: time = 8 weeks, cost = $430,000.
Crash point: time = 6 weeks, cost = $490,000.
Maximum reduction in time = 8 − 6 = 2 weeks.

$$\text{Crash cost per week saved} \quad = \frac{\$490,000 - \$430,000}{2}$$

$$= \$30,000.$$

After investigating the time-cost trade-off for each of the other activities in the same way, Table 22.7 gives the corresponding data obtained for all the activities.

Which Activities Should Be Crashed?

Summing the *normal cost* and *crash cost* columns of Table 22.7 gives

Sum of normal costs = $4.55 million,
Sum of crash costs   = $6.15 million.

---

[1]This is a convenient assumption, but it often is only a rough approximation since the underlying assumptions of proportionality and divisibility may not hold completely. If, in fact, the true time-cost graph is nonlinear, but also is convex, linear programming can still be employed by using a piecewise linear approximation and then applying the separable programming technique described in Sec. 13.8.

■ **TABLE 22.7** Time-cost trade-off data for the activities of Reliable's project

	Time		Cost		Maximum Reduction in Time	Crash Cost per Week Saved
Activity	Normal	Crash	Normal	Crash		
A	2 weeks	1 week	$180,000	$ 280,000	1 week	$100,000
B	4 weeks	2 weeks	320,000	420,000	2 weeks	50,000
C	10 weeks	7 weeks	620,000	860,000	3 weeks	80,000
D	6 weeks	4 weeks	260,000	340,000	2 weeks	40,000
E	4 weeks	3 weeks	410,000	570,000	1 week	160,000
F	5 weeks	3 weeks	180,000	260,000	2 weeks	40,000
G	7 weeks	4 weeks	900,000	1,020,000	3 weeks	40,000
H	9 weeks	6 weeks	200,000	380,000	3 weeks	60,000
I	7 weeks	5 weeks	210,000	270,000	2 weeks	30,000
J	8 weeks	6 weeks	430,000	490,000	2 weeks	30,000
K	4 weeks	3 weeks	160,000	200,000	1 week	40,000
L	5 weeks	3 weeks	250,000	350,000	2 weeks	50,000
M	2 weeks	1 week	100,000	200,000	1 week	100,000
N	6 weeks	3 weeks	330,000	510,000	3 weeks	60,000

Recall that the company will be paid $5.4 million for doing this project. (This figure excludes the $150,000 bonus for finishing within 40 weeks and the $300,000 penalty for not finishing within 47 weeks.) This payment needs to cover some *overhead costs* in addition to the costs of the activities listed in the table, as well as provide a reasonable profit to the company. When developing the winning bid of $5.4 million, Reliable's management felt that this amount would provide a reasonable profit as long as the total cost of the activities could be held fairly close to the normal level of about $4.55 million. Mr. Perty understands very well that it is his responsibility to keep the project as close to both budget and schedule as possible.

As found previously in Fig. 22.5, if all the activities are performed in the normal way, the anticipated duration of the project would be 44 weeks (if delays can be avoided). If *all* the activities were to be *fully crashed* instead, then a similar calculation would find that this duration would be reduced to only 28 weeks. But look at the prohibitive cost ($6.15 million) of doing this! Fully crashing all activities clearly is not a viable option.

However, Mr. Perty still wants to investigate the possibility of partially or fully crashing just a few activities to reduce the anticipated duration of the project to 40 weeks.

The problem: What is the least expensive way of crashing some activities to reduce the (estimated) project duration to the specified level (40 weeks)?

One way of solving this problem is **marginal cost analysis,** which uses the last column of Table 22.7 (along with Fig. 22.5 in Sec. 22.3) to determine the least expensive way to reduce project duration 1 week at a time. The easiest way to conduct this kind of analysis is to set up a table like Table 22.8 that lists all the paths through the project network and the current length of each of these paths. To get started, this information can be copied directly from Table 22.2.

Since the fourth path listed in Table 22.8 has the longest length (44 weeks), the only way to reduce project duration by a week is to reduce the duration of the activities on this particular path by a week. Comparing the crash cost per week saved given in the last column of Table 22.7 for these activities, the smallest cost is $30,000 for activity *J.* (Note that activity *I* with this same cost is not on this path.) Therefore, the first change is to crash activity *J* enough to reduce its duration by a week.

■ **TABLE 22.8** The initial table for starting marginal cost analysis of Reliable's project

Activity to Crash	Crash Cost	Length of Path					
		ABCDGHM	ABCEHM	ABCEFJKN	ABCEFJLN	ABCIJKN	ABCIJLN
		40	31	43	44	41	42

■ **TABLE 22.9** The final table for performing marginal cost analysis on Reliable's project

Activity to Crash	Crash Cost	Length of Path					
		ABCDGHM	ABCEHM	ABCEFJKN	ABCEFJLN	ABCIJKN	ABCIJLN
		40	31	43	44	41	42
J	$30,000	40	31	42	43	40	41
J	$30,000	40	31	41	42	39	40
F	$40,000	40	31	40	41	39	40
F	$40,000	40	31	39	40	39	40

This change results in reducing the length of each path that includes activity *J* (the third, fourth, fifth, and sixth paths in Table 22.8) by a week, as shown in the second row of Table 22.9. Because the fourth path still is the longest (43 weeks), the same process is repeated to find the least expensive activity to shorten on this path. This again is activity *J*, since the next-to-last column in Table 22.7 indicates that a maximum reduction of 2 weeks is allowed for this activity. This second reduction of a week for activity *J* leads to the third row of Table 22.9.

At this point, the fourth path still is the longest (42 weeks), but activity *J* cannot be shortened any further. Among the other activities on this path, activity *F* now is the least expensive to shorten ($40,000 per week) according to the last column of Table 22.7. Therefore, this activity is shortened by a week to obtain the fourth row of Table 22.9, and then (because a maximum reduction of 2 weeks is allowed) is shortened by another week to obtain the last row of this table.

The longest path (a tie between the first, fourth, and sixth paths) now has the desired length of 40 weeks, so we don't need to do any more crashing. (If we did need to go further, the next step would require looking at the activities on all three paths to find the least expensive way of shortening all three paths by a week.) The total cost of crashing activities *J* and *F* to get down to this project duration of 40 weeks is calculated by adding the costs in the second column of Table 22.9—a total of $140,000. Figure 22.10 shows the resulting project network, where the darker arrows show the critical paths.

Since $140,000 is slightly less than the bonus of $150,000 for finishing within 40 weeks, it might appear that Mr. Perty should proceed with this solution. However, because of uncertainties about activity durations, he concludes that he probably should not crash the project at all. (We will discuss this further at the end of the section.)

Figure 22.10 shows that reducing the durations of activities *F* and *J* to their crash times has led to now having *three* critical paths through the network. The reason is that, as we found earlier from the last row of Table 22.9, the three paths tie for being the longest, each with a length of 40 weeks.

With larger networks, marginal cost analysis can become quite unwieldy. A more efficient procedure would be desirable for large projects. For this reason, the standard CPM
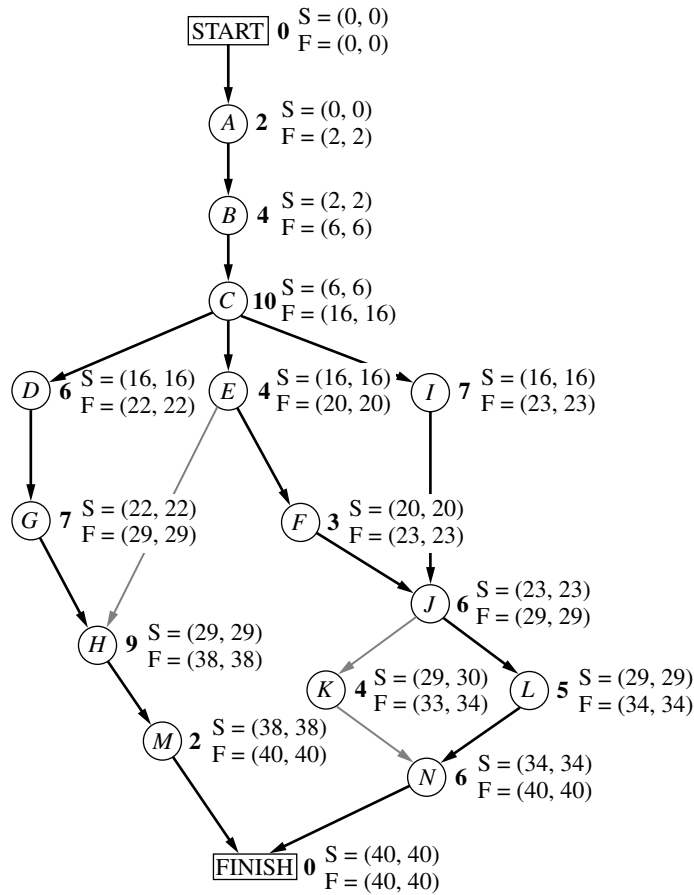
■ FIGURE 22.10
The project network if
activities *J* and *F* are fully
crashed (with all other
activities normal) for
Reliable's project. The darker
arrows show the various
critical paths through the
project network.

procedure is to apply *linear programming* instead (commonly with a customized software
package that exploits the special structure of this network optimization model).

Using Linear Programming to Make Crashing Decisions

The problem of finding the least expensive way of crashing activities can be rephrased in
a form more familiar to linear programming as follows.

Restatement of the problem: Let Z be the total cost of crashing activities. The
problem then is to minimize Z, subject to the constraint that project duration must
be less than or equal to the time desired by the project manager.

The natural decision variables are

x_j = reduction in the duration of activity j due to crashing this activity,
for $j = A, B \ldots , N$.

By using the last column of Table 22.7, the objective function to be minimized then is

$Z = 100{,}000x_A + 50{,}000x_B + \cdots + 60{,}000x_N.$

Each of the 14 decision variables on the right-hand side needs to be restricted to nonnega-
tive values that do not exceed the maximum given in the next-to-last column of Table 22.7.

To impose the constraint that project duration must be less than or equal to the desired value (40 weeks), let

y_{FINISH} = project duration, i.e., the time at which the FINISH node in the project
    network is reached.

The constraint then is

$y_{\text{FINISH}} \leq 40.$

To help the linear programming model assign the appropriate value to y_{FINISH}, given the values of $x_A, x_B, \ldots, x_N$, it is convenient to introduce into the model the following additional variables.

y_j = start time of activity j (for $j = B, C, \ldots, N$), given the values of $x_A, x_B, \ldots, x_N$.

(No such variable is needed for activity A, since an activity that begins the project is automatically assigned a value of 0.) By treating the FINISH node as another activity (albeit one with zero duration), as we now will do, this definition of y_j for activity FINISH also fits the definition of y_{FINISH} given in the preceding paragraph.

The start time of each activity (including FINISH) is directly related to the start time and duration of each of its immediate predecessors as summarized below.

For each activity ($B, C, \ldots, N$, FINISH) and each of its immediate predecessors,
Start time of this activity $\geq$ (start time + duration) for this immediate predecessor.

Furthermore, by using the normal times from Table 22.7, the duration of each activity is given by the following formula:

Duration of activity j = its normal time $- x_j$,

To illustrate these relationships, consider activity F in the project network (Fig. 22.5 or 22.10).

Immediate predecessor of activity F:
Activity E, which has duration $= 4 - x_E$.

Relationship between these activities:

$y_F \geq y_E + 4 - x_E.$

Thus, activity F cannot start until activity E starts and then completes its duration of $4 - x_E$. Now consider activity J, which has two immediate predecessors.

Immediate predecessors of activity J:
Activity F, which has duration $= 5 - x_F$.
Activity I, which has duration $= 7 - x_I$.

Relationships between these activities:

$y_J \geq y_F + 5 - x_F,$
$y_J \geq y_I + 7 - x_I.$

These inequalities together say that activity j cannot start until both of its predecessors finish.

By including these relationships for all the activities as constraints, we obtain the complete linear programming model given below.

Minimize $Z = 100{,}000x_A + 50{,}000x_B + \cdots + 60{,}000x_N,$

subject to the following constraints:

1. Maximum reduction constraints:
Using the next-to-last column of Table 22.7,

$$x_A \le 1, \; x_B \le 2, \ldots, x_N \le 3.$$

2. Nonnegativity constraints:

$$x_A \ge 0, \; x_B \ge 0, \ldots, x_N \ge 0$$
$$y_B \ge 0, \; y_C \ge 0, \ldots, y_N \ge 0, \; y_{\text{FINISH}} \ge 0.$$

3. Start-time constraints:
As described above the objective function, except for activity *A* (which starts the project), there is one such constraint for each activity with a single immediate predecessor (activities *B, C, D, E, F, G, I, K, L, M*) and two constraints for each activity with two immediate predecessors (activities *H, J, N,* FINISH), as listed below.

One immediate predecessor	Two immediate predecessors
$y_B \ge 0 + 2 - x_A$	$y_H \ge y_G + 7 - x_G$
$y_C \ge y_B + 4 - x_B$	$y_H \ge y_E + 4 - x_E$
$y_D \ge y_C + 10 - x_C$	$\vdots$
$\vdots$	$y_{\text{FINISH}} \ge y_M + 2 - x_M$
$y_M \ge y_H + 9 - x_H$	$y_{\text{FINISH}} \ge y_N + 6 - x_N$

(In general, the number of start-time constraints for an activity equals its number of immediate predecessors since each immediate predecessor contributes one start-time constraint.)

4. Project duration constraint:

$$y_{\text{FINISH}} \le 40.$$

Figure 22.11 shows how this problem can be formulated as a linear programming model on a spreadsheet. The decisions to be made are shown in the changing cells, StartTime (I6:I19), TimeReduction (J6:J19), and ProjectFinishTime (I22). Columns B to H correspond to the columns in Table 22.8. As the equations in the bottom half of the figure indicate, columns G and H are calculated in a straightforward way. The equations for column K express the fact that the finish time for each activity is its start time *plus* its normal time *minus* its time reduction due to crashing. The equation entered into the target cell TotalCost (I24) adds all the normal costs plus the extra costs due to crashing to obtain the total cost.

The last set of constraints in the Solver dialogue box, TimeReduction (J6:J19) $\le$ MaxTimeReduction (G6:G19), specifies that the time reduction for each activity cannot exceed its maximum time reduction given in column G. The two preceding constraints, ProjectFinishTime (I22) $\ge$ Mfinish (K18) and ProjectFinishTime (I22) $\ge$ Nfinish (K19), indicate that the project cannot finish until each of the two immediate predecessors (activities *M* and *N*) finish. The constraint that ProjectFinishTime (I22) $\le$ MaxTime (K22) is a key one that specifies that the project must finish within 40 weeks.

The constraints involving StartTime (I6:I19) all are *start-time constraints* that specify that an activity cannot start until each of its immediate predecessors has finished. For example, the first constraint shown, BStart (I7) $\ge$ AFinish (K6), says that activity *B* cannot start until activity *A* (its immediate predecessor) finishes. When an activity has more than one immediate predecessor, there is one such constraint for each of them. To illustrate, activity *H* has both activities *E* and *G* as immediate predecessors. Consequently, activity *H* has two start-time constraints, HStart (I13) $\ge$ EFinish (K10) and HStart (I13) $\ge$ GFinish (K12).

	A	B	C	D	E	F	G	H	I	J	K
1	**Reliable Construction Co. Project Scheduling Problem with Time-Cost Trade-offs**										
2											
3							Maximum	Crash Cost			
4			Time		Cost		Time	per Week	Start	Time	Finish
5		Activity	Normal	Crash	Normal	Crash	Reduction	saved	Time	Reduction	Time
6		A	2	1	$180,000	$280,000	1	$100,000	0	0	2
7		B	4	2	$320,000	$420,000	2	$50,000	2	0	6
8		C	10	7	$620,000	$860,000	3	$80,000	6	0	16
9		D	6	4	$260,000	$340,000	2	$40,000	16	0	22
10		E	4	3	$410,000	$570,000	1	$160,000	16	0	20
11		F	5	3	$180,000	$260,000	2	$40,000	20	2	23
12		G	7	4	$900,000	$1,020,000	3	$40,000	22	0	29
13		H	9	6	$200,000	$380,000	3	$60,000	29	0	38
14		I	7	5	$210,000	$270,000	2	$30,000	16	0	23
15		J	8	6	$430,000	$490,000	2	$30,000	23	2	29
16		K	4	3	$160,000	$200,000	1	$40,000	30	0	34
17		L	5	3	$250,000	$350,000	2	$50,000	29	0	34
18		M	2	1	$100,000	$200,000	1	$100,000	38	0	40
19		N	6	3	$330,000	$510,000	3	$60,000	34	0	40
20											
21											Max Time
22								Project Finish Time	40	<=	40
23											
24								Total Cost	$4,690,000		

Solver Parameters

Set Objective Cell: TotalCost
To: Min
By Changing Variable Cells:
   StartTime, TimeReduction, ProjectFinishTime
Subject to the Constraints:
   BStart >= AFinish   CStart >= BFinish
   DStart >= CFinish   EStart >= CFinish
   FStart >= EFinish   GStart >=DFinish
   HStart >= EFinish   HStart >= GFinish
   IStart >= CFinish   JStart >= FFinish
   JStart >= IFinish   KStart >= JFinish
   LStart >= JFinish   MStart >= HFinish
   NStart >= KFinish   NStart >= LFinish
   ProjectFinishTime <= MaxTime
   ProjectFinishTime >= MFinish
   ProjectFinishTime >= NFinish
   TimeReduction <= MaxTimeReduction
Solver Options:
   Make Variables Nonnegative
   Solving Method: Simplex LP

	Range Name	Cells
	AFinish	K6
	AStart	I6
	BFinish	K7
	BStart	I7
	CFinish	K8
	CrashCost	F6:F19
	CrashCostPerWeekSaved	H6:H19
	CrashTime	D6:D19
	CStart	I8
	DFinish	K9
	DStart	I9
	EFinish	K10
	EStart	I10
	FFinish	K11
	FinishTime	K6:K19
	FStart	I11
	GFinish	K12
	GStart	I12
	HFinish	K13
	HStart	I13
	IFinish	K14
	IStart	I14
	JFinish	K15
	JStart	I15
	KFinish	K16
	KStart	I16
	LFinish	K17
	LStart	I17
	MaxTime	K22
	MaxTimeReduction	G6:G19
	MFinish	K18
	MStart	I18
	NFinish	K19
	NormalCost	E6:E19
	NormalTime	C6:C19
	NStart	I19
	ProjectFinishTime	I22
	StartTime	I6:I19
	TimeReduction	J6:J19
	TotalCost	I24

	G	H
3	Maximum	Crash Cost
4	Time	per Week
5	Reduction	saved
6	=NormalTime-CrashTime	=(CrashCost-NormalCost)/MaxTimeReduction
7	=NormalTime-CrashTime	=(CrashCost-NormalCost)/MaxTimeReduction
8	=NormalTime-CrashTime	=(CrashCost-NormalCost)/MaxTimeReduction
9	=NormalTime-CrashTime	=(CrashCost-NormalCost)/MaxTimeReduction
10	:	:
11	:	:

	K
4	Finish
5	Time
6	=StartTime+NormalTime-TimeReduction
7	=StartTime+NormalTime-TimeReduction
8	=StartTime+NormalTime-TimeReduction
9	=StartTime+NormalTime-TimeReduction
10	:
11	:

	H	I
24	Total Cost	=SUM(NormalCost)+SUMPRODUCT(CrashCostPerWeekSaved,TimeReduction)

FIGURE 22.11

The spreadsheet displays the application of the CPM method of time-cost trade-offs to Reliable's project, where columns I and J show the optimal solution obtained by using Solver with the entries shown in the Solver parameters box.

You may have noticed that the $\geq$ form of the *start-time constraints* allows a delay in starting an activity after all its immediate predecessors have finished. Although such a delay is feasible in the model, it cannot be optimal for any activity on a critical path, since this needless delay would increase the total cost (by necessitating additional crashing to meet the project duration constraint). Therefore, an optimal solution for the model will not have any such delays, except possibly for activities not on a critical path.

Columns I and J in Fig. 22.11 show the optimal solution obtained after having clicked on the Solve button. (Note that this solution involves one delay—activity *K* starts at 30 even though its only immediate predecessor, activity *J,* finishes at 29—but this doesn't matter since activity *K* is not on a critical path.) This solution corresponds to the one displayed in Fig. 22.10 that was obtained by marginal cost analysis.

If you would like to see **another example** that illustrates both the marginal cost analysis approach and the linear programming approach to applying the CPM method of time-cost trade-offs, the Chapter 10 portion of the Solved Examples section of the book's website provides one.

Mr. Perty's Conclusions

Mr. Perty always keeps a sharp eye on the bottom line. Therefore, when his staff brings him the above plan for crashing the project to try to reduce its duration from about 44 weeks to about 40 weeks, he first looks at the estimated total cost of $4.69 million. Since the estimated total cost without any crashing is $4.55 million, the additional cost from the crashing would be about $140,000. This is $10,000 less than the bonus of $150,000 that the company would earn by finishing within 40 weeks.

However, Mr. Perty knows from long experience what we discussed in the preceding section, namely, that there is considerable uncertainty about how much time actually will be needed for each activity and so for the overall project. Recall that the PERT three-estimate approach led to having a *probability distribution* for project duration. Without crashing, this probability distribution has a *mean* of 44 weeks but such a large *variance* that there is even a substantial probability (roughly 0.2) of not even finishing within 47 weeks (which would trigger a penalty of $300,000). With the new crashing plan reducing the mean to 40 weeks, there is as much chance that the actual project duration will turn out to exceed 40 weeks as being within 40 weeks. Why spend an extra $140,000 to obtain a 50 percent chance of earning the bonus of $150,000?

> **Conclusion 1:** The plan for crashing the project only provides a probability of 0.5 of actually finishing the project within 40 weeks, so the extra cost of the plan ($140,000) is not justified. Therefore, Mr. Perty rejects any crashing at this stage.

Mr. Perty does note that the two activities that had been proposed for crashing (*F* and *J*) come about halfway through the project. Therefore, if the project is well ahead of schedule before reaching activity *F,* then implementing the crashing plan almost certainly would enable finishing the project within 40 weeks. Furthermore, Mr. Perty knows that it would be good for the company's reputation (as well as a feather in his own cap) to finish this early.

> **Conclusion 2:** The extra cost of the crashing plan can be justified if it almost certainly would earn the bonus of $150,000 for finishing the project within 40 weeks. Therefore, Mr. Perty will hold the plan in reserve to be implemented if the project is running well ahead of schedule before reaching activity *F.*

Mr. Perty is more concerned about the possibility that the project will run so far behind schedule that the penalty of $300,000 will be incurred for not finishing within 47 weeks. If this becomes likely without crashing, Mr. Perty sees that it probably can be avoided by

crashing activity *J* (at a cost of $30,000 per week saved) and, if necessary, crashing activity *F* as well (at a cost of $40,000 per week saved). This will hold true as long as these activities remain on the critical path (as is likely) after the delays occurred.

> **Conclusion 3:** The extra cost of part or all of the crashing plan can be easily justified if it likely would make the difference in avoiding the penalty of $300,000 for not finishing the project within 47 weeks. Therefore, Mr. Perty will hold the crashing plan in reserve to be partially or wholly implemented if the project is running far behind schedule before reaching activity *F* or activity *J.*

In addition to carefully monitoring the schedule as the project evolves (and making a later decision about any crashing), Mr. Perty will be closely watching the costs to try to keep the project within budget. The next section describes how he plans to do this.

■ 22.6 SCHEDULING AND CONTROLLING PROJECT COSTS

Any good project manager like Mr. Perty carefully plans and monitors both the *time* and *cost* aspects of the project. Both schedule and budget are important.

Sections 22.3 and 22.4 have described how PERT/CPM deals with the *time* aspect in developing a schedule and taking uncertainties in activity or project durations into account. Section 22.5 then placed an equal emphasis on time and cost by describing the CPM method of time-cost trade-offs.

Mr. Perty now is ready to turn his focus to *costs* by addressing the last of his questions posed at the end of Sec. 22.1.

Question 9: How should ongoing costs be monitored to try to keep the project within budget?

Mr. Perty recalls that the PERT/CPM technique known as PERT/Cost is specifically designed for this purpose.

> **PERT/Cost** is a systematic procedure (normally computerized) to help the project manager plan, schedule, and control project costs.

The PERT/Cost procedure begins with the hard work of developing an estimate of the cost of each activity when it is performed in the planned way (including any crashing). At this stage, Mr. Perty does not plan on any crashing, so the estimated costs of the activities in Reliable's project are given in the *normal cost* column of Table 22.7 in the preceding section. These costs then are displayed in the *project budget* shown in Table 22.10. This table also includes the estimated duration of each activity (as already given in Table 22.1 or in Figs. 22.1 to 22.5 or in the *normal time* column of Table 22.7). Dividing the cost of each activity by its duration gives the amount in the rightmost column of Table 22.10.

> **Assumption:** A common assumption when using PERT/Cost is that the costs of performing an activity are incurred at a constant rate throughout its duration. Mr. Perty is making this assumption, so the estimated cost during each week of an activity's duration is given by the rightmost column of Table 22.10.

When applying PERT/Cost to larger projects with numerous activities, it is common to combine each group of related activities into a "work package." Both the project budget and the schedule of project costs (described next) then are developed in terms of these work packages rather than the individual activities. Mr. Perty has chosen not to do this, since his project has only 14 activities.

Scheduling Project Costs

Mr. Perty needs to know how much money is required to cover project expenses week by week. PERT/Cost provides this information by using the rightmost column of Table 22.10 to develop a weekly schedule of expenses when the individual activities begin at their earliest start times. Then, to indicate how much flexibility is available for delaying expenses, PERT/Cost does the same thing when the individual activities begin at their latest start times instead.

To do this, this chapter's Excel files in your OR Courseware includes an Excel template (labeled PERT Cost) for generating a project's schedule of costs for up to 45 time periods. Figure 22.12 shows this Excel template (including the equations entered into its output cells) for the beginning of Reliable's project, based on earliest start times (column E) as first obtained in Fig. 22.3, where columns B, C, and D come directly from Table 22.10. Figure 22.13 jumps ahead to show this same template for weeks 17 to 25. Since activities D, E, and I all have earliest start times of 16 (16 weeks after the commencement of the project), they all start in week 17, while activities F and G commence later during the period shown. Columns W through AE give the weekly cost (in dollars) of each of these activities, as obtained from column F (see Fig. 22.12), for the duration of the activity (given by column C). Row 21 shows the sum of the weekly activity costs for each week.

Row 22 of this template gives the total project cost from week 1 on up to the indicated week. For example, consider week 17. Prior to week 17, activities A, B, and C all have been completed but no other activities have begun, so the total cost for the first 16 weeks (from the third column of Table 22.10) is $180,000 + $320,000 + $620,000 = $1,120,000. Adding the weekly project cost for week 17 then gives $1,120,000 + $175,833 = $1,295,833.

Thus, Fig. 22.13 (and its extension to earlier and later weeks) shows Mr. Perty just how much money he will need to cover each week's expenses, as well as the cumulative amount, assuming the project can stick to the earliest start time schedule.

Next, PERT/Cost uses the same procedure to develop the corresponding information when each activity begins at its *latest* start times instead. These latest start times were first obtained in Fig. 22.4 and are repeated here in column E of Fig. 22.14. The rest of this figure then is generated in the same way as for Fig. 22.13. For example, since activity D has a latest start time of 20 (versus an earliest start time of 16), its weekly cost of $43,333

■ **TABLE 22.10** The project budget for Reliable's project

Activity	Estimated Duration	Estimated Cost	Cost per Week of Its Duration
A	2 weeks	$180,000	$ 90,000
B	4 weeks	320,000	80,000
C	10 weeks	620,000	62,000
D	6 weeks	260,000	43,333
E	4 weeks	410,000	102,500
F	5 weeks	180,000	36,000
G	7 weeks	900,000	128,571
H	9 weeks	200,000	22,222
I	7 weeks	210,000	30,000
J	8 weeks	430,000	53,750
K	4 weeks	160,000	40,000
L	5 weeks	250,00	50,000
M	2 weeks	100,000	50,000
N	6 weeks	330,000	55,000

	A	B	C	D	E	F	G	H	I	J
1		**Template for PERT/Cost**								
2										
3			Estimated							
4			Duration	Estimated	Start	Cost Per Week	Week	Week	Week	Week
5		Activity	(weeks)	Cost	Time	of Its Duration	1	2	3	4
6		A	2	$180,000	0	$90,000	$90,000	$90,000	$0	$0
7		B	4	$320,000	2	$80,000	$0	$0	$80,000	$80,000
8		C	10	$620,000	6	$62,000	$0	$0	$0	$0
9		D	6	$260,000	16	$43,333	$0	$0	$0	$0
10		E	4	$410,000	16	$102,500	$0	$0	$0	$0
11		F	5	$180,000	20	$36,000	$0	$0	$0	$0
12		G	7	$900,000	22	$128,571	$0	$0	$0	$0
13		H	9	$200,000	29	$22,222	$0	$0	$0	$0
14		I	7	$210,000	16	$30,000	$0	$0	$0	$0
15		J	8	$430,000	25	$53,750	$0	$0	$0	$0
16		K	4	$160,000	33	$40,000	$0	$0	$0	$0
17		L	5	$250,000	33	$50,000	$0	$0	$0	$0
18		M	2	$100,000	38	$50,000	$0	$0	$0	$0
19		N	6	$330,000	38	$55,000	$0	$0	$0	$0
20										
21						Weekly Project Cost	$90,000	$90,000	$80,000	$80,000
22						Cumulative Project Cost	$90,000	$180,000	$260,000	$340,000

	F	G	H
4	Cost Per Week	Week	Week
5	of Its Duration	1	2
6	=EstimatedCost/EstimatedDuration	=IF(AND(Week>StartTime,Week<=StartTime+EstimatedDuration),CostPerWeek,0)	…
7	=EstimatedCost/EstimatedDuration	=IF(AND(Week>StartTime,Week<=StartTime+EstimatedDuration),CostPerWeek,0)	…
8	=EstimatedCost/EstimatedDuration	=IF(AND(Week>StartTime,Week<=StartTime+EstimatedDuration),CostPerWeek,0)	…
9	=EstimatedCost/EstimatedDuration	:	
10	=EstimatedCost/EstimatedDuration	:	

	F	G	H	I	J
21	Weekly Project Cost	=SUM(G6:G19)	=SUM(H6:H19)	=SUM(I6:I19)	…
22	Cumulative Project Cost	=G21	=G22+H21	=H22+I21	…

Range Name	Cells
Activity	B6:B19
CostPerWeek	F6:F19
CumulativeProjectCost	G22:AY22
EstimatedCost	D6:D19
EstimatedDuration	C6:C19
StartTime	E6:E19
Week	G5:AY5
WeeklyProjectCost	G21:AY21

FIGURE 22.12

This Excel template in your OR Courseware enables efficient application of the PERT/Cost procedure, as illustrated here for the beginning of Reliable's project when using earliest start times.

	A	B	E	W	X	Y	Z	AA	AB	AC	AD	AE
1		**Template for PERT/Cost**										
2												
3												
4			Start	Week	Week	Week	Week	Week	Week	Week	Week	Week
5		Activity	Time	17	18	19	20	21	22	23	24	25
6		A	0	$0	$0	$0	$0	$0	$0	$0	$0	$0
7		B	2	$0	$0	$0	$0	$0	$0	$0	$0	$0
8		C	6	$0	$0	$0	$0	$0	$0	$0	$0	$0
9		D	16	$43,333	$43,333	$43,333	$43,333	$43,333	$43,333	$0	$0	$0
10		E	16	$102,500	$102,500	$102,500	$102,500	$0	$0	$0	$0	$0
11		F	20	$0	$0	$0	$0	$36,000	$36,000	$36,000	$36,000	$36,000
12		G	22	$0	$0	$0	$0	$0	$0	$128,571	$128,571	$128,571
13		H	29	$0	$0	$0	$0	$0	$0	$0	$0	$0
14		I	16	$30,000	$30,000	$30,000	$30,000	$30,000	$30,000	$30,000	$0	$0
15		J	25	$0	$0	$0	$0	$0	$0	$0	$0	$0
16		K	33	$0	$0	$0	$0	$0	$0	$0	$0	$0
17		L	33	$0	$0	$0	$0	$0	$0	$0	$0	$0
18		M	38	$0	$0	$0	$0	$0	$0	$0	$0	$0
19		N	38	$0	$0	$0	$0	$0	$0	$0	$0	$0
20												
21				$175,833	$175,833	$175,833	$175,833	$109,333	$109,333	$194,571	$164,571	$164,571
22				$1,295,833	$1,471,667	$1,647,500	$1,823,333	$1,932,667	$2,042,000	$2,236,571	$2,401,143	$2,565,714

■ **FIGURE 22.13**

This spreadsheet extends the template in Fig. 22.12 to weeks 17 to 25.

	A	B	C	W	X	Y	Z	AA	AB	AC	AD	AE
1		**Reliable's Late Start Schedule of Costs**										
2												
3			Estimated									
4			Duration	Week	Week	Week	Week	Week	Week	Week	Week	Week
5		Activity	(weeks)	17	18	19	20	21	22	23	24	25
6		A	2	$0	$0	$0	$0	$0	$0	$0	$0	$0
7		B	4	$0	$0	$0	$0	$0	$0	$0	$0	$0
8		C	10	$0	$0	$0	$0	$0	$0	$0	$0	$0
9		D	6	$0	$0	$0	$0	$43,333	$43,333	$43,333	$43,333	$43,333
10		E	4	$102,500	$102,500	$102,500	$102,500	$0	$0	$0	$0	$0
11		F	5	$0	$0	$0	$0	$36,000	$36,000	$36,000	$36,000	$36,000
12		G	7	$0	$0	$0	$0	$0	$0	$0	$0	$0
13		H	9	$0	$0	$0	$0	$0	$0	$0	$0	$0
14		I	7	$0	$0	$30,000	$30,000	$30,000	$30,000	$30,000	$30,000	$30,000
15		J	8	$0	$0	$0	$0	$0	$0	$0	$0	$0
16		K	4	$0	$0	$0	$0	$0	$0	$0	$0	$0
17		L	5	$0	$0	$0	$0	$0	$0	$0	$0	$0
18		M	2	$0	$0	$0	$0	$0	$0	$0	$0	$0
19		N	6	$0	$0	$0	$0	$0	$0	$0	$0	$0
20												
21				$102,500	$102,500	$132,500	$132,500	$109,333	$109,333	$109,333	$109,333	$109,333
22				$1,222,500	$1,325,000	$1,457,500	$1,590,000	$1,699,333	$1,808,667	$1,918,000	$2,027,333	$2,136,667

■ **FIGURE 22.14**
The application of the PERT/Cost procedure to weeks 17 to 25 of Reliable's project when using latest start times.

now begins in week 21 rather than week 17. Similarly, activity *G* has a latest start time of 26, so it has no entries for the weeks considered in this figure.

Figure 22.14 (and its extension to earlier and later weeks) tells Mr. Perty what his weekly and cumulative expenses would be if he postpones each activity as long as possible without delaying project completion (assuming no unexpected delays occur). Comparing row 22 of Figs. 22.13 and 22.14 indicates that fairly substantial *temporary* savings can be achieved by such postponements, which is very helpful if the company is incurring cash shortages. (However, such postponements would only be used reluctantly since they would remove any latitude for avoiding a delay in the completion of the project if any activities incur unexpected delays.)

To better visualize the comparison between row 22 of Figs. 22.13 and 22.14, it is helpful to graph these two rows together over all 44 weeks of the project as shown in Fig. 22.15. Since the earliest start times and latest start times are the same for the first three activities (*A, B, C*), which encompass the first 16 weeks, the cumulative project cost is the same for the two kinds of start times over this period. After week 16, we obtain two distinct cost curves by plotting the values in row 22 of Figs. 22.13 and 22.14 (and their extensions to later weeks). Since sticking to either earliest start times or latest start times leads

■ **FIGURE 22.15**
The schedule of cumulative project costs when all activities begin at their earliest start times (the top cost curve) or at their latest start times (the bottom cost curve).
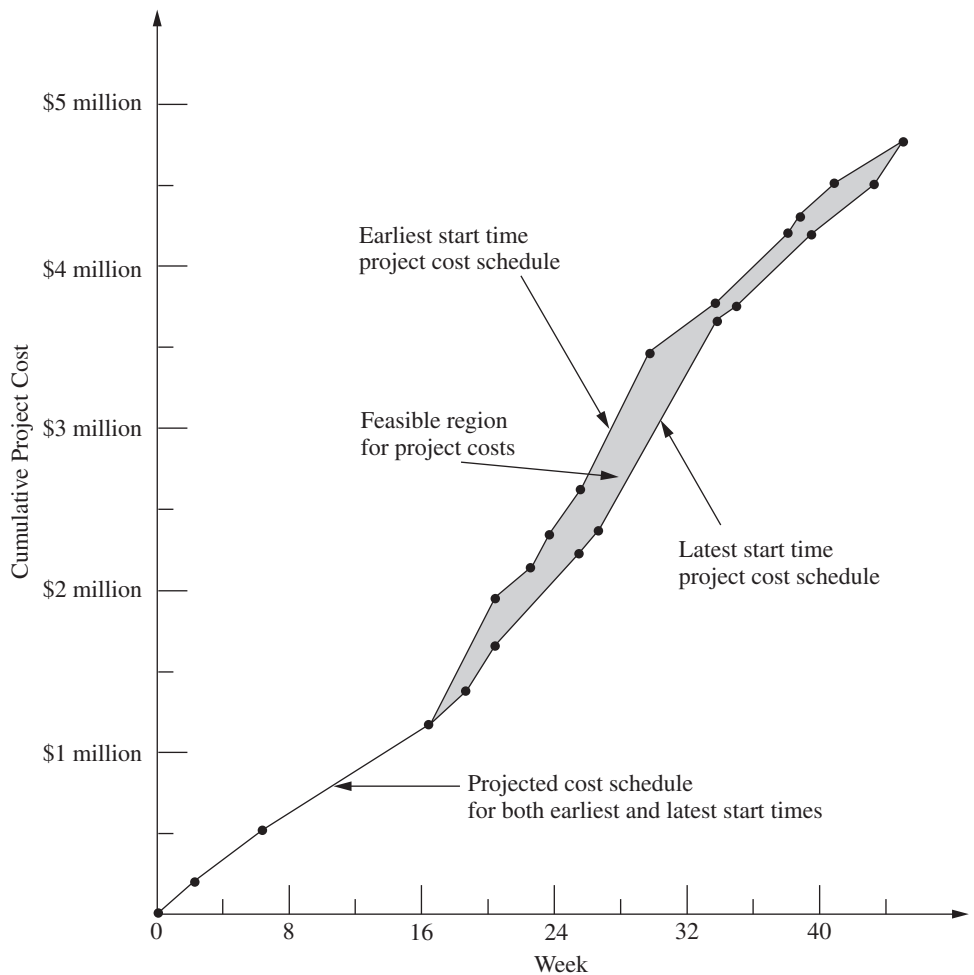
to project completion at the end of 44 weeks, the two cost curves come together again at that point with a total project cost of $4.55 million. The dots on either curve are the points at which the weekly project costs change.

Naturally, the start times and activity costs that lead to Fig. 22.15 are only estimates of what actually will transpire. However, the figure provides a *best forecast* of cumulative project costs week by week when following a work schedule based on either earliest or latest start times. If either of these work schedules is selected, this best forecast then becomes a *budget* to be followed as closely as possible. A budget in the shaded area between the two cost curves also can be obtained by selecting a work schedule that calls for beginning each activity somewhere between its earliest and latest start times. The only *feasible* budgets for scheduling project completion at the end of week 44 (without any crashing) lie in this shaded area or on one of the two cost curves.

Reliable Construction Co. has adequate funds to cover expenses until payments are received. Therefore, Mr. Perty has selected a work schedule based on earliest start times to provide the best chance for prompt completion. (He is still nervous about the significant probability of incurring the penalty of $300,000 for not finishing within 47 weeks.) Consequently, his budget is provided by the top cost curve in Fig. 22.15.

Controlling Project Costs

Once the project is under way, Mr. Perty will need to carefully monitor actual costs and take corrective action as needed to avoid serious cost overruns. One important way of monitoring costs is to compare actual costs to date with his budget provided by the top curve in Fig. 22.15.

However, since deviations from the planned work schedule may occur, this method of monitoring costs is not adequate by itself. For example, suppose that individual activities have been costing more than budgeted, but delays have prevented some activities from beginning when scheduled. These delays might cause the total cost to date to be less than the budgeted cumulative project cost, thereby giving the illusion that project costs are well under control. Furthermore, regardless of whether the cost performance of the project as a whole seems satisfactory, Mr. Perty needs information about the cost performance of individual activities in order to identify trouble spots where corrective action is needed.

Therefore, PERT/Cost periodically generates a report that focuses on the cost performance of the individual activities. To illustrate, Table 22.11 shows the report that Mr. Perty received after the completion of week 22 (halfway through the project schedule). The first column lists the activities that have at least begun by this time. The next column gives the budgeted total cost of each activity (as given previously in the third column of Table 22.10). The third column indicates what percentage of the activity now has been completed.

■ **TABLE 22.11** PERT/Cost report after week 22 of Reliable's project

Activity	Budgeted Cost	Percent Completed	Value Completed	Actual Cost to Date	Cost Overrun to Date
A	$ 180,000	100%	$ 180,000	$ 200,000	$20,000
B	320,000	100	320,000	330,000	10,000
C	620,000	100	620,000	600,000	− 20,000
D	260,000	75	195,000	200,000	5,000
E	410,000	100	410,000	400,000	− 10,000
F	180,000	25	45,000	60,000	15,000
I	210,000	50	105,000	130,000	25,000
Total	$2,180,000		$1,875,000	$1,920,000	$45,000

Multiplying the second and third columns then gives the fourth column, which thereby represents the budgeted value of the work completed on the activity.

The fourth column is the one that Mr. Perty wants to compare to the *actual cost* to date given in the fifth column. Subtracting the fourth column from the fifth gives the *cost overrun* to date of each activity, as shown in the rightmost column. (A negative number in the cost overrun column indicates a *cost underrun.*)

Mr. Perty pays special attention in the report to the activities that are not yet completed, since these are the ones that he can still affect. (He used earlier reports to monitor activities *A, B, C,* and *E* while they were under way, which led to meeting the total budget for these four activities.) Activity *D* is barely over budget (less than 3 percent), but Mr. Perty is very concerned about the large cost overruns to date for activities *F* and *I.* Therefore, he next will investigate these two activities and work with the supervisors involved to improve their cost performances.

Note in the bottom row of Table 22.11 that the cumulative project cost after week 22 is $1.92 million. This is considerably less than Mr. Perty's *budgeted* cumulative project cost of $2.042 million given in cell AB22 of Fig. 22.13. Without any further information, this comparison would suggest an excellent cost performance for the project so far. However, the real reason for being under budget is that the current activities all are behind schedule and so have not yet incurred some expenses that had been scheduled to occur earlier. Fortunately, the PERT/Cost report provides valuable additional information that paints a truer picture of cost performance to date. By focusing on individual activities rather than the overall project, the report identifies the current trouble spots (activities *F* and *I*) that require Mr. Perty's immediate attention. Thus, the report enables him to take corrective action while there is still time to reverse these cost overruns.

■ 22.7   AN EVALUATION OF PERT/CPM

PERT/CPM has stood the test of time. Despite being over 60 years old, it continues to be one of the most widely used OR techniques. It is a standard tool of project managers.

The Value of PERT/CPM

Much of the value of PERT/CPM derives from the basic framework it provides for planning a project. Recall its planning steps: (1) Identify the activities that are needed to carry out the project. (2) Estimate how much time will be needed for each activity. (3) Determine the activities that must immediately precede each activity. (4) Develop the project network that visually displays the relationships between the activities. The discipline of going through these steps forces the needed planning to be done.

The scheduling information generated by PERT/CPM also is vital to the project manager. When can each activity begin if there are no delays? How much delay in an activity can be tolerated without delaying project completion? What is the critical path of activities where no delay can be tolerated? What is the effect of uncertainty in activity times? What is the probability of meeting the project deadline under the current plan? PERT/CPM provides the answers.

PERT/CPM also assists the project manager in other ways. Schedule and budget are key concerns. The CPM method of time-cost trade-offs enables investigating ways of reducing the duration of the project at an additional cost. PERT/Cost provides a systematic procedure for planning, scheduling, and controlling project costs.

In many ways, PERT/CPM exemplifies the application of OR at its finest. Its modeling approach focuses on the key features of the problem (activities, precedence

relationships, time, and cost) without getting mired down in unimportant details. The resulting model (a project network and an optional linear programming formulation) are easy to understand and apply. It addresses the issues that are important to management (planning, scheduling, dealing with uncertainty, time-cost trade-offs, and controlling costs). It assists the project manager in dealing with these issues in useful ways and in a timely manner.

Using the Computer

PERT/CPM continues to evolve to meet new needs. At its inception in the late 1950s, it was largely executed manually. The project network sometimes was spread out over the walls of the project manager. Recording changes in the plan became a major task. Communicating changes to crew supervisors and subcontractors was cumbersome. The computer has changed all of that.

For many years now, PERT/CPM has become highly computerized. There has been a remarkable growth in the number and power of software packages for PERT/CPM that run on personal computers or workstations. *Project management software* (for example, Microsoft Project) now is a standard tool for project managers. This has enabled applications to numerous projects that each involve many millions of dollars and perhaps even thousands of activities. Possible revisions in the project plan now can be investigated almost instantaneously. Actual changes and the resulting updates in the schedule, etc., are recorded virtually effortlessly. Communications to all parties involved through computer networks and telecommunication systems also have become quick and easy.

Nevertheless, PERT/CPM still is not a panacea. It has certain major deficiencies for some applications. We briefly describe each of these deficiencies below along with how it is being addressed through research on improvements or extensions to PERT/CPM.

Approximating the Means and Variances of Activity Durations

The PERT three-estimate approach described in Sec. 22.4 provides a straightforward procedure for approximating the mean and variance of the probability distribution of the duration of each activity. Recall that this approach involved obtaining a most likely estimate, an optimistic estimate, and a pessimistic estimate of the duration. Given these three estimates, simple formulas were given for approximating the mean and variance. The means and variances for the various activities then were used to estimate the probability of completing the project by a specified time.

Unfortunately, considerable subsequent research has shown that this approach tends to provide a pretty rough approximation of the mean and variance. Part of the difficulty lies in aiming the optimistic and pessimistic estimates at the *endpoints* of the probability distribution. These endpoints correspond to very rare events (the best and worst that could ever occur) that typically are outside the estimator's realm of experience. The accuracy and reliability of such estimates are not as good as for points that are not at the extremes of the probability distribution. For example, research has demonstrated that much better estimates can be obtained by aiming them at the 10 and 90 percent points of the probability distribution. The optimistic and pessimistic estimates then would be described in terms of having 1 chance in 10 of doing better or 1 chance in 10 of doing worse. The middle estimate also can be improved by aiming it at the 50 percent point (the median value) of the probability distribution.

Revising the definitions of the three estimates along these lines leads to considerably more complicated formulas for the mean and variance of the duration of an activity. However, this is no problem since the analysis is computerized anyway. The important

consideration is that much better approximations of the mean and variance are obtained in this way.[1]

Approximating the Probability of Meeting the Deadline

Of all the assumptions and simplifying approximations made by PERT/CPM, one is particularly controversial. This is Simplifying Approximation 1 in Sec. 22.4, which assumes that the *mean critical path* will turn out to be the longest path through the project network. This approximation greatly simplifies the calculation of the approximate probability of completing the project by a specified deadline. Unfortunately, in reality, there usually is a significant chance, and sometimes a very substantial chance, that some other path or paths will turn out to be longer than the mean critical path. Consequently, the calculated probability of meeting the deadline usually overstates the true probability somewhat. PERT/CPM provides no information on the likely size of the error. (Research has found that the error often is modest, but can be very large.) Thus, the project manager who relies on the calculated probability can be badly misled.

Considerable research has been conducted to develop more accurate (albeit more complicated) analytical approximations of this probability. Of special interest are methods that provide both upper and lower bounds on the probability.[2]

Another alternative is to use the technique of simulation described in Chap. 20 to approximate this probability. This appears to be the most commonly used method in practice (when any is used) to improve upon the PERT/CPM approximation. We describe in Sec. 28.2 how this would be done for the Reliable Construction Co. project.

Dealing with Overlapping Activities

Another key assumption of PERT/CPM is that an activity cannot begin until all its immediate predecessors are completely finished. Although this may appear to be a perfectly reasonable assumption, it too is sometimes only a rough approximation of reality.

For example, in the Reliable Construction Co. project, consider activity *H* (do the exterior painting) and its immediate predecessor, activity *G* (put up the exterior siding). Naturally, this painting cannot begin until the exterior siding is there on which to paint. However, it certainly is possible to begin painting on one wall while the exterior siding still is being put up to form the other walls. Thus, activity *H* actually can begin before activity *G* is completely finished. Although careful coordination is needed, this possibility to overlap activities can significantly reduce project duration below that predicted by PERT/CPM.

The **precedence diagramming method (PDM)** has been developed as an extension of PERT/CPM to deal with such overlapping activities.[3] PDM provides four options for the relationship between an activity and any one of its immediate predecessors:

Option 1: The activity cannot begin until the immediate predecessor has been in progress a certain amount of time.

Option 2: The activity cannot finish until a certain amount of time after the immediate predecessor has finished.

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[1]For further information, see, for example, D. L. Keefer and W. A. Verdini, "Better Estimation of PERT Activity Time Parameters," *Management Science,* **39:** 1086–1091, Sept. 1993. Also see A. H.-L. Lau, H.-S. Lau, and Y. Zhang, "A Simple and Logical Alternative for Making PERT Time Estimates," *IIE Transactions,* **28:** 183–192, March 1996, R. H. Pleguezuelo, J. G. Pérez, and S. C. Ramband, "Note on the Reasonableness of PERT Hypotheses," *Operations Research Letters,* **31:** 60–62, Jan. 2003, and S. Koltz and J. R. van Dorp, "A Novel Method for Fitting Unimodal Continouous Distributions on a Bounded Domain Utilizing Expert Judgment Estimates, *IIE Transactions,* **38:** 421–436, May 2006.

[2]See, for example, J. Kamburowski, "Bounding the Distribution of Project Duration in PERT Networks," *Operations Research Letters,* **12:** 17–22, July 1992. Also see T. Iida, "Computing Bounds on Project Duration Distributions for Stochastic PERT Networks," *Naval Research Logistics,* **47:** 559–580, Oct. 2000.

[3]See Selected Reference 1 for further information about PDM.

Option 3: The activity cannot finish until a certain amount of time after the immediate predecessor has started.

Option 4: The activity cannot begin until a certain amount of time after the immediate predecessor has finished. (Rather than overlapping the activities, note that this option creates a lag between them such as, for example, waiting for the paint to dry before beginning the activity that follows painting.)

Alternatively, the *certain amount of time* mentioned in each option also can be expressed as a certain percentage of the work content of the immediate predecessor.

After incorporating these options, PDM can be used much like PERT/CPM to determine earliest start times, latest start times, and the critical path and to investigate time-cost trade-offs, etc.

Although it adds considerable flexibility to PERT/CPM, PDM is neither as well known nor as widely used as PERT/CPM. This should gradually change.

Incorporating the Allocation of Resources to Activities

PERT/CPM assumes that each activity has available all the resources (money, personnel, equipment, etc.) needed to perform the activity in the normal way (or on a crashed basis). In actuality, many projects have only limited resources for which the activities must compete. A major challenge in planning the project then is to determine how the resources should be allocated to the activities.

Once the resources have been allocated, PERT/CPM can be applied in the usual way. However, it would be far better to combine the allocation of the resources with the kind of planning and scheduling done by PERT/CPM so as to strive simultaneously toward a desired objective. For example, a common objective is to allocate the resources so as to minimize the duration of the project.

Much research has been conducted (and is continuing) to develop the methodology for simultaneously allocating resources and scheduling the activities of a project. This subject is beyond the scope of this book, but considerable reading is available elsewhere.[1]

The Future

Despite its deficiencies, PERT/CPM undoubtedly will continue to be widely used for the foreseeable future. It provides the project manager with most of what he or she wants: structure, scheduling information, tools for controlling schedule (latest start times, slacks, the critical path, etc.) and controlling costs (PERT/Cost), as well as the flexibility to investigate time-cost trade-offs.

Even though some of the approximations involved with the PERT three-estimate approach are questionable, these inaccurances ultimately may not be too important. Just the process of developing estimates of the duration of activities encourages effective interaction between the project manager and subordinates that leads to setting mutual goals for start times, activity durations, project duration, etc. Striving together toward these goals may make them self-fulfilling prophecies despite inaccuracies in the underlying mathematics that led to these goals.

Similarly, possibilities for a modest amount of overlapping of activities need not invalidate a schedule by PERT/CPM, despite its assumption that no overlapping can occur.

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[1]See, for example, Selected Reference 1. Also see L. Özdamar and G. Ulusay, "A Survey on the Resource-Constrained Project Scheduling Problem," *IIE Transactions,* **27:** 574–586, Oct. 1995 and G. Zhu, J. F. Bard, and G. Yu, "A Branch-and-Cut Procedure for the Multimode Resource-Constrained Project Scheduling Problem," *INFORMS Journal on Computing,* **18:** 377–390, Summer 2006, as well as Selected References 2, 3, 4, and 5.

Actually having a small amount of overlapping may just provide the slack needed to compensate for the "unexpected" delays that inevitably seem to slip into a schedule.

Even when needing to allocate resources to activities, just using common sense in this allocation and then applying PERT/CPM should be quite satisfactory for some projects.

Nevertheless, it is unfortunate that the kinds of improvements and extensions to PERT/CPM described in this section have not been incorporated much into practice to date. Old comfortable methods that have proved their value are not readily discarded, and it takes awhile to learn about and gain confidence in new, better methods. However, we anticipate that these improvements and extensions gradually will come into more widespread use as they prove their value as well. We also expect that the recent and current extensive research on techniques for project management and scheduling (much of it in Europe) will continue and will lead to further improvements in the future.

■ 22.8   CONCLUSIONS

Ever since their inception in the late 1950s, PERT and CPM have been used extensively to assist project managers in planning, scheduling, and controlling their projects. Over time, these two techniques gradually have merged.

The application of PERT/CPM begins by breaking the project down into its individual activities, identifying the immediate predecessors of each activity, and estimating the duration of each activity. A project network then is constructed to visually display all this information. The type of network that is becoming increasingly popular for this purpose is the activity-on-node (AON) project network, where each activity is represented by a node.

PERT/CPM generates a great deal of useful scheduling information for the project manager, including the earliest start time, the latest start time, and the slack for each activity. It also identifies the critical path of activities such that any delay along this path will delay project completion. Since the critical path is the longest path through the project network, its length determines the duration of the project, assuming all activities remain on schedule.

However, it is difficult for all activities to remain on schedule because there frequently is considerable uncertainty about what the duration of an activity will turn out to be. The PERT three-estimate approach addresses this situation by obtaining three different kinds of estimates (most likely, optimistic, and pessimistic) for the duration of each activity. This information is used to approximate the mean and variance of the probability distribution of this duration. It then is possible to approximate the probability that the project will be completed by the deadline.

The CPM method of time-cost trade-offs enables the project manager to investigate the effect on total cost of changing the estimated duration of the project to various alternative values. The data needed for this activity are the time and cost for each activity when it is done in the normal way and then when it is fully crashed (expedited). Either marginal cost analysis or linear programming can be used to determine how much (if any) to crash each activity in order to minimize the total cost of meeting any specified deadline for the project.

The PERT/CPM technique called PERT/Cost provides the project manager with a systematic procedure for planning, scheduling, and controlling project costs. It generates a complete schedule for what the project costs should be in each time period when activities begin at either their earliest start times or latest start times. It also generates periodic reports that evaluate the cost performance of the individual activities, including identifying those where cost overruns are occurring.

PERT/CPM does have some important deficiencies. These include questionable approximations made when estimating the mean and variance of activity durations as well as when estimating the probability that the project will be completed by the deadline.

Another deficiency is that it does not allow an activity to begin until all its immediate predecessors are completely finished, even though some overlap is sometimes possible. In addition, PERT/CPM does not address the important issue of how to allocate limited resources to the various activities.

Nevertheless, PERT/CPM has stood the test of time in providing project managers with most of the help they want. Furthermore, much progress is being made in developing improvements and extensions to PERT/CPM (such as the precedence diagramming method for dealing with overlapping activities) that addresses these deficiencies.

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■ LEARNING AIDS FOR THIS CHAPTER ON THIS WEBSITE

"Ch. 22—Project Management" Files:

Excel Files
LINGO/LINDO File
MPL/CPLEX File

Excel Templates in Excel Files:

Template for PERT Three-Estimate Approach (labeled PERT)
Template for PERT/Cost (labeled PERT Cost)

An Excel Add-in:

Analytic Solver Platform for Education (ASPE)

See Appendix 1 for documentation of the software.

■ PROBLEMS

The symbols to the left of some of the problems (or their parts) have the following meaning:

T: The corresponding template listed above may be helpful.

C: Use the computer with any of the software options available to you (or as instructed by your instructor) to solve the problem.

22.2-1. Christine Phillips is in charge of planning and coordinating next spring's sales management training program for her company. Christine has listed the following activity information for this project:

Activity	Activity Description	Immediate Predecessors	Estimated Duration
A	Select location	—	2 weeks
B	Obtain speakers	—	3 weeks
C	Make speaker travel plans	A, B	2 weeks
D	Prepare and mail brochure	A, B	2 weeks
E	Take reservations	D	3 weeks

Construct the project network for this project.

22.2-2. Reconsider Prob. 22.2-1. Christine has done more detailed planning for this project and so now has the following expanded activity list:

Activity	Activity Description	Immediate Predecessors	Estimated Duration
A	Select location	—	2 weeks
B	Obtain keynote speaker	—	1 week
C	Obtain other speakers	B	2 weeks
D	Make travel plans for keynote speaker	A, B	2 weeks
E	Make travel plans for other speakers	A, C	3 weeks
F	Make food arrangements	A	2 weeks
G	Negotiate hotel rates	A	1 week
H	Prepare brochure	C, G	1 week
I	Mail brochure	H	1 week
J	Take reservations	I	3 weeks
K	Prepare handouts	C, F	4 weeks

Construct the new project network.

22.2-3. Construct the project network for a project with the following activity list.

Activity	Immediate Predecessors	Estimated Duration
A	—	1 month
B	A	2 months
C	B	4 months
D	B	3 months
E	B	2 months
F	C	3 months
G	D, E	5 months
H	F	1 month
I	G, H	4 months
J	I	2 months
K	I	3 months
L	J	3 months
M	K	5 months
N	L	4 months

22.3-1. You and several friends are about to prepare a lasagna dinner. The tasks to be performed, their immediate predecessors, and their estimated durations are as follows:

Task	Task Description	Tasks that Must Precede	Time
A	Buy the mozzarella cheese*		30 minutes
B	Slice the mozzarella	A	5 minutes
C	Beat 2 eggs		2 minutes
D	Mix eggs and ricotta cheese	C	3 minutes
E	Cut up onions and mushrooms		7 minutes
F	Cook the tomato sauce	E	25 minutes
G	Boil large quantity of water		15 minutes
H	Boil the lasagna noodles	G	10 minutes
I	Drain the lasagna noodles	H	2 minutes
J	Assemble all the ingredients	I, F, D, B	10 minutes
K	Preheat the oven		15 minutes
L	Bake the lasagna	J, K	30 minutes

*There is none in the refrigerator.

(a) Construct the project network for preparing this dinner.
(b) Find all the paths and path lengths through this project network. Which of these paths is a critical path?
(c) Find the earliest start time and earliest finish time for each activity.
(d) Find the latest start time and latest finish time for each activity.
(e) Find the slack for each activity. Which of the paths is a critical path?
(f) Because of a phone call, you were interrupted for 6 minutes when you should have been cutting the onions and mushrooms. By how much will the dinner be delayed? If you use your food processor, which reduces the cutting time from 7 to 2 minutes, will the dinner still be delayed?

22.3-2. Consider Christine Phillip's project involving planning and coordinating next spring's sales management training program for her company as described in Prob. 22.2-1. After constructing the project network, she now is ready for the following steps.
(a) Find all the paths and path lengths through this project network. Which of these paths is a critical path?
(b) Find the earliest times, latest times, and slack for each activity. Use this information to determine which of the paths is a critical path.
(c) It is now one week later, and Christine is ahead of schedule. She has already selected a location for the sales meeting, and all the other activities are right on schedule. Will this shorten the length of the project? Why or why not?

22.3-3. Refer to the activity list given in Prob. 22.2-2 as Christine Phillips does more detailed planning for next spring's sales management training program for her company. After constructing the project network, she now is ready for the following steps.
(a) Find all the paths and path lengths through this project network. Which of these paths is a critical path?
(b) Find the earliest times, latest times, and slack for each activity. Use this information to determine which of the paths is a critical path.
(c) It is now one week later, and Christine is ahead of schedule. She has already selected a location for the sales meeting, and

all the other activities are right on schedule. Will this shorten the length of the project? Why or why not?

22.3-4. Ken Johnston, the data processing manager for Stanley Morgan Bank, is planning a project to install a new management information system. He now is ready to start the project, and wishes to finish in 20 weeks. After identifying the 14 separate activities needed to carry out this project, as well as their precedence relationships and estimated durations (in weeks), Ken has constructed the following project network:
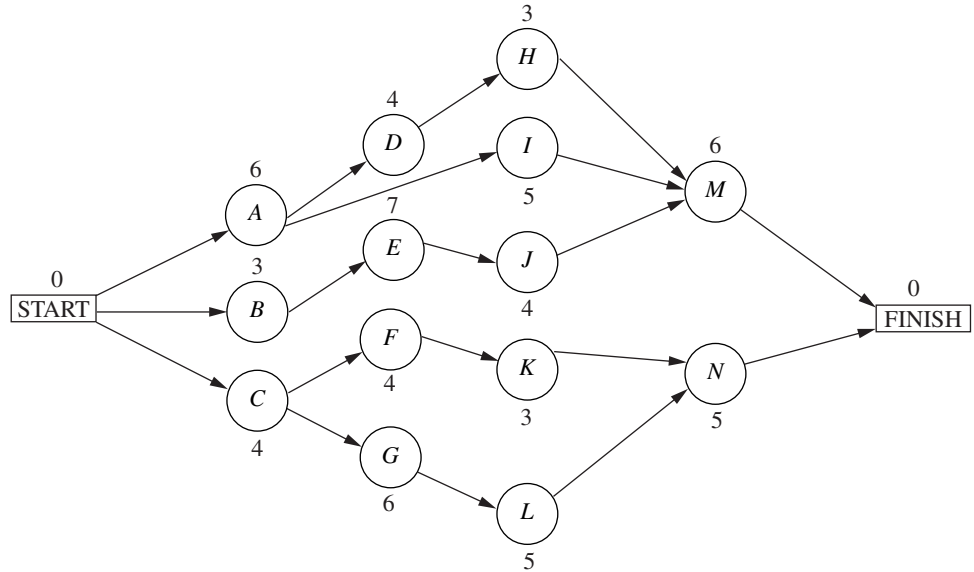
(a) Find all the paths and path lengths through this project network. Which of these paths is a critical path?

(b) Find the earliest times, latest times, and slack for each activity. Will Ken be able to meet his deadline if no delays occur?

(c) Use the information from part (b) to determine which of the paths is a critical path. What does this tell Ken about which activities he should focus most of his attention on for staying on schedule?

(d) Use the information from part (b) to determine what the duration of the project would be if the only delay is that activity I takes 2 extra weeks. What if the only delay is that activity H takes 2 extra weeks? What if the only delay is that activity J takes 2 extra weeks?

22.3-5. You are given the following information about a project consisting of six activities:

Activity	Immediate Predecessors	Estimated Duration
A	—	5 months
B	—	1 month
C	B	2 months
D	A, C	4 months
E	A	6 months
F	D, E	3 months

(a) Construct the project network for this project.

(b) Find the earliest times, latest times, and slack for each activity. Which of the paths is a critical path?

(c) If all other activities take the estimated amount of time, what is the maximum duration of activity D without delaying the completion of the project?

22.3-6. Reconsider the Reliable Construction Co. project introduced in Sec. 22.1, including the complete project network obtained in Fig. 22.5 at the end of Sec. 22.3. Note that the estimated durations of the activities in this figure turn out to be the same as the mean durations given in Table 22.4 (Sec. 22.4) when using the PERT three-estimate approach.

Now suppose that the *pessimistic* estimates in Table 22.4 are used instead to provide the estimated durations in Fig. 22.5. Find the new earliest times, latest times, and slacks for all the activities in this project network. Also identify the critical path and the total estimated duration of the project. (Table 22.5 provides some clues.)

22.3-7. Follow the instructions for Prob. 22.3-6 except use the *optimistic* estimates in Table 22.4 instead.

22.3-8. Follow the instructions for Prob. 22.3-6 except use the *crash times* given in Table 22.7 (Sec. 22.5) instead.

22.4-1. Using the PERT three-estimate approach, the three estimates for one of the activities are as follows: optimistic estimate = 30 days, most likely estimate = 36 days, pessimistic estimate = 48 days. What are the resulting estimates of the mean and variance of the duration of the activity?

22.4-2. Alfred Lowenstein is the president of the research division for Better Health, Inc., a major pharmaceutical company. His most important project coming up is the development of a new drug to

combat AIDS. He has identified 10 groups in his division which will need to carry out different phases of this research and development project. Referring to the work to be done by the respective groups as activities *A, B, . . . , J,* the precedence relationships for when these groups need to do their work are shown in the following project network.
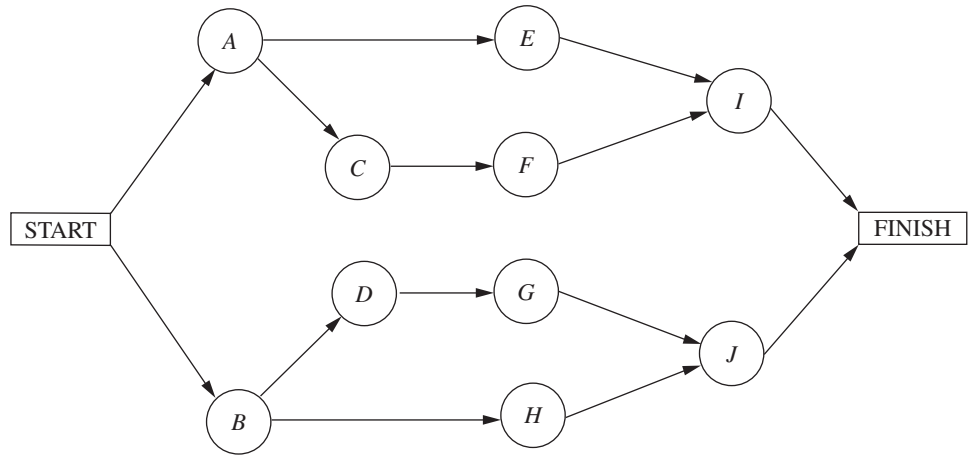
To beat the competition, Better Health's CEO has informed Alfred that he wants the drug ready within 22 months if possible.

Alfred knows very well that there is considerable uncertainty about how long each group will need to do its work. Using the PERT three-estimate approach, the manager of each group has provided a most likely estimate, an optimistic estimate, and a pessimistic estimate of the duration of that group's activity. Using PERT formulas, these estimates now have been converted into estimates of the mean and variance of the probability distribution of the duration of each group's activity, as given in the following table (after rounding to the nearest integer).

Activity	Duration	
	Estimated Mean	**Estimated Variance**
A	4 months	5 months
B	6 months	10 months
C	4 months	8 months
D	3 months	6 months
E	8 months	12 months
F	4 months	6 months
G	3 months	5 months
H	7 months	14 months
I	5 months	8 months
J	5 months	7 months

T **(a)** Find the mean critical path for this project.

T **(b)** Use this mean critical path to find the approximate probability that the project will be completed within 22 months.

T **(c)** Now consider the other three paths through this project network. For each of these paths, find the approximate probability that the path will be completed within 22 months.

(d) What should Alfred tell his CEO about the likelihood that the drug will be ready within 22 months?

T **22.4-3.** Reconsider Prob. 22.4-2. For each of the 10 activities, here are the three estimates that led to the estimates of the mean and variance of the duration of the activity (rounded to the nearest integer) given in the table for Prob. 22.4-2.

Activity	Optimistic Estimate	Most Likely Estimate	Pessimistic Estimate
A	1.5 months	2 months	15 months
B	2 months	3.5 months	21 months
C	1 month	1.5 months	18 months
D	0.5 month	1 month	15 months
E	3 months	5 months	24 months
F	1 month	2 months	16 months
G	0.5 month	1 month	14 months
H	2.5 months	3.5 months	25 months
I	1 month	3 months	18 months
J	2 months	3 months	18 months

(Note how the great uncertainty in the duration of these research activities causes each pessimistic estimate to be several times larger than either the optimistic estimate or the most likely estimate.)

Now use the Excel template in your OR Courseware (as depicted in Fig. 22.8) to help you carry out the instructions for Prob. 22.4-2. In particular, enter the three estimates for each activity, and the template immediately will display the estimates of the means and variances of the activity durations. After indicating each path of interest, the template also will display the approximate probability that the path will be completed within 22 months.

22.4-4. Bill Fredlund, president of Lincoln Log Construction, is considering placing a bid on a building project. Bill has determined that five tasks would need to be performed to carry out the project. Using the PERT three-estimate approach, Bill has obtained the estimates in the next table for how long these tasks will take. Also shown are the precedence relationships for these tasks.

| | Time Required | | | |
Task	Optimistic Estimate	Most Likely Estimate	Pessimistic Estimate	Immediate Predecessors
A	3 weeks	4 weeks	5 weeks	—
B	2 weeks	2 weeks	2 weeks	A
C	3 weeks	5 weeks	6 weeks	B
D	1 week	3 weeks	5 weeks	A
E	2 weeks	3 weeks	5 weeks	B, D

There is a penalty of $500,000 if the project is not completed in 11 weeks. Therefore, Bill is very interested in how likely it is that his company could finish the project in time.

(a) Construct the project network for this project.

T (b) Find the estimate of the mean and variance of the duration of each activity.

(c) Find the mean critical path.

T (d) Find the approximate probability of completing the project within 11 weeks.

(e) Bill has concluded that the bid he would need to make to have a realistic chance of winning the contract would earn Lincoln Log Construction a profit of about $250,000 if the project is completed within 11 weeks. However, because of the penalty for missing this deadline, his company would lose about $250,000 if the project takes more than 11 weeks. Therefore, he wants to place the bid only if he has at least a 50 percent chance of meeting the deadline. How would you advise him?

22.4-5. Sharon Lowe, vice president for marketing for the Electronic Toys Company, is about to begin a project to design an advertising campaign for a new line of toys. She wants the project completed within 57 days in time to launch the advertising campaign at the beginning of the Christmas season.

Sharon has identified the six activities (labeled A, B, . . . , F) needed to execute this project. Considering the order in which these activities need to occur, she also has constructed the following project network.
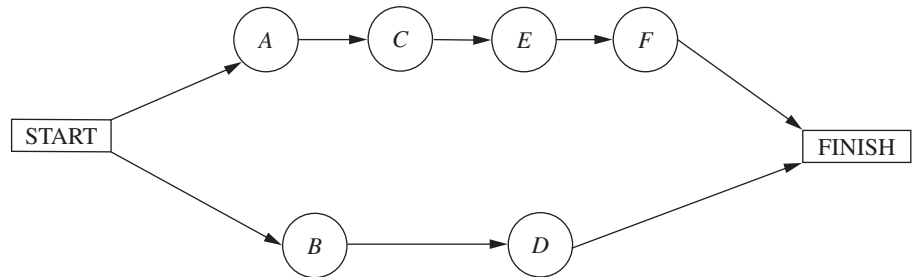
Using the PERT three-estimate approach, Sharon has obtained the following estimates of the duration of each activity.
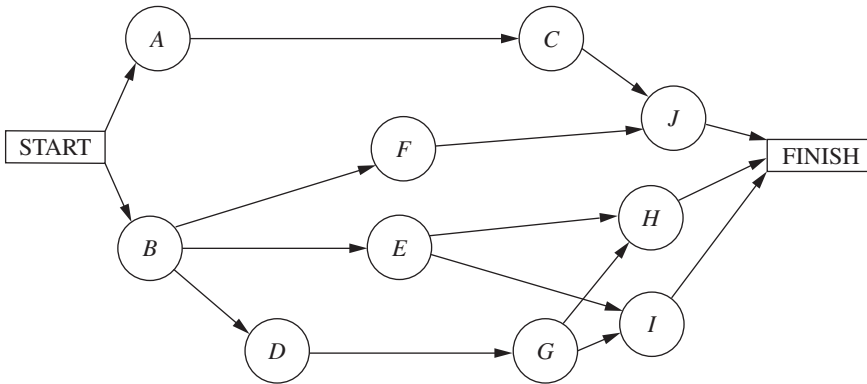
Activity	Optimistic Estimate	Most Likely Estimate	Pessimistic Estimate
A	12 days	12 days	12 days
B	15 days	21 days	39 days
C	12 days	15 days	18 days
D	18 days	27 days	36 days
E	12 days	18 days	24 days
F	2 days	5 days	14 days

T (a) Find the estimate of the mean and variance of the duration of each activity.

(b) Find the mean critical path.

T (c) Use the mean critical path to find the approximate probability that the advertising campaign will be ready to launch within 57 days.

T (d) Now consider the other path through the project network. Find the approximate probability that this path will be completed within 57 days.

(e) Since these paths do not overlap, a better estimate of the probability that the project will finish within 57 days can be obtained as follows. The project will finish within 57 days if *both* paths are completed within 57 days. Therefore, the approximate probability that the project will finish within 57 days is the *product* of the probabilities found in parts (c) and (d). Perform this calculation. What does this answer say about the accuracy of the standard procedure used in part (c)?

22.4-6. The Lockhead Aircraft Co. is ready to begin a project to develop a new fighter airplane for the U.S. Air Force. The company's contract with the Department of Defense calls for project completion within 100 weeks, with penalties imposed for late delivery.

The project involves 10 activities (labeled A, B, . . . , J), where their precedence relationships are shown in the following project network.

Using the PERT three-estimate approach, the usual three estimates of the duration of each activity have been obtained as given below.

Activity	Optimistic Estimate	Most Likely Estimate	Pessimistic Estimate
A	28 weeks	32 weeks	36 weeks
B	22 weeks	28 weeks	32 weeks
C	26 weeks	36 weeks	46 weeks
D	14 weeks	16 weeks	18 weeks
E	32 weeks	32 weeks	32 weeks
F	40 weeks	52 weeks	74 weeks
G	12 weeks	16 weeks	24 weeks
H	16 weeks	20 weeks	26 weeks
I	26 weeks	34 weeks	42 weeks
J	12 weeks	16 weeks	30 weeks

T (a) Find the estimate of the mean and variance of the duration of each activity.
(b) Find the mean critical path.

T (c) Find the approximate probability that the project will finish within 100 weeks.
(d) Is the approximate probability obtained in part (c) likely to be higher or lower than the true value?

22.4-7. Label each of the following statements about the PERT three-estimate approach as true or false, and then justify your answer by referring to specific statements (with page citations) in the chapter.
(a) Activity durations are assumed to be no larger than the optimistic estimate and no smaller than the pessimistic estimate.
(b) Activity durations are assumed to have a normal distribution.
(c) The mean critical path is assumed to always require the minimum elapsed time of any path through the project network.

22.5-1. Do Prob. 10.8-1.

22.5-2. Do Prob. 10.8-2.

22.5-3. Reconsider the Electronic Toys Co. problem presented in Prob. 22.4-5. Sharon Lowe is concerned that there is a significant chance that the vitally important deadline of 57 days will not be met. Therefore, to make it virtually certain that the deadline will be met, she has decided to crash the project, using the CPM method of time-cost trade-offs to determine how to do this in the most economical way.

Sharon now has gathered the data needed to apply this method, as given below.

Activity	Normal Time	Crash Time	Normal Cost	Crash Cost
A	12 days	9 days	$210,000	$270,000
B	23 days	18 days	410,000	460,000
C	15 days	12 days	290,000	320,000
D	27 days	21 days	440,000	500,000
E	18 days	14 days	350,000	410,000
F	6 days	4 days	160,000	210,000

The normal times are the estimates of the means obtained from the original data in Prob. 22.4-5. The mean critical path gives an estimate that the project will finish in 51 days. However, Sharon knows from the earlier analysis that some of the pessimistic estimates are far larger than the means, so the project duration might be considerably longer than 51 days. Therefore, to better ensure that the project will finish within 57 days, she has decided to require that the estimated project duration based on means (as used throughout the CPM analysis) must not exceed 47 days.

(a) Consider the lower path through the project network. Use marginal cost analysis to determine the most economical way of reducing the length of this path to 47 days.

(b) Repeat part (*a*) for the upper path through the project network. What is the total crashing cost for the optimal way of decreasing estimated project duration of 47 days?

C (c) Use Excel to solve the problem.

C (d) Use another software option to solve the problem.

22.5-4. Consider the scenario described in Prob. 10.8-3.

(a) To prepare for analyzing the effect of crashing, find the earliest times, latest times, and slack for each activity when they are done in the normal way. Also identify the corresponding critical path(s) and project duration.

(b) Use marginal cost analysis to determine which activities should be crashed and by how much to minimize the overall cost of the project. Under this plan, what is the duration and cost of each activity? How much money is saved by doing this crashing?

(c) Now use the linear programming approach to do part (*b*) by shortening the deadline 1 week at a time from the project duration found in part (*a*).

22.5-5. Do Prob. 10.8-4.

22.5-6. Do Prob 10.8-5.

22.6-1 Reconsider Prob. 22.5-4 involving the Good Homes Construction Co. project to construct a large new home. Michael Dean now has generated the plan for how to crash this project. Since this plan causes all three paths through the project network to be critical paths, the earliest start time for each activity also is its latest start time.

Michael has decided to use PERT/Cost to schedule and control project costs.

(a) Find the earliest start time for each activity and the earliest finish time for the completion of the project.

(b) Construct a table like Table 22.10 to show the budget for this project.

(c) Construct a table like Fig. 22.13 (by hand) to show the schedule of costs based on earliest times for each of the 8 weeks of the project.

T (d) Now use the corresponding Excel template in your OR Courseware to do parts (b) and (c) on a single spreadsheet.

(e) After 4 weeks, activity A has been completed (with an actual cost of $65,000), and activity B has just now been completed (with an actual cost of $55,000), but activity C is just 33 percent completed (with an actual cost to date of $44,000). Construct a PERT/Cost report after week 4. Where should Michael concentrate his efforts to improve cost performances?

22.6-2. The P-H Microchip Co. needs to undertake a major maintenance and renovation program to overhaul and modernize its facilities for wafer fabrication. This project involves six activities (labeled A, B, . . . , F) with the precedence relationships shown in the following network.
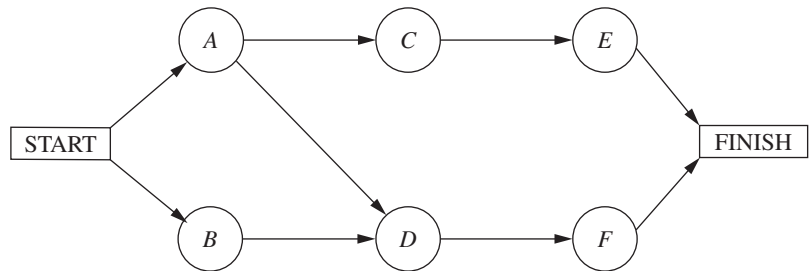
The estimated durations and costs of these activities are shown below in the left column.

Activity	Estimated Duration	Estimated Cost
A	6 weeks	$420,000
B	2 weeks	180,000
C	4 weeks	540,000
D	5 weeks	360,000
E	7 weeks	590,000
F	9 weeks	630,000

(a) Find the earliest times, latest times, and slack for each activity. What is the earliest finish time for the completion of the project?

T (b) Use the Excel template for PERT/Cost in your OR Courseware to display the budget and schedule of costs based on earliest start times for this project on a single spreadsheet.

T (c) Repeat part (b) except based on latest start times.

(d) Use these spreadsheets to draw a figure like Fig. 22.15 to show the schedule of cumulative project costs when all activities begin at their earliest start times or at their latest start times.

(e) After 4 weeks, activity B has been completed (with an actual cost of $200,000), activity A is 50 percent completed (with an actual cost to date of $200,000), and activity D is 50 percent completed (with an actual cost to date of $210,000). Construct a PERT/Cost report after week 4. Where should the project manager focus her attention to improve cost performances?

22.6-3. Reconsider Prob. 22.3-4 involving a project at Stanley Morgan Bank to install a new management information system. Ken Johnston already has obtained the earliest times, latest times,

and slack for each activity. He now is getting ready to use PERT/Cost to schedule and control the costs for this project. The estimated durations and costs of the various activities are given in the table on the right.

Activity	Estimated Duration	Estimated Cost
A	6 weeks	$180,000
B	3 weeks	75,000
C	4 weeks	120,000
D	4 weeks	140,000
E	7 weeks	175,000
F	4 weeks	80,000
G	6 weeks	210,000
H	3 weeks	45,000
I	5 weeks	125,000
J	4 weeks	100,000
K	3 weeks	60,000
L	5 weeks	50,000
M	6 weeks	90,000
N	5 weeks	150,000

T (a) Use the Excel template for PERT/Cost in your OR Courseware to display the budget and schedule of costs based on earliest start times for this project on a single spreadsheet.

T (b) Repeat part (a) except based on latest start times.

(c) Use these spreadsheets to draw a figure like Fig. 22.15 to show the schedule of cumulative project costs when all activities begin at their earliest start times or at their latest start times.

(d) After 8 weeks, activities A, B, and C have been completed with actual costs of $190,000, $70,000, and $150,000, respectively.

Activities *D, E, F, G,* and *I* are under way, with the percent completed being 40, 50, 60, 25, and 20 percent, respectively. Their actual costs to date are $70,000, $100,000, $45,000, $50,000, and $35,000, respectively. Construct a PERT/Cost report after week 8. Which activities should Ken Johnston investigate to try to improve their cost performances?

■ CASE

CASE 22.1 "School's Out Forever . . ." Alice Cooper

Brent Bonnin begins his senior year of college filled with excitement and a twinge of fear. The excitement stems from his anticipation of being done with it all—professors, exams, problem sets, grades, group meetings, all-nighters. . . . The list could go on and on. The fear stems from the fact that he is graduating in December and has only 4 months to find a job.

Brent is a little unsure about how he should approach the job search. During his sophomore and junior years, he had certainly heard seniors talking about their strategies for finding the perfect job, and he knows that he should first visit the Campus Career Planning Center to devise a search plan.

On Sept. 1, the fist day of school, he walks through the doors of the Campus Career Planning Center and meets Elizabeth Merryweather, a recent graduate overflowing with

energy and comforting smiles. Brent explains to Elizabeth that since he is graduating in December and plans to begin work in January, he wants to leave all of November and December open for interviews. Such a plan means that he has to have all his preliminary materials, such as cover letters and résumés, submitted to the companies where he wants to work by Oct. 31.

Elizabeth recognizes that Brent has to follow a very tight schedule, if he wants to meet his goal within the next 60 days. She suggests that the two of them sit down together and decide the major milestones that need to be completed in the job search process. Elizabeth and Brent list the 19 major milestones. For each of the 19 milestones, they identify the other milestones that must be accomplished directly before Brent can begin this next milestone. They also estimate the time needed to complete each milestone. The list is shown below.

Milestone	Milestones Directly Preceding Each Milestone	Time to Complete Each Milestone
A. Complete and submit an on-line registration form to the career center.	None.	2 days (This figure includes the time needed for the career center to process the registration form.)
B. Attend the career center orientation to learn about the resources available at the center and the campus recruiting process.	None.	5 days (This figure includes the time Brent must wait before the career center hosts an orientation.)
C. Write an initial résumé that includes all academic and career experiences.	None.	7 days
D. Search the Internet to find job opportunities available outside of campus recruiting.	None.	10 days
E. Attend the company presentations hosted during the fall to understand the cultures of companies and to meet with company representatives.	None.	25 days

(*Continued*)

Milestone	Milestones Directly Preceding Each Milestone	Time to Complete Each Milestone
F. Review the industry resources available at the career center to understand the career and growth opportunities available in each industry. Take career test to understand the career that provides the best fit with your skills and interests. Contact alumni listed in the career center directories to discuss the nature of a variety of jobs.	Complete and submit an on-line registration form to the career center. Attend the career center orientation.	7 days
G. Attend a mock interview hosted by the career center to practice interviewing and to learn effective interviewing styles.	Complete and submit an on-line registration form to the career center. Attend the career center orientation. Write the initial résumé.	4 days (This figure includes the time that elapses between the day that Brent signs up for the interview and the day that the interview takes place.)
H. Submit the initial résumé to the career center for review.	Complete and submit an on-line registration form to the career center. Attend the career center orientation. Write the initial résumé.	2 days (This figure includes the time the career center needs to review the résumé.)
I. Meet with a résumé expert to discuss improvements to the initial résumé.	Submit the initial résumé to the career center for review.	1 day
J. Revise the initial résumé.	Meet with a résumé expert to discuss improvements.	4 days
K. Attend the career fair to gather company literature, speak to company representatives, and submit résumés.	Revise the initial résumé.	1 day
L. Search campus job listings to identify the potential jobs that fit your qualifications and interests.	Review the industry resources, take the career test, and contact alumni.	5 days
M. Decide which jobs you will pursue given the job opportunities you found on the Internet, at the career fair, and through the campus job listings.	Search the Internet. Search the campus job listings. Attend the career fair.	3 days

Milestone	Milestones Directly Preceding Each Milestone	Time to Complete Each Milestone
N. Bid to obtain job interviews with companies that recruit through the campus career center and have open interview schedules.*	Decide which jobs you will pursue.	3 days
O. Write cover letters to seek jobs with companies that either do not recruit through the campus career center or recruit through the campus career center but have closed interview schedules.† Tailor each cover letter to the culture of each company.	Decide which jobs you will pursue. Attend company presentations.	10 days
P. Submit the cover letters to the career center for review.	Write the cover letters.	4 days (This figure includes the time the career center needs to review the cover letters.)
Q. Revise the cover letters.	Submit the cover letters to the career center for review.	4 days
R. For the companies that are not recruiting through the campus career center, mail the cover letter and résumé to the company's recruiting department.	Revise the cover letters.	6 days (This figure includes the time needed to print and package the application materials and the time needed for the materials to reach the companies.)
S. For the companies that recruit through the campus career center but that hold closed interview schedules, drop the cover letter and résumé at the career center.	Revise the cover letters	2 days (This figure includes the time needed to print and package the application materials).

*An open interview schedule occurs when the company does not select the candidates that it wants to interview. Any candidate may interview, but since the company has only a limited number of interview slots, interested candidates must bid points (out of their total allocation of points) for the interviews. The candidates with the highest bids win the interview slots.

†Closed interview schedules occur when a company requires candidates to submit their cover letters, résumés, and test scores so that the company is able to select the candidates it wants to interview.

In the evening after his meeting with Elizabeth, Brent meets with his buddies at the college coffeehouse to chat about their summer endeavors. Brent also tells his friends about the meeting he had earlier with Elizabeth. He describes the long to-do list he and Elizabeth developed and says that he is really worried about keeping track of all the major milestones and getting his job search organized. One of his friends reminds him of the cool OR class they all took together in the first semester of Brent's junior year, and how they had learned about some techniques to organize large projects. Brent remembers this class fondly, since he was able to use a number of the methods he studied in that class in his last summer job.

(a) Draw the project network for completing all milestones before the interview process. If everything stays on schedule, how

long will it take Brent until he can start with the interviews? What are the critical steps in the process?

(b) Brent realizes that there is a lot of uncertainty in the times it will take him to complete some of the milestones. He expects to get really busy during his senior year, in particular since he is taking a demanding course load. Also, students sometimes have to wait quite a while before they get appointments with the counselors at the career center. In addition to the list estimating the most likely times that he and Elizabeth wrote down, he makes a list of optimistic and pessimistic estimates of how long the various milestones might take.

Milestone	Optimistic Estimate	Pessimistic Estimate
A	1 day	4 days
B	3 days	10 days
C	5 days	14 days
D	7 days	12 days
E	20 days	30 days
F	5 days	12 days
G	3 days	8 days
H	1 day	6 days
I	1 day	1 day
J	3 days	6 days
K	1 day	1 day
L	3 days	10 days
M	2 days	4 days
N	2 days	8 days
O	3 days	12 days
P	2 days	7 days
Q	3 days	9 days
R	4 days	10 days
S	1 day	3 days

How long will it take Brent to get done under the worst-case scenario? How long will it take if all his optimistic estimates are correct?

(c) Determine the mean critical path for Brent's job search process. What is the variance of the project duration?

(d) Give a rough estimate of the probability that Brent will be done within 60 days.

(e) Brent realizes that he has made a serious mistake in his calculations so far. He cannot schedule the career fair to fit his schedule. Brent read in the campus newspaper that the fair has been set 24 days from today on Sept. 25. Draw a revised project network that takes into account this complicating fact.

(f) What is the mean critical path for the new network? What is the probability that Brent will complete his project within 60 days?

(*Note:* A data file for this case is provided on the book's website for your convenience.)