CHAPTER 25

Reliability

The many definitions of reliability that exist depend upon the viewpoint of the user. However, they all have a common core that contains the statement that reliability, $R(t)$, is the probability that a device performs adequately over the interval $[0, t]$. In general, it is assumed that unless repair or replacement occurs, adequate performance at time t implies adequate performance during the interval $[0, t]$. The device under consideration may be an entire system, a subsystem, or a component.[1] Although this definition is simple, the systems to which it is applied are generally very complex. In principle, it is possible to break down the system into black boxes, with each black box being in one of two states: good or bad. Mathematical models of the system can then be abstracted from the physical processes and the theory of combinatorial probability used to predict the reliability of the system. The black boxes may be independent of, or be very dependent upon, each other. For any reasonable system, such a probability analysis generally becomes so cumbersome that it must be considered impractical. Hence, we seek other methods that either simplify the calculations or provide bounds on the reliability of the entire complex system.

As an example, consider an automobile. There are a large number of functional parts, wiring, and joints. These may be broken into subsystems, with each subsystem having a reliability associated with it. Possible subsystems are the engine, transmission, exhaust, body, carburetor, and brakes. A mathematical model of the automobile system can be abstracted and the theory of combinatorial probability used to predict the reliability of the automobile.

25.1 STRUCTURE FUNCTION OF A SYSTEM

Suppose an automobile can be divided into n components (subsystems). The performance of each component can be denoted by a random variable, X_i, that takes on the value $x_i = 1$ if the component performs satisfactorily for the desired time and $x_i = 0$ if the component fails during this time. In general, then, X_i is a binary random variable defined by

$$X_i = \begin{cases} 1, & \text{if component } i \text{ performs satisfactorily during time } [0, t] \\ 0, & \text{if component i fails during time } [0, t]. \end{cases}$$

[1]A subsystem can be viewed as containing one or more components.

The performance of the system is measured by the binary random variable[1] $\phi(X_1, X_2, \ldots, X_n)$, where

$$\phi(X_1, X_2, \ldots, X_n) = \begin{cases} 1, & \text{if system performs satisfactorily during time } [0, t] \\ 0, & \text{if system fails during time } [0, t]. \end{cases}$$

The function ϕ is called the **structure function** of the system and is just a function of the n-component random variables. Thus, the performance of the automobile is a function of its n components and takes on the value 1 if the automobile functions properly for the desired time and 0 if it does not. Because the performance of each component in the automobile takes on the value 1 or 0, the function ϕ is defined over 2^n points, with each point resulting in a 1 if the automobile performs satisfactorily and a 0 if the automobile fails.

There are several important structure functions to consider, depending upon how the components are assembled. Three structure functions will be discussed in detail.

Series System

The series system is the simplest and most common of all the configurations. For a **series system,** the system fails if any component of the system fails; i.e., it performs satisfactorily if and only if all the components perform satisfactorily. The structure function for a series system is given by

$$\phi(X_1, X_2, \ldots, X_n) = X_1X_2 \cdots X_n = \min\{X_1, X_2, \ldots . X_n\}.$$

This equation holds because each X_i is either 1 or 0. Hence, the structure function takes on the value 1 if each X_i equals 1 or, equivalently, if the minimum of the X_i equals 1. For example, suppose the automobile is divided into only two components: the engine (X_1) and the transmission (X_2). Then it is reasonable to assume that the automobile will perform satisfactorily for the desired time period if and only if the engine and the transmission both perform satisfactorily. Hence,

$$\phi(X_1, X_2) = X_1X_2,$$

and

$$\phi(1, 1) = 1, \qquad \phi(1, 0) = \phi(0, 1) = \phi(0, 0) = 0.$$

Parallel System

A **parallel system** of n components is defined to be a *system that fails if all components fail,* or alternatively, a *system that performs satisfactorily if at least one of the n components performs satisfactorily* (with all n components operating simultaneously). This property of parallel systems is often called *redundancy* (i.e, there are alternative components, existing within the system, to help the system operate successfully in case of failure of one or more components). The structure function for a parallel system is given by

$$\begin{aligned}\phi(X_1, X_2, \ldots, X_n) &= 1 - (1 - X_1)(1 - X_2) \cdots (1 - X_n) \\ &= \max\{X_1, X_2, \ldots, X_n\}.\end{aligned}$$

This equation again follows because each X_i is either 1 or 0. The structure function takes on the value 1 if at least one of the X_i equals 1 or, equivalently, if the largest X_i equals 1. In the automobile example, the car is equipped with front disk (X_1) and rear drum (X_2) brakes.

[1]Note that X_i and ϕ are functions of the time t, but t will be suppressed for each of notation.

The automobile will perform successfully if either the front or rear brakes operate properly.[1] If one is concerned with the structure function of the brake subsystem, then

$$\phi(X_1\, X_2) = 1 - (1 - X_1)(1 - X_2) = X_1 + X_2 - X_1X_2,$$

and

$$\phi(1, 1) = \phi(1, 0) = \phi(0, 1) = 1, \qquad \phi(0, 0) = 0.$$

k Out of *n* System

Some systems are assembled such that the system operates if k out of n components function properly. Note that the series system is a k out of n system, with $k = n$, and the parallel system is a k out of n system, with $k = 1$. The structure function for a k out of n system is given by

$$\phi(X_1, X_2, \ldots, X_n) = \begin{cases} 1, & \text{if } \sum_{i=1}^{n} X_i \geq k \\ 0, & \text{if } \sum_{i=1}^{n} X_i < k. \end{cases}$$

In the automobile example, consider a large truck equipped with eight tires. The structure function for the tire system is an example of a four-out-of-eight system. (Although the system's performance may be degraded if fewer than eight tires are operating, rearrangement of the tire configuration will result in adequate performance as long as at least four tires are usable.)

It is reasonable to expect the performance of an automobile to improve if the performance of one or more components is improved. This improvement can be reflected in the characterization of the structure function, where, for example, one would expect $\phi(1, 0, 0, 1)$ to be no less than $\phi(1, 0, 0, 0)$. Hence, it will be assumed that if $x_i \leq y_i$, for $i = 1, 2, \ldots n$, then

$$\phi(y_1, y_2, \ldots, y_n) \geq \phi(x_1, x_2, \ldots, x_n).$$

A system possessing this property (ϕ is an increasing function of x) is called a **coherent** (or **monotone**) **system.**

25.2 SYSTEM RELIABILITY

The structure function of a system containing n components is a binary random variable that takes on the value 1 or 0. Furthermore, the **reliability** of this system can be expressed as[2]

$$R = P\{\phi(X_1, X_2, \ldots, X_n) = 1\}.$$

Thus, for a series system, the reliability is given by

$$R = P\{X_1X_2 \cdots X_n = 1\} = P\{X_1 = 1, X_2 = 1, \ldots, X_n = 1\}.$$

When the usual terms for conditional probability are employed,

$$R = P\{X_1 = 1\}P\{X_2 = 1 | X_1 = 1\}P\{X_3 = 1 | X_1 = 1, X_2 = 1\} \ldots P\{X_n = 1 | X_1 = 1, \ldots, X_{n-1} = 1\}.$$

[1]It is evident that the loss of the front or rear brakes will affect the braking capability of the automobile, but the definition of "perform successfully" may allow for either set working.

[2]The time t is now suppressed in the notation. Recall that the time is implicitly included in determining whether or not the ith component performs satisfactorily.

In general, such conditional probabilities require careful analysis. For example, $P\{X_2 = 1 | X_1 = 1\}$ is the probability that component 2 will perform successfully, given that component 1 performs successfully. Consider a system where the heat from component 1 affects the temperature of component 2 and thereby its probability of success. The performance of these components is then *dependent,* and the evaluation of the conditional probability is extremely difficult. If, on the other hand, the performance characteristics of these components do not interact, e.g., the temperature of one component does not affect the performance of the other component, then the components can be said to be *independent.* The expression for the reliability then simplifies and becomes

$$R = P\{X_1 = 1\}P\{X_2 = 1\} \cdots P\{X_n = 1\}.$$

When the components of a series system are assumed to be independent, it should be noted that the reliability is a function of the probability distribution of the X_i. This phenomenon is true for any system structure.

Unless otherwise specified, it will be assumed throughout the remainder of this chapter that the component performances are independent. Hence, the probability distribution of the binary random variables X_i can be expressed as

$$P\{X_i = 1\} = p_i,$$

and

$$P\{X_i = 0\} = 1 - p_i,$$

Thus, for systems composed of independent components, the reliability becomes a function of the p_i; that is.

$$R = R(p_1, p_2, \ldots, p_n).$$

Reliability of Series Systems

As previously indicated, for a series structure,

$$\begin{aligned} R(p_1, p_2, \ldots, p_n) &= P\{\phi(X_1, X_2, \ldots, X_n) = 1\} \\ &= P\{X_1X_2 \cdots X_n = 1\} \\ &= P\{X_1 = 1, X_2 = 1, \ldots, X_n = 1\} \\ &= P\{X_1 = 1\}P\{X_2 = 1\} \cdots P\{X_n = 1\} \\ &= p_1p_2 \cdots p_n. \end{aligned}$$

Thus, returning to the automobile example, if the probability that the engine performs satisfactorily is 0.95 and the probability that the transmission performs satisfactorily is 0.99, then the reliability of this automobile series subsystem is given by $R = (0.95)(0.99) = 0.94$.

Reliability of Parallel Systems

The structure function for a parallel system is

$$\phi(X_1, X_2, \ldots, X_n) = \max(X_1, X_2, \ldots, X_n),$$

and the reliability is given by

$$\begin{aligned} R(p_1, p_2, \ldots, p_n) &= P\{\max(X_1, X_2, \ldots, X_n) = 1\} \\ &= 1 - P\{\text{all } X_i = 0\} \\ &= 1 - P\{X_1 = 0, X_2 = 0, \ldots, X_n = 0\} \\ &= 1 - (1 - p_1)(1 - p_2) \cdots (1 - p_n). \end{aligned}$$

Thus, if the probability that the front disk brakes and the rear drum brakes perform satisfactorily is 0.99 for each, the subsystem reliability is given by

$$R = 1 - (0.01)(0.01) = 0.9999.$$

Reliability of *k* Out of *n* Systems

The structure function for a k out of n system is

$$\phi(X_1, X_2, \ldots, X_n) = \begin{cases} 1, & \text{if } \sum_{i=1}^{n} X_i \geq k \\ 0, & \text{if } \sum_{i=1}^{n} X_i < k, \end{cases}$$

and the reliability is given by

$$R(p_1, p_2, \ldots, p_n) = P\left\{\sum_{i=1}^{n} X_i \geq k\right\}.$$

The evaluation of this expression is, in general, quite difficult except for the case of $p_1 = p_2 = \cdots = p_n = p$. Under this assumption, $\Sigma_{i=1}^{n} X_i$ has a binomial distribution with parameters n and p, so that

$$R(p, p, \ldots, p) = \sum_{i=k}^{n} \binom{n}{i} = p^i(1 - p)^{n-i}.$$

For the truck tire example, if each tire has a probability of 0.95 of performing satisfactorily, then the reliability of a four-out-of-eight system is given by

$$R = \sum_{i=4}^{8} \binom{8}{i} (0.95)^i(0.05)^{8-i} = 0.9999.$$

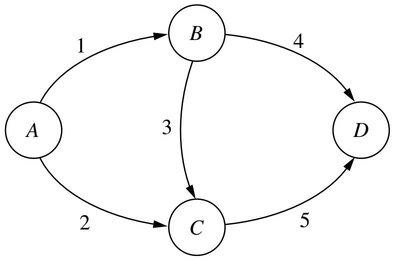
For general structures, the system reliability calculations can become quite tedious. A technique for computing reliabilities for this general case will be presented in the next section. However, the final result of this section is to indicate that the reliability function of a system of independent components can be shown to be an increasing function of the p_i; that is, if $p_i \leq q_i$ for $i = 1, 2, \ldots, n$, then

$$R(q_1, q_2, \ldots, q_n) \geq R\,(p_1, p_2, \ldots, p_n).$$

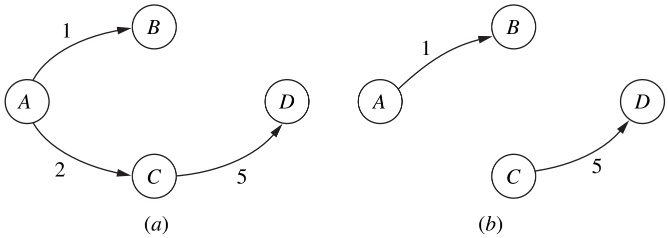
This result is analogous to, and dependent upon, the assumption that the structure function of the system is **coherent**. The implication of this intuitive result is that the reliability of the automobile will improve if the reliability of one or more components is improved.

25.3 CALCULATION OF EXACT SYSTEM RELIABILITY

A representation of the structure of a system can be expressed in terms of a network, and some of the material presented in Chap. 10 is relevant. For example, consider the system that can be represented by the network in Fig. 25.1. This system consists of five components, connected in a somewhat complex manner. According to the network diagram, the system will operate successfully if there exists a flow from A (the source) to D (the sink) through the directed graph, i.e., if components 1 and 4 operate successfully, or components 2 and 5 operate



■ **FIGURE 25.1**
A five-component system.



■ **FIGURE 25.2**
(*a*) System with components 3 and 4 failed; (*b*) system with components 2, 3, and 4 failed.

successfully, or components 1, 3, and 5 operate successfully. In fact, each arc can be viewed as having capacity 1 or 0, depending upon whether or not the component is operating. If an arc has a 0 attached to it (the component fails), then the network would lose that arc, and the system would operate successfully if and only if there is a path from the source to the sink in the resultant network. This situation is illustrated in Fig. 25.2, where the system still operates if components 3 and 4 fail but becomes inoperable if components 2, 3, and 4 fail. This suggests a possible method for computing the exact system reliability. Again, denote the performance of the *i*th component by the binary random variable X_i. Then X_i takes on the value 1 with probability p_i and 0 with probability $(1 - p_i)$. For each realization, $X_1 = x_1$, $X_2 = x_2$, $X_3 = x_3$, $X_4 = x_4$ and $X_5 = x_5$ (there are 2^5 such realizations), it is determined whether or not the system will operate, i.e., whether or not the structure function equals 1. The network consisting of those arcs with X_i equal to 1 contains at least one path if and only if the corresponding structure function equals 1. If a path is formed, the probability of obtaining this configuration is obtained. For the realization in Fig. 21.2*a*. a path is formed, and

$$P\{X_1 = 1, X_2 = 1, X_3 = 0, X_4 = 0, X_5 = 1\} = p_1 p_2 (1 - p_3)(1 - p_4) p_5.$$

Because each realization is disjoint, the system reliability is just the sum of the probabilities of those realizations that contain a path. Unfortunately, even for this simple system, 32 different realizations must be evaluated, and other techniques are desirable.

Another possible procedure for finding the exact reliability is to note that the reliability $R(p_1, p_2, \ldots, p_n)$ can be expressed as

$$R(p_1, p_2, \ldots, p_n) = P\{\text{maximum flow from source to sink} \geq 1\}.$$

This identity allows the concept of paths and cuts presented in Chap. 10 to be used. In reliability theory, the terminology of minimal paths and minimal cuts is introduced. A **minimal path** is a *minimal set of components that, by functioning, ensures the successful operation of the system*. For the example in Fig. 25.1. components 2 and 5 are a minimal path. A **minimal cut** is a *minimal set of components that, by failing, ensures the failure of the system.* In Fig. 25.1, components 1 and 2 are a minimal cut. For the system given in Fig. 25.1, the minimal paths and cuts are

Minimal Paths	**Minimal Cuts**
X_1X_4	X_1X_2
$X_1X_3X_5$	X_4X_5
X_2X_5	$X_2X_3X_4$
	X_1X_5

If we use all the *minimal paths,* there are *two ways* to obtain the *exact system reliability*. Because the system will operate if all the components in at least one of the minimal paths operate, the system reliability can be expressed as

$$\begin{aligned} R(p_1, p_2, p_3, p_4, p_5) &= P\{\phi(X_1, X_2, X_3, X_4, X_5) = 1\} \\ &= P\{(X_1X_4 = 1) \cup (X_1X_3X_5 = 1) \cup (X_2X_5 = 1)\}. \end{aligned}$$

Using the algebra of sets,

$$\begin{aligned} R(p_1, p_2, p_3, p_4, p_5) &= P\{X_1X_4 = 1\} + P\{X_1X_3X_5 = 1\} \\ &\quad + P\{X_2X_5 = 1\} - P\{X_1X_3X_4X_5 = 1\} \\ &\quad - P\{X_1X_2X_4X_5 = 1\} - P\{X_1X_2X_3X_5 = 1\} \\ &\quad + P\{X_1X_2X_3X_4X_5 = 1) \\ &= p_1p_4 + p_1p_3p_5 + p_2p_5 - p_1p_3p_4p_5 \\ &\quad - p_1p_2p_4p_5 - p_1p_2p_3p_5 + p_1p_2p_3p_4p_5 \\ &= 2p^2 + p^3 - 3p^4 + p^5, \qquad \text{when } p_i = p. \end{aligned}$$

Notice that there are $2^3 - 1 = 7$ terms in the expansion of the reliability function (in general, if there are r paths, then there are $2^r - 1$ terms in the expansion), so that this calculation is not simple.

The second method of determining the system reliability from paths is as follows: For the minimal path containing components 1 and 4, $X_1X_4 = 1$ if and only if both components function. This fact is similarly true for the other two minimal paths. However, the system will operate if all the components in at least one of the minimal paths operate. Hence, paths operate as a parallel system, so that

$$\begin{aligned} \phi(X_1, X_2, X_3, X_4, X_5) &= \max[X_1X_4, X_1X_3X_5, X_2X_5] \\ &= 1 - (1 - X_1X_4)(1 - X_1X_3X_5)(1 - X_2X_5). \end{aligned}$$

Because $X_i^2 = X_i$, then

$$\begin{aligned} \phi(X_1, X_2, X_3, X_4, X_5) &= X_1X_4 + X_1X_3X_5 + X_2X_5 - X_1X_3X_4X_5 - X_1X_2X_4X_5 \\ &\quad - X_1X_2X_3X_5 + X_1X_2X_3X_4X_5. \end{aligned}$$

Noting that ϕ is a binary random variable taking on the value 1 and 0,

$$\begin{aligned} E[\phi(X_1, X_2, X_3, X_4, X_5)] &= P\{\phi(X_1, X_2, X_3, X_4, X_5) = 1\} \\ &= R(p_1, p_2, p_3, p_4, p_5). \end{aligned}$$

Therefore,

$$\begin{aligned} &R(p_1, p_2, p_3, p_4, p_5) \\ &\quad = E[X_1X_4 + X_1X_3X_5 + X_2X_5 - X_1X_3X_4X_5 - X_1X_2X_4X_5 \\ &\qquad - X_1X_2X_3X_5 + X_1X_2X_3X_4X_5] \\ &\quad = p_1p_4 + p_1p_3p_5 + p_2p_5 - p_1p_3p_4p_5 - p_1p_2p_4p_5 - p_1p_2p_3p_5 \\ &\qquad + p_1p_2p_3p_4p_5. \end{aligned}$$

This result is the same as the one obtained earlier and requires essentially the same amount of calculation.

If we use all the *minimal cuts,* there are also *two ways* to obtain the *exact system reliability*. Because the system will fail if and only if all the components in at least one of the minimal cuts fail, the system reliability can be expressed as

$$\begin{aligned}
R(p_1, p_2, p_3, p_4, p_5) &= 1 - P\{\phi(X_1, X_2, X_3, X_4, X_5) = 0\} \\
&= 1 - P\{X_1 = 0, X_2 = 0) \cup (X_4 = 0, X_5 = 0) \\
&\quad \cup (X_2 = 0, X_3 = 0, X_4 = 0) \cup (X_1 = 0, X_5 = 0) \\
&= 1 - P\{X_1 = 0, X_2 = 0\} - P\{X_4 = 0, X_5 = 0\} \\
&\quad - P\{X_2 = 0, X_3 = 0, X_4 = 0\} - P\{X_1 = 0, X_5 = 0\} \\
&\quad + P\{X_1 = 0, X_2 = 0, X_4 = 0, X_5 = 0\} \\
&\quad + P\{X_1 = 0, X_2 = 0, X_3 = 0, X_4 = 0\} \\
&\quad + P\{X_1 = 0, X_2 = 0, X_5 = 0\} \\
&\quad + P\{X_2 = 0, X_3 = 0, X_4 = 0, X_5 = 0\} \\
&\quad + P\{X_1 = 0, X_4 = 0, X_5 = 0\} \\
&\quad + P\{X_1 = 0, X_2 = 0, X_3 = 0, X_4 = 0, X_5 = 0\} \\
&\quad - P\{X_1 = 0, X_2 = 0, X_3 = 0, X_4 = 0, X_5 = 0\} \\
&\quad - P\{X_1 = 0, X_2 = 0, X_4 = 0, X_5 = 0\} \\
&\quad - P\{X_1 = 0, X_2 = 0, X_3 = 0, X_4 = 0, X_5 = 0\} \\
&\quad - P\{X_1 = 0, X_2 = 0, X_3 = 0, X_4 = 0, X_5 = 0\} \\
&\quad + P\{X_1 = 0, X_2 = 0, X_3 = 0, X_4 = 0, X_5 = 0\} \\
&= 1 - q_1q_2 - q_4q_5 - q_2q_3q_4 - q_1q_5 + q_1q_2q_3q_4 \\
&\quad + q_1q_2q_5 + q_2q_3q_4q_5 + q_1q_4q_5 - q_1q_2q_3q_4q_5,
\end{aligned}$$

where

$$q_i = 1 - p_i.$$

This result is, of course, algebraically equivalent to the one obtained previously, and it involves $2^4 - 1 = 15$ terms in the expansion of the reliability function. In general, if there are s cuts, there are $2^s - 1$ terms in the expansion.

The second method of determining the system reliability from cuts is: For the minimal cut containing components 1 and 2, $1 - (1 - X_1)(1 - X_2) = 0$ if and only if both components fail. This fact is similarly true for the other three cuts. However, the system will operate if at least one of the components in *each* cut operates. Hence, cuts operate as a series system, so that

$$\begin{aligned}
&\phi(X_1, X_2, X_3, X_4, X_5) \\
&= \min[1 - (1 - X_1)(1 - X_2), 1 - (1 - X_4)(1 - X_5), \\
&\quad 1 - (1 - X_2)(1 - X_3)(1 - X_4), 1 - (1 - X_1)(1 - X_5)] \\
&= ([1 - (1 - X_1)(1 - X_2)][1 - (1 - X_4)(1 - X_5)] \\
&\quad [1 - (1 - X_2)(1 - X_3)(1 - X_4)][1 - (1 - X_1)(1 - X_5)]) \\
&= 1 - (1 - X_1)(1 - X_2) - (1 - X_4)(1 - X_5) \\
&\quad - (1 - X_2)(1 - X_3)(1 - X_4) - (1 - X_1)(1 - X_5) \\
&\quad + (1 - X_1)(1 - X_2)(1 - X_3)(1 - X_4) \\
&\quad + (1 - X_1)(1 - X_2)(1 - X_5) \\
&\quad + (1 - X_2)(1 - X_3)(1 - X_4)(1 - X_5) \\
&\quad + (1 - X_1)(1 - X_4)(1 - X_5) \\
&\quad - (1 - X_1)(1 - X_2)(1 - X_3)(1 - X_4)(1 - X_5).
\end{aligned}$$

Taking expectations on both sides leads to the desired expression for the reliability. *Again,* this method requires essentially the same amount of calculation as required for the first procedure using cuts.

Although the results presented in this section were based upon the example, an extension to any system can be easily obtained. All minimal paths and/or cuts must be found and one of the four methods presented chosen.

As previously mentioned, if there are r paths and s cuts in the network, then calculating the exact reliability using paths will involve summing $2^r - 1$ terms, and using cuts will involve $2^s - 1$ terms. Hence, the method using paths should be used if and only if $r \le s$. Generally, however, it is simpler to find minimal paths rather than minimal cuts, so that the method using paths may have to be used because finding all cuts may be computationally infeasible. It is evident that finding the exact reliability of a system is quite difficult and that bounds are desirable, provided that the calculations are substantially reduced.

25.4 BOUNDS ON SYSTEM RELIABILITY

It is evident that the calculations required to compute exact system reliability are numerous, and that other methods, such as obtaining upper and lower bounds, are desirable.

To obtain bounds, the following result concerning binary random variables is very useful.

If $X_1, X_2, \ldots, X_n$ are independent binary random variables that take on the value 1 or 0, and $Y_i = \Pi_{j \in J_i} X_j$, where the product ranges over all j that are elements in the set J_i, $i = 1, 2, \ldots, r$, then

$$P\{Y_1 = 0, Y_2 = 0, \ldots, Y_i = 0\} \ge P\{Y_1 = 0\}P\{Y_2 = 0\} \cdots P\{Y_i = 0\}.$$

Returning to the example of Sec. 25.3, it was pointed out that the system will operate if all the components in at least one of the minimal paths operate, so that

$$\begin{aligned} R(p_1, p_2, p_3, p_4, p_5) &= P\{\phi(X_1, X_2, X_3, X_4, X_5) = 1\} \\ &= 1 - P\{\text{all paths fail}\} \\ &= 1 - P\{X_1X_4 = 0, X_1X_3X_5 = 0, X_2X_5 = 0\}. \end{aligned}$$

From the result on binary random variables,

$$\begin{aligned} R(p_1, p_2, p_3, p_4, p_5) &\le 1 - P\{X_1X_4 = 0\}P\{X_1X_3X_5 = 0\}P\{X_2X_5 = 0\} \\ &= 1 - (1 - p_1p_4)(1 - p_1p_3p_5)(1 - p_2p_5) \\ &= 1 - (1 - p^2)^2(1 - p^3). \end{aligned}$$

when

$$p_i = p,$$

so that an upper bound is obtained.

Similarly, in Sec. 25.3, it was pointed out that the system will operate if at least one of the components in *each* cut operates, so that

$$\begin{aligned} &R(p_1, p_2, p_3, p_4, p_5) \\ &\quad = P\{\phi(X_1, X_2, X_3, X_4, X_5) = 1\} = P\{\text{at least one of } X_1, X_2 \text{ operates; at least one} \\ &\qquad \text{of } X_4, X_5 \text{ operates; at least one of } X_2, X_3, X_4 \text{ operates; at least one of } X_1, X_5 \\ &\qquad \text{operates}\} \\ &\quad = P\{[1 - (1 - X_1)(1 - X_2)] = 1, [1 - (1 - X_4)(1 - X_5)] = 1, \\ &\qquad [1 - (1 - X_2)(1 - X_3)(1 - X_4)] = 1, [1 - (1 - X_1)(1 - X_5)] = 1\} \\ &\quad = P\{[1 - X_1)(1 - X_2) = 0, (1 - X_4)(1 - X_5) = 0, \\ &\qquad (1 - X_2)(1 - X_3)(1 - X_4) = 0, (1 - X_1)(1 - X_5) = 0\}. \end{aligned}$$

Now $(1 - X_i)$ are independent binary random variables that take on the values 1 and 0, so that the result on binary random variables is again applicable; that is.

$$\begin{aligned} &R(p_1, p_2, p_3, p_4, p_5) \\ &\quad \geq (P\{(1 - X_1)(1 - X_2) = 0\}P\{(1 - X_4)(1 - X_5) = 0\} \\ &\qquad\qquad P\{(1 - X_2)(1 - X_3)(1 - X_4) = 0\}P(1 - X_1)(1 - X_5) = 0\}) \\ &\quad = ([1 - (1 - p_1)(1 - p_2)][1 - (1 - p_4)(1 - p_5)] \\ &\qquad\qquad [1 - (1 - p_2)(1 - p_3)(1 - p_4)][1 - (1 - p_1)(1 - p_5)]) \\ &\quad = [1 - (1 - p)^2]^3[1 - (1 - p)^3], \end{aligned}$$

when

$$p_i = p,$$

so that a lower bound is obtained.

Thus, we obtain an upper bound on the reliability based upon paths and a lower bound based upon cuts. For example, if $p_i = p = 0.9$, then

$$\begin{aligned} 0.9693 = [1 - (0.1)^2]^3[1 - (0.1)^3] &\leq R(0.9, 0.9, 0.9, 0.9, 0.9) \\ &\leq 1 - [1 - (0.9)^2]^2[1 - (0.9)^3] = 0.9902. \end{aligned}$$

Furthermore, the exact reliability obtained from the expressions in Sec. 25.3 is given by

$$R(0.9, 0.9, 0.9, 0.9, 0.9) = (0.9)^2 + (0.9)^3 - 3(0.9)^4 + (0.9)^5 = 0.9712.$$

In general, this technique provides useful results in that the bounds are frequently quite narrow.

25.5 BOUNDS ON RELIABILITY BASED UPON FAILURE TIMES

The previous sections considered systems that performed successfully during a designated period or failed during this same period. An alternative way of viewing systems is to view their performance as a function of time.

Consider a component (or system) and its associated random variable, the time to failure, T. Denote the cumulative distribution function of the time to failure of the component by F and its density function by f. In terms of the previous discussion, the random variables X and T are related in that X takes on the values

$$\begin{aligned} &1, \quad \text{if } T \geq t \\ &0, \quad \text{if } T < t. \end{aligned}$$

Then

$$R(t) = P\{X = 1\} = 1 - F(t) = \int_t^x f(y)\, dy.$$

An appealing intuitive property in reliability is the failure rate. For those values of t for which $F(t) < 1$, the **failure rate** $r(t)$ is defined by

$$r(t) = \frac{f(t)}{R(t)}.$$

This function has a useful probabilistic interpretation, namely, $r(t)\,dt$ represents the conditional probability that an object surviving to age t will fail in the interval $[t, t + dt]$. This function is sometimes called the **hazard rate**.

In many applications, there is every reason to believe that the failure rate tends to increase because of the inevitable deterioration that occurs. Such a failure rate that remains constant or increases with age is said to have an **increasing failure rate** (IFR).

In some applications, the failure rate tends to decrease. It would be expected to decrease initially, for instance, for materials that exhibit the phenomenon of work hardening. Certain solid-state electronic devices are also believed to have a decreasing failure rate. Thus, a failure rate that remains constant or decreases with age is said to have a **decreasing failure rate** (DFR).

The failure rate possesses some interesting properties. The time to failure distribution is completely determined by the failure rate. In particular, it is easily shown that

$$R(t) = 1 - F(t) = \exp\left[-\int_0^t r(\xi)\, d\xi\right].$$

Thus, an assumption made about the failure rate has direct implications on the time to failure distribution. As an example, consider a component whose failure distribution is given by the exponential distribution, i.e.,

$$F(t) = P\{T \le t\} = 1 - e^{-t/\theta}.$$

Thus, $R(t)$ is given by $e^{-t/\theta}$, and the failure rate is given by

$$r(t) = \frac{(1/\theta)e^{-t/\theta}}{e^{-t/\theta}} = \frac{1}{\theta}.$$

Note that the exponential distribution has a constant failure rate and hence has both IFR and DFR. In fact, using the expression relating the time to failure distribution and the failure rate, it is evident that a component having a constant failure rate must have a time to failure distribution that is exponential.

Bounds for IFR Distributions

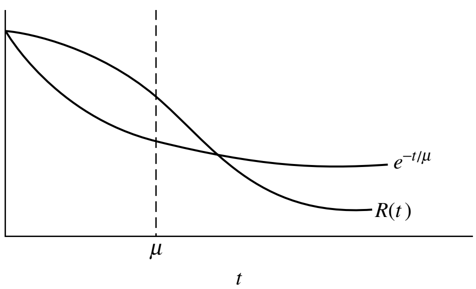
Under either the IFR or DFR assumption, it is possible to obtain sharp bounds on the reliability in terms of moments and percentiles: In particular, such bounds can be derived from statements based upon the *mean time to failure*. This fact is particularly important because many design engineers present specifications in terms of mean time to failure.

Because the exponential distribution with constant failure rate is the boundary distribution between IFR and DFR distributions, it provides natural bounds on the survival probability of IFR and DFR distributions. In particular, it can be shown that if all that is known about the failure distribution is that it is IFR and has mean μ, then the greatest lower bound on the reliability that can be given is

$$R(t) \ge \begin{cases} e^{-t/\mu}, & \text{for } t < \mu \\ 0, & \text{for } t \ge \mu, \end{cases}$$

and the inequality is sharp; i.e., the exponential distribution with mean μ attains the lower bound for $t < \mu$, and the degenerate distribution concentrating at μ attains the lower bound for $t \ge \mu$. This situation can be represented graphically as shown in Fig. 25.3.

■ **FIGURE 25.3**
A lower bound on reliability for IFR distributions.



The least upper bound on $R(t)$ that can be obtained if we know only that F is IFR with mean μ is given by

$$R(t) \le \begin{cases} 1, & \text{for } t \le \mu \\ e^{-\omega t}, & \text{for } t > \mu, \end{cases}$$

where ω depends on t and satisfies $1 - \omega\mu = e^{-\omega t}$. It is important to note that the ω in the term $e^{-\omega t}$ is a function of t, so that a different ω must be found for each t. For fixed t and μ, this ω is obtained by finding the intersection of the linear function $(1 - \omega\mu)$ and the exponential function $e^{-\omega t}$. It can be shown that for $t > \mu$, such an intersection always exists.

Thus, $R(t)$ for an IFR distribution with mean μ can be bounded above and below, as shown in Fig. 25.4. Note that the lower bound is the only one of consequence for $t < \mu$, and that the upper bound is the only one of consequence for $t > \mu$.

Increasing Failure Rate Average

Now that bounds on the reliability of a component have been obtained, what can be said about the preservation of *monotone failure rate;* i.e., what structures have the IFR property when their individual components have this property? Series structures of independent IFR (DFR) components are also IFR (DFR), k out of n structures consisting of n identical independent components, each having an IFR failure distribution, are also IFR; however, parallel structures of independent IFR components are not IFR unless they are composed of identical components. Thus, it is evident that, even for some simple systems, there may not be a preservation of the monotone failure rate.

Instead of using the failure rate as a means for characterizing the reliability,

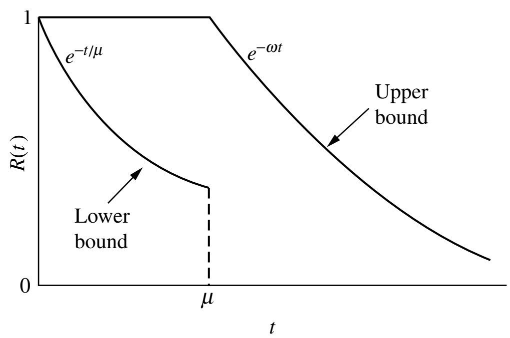
$$R(t) = \exp\left[-\int_0^t r(\xi)\, d\xi\right],$$

a somewhat less appealing characterization can be obtained from the failure-rate average function,

$$\int_0^t \frac{r(\xi)\, d\xi}{t} = -\frac{\log R(t)}{t}.$$

A time-to-failure distribution such that $F(0) = 0$ is called **increasing failure rate average** (IFRA) if and only if

$$\int_0^t \frac{r(\xi)\, d\xi}{t}$$



■ **FIGURE 25.4** Upper and lower bounds on reliability for IFR distributions.

is nondecreasing in $t \geq 0$. A similar definition is given for DFRA. It can be shown that a coherent system of independent components, each of which has an IFRA failure distribution, has a system failure distribution that is also IFRA.

As with IFR systems, there are bounds for IFRA systems. It can be easily shown that IFR distributions are also IFRA distributions (but not the reverse), and the same upper bound as given for IFR distributions is applicable here. A sharp lower bound for IFRA distributions with mean μ is given by

$$R(t) \geq \begin{cases} 0, & \text{for } t \geq \mu \\ e^{-bt}, & \text{for } t < \mu. \end{cases}$$

where b depends upon t and is defined by $e^{-bt} = b(\mu - t)$.

As an example, a monotone system containing only independent components, each of which is exponential (thereby IFRA), is itself IFRA, and the aforementioned bounds are applicable. Furthermore, these bounds are dependent only upon the system mean time to failure.

25.6 CONCLUSIONS

In recent decades, the delivery of systems that perform adequately for a specified period of time in a given environment has become an important goal for both industry and government. In the space program, higher system reliability means the difference between life and death. In general, the cost of maintaining and/or repairing electronic equipment during the first year of operation often exceeds the purchase cost, giving impetus to the study and development of reliability techniques.

This chapter has been concerned with determining system reliability (or bounds) from a knowledge of component reliability or characteristics of components, such as failure rate or mean time to failure. Even the desirable state of knowing these values may lead to cumbersome and sometimes crude results. However, it must be emphasized that these values, e.g., component reliability or mean time to failure, may *not* be known and are often just the design engineers' educated guesses. Furthermore, except in the case of the exponential distribution, knowledge of the mean time to failure leads to nothing but bounds. Also, it is evident that the reliability of components or systems depends heavily upon the failure rate, and the assumption of constant failure rate, which appears to be used frequently in practice, should not be made without careful analysis.

The contents of the chapter have not been concerned with the statistical aspects of reliability, i.e., estimating reliability from test data. This subject was omitted because the book's emphasis is on probability models, but this is not a reflection on its importance. The statistical aspects of reliability may very well be the important problem. Statistical estimation of component reliability is well in hand, but estimation of system reliability from component data is virtually an unsolved problem.

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PROBLEMS

25.1-1. Show that the structure function for a three-component system that functions if and only if component 1 functions *and* at least one of components 2 or 3 functions is given by

$$\begin{aligned}\phi(X_1X_2X_3) &= X_1 \max(X_2, X_3)\\ &= X_1\,[1 - (1 - X_2)(1 - X_3)].\end{aligned}$$

25.1-2. Show that the structure function for a four-component system that functions if and only if components 1 and 2 function *and* at least one of components 3 or 4 functions is given by

$$\phi(X_1, X_2, X_3, X_4) = X_1X_2 \max(X_3, X_4).$$

25.2-1. Find the reliability of the structure function given in Prob. 25.1-1 when each component has probability p_i of performing successfully and the components are independent.

25.2-2. Find the reliability of the structure function given in Prob. 25.1-2 when each component has probability p_i of performing successfully and the components are independent.

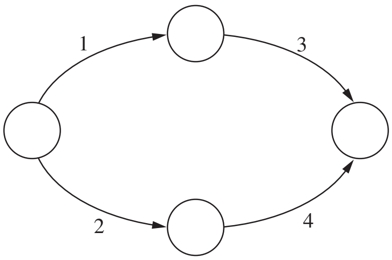
25.3-1. Consider a system consisting of three components (labeled 1, 2, 3) that operate simultaneously. The system is able to function satisfactorily as long as *any two* of the three components are still functioning satisfactorily. The goal is for the system to function satisfactorily for a length of time t, so the system's reliability, $R(t)$, is the probability that this will occur. The times until failure of the individual components are independently (but not identically) distributed, where p_i is the probability that the time until failure of component i exceeds t, for $i = 1, 2, 3$.
(a) Is this a k out of n system? If so, what are k and n?
(b) Draw a network representation of this system.
(c) Develop an explicit expression for the structure function of this system.
(d) Find $R(t)$ as a function of the p_i's.

25.3-2. Consider a system consisting of five components, labeled 1, 2, 3, 4, 5. The system is able to function satisfactorily as long as *at least one* of the following three combinations of components has *every* component in that combination functioning satisfactorily:
(1) Components 1 and 4;
(2) Components 2 and 5;
(3) Components 2, 3, and 4.

For a given amount of time t, let $R_i(t)$ be the known reliability of component i ($i = 1, 2, 3, 4, 5$), that is, the probability that this component will function satisfactorily for this length of time. Assume that the times until failure of the individual components are independently distributed. Let $R(t)$ be the unknown reliability of the overall system.
(a) Draw a network representation of this system.
(b) Develop an explicit expression for the structure function of this system.
(c) Find $R(t)$ as a function of the $R_i(t)$.

25.3-3. Suppose that there exist three different types of components, with two units of each type. Each unit operates independently, and each type has probability p_i of performing successfully. Either one or two systems can be built. One system can be assembled as follows: The two units of each type of component are put together in parallel, and the three types are then assembled to operate in series. Alternatively, two subsystems are assembled, each consisting of the three different types of components assembled in series. The final system is obtained by putting the two subsystems together in parallel. Which system has higher reliability?

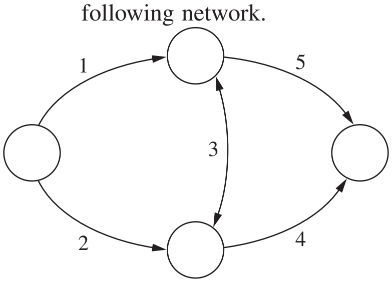
25.4-1. Consider the following network.



Assume that each component is independent with probability p_i of performing satisfactorily.

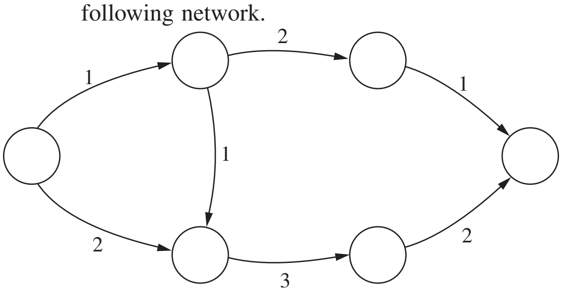
(a) Find all the minimal paths and cuts.
(b) Compute the exact system reliability, and evaluate it when $p_i = p = 0.90$.
(c) Find upper and lower bounds on the reliability, and evaluate them when $p_i = p = 0.90$.

25.4-2. Follow the instructions of Prob. 25.4-1 when using the following network.

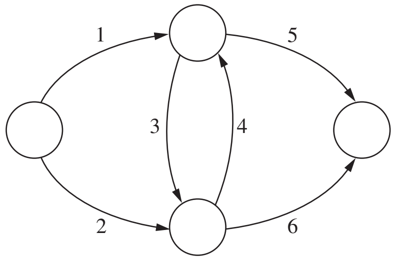


Note that component 3 flows in both directions.

25.4-3. Follow the instructions of Prob. 25.4-1 when using the following network.



25.4-4. Follow the instructions of Prob. 25.4-1 when using the following network.



25.5-1. Suppose F is IFR, with $\mu = 0.5$. Find upper and lower bounds on $R(t)$ for $(a) t = \frac{1}{4}$ and (b) $t = 1$.

25.5-2. A time-to-failure distribution is said to have a Weibull distribution if the cumulative distribution function is given by

$$F(t) = 1 - e^{-t^{\beta}/\eta}, \qquad \text{where } \eta, \beta > 0.$$

Find the failure rate, and show that the Weibull distribution is IFR when $\beta \geq 1$ and DFR when $0 < \beta \leq 1$.

25.5-3. Suppose that a system consists of two different, but independent, components, arranged into a series system. Further assume that the time to failure for each component has an exponential distribution with parameter θ_i, $i = 1, 2$. Show that the distribution of the time to failure of the system is IFR.

25.5-4. Consider a parallel system consisting of two independent components whose time to failure distributions are exponential with parameters μ_1 and μ_2, respectively ($\mu_1 \neq \mu_2$). Show that the time to failure distribution of the system is not IFR.

$$R(t) = P\{T_1 > t \text{ or } T_2 > t\} = 1 - P\{T_1 \leq t \text{ and } T_2 \leq t\}$$
$$= 1 - (1 - e^{-t/\mu_1})(1 - e^{-t/\mu_2}).$$

25.5-5. For Prob. 25.5-4, show that the time to failure distribution is IFRA.