Chapter 8

Options

Learning Objectives

After reading this chapter, you should be able to:

- explain the basic put–call parity formula, how it relates to the value of a forward contract, and the types of option to which put–call parity applies
- relate put–call parity to a boundary condition for the minimum call price, and know the implications of this boundary condition for pricing American call options and determining when they should be exercised
- gather the information needed to price a European option with (a) the binomial model and (b) the Black–Scholes model, and then implement these models to price an option
- understand the Black–Scholes formula in the following form

  $$S_0N(d_1) - \mathrm{PV}(K)N(d_1 - \sigma\sqrt{\mathrm{T}})$$

  and interpret $N(d_1)$
- provide examples that illustrate why an American call on a dividend-paying stock or an American put (irrespective of dividend policy) might be exercised prematurely
- understand the effect of volatility on option prices and premature exercise.

In late 2005, a new controversy hit the financial markets and the corporate world – the backdating of executive share options. This practice involved setting the date of an option award to a time when the company share price was lower than normal, even though the option was granted at a later date. Using this technique, the profit an executive can make when exercising the option is much larger than it should be. Option backdating was found to be widespread, especially among technology firms, and by the end of 2006 had led to the resignation of more than 50 US executives who were alleged to have knowingly used backdating to inflate executive salaries. Other forms of backdating, not necessarily illegal, have also become known, including 'spring loading' and 'bullet dodging', which involve backdating the option award to just before the announcement of good and bad news respectively. While the option backdating controversy has been largely a US issue, the practice could also naturally happen in other countries. For example, in the UK, listed companies can set the date of their option grant, but there is less scope to manipulate the awards, because they can grant options only within 42 days of the annual earnings announcement.

One of the most important applications of the theory of derivatives valuation is the pricing of options. Options, introduced earlier in the text, are ubiquitous in financial markets and, as seen in this chapter and in much of the remainder of this text, are often found implicitly in numerous corporate financial decisions and corporate securities.

Options have long been important to portfolio managers. In recent years they have become increasingly important to corporate treasurers. There are several reasons for this. For one, a corporate treasurer needs to be educated about options for proper oversight of the pension funds of the corporation. In addition, the treasurer needs to understand that options can be useful in corporate hedging,[1] and are implicit in many securities that the corporation issues, such as convertible bonds, mortgages and callable bonds.[2] Options are also an important form of executive compensation and, in some cases, a form of financing for the firm.

This chapter applies the no-arbitrage tracking portfolio approach to derive the 'fair market' values of options, using both the binomial approach and a continuous-time approach known as the Black–Scholes model. It then discusses several practical considerations and limitations of the binomial and Black–Scholes option valuation models. Among these is the problem of estimating volatility. The chapter generalizes the Black–Scholes model to a variety of underlying securities, including bonds and commodities. It concludes with a discussion of the empirical evidence for the Black–Scholes model.

8.1 A Description of Options and Options Markets

There are two basic types of option: call options and put options. The next subsection elaborates on an additional important classification of options.

European and American Options

- A **European** call (put) **option** is the right to buy (sell) a unit of the underlying asset at the strike price, *at a specific point in time*, the expiration or exercise date.
- An **American** call (put) **option** is the right to buy (sell) a unit of the underlying asset *at any time on or before* the expiration date of the option.

Most of the options that trade on organized exchanges are American options. For example, Euronext.liffe, a European derivatives exchange, trades American options, primarily on equities and equity indexes such as the FTSE 100, CAC40 and Bel-20. European options are available, but they are not as popular as their American counterparts, because they can be exercised only on one date, the expiration date.

European options are easier to value than American options, because the analyst need be concerned only about their value at one future date. However, in some circumstances, which will be discussed shortly, European and American options have the same value. Because of their relative simplicity, and because the understanding of European option valuation is often the springboard to understanding the process of valuing the more popular American options, this chapter devotes more space to the valuation of European options than to American options.

The Four Features of Options

Four features characterize a simple option, the offer of the right to buy (call) or the right to sell (put):

1 an underlying risky asset that determines the option's value at some future date[3]
2 a strike price

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[1] Hedging is discussed in Chapter 22.
[2] Callable bonds give the corporation the right to redeem the outstanding bonds before maturity by paying a premium to bondholders. Therefore it is an option that the firm can choose to exercise by 'calling' the bonds. Typically, this will occur when interest rates are low. See Chapter 2.
[3] Typically, the underlying asset is ordinary equity, a portfolio of equities, foreign currencies or futures contracts, but there are many other assets or portfolios of assets on which options can be written. We use the term 'asset' loosely here to mean anything that has an uncertain value over time, be it an asset, a liability, a contract or a commodity. There is a vast over-the-counter market between financial institutions in which options of almost any variety on virtually any underlying asset or portfolio of assets are traded.

3 an **exercise commencement date**, before which the option cannot be exercised
4 an expiration date beyond which the option can no longer be exercised.

Because European options can be exercised only on their expiration dates, their commencement and expiration dates are the same. American options, which can be exercised at any time on or before their expiration dates, have commencement dates that coincide with their dates of initial issuance.[4]

8.2 Option Expiration

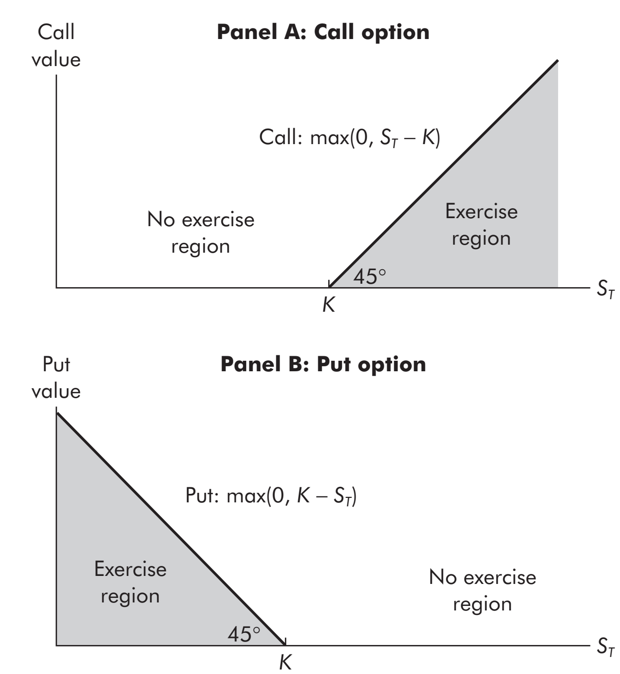
Exhibit 8.1, first seen in Chapter 7, graphs the value of a call and put option at the expiration date against the value of the underlying asset at expiration. In this chapter we attach some algebra to the graphs of call and put values. For expositional simplicity, we shall often refer to the underlying asset as a share of ordinary equity, but our results also apply to options on virtually any financial instrument.

The uncertain future share price at the expiration date, T, is denoted by S_T. The strike or exercise price is denoted by K. The expiration value for the call option is the larger of zero and the difference between the share price at the expiration date and the strike price, denoted as $\max(0, S_T - K)$. For the put option, the expiration value is $\max(0, K - S_T)$.

Note that the two graphs in Exhibit 8.1 never lie below the horizontal axis. They either coincide with the axis or lie above it on the 45° line. In algebraic terms, the expressions for the future call value, $\max(0, S_T - K)$, and the future put value, $\max(0, K - S_T)$, are never negative. Recall from Chapter 7 that options can never have a negative value, because options expire unexercised if option exercise hurts the option holder. The absence of a negative future value for the option and the possibility of a positive future value make paying for an option worth while.

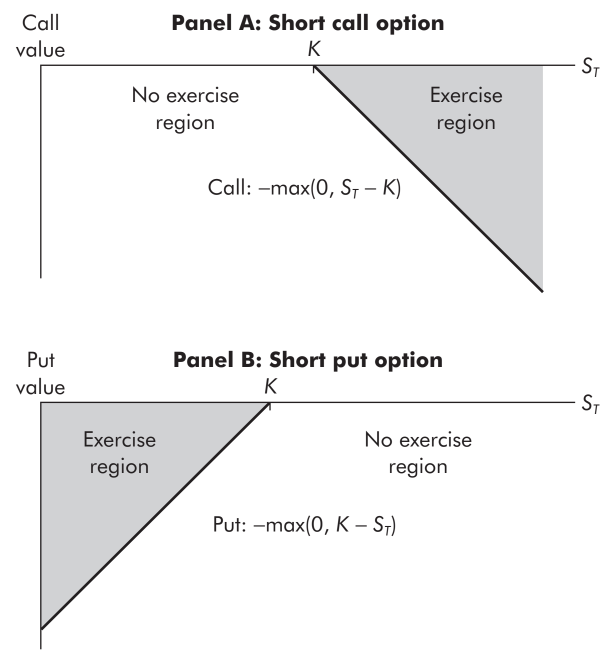
Future cash flows are never positive when writing an option. Exhibit 8.2 illustrates the value at expiration of the short position generated by writing an option. When the call's strike price, K, exceeds the future share price S_T (or S_T exceeds K for the put), the option expires unexercised. On the other hand, if S_T exceeds K, the call writer has to sell a share of equity for less than its fair value. Similarly, if K exceeds S_T, the put writer has to buy a share of equity for more than it is worth. In all cases, there is no positive future

Exhibit 8.1 The Value of a Call Option and a Put Option at Expiration



[4] Deferred American options, not discussed here, have issue dates that precede their commencement dates.

Exhibit 8.2 The Value of Short Positions in Call and Put Options at Expiration



cash flow to the option writer. To compensate the option writer for these future adverse consequences, the option buyer pays money to the writer to acquire the option.

Finally, observe that the non-random number S_0, which denotes the current share price, does not appear in Exhibits 8.1 and 8.2 because the focus is only on what happens at option expiration. One of the goals of this chapter is to translate the future relation between the equity value and the option value into a relation between the current value of the equity and the current value of the option. The next section illustrates the type of reasoning used to derive such a relation.

8.3 Put–Call Parity

With some rudimentary understanding of the institutional features of options behind us, we now move on to analyse their valuation. One of the most important insights in option pricing, developed by Stoll (1969), is known as the **put–call parity formula**. This equation relates the prices of European calls to the prices of European puts. However, as this section illustrates, this formula also is important because it has a number of implications that go beyond relating call prices to put prices.

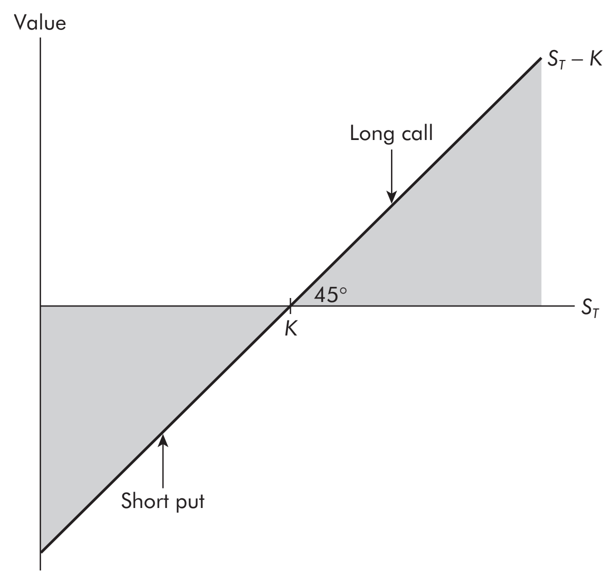
Put–Call Parity and Forward Contracts: Deriving the Formula

Exhibit 8.3 illustrates the value of a long position in a European call option at expiration and a short position in an otherwise identical put option. This combined pay-off is identical to the pay-off of a forward contract, which is the obligation (not the option) to buy the underlying asset for K at the expiration date.[5]

Result 7.1 in Chapter 7 indicated that the value of a forward contract with a strike price of K on a non-dividend-paying stock is $S_0 - K/(1 + r_f)^T$, the difference between the current share price and the present value of the strike price (which we also denote as PV(K)), obtained by discounting K at the risk-free rate. The forward contract has this value because it is possible to track this pay-off perfectly by purchasing one

[5] In contrast to options, forward contracts – as obligations to purchase at pre-specified prices – can have positive or negative values. Typically, the strike price of a forward contract is set initially, so that the contract has zero value. In this case, the pre-specified price is known as the forward price. See Chapter 7 for more detail.

Exhibit 8.3 The Value at Expiration of a Long Position in a Call Option and a Short Position in a Put Option with the Same Features



share of equity and borrowing the present value of K.[6] Since the tracking portfolio for the forward contract, with future (random) value $\tilde{S}_T - K$, also tracks the future value of a long position in a European call and a short position in a European put (see Exhibit 8.3), to prevent arbitrage the tracking portfolio and the tracked investment must have the same value today.

Results

Result 8.1

(*The put–call parity formula*.) If no dividends are paid to holders of the underlying equity prior to expiration, then, assuming no arbitrage,

$$c_0 - p_0 = S_0 - \text{PV}(K) \tag{8.1}$$

That is, a long position in a call and a short position in a put sells for the current share price less the strike price discounted at the risk-free rate.

Using Tracking Portfolios to Prove the Formula

Exhibit 8.4a, which uses algebra, and 8.4b, which uses numbers, illustrate this point further. The columns in the exhibits compare the value of a forward contract implicit in buying a call and writing a put (investment 1) with the value from buying the equity and borrowing the present value of K (investment 2, the tracking portfolio), where K, the strike price of both the call and put options, is assumed to be 50 in Exhibit 8.4b. Column (1) makes this comparison at the current date. In this column, C_0 denotes the current value of a European call and p_0 the current value of a European put, each with the same time to expiration. Columns (2) and (3) in Exhibit 8.4a make the comparison between investments 1 and 2 at expiration, comparing the future values of investments 1 and 2 in the exercise and no exercise regions of the two European options. To represent the uncertainty of the future share price, Exhibit 8.4b assumes three possible future share prices at expiration: $S_T = 45$, $S_T = 52$ and $S_T = 59$, which correspond to columns (2), (3) and (4), respectively. In Exhibit 8.4b, columns (3) and (4) have call option exercise, while column (2) has put option exercise.

[6] Example 7.4 in Chapter 7 proves this.

Exhibit 8.4a Comparing Two Investments (Algebra)

Investment	Cost of acquiring the investment today (1)	Cash flows at the expiration date if S at that time is:	
		$\tilde{S}_T < K$ (2)	$\tilde{S}_T > K$ (3)
Investment 1: Buy call and write put = Long position in a call and short position in a put	$C_0 - P_0$ $= C_0$ $-P_0$	$\tilde{S}_T - K$ $= 0$ $-(K - \tilde{S}_T)$	$\tilde{S}_T - K$ $= \tilde{S}_T - K$ $- 0$
Investment 2: Tracking portfolio = Buy equity and borrow present value of K	$S_0 - \text{PV}(K)$ $= S_0$ $-\text{PV}(K)$	$\tilde{S}_T - K$ $= \tilde{S}_T$ $-K$	$\tilde{S}_T - K$ $= \tilde{S}_T$ $- K$

Exhibit 8.4b Comparing Two Investments (Numbers) with K = €50

Investment	Cost of acquiring the investment today (1)	Cash flows at the expiration date if S at that time is:		
		€45 (2)	€52 (3)	€59 (4)
Investment 1: Buy call and write put = Long position in a call and short position in a put	$C_0 - P_0$ $= C_0$ $-P_0$	–€5 = 0 –(€50 – €45)	€2 = (€52 – €50) –0	€9 = (€59 – €50) –0
Investment 2: Tracking portolio = Buy equity and borrow present value of K	S_0 – PV(€50) = S_0 –PV(€50)	–€5 = €45 –€50	€2 = €52 –€50	€9 = €59 –€50

Since the cash flows at expiration from investments 1 and 2 are both $\tilde{S}_T - K$, irrespective of the future value realized by $\tilde{S}_T$, the date 0 values of investments 1 and 2 observed in column (1) have to be the same if there is no arbitrage. The algebraic statement of this is equation (8.1).

An Example Illustrating How to Apply the Formula

Example 8.1 illustrates an application of Result 8.1.

Example 8.1

Comparing Prices of At-the-Money Calls and Puts

An at-the-money option has a strike price equal to the current share price. Assuming no dividends, what sells for more: an at-the-money European put or an at-the-money European call?

Answer: From put–call parity, $c_0 - p_0 = S_0 - \text{PV}(K)$. If $S_0 = K$, $c_0 - p_0 = K - \text{PV}(K) > 0$. Thus the call sells for more.

Arbitrage When the Formula is Violated

Example 8.2 demonstrates how to achieve arbitrage when equation (8.1) is violated.

Put–Call Parity and a Minimum Value for a Call

The put–call parity formula provides a lower bound for the current value of a call, given the current share price. Because it is necessary to subtract a non-negative current put price, p_0, from the current call price, c_0, in order to make $c_0 - p_0 = S_0 - \text{PV}(K)$, it follows that

$$C_0 \geq S_0 - \mathrm{PV}(K) \qquad (8.1a)$$

Thus the minimum value of a call is the current price of the underlying equity, S_0, less the present value of the strike price, PV(K).

Example 8.2

Generating Arbitrage Profits When there is a Violation of Put–Call Parity

Assume that a one-year European call on a €45 share of equity with a strike price of €44 sells for c_0 – €2 and a European put on the same equity (S_0 = €45), with the same strike price (€44) and time to expiration (one year), sells for P_0 = €1. If the one-year, risk-free interest rate is 10 per cent, the put–call parity formula is violated, and there is an arbitrage opportunity. Describe it.

Answer: PV(K) = €44/1.1 = €40. Thus $C_0 - P_0$ = €1 is less than $S_0 - \mathrm{PV}(K)$ = €5. Therefore buying a call and writing a put is a cheaper way of producing the cash flows from a forward contract than buying equity and borrowing the present value of K. Pure arbitrage arises from buying the cheap investment and writing the expensive one: that is,

1. Buy the call for €2.
2. Write the put for €1.
3. Sell short the equity and receive €45.
4. Invest €40 in the 10 per cent risk-free asset.

This strategy results in an initial cash inflow of €4.

- If the share price exceeds the strike price at expiration, exercise the call, using the proceeds from the risk-free investment to pay for the €44 strike price. Close out the short position in the equity with the share received at exercise. The worthless put expires unexercised. Hence there are no cash flows and no positions left at expiration, but the original €4 is yours to keep.
- If the share price at expiration is less than the €44 strike price, the put will be exercised and you, as the put writer, will be forced to receive a share of equity and pay €44 for it. The share you acquire can be used to close out your short position in the equity, and the €44 strike price comes out of your risk-free investment. The worthless call expires unexercised. Again, there are no cash flows or positions left at expiration. The original €4 is yours to keep.

Put–Call Parity and the Pricing and Premature Exercise of American Calls

This subsection uses equation (8.1) to demonstrate that the values of American and European call options on non-dividend-paying equities are the same. This is a remarkable result and, at first glance, a bit surprising. American options have all the rights of European options, plus more: thus values of American options should equal or exceed the values of their otherwise identical European counterparts. In general, American options may be worth more than their European counterparts, as this chapter later demonstrates with puts. What is surprising is that they are *not always* worth more.

Premature Exercise of American Call Options on Equities with No Dividends before Expiration

American options are worth more than European options only if the right of premature exercise has value. Before expiration, however, the present value of the strike price of any option, European or American, is less than the strike price itself: that is, $\mathrm{PV}(K) < K$. Inequality (8.1a) – which, as an extension of the put–call parity formula, assumes no dividends before expiration – is therefore the strict inequality

$$c_0 > S_0 - K \qquad (8.1b)$$

before expiration, which must hold for both European and American call options. Inequality (8.1b) implies Result 8.2.

Result 8.2

It never pays to exercise an American call option prematurely on an equity that pays no dividends before expiration.

Results

One should never prematurely exercise an American call option on an equity that pays no dividends before expiration because exercising generates cash of $S_0 - K$, while selling the option gives the seller cash of c_0, which is larger by inequality (8.1b). This suggests that waiting until the expiration date always has some value, at which point exercise of an in-the-money call option should take place.

What if the market price of the call happens to be the same as the call option's exercise value? Example 8.3 shows that there is arbitrage if an American call option on a non-dividend-paying equity does not sell for (strictly) more than its premature exercise value before expiration.

Example 8.3

Arbitrage When a Call Sells for its Exercise Value

Consider an American call option with a £4.00 strike price on Pipex Communications plc, a British telecommunications and Internet services firm. Assume that the equity sells for £4.50 a share, and pays no dividends to expiration. The option sells for £0.50 one year before expiration. Describe an arbitrage opportunity, assuming the interest rate is 10 per cent per year.

Answer: Sell short a share of Pipex equity and use the £4.50 you receive to buy the option for £0.50 and place the remaining £4.00 in a savings account. The initial cash flow from this strategy is zero. If the equity is selling for more than £4.00 at expiration, exercise the option and use your savings account balance to pay the strike price. Although the equity acquisition is used to close out your short position, the £0.40 interest (£4.00 × 0.1) on the savings account is yours to keep. If the share price is less than £4.00 at expiration, buy the equity with funds from the savings account to cancel the short position. The £0.40 interest in the savings account and the difference between the £4.00 (initial principal in the savings account) and the share price is yours to keep.

Holding onto an American call option instead of exercising it prematurely is like buying the option at its current exercise value in exchange for its future exercise value: that is, not exercising the option means giving up the exercise value today in order to maintain the value you will get from exercise at a later date. Hence the cost of not exercising prematurely (that is, waiting) is the lost exercise value of the option.

Example 8.3 shows that the option to exercise at a later date is more valuable than the immediate exercise of an option. An investor can buy the option for £0.50, sell the equity short and gain an arbitrage opportunity by exercising the option in the future if it pays to do so. Thus the option has to be worth more than the £0.50 it costs. It follows that holding on to an option already owned has to be worth more than the £0.50 received from early exercise (see exercise 8.1 at the end of the chapter).

When Premature Exercise of American Call Options Can Occur

Result 8.2 does not necessarily apply to an underlying security that pays cash prior to expiration. This can clearly be seen in the case of an equity that is about to pay a liquidating dividend. An investor needs to exercise an in-the-money American call option on the equity before the **ex-dividend date** of a liquidating dividend, which is the last date one can exercise the option and still receive the dividend.[7] The option is worthless

[7] Ex-dividend dates, which are ex-dates for dividends (see Chapter 2 for a general definition of ex-dates), are determined by the record dates for dividend payments and the settlement procedures of the securities market. The record date is the date when the legal owner of the equity is put on record for purposes of receiving a corporation's dividend payment. On NYSE Euronext, an investor is not the legal owner of an equity until three business days after the order has been executed. This makes the ex-date three business days before the record date.

thereafter, since it represents the right to buy a share of a company that has no assets. All dividends dissipate some of a company's assets. Hence, for similar reasons, even a small dividend with an ex-dividend date shortly before the expiration date of the option could trigger an early exercise of the option.

Early exercise is also possible when the option cannot be valued by the principle of no arbitrage, as would be the case if the investor was prohibited from selling short the tracking portfolio. This issue arises in many executive share options. Executive share options awarded to the company's CEO may make the CEO's portfolio more heavily weighted towards the company than prudent mean-variance analysis would dictate it should be. One way to eliminate this diversifiable risk is to sell short the company's equity (a part of the tracking portfolio), but the CEO and most other top corporate executives who receive such options are prohibited from doing this. Another way is to sell the options, but this too is prohibited. The only way for an executive to diversify is to exercise the option, take the equity, and then sell the shares. Such suboptimal exercise timing, however, does not capture the full value of the executive share option. Nevertheless, the CEO may be willing to lose a little value to gain some diversification.

Except for these two cases – one in which the underlying asset pays cash before expiration and the other, executive share options, for which arbitraging away violations of inequality (8.1b) is not possible because of market frictions – one should not prematurely exercise a call option.

When dividends or other forms of cash on the underlying asset are paid, the appropriate timing for early exercise is described in the following generalization of Result 8.2.

Results

Result 8.3

An investor does not capture the full value of an American call option by exercising between ex-dividend or (in the case of a bond option) ex-coupon dates.

Only at the ex-dividend date, just before the drop in the price of the security caused by the cash distribution, is such early exercise worth while.

To understand Result 8.3, consider the following example. Suppose it is your birthday, and Aunt Michelle sends you one share of Allianz equity, an insurance firm listed on the Frankfurt Stock Exchange. Uncle Kevin, however, gives you a gift certificate entitling you to one share of Allianz equity on your next birthday, one year hence. Provided that Allianz pays no dividends within the next year, the values of the two gifts are the same. The value of the deferred gift is the cost of Allianz equity on the date of your birthday. Putting it differently, in the absence of a dividend, the only right obtained by receiving a security early is that you will have it at that later date. You might retort that if you wake up tomorrow and think Allianz's share price is going down, you will sell Aunt Michelle's Allianz share, but you are forced to receive Uncle Kevin's share. However, this is not really so, because at any time you think Allianz's share price is headed down, you can sell short Allianz and use Uncle Kevin's gift to close out your short position. In this case, the magnitude and timing of the cash flows from Aunt Michelle's gift of Allianz, which you sell tomorrow, are identical to those from the combination of Uncle Kevin's gift certificate and the Allianz short position that you execute tomorrow. Hence you *do not* create value by receiving shares on non-dividend-paying equities early, but you *do* create value by deferring the payment of the strike price – increasing wealth by the amount of interest collected in the period before exercise.

Thus with an option, or even with a forward contract on a non-dividend-paying security, paying for the security at the latest date possible makes sense. With an option, however, you have a further incentive to wait: if the security later goes down in value, you can choose not to acquire the security by not exercising the option, and if it goes up, you can exercise the option and acquire the security.

Premature Exercise of American Put Options

The value from waiting to see how the security turns out also applies to a put option. With a put, however, the option holder receives rather than pays out cash upon exercise, and the earlier the receipt of cash, the better. With a put, an investor trades off the interest earned from receiving cash early against the value gained from waiting to see how things will turn out. A put on an equity that sells for pennies with a strike price of £4.00 probably should be exercised. At best, waiting can provide only a few more pennies (the equity cannot sell for less than zero), which should easily be covered by the interest on the £4.00 received.

Relating the Price of an American Call Option to an Otherwise Identical European Call Option

If it is never prudent to exercise an American call option prior to expiration, then the right of premature exercise has no value. Thus in cases where this is true (for example, no dividends), the no-arbitrage prices of American and European call options are the same.

Result 8.4

If the underlying equity pays no dividends before expiration, then the no-arbitrage values of American and European call options with the same features are the same.

Results

Put–Call Parity for European Options on Dividend-Paying Equities

If there are riskless dividends, it is possible to modify the put–call parity relation. In this case, the forward contract with price $c_0 - p_0$ is worth less than the equity minus the present value of the strike price, $S_0 -$ PV(K). Buying a share of equity and borrowing PV(K) now also generates dividends that are not received by the holder of the forward contract (or, equivalently, the long call plus short put position). Hence, although the tracking portfolio and the forward contract have the same value at expiration, their intermediate cash flows do not match. Only the tracking portfolio receives a dividend prior to expiration. Hence the value of the tracking portfolio (investment 2 in Exhibit 8.4) exceeds the value of the forward contract (investment 1) by the present value of the dividends, denoted PV(div) (with PV(div) obtained by discounting each dividend at the risk-free rate and summing the discounted values).

Result 8.5

(*Put–call parity formula generalized.*) $c_0 - p_0 - S_0 - \text{PV}(K) - \text{PV}(\text{div})$. The difference between the no-arbitrage values of a European call and a European put with the same features is the current share price less the sum of the present value of the strike price and the present value of all dividends to expiration.

Results

Example 8.4 applies this formula to illustrate how to compute European put values in relation to the known value of a European call on a dividend-paying equity.

Put–Call Parity and Corporate Securities as Options

Important option-based interpretations of corporate securities can be derived from put–call parity. Equity can be thought of as a call option on the assets of the firm.[8] This arises because of the limited liability of corporate equity holders. Consider a simple two-date model (dates 0 and 1) in which a firm has debt with a face value of K to be paid at date 1, assets with a random pay-off at date 1, and no dividend payment at or before date 1. In this case, equity holders have a decision to make at date 1. If they pay the face value of the debt (the strike price), they receive the date 1 cash flows of the assets. On the other hand, if the assets at date 1 are worth less than the face value of the debt, the firm is bankrupt, and the equity holders walk away from the firm with no personal liability. Viewed from date 0, this is simply a call option to buy the firm's assets from the firm's debt holders.[9]

[8] See Chapter 16 for more information on this subject.

[9] This analysis is easily modified to accommodate riskless dividends. In this case, equity holders have a claim to a riskless dividend plus a call option on the difference between the firm's assets and the present value of the dividend. The dividend-inclusive put–call parity formula can then be used to interpret corporate debt.

Example 8.4

Inferring Put Values from Call Values on a Dividend-Paying Stock

Assume that the shares of Accor, a French hotel group, currently sell for €100. A European call on Accor has a strike price of €121, expires two years from now, and currently sells for €20. What is the value of the comparable European put? Assume the risk-free rate is 10 per cent per year and that Accor is certain to pay a dividend of €2.75 one year from now.

Answer: From put–call parity

$$€20 - p_0 = €100 - \frac{€121}{1.1^2} - \frac{€2.75}{1.1}$$

or

$$p_0 = -(€100 - €100 - €2.50 - €20) = €22.50$$

Since the value of debt plus the value of equity adds up to the total value of assets at date 0, one also can view corporate bonds as a long position in the firm's assets and a short position in a call option on the firm's assets. With S_0 now denoting the current value of the firm's assets, c_0 denoting the current value of its equity, K denoting the face value of the firm's debt, and D_0 as the market (current) value of its debt, the statement that bonds are assets less a call option is represented algebraically as

$$D_0 = S_0 - c_0$$

However, when using the no-dividend put–call parity formula, $c_0 - p_0 = S_0 - \text{PV}(K)$, to substitute for c_0 in this expression, risky corporate debt is

$$D_0 = \text{PV}(K) - p_0$$

One can draw the following conclusion, which holds even if dividends are paid prior to the debt maturity date.

Results

Result 8.6

It is possible to view equity as a call option on the assets of the firm and to view risky corporate debt as riskless debt worth PV(K) plus a short position in a put option on the assets of the firm ($-p_0$) with a strike price of K.

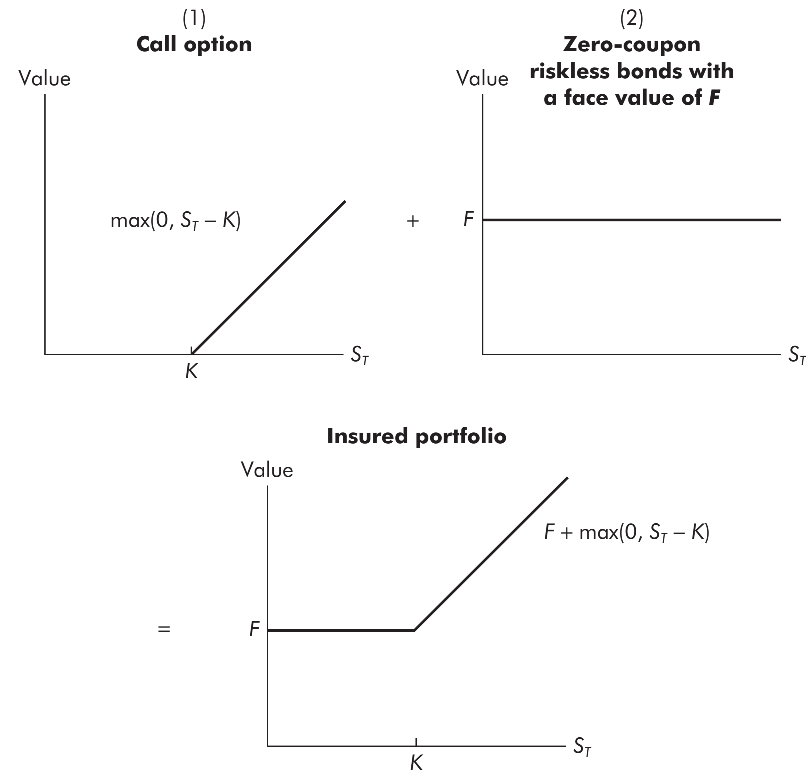
Because corporate securities are options on the firm's assets, any characteristic of the assets of the firm that affects option values will alter the values of debt and equity. One important characteristic of the underlying asset that affects option values, presented later in this chapter, is the variance of the asset return.

Result 8.6 also implies that the more debt a firm has, the less in the money is the implicit option in equity. Thus knowing how option risk is affected by the degree to which the option is in or out of the money may shed light on how the mix of debt and equity affects the risk of the firm's debt and equity securities.

Finally, because equity is an option on the assets of the firm, a call option on the equity of a firm is really an option on an option, or a **compound option**.[10] The binomial derivatives valuation methodology developed in Chapter 7 can be used to value compound options.

[10] The valuation of compound options was first developed in Geske (1979).

Exhibit 8.5 Horizon Date Values of the Two Components of an Insured Portfolio



Put–Call Parity and Portfolio Insurance

In the mid-1980s the firm Leland, O'Brien and Rubinstein, or LOR, developed a successful financial product known as portfolio insurance. **Portfolio insurance** is an option-based investment that, when added to the existing investments of a pension fund or mutual fund, protects the fund's value at a target horizon date against drastic losses.

LOR noticed that options have the desirable feature of unlimited upside potential with limited downside risk. Exhibit 8.5 demonstrates that if portfolios are composed of

- a call option, expiring on the future horizon date, with a strike price of K, and
- riskless zero-coupon bonds worth F, the floor amount, at the option expiration date

the portfolio's value at the date the options expire would never fall below the value of the riskless bonds, the floor amount, at that date. If the underlying asset of the call option performed poorly, the option would expire unexercised; however, because the call option value in this case is zero, the portfolio value would be the value of the riskless bonds. If the underlying asset performed well, the positive value of the call option would enhance the value of the portfolio beyond its floor value. In essence, this portfolio is insured.

The present value of the two components of an insured portfolio is

$$c_0 + \text{PV}(F)$$

where PV(F) is the floor amount, discounted at the risk-free rate.

The problem is that the portfolios of pension funds and mutual funds are not composed of riskless zero-coupon bonds and call options. The challenge is how to turn them into something with similar payoffs. As conceived by LOR, portfolio insurance is the acquisition of a put on an equity index. The put's strike price determines an absolute floor on losses due to movements in the equity index. The put can be either purchased directly or produced synthetically by creating the put's tracking portfolio (see Chapter 7). Because of the lack of liquidity in put options on equity indexes at desired strike prices and maturities, the tracking portfolio is typically constructed from a dynamic strategy in stock index futures. For a fee, LOR's computers would tailor a strategy to meet a fund's insurance objectives.[11]

To understand how portfolio insurance works, note that the extended put–call parity formula in Result 8.5, $c_0 - p_0 = S_0 - \text{PV}(K) - \text{PV}(\text{div})$, implies that the *present value* of the desired insured portfolio is

$$c_0 + \text{PV}(F) = S_0 + p_0 - [\text{PV}(\text{div}) + \text{PV}(K) - \text{PV}(F)]$$

where

S_0 = the current value of the uninsured equity portfolio

p_0 = the cost of a put with a strike price of K

PV(div) = the present value of the uninsured equity portfolio's dividends.

The left-hand side of the equation is the present value of a desired insured portfolio with a floor of F. The right-hand side implies that if an investor starts with an uninsured equity portfolio at a value of S_0, he or she must acquire a put.

If there is to be costless portfolio insurance (that is, no liquidation of the existing portfolio to buy the portfolio insurance), the left-hand side of the equation must equal S_0. With such costless insurance, the expression in brackets above must equal the cost of the put. This implies that the floor amount, F, and the strike price of the put, K, which also affects p_0, must be chosen judiciously.

8.4 Binomial Valuation of European Options

Chapter 7 illustrated how to value any derivative security with the risk-neutral valuation method. Valuing European options with this method is simply a matter of applying the risk-neutral probabilities to the expiration date values of the option and discounting the risk-neutral weighted average at the risk-free rate. This section applies the risk-neutral valuation method to algebraic symbols that represent the binomial expiration date values of a European call option in order to derive an analytic formula for valuing European call options. (Put–call parity can be used to obtain the European put formula.) To simplify the algebra, assume that the one-period risk-free rate is constant, and that the ratio of price in the next period to price in this period is always u or d.

This section first analyses the problem of valuing a European call one period before expiration. It then generalizes the problem to one of valuing a call T periods before expiration. According to Result 8.4, if there are no dividends, the formula obtained also applies to the value of an American call.

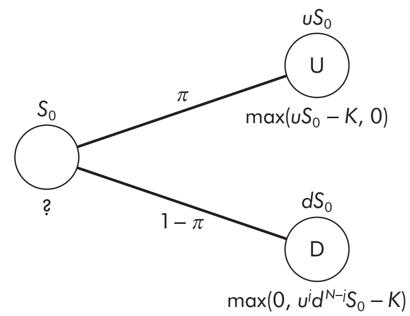
Exhibit 8.6 illustrates the investor's view of the tree diagram one period before the option expiration date. Both the values of the equity (above the nodes) and the call option (below the nodes) are represented.[12] We noted in Chapter 7 that it is possible to value the cash flows of any derivative security after computing the risk-neutral probabilities for the equity. The hypothetical probabilities that would exist in a risk-neutral world must make the expected return on the equity equal the risk-free rate. The risk-neutral probabilities for the up and down moves that do this, π and $1 - \pi$, respectively, satisfy

$$\pi = \frac{1 + r_f - d}{u - d}$$

[11] Proper risk management is important in the management of portfolios and corporations. A casual reading of the business press would have you believe that all derivative securities are extremely risky. Here, however, the acquisition of protective puts can reduce risk by placing a floor on one's losses.

[12] The expiration date prices of the equity and call are at the two circular nodes on the right-hand side of the tree diagram, U and D, in Exhibit 8.6. The prices one period before are at the single node on the left-hand side.

Exhibit 8.6 Tree Diagram for the Value of the Equity (Above Node) and the Call Option (Below Node) Near the Expiration Date



and

$$1 - \pi = \frac{u - 1 - r_f}{u - d}$$

where

u = ratio of next period's share price to this period's price if the up state occurs

d = ratio of next period's share price to this period's price if the down state occurs

r_f = risk-free rate.

Discounting the risk-neutral expected value of the expiration value of the call at the risk-free rate yields the proper no-arbitrage call value, c_0, as a function of the share price, S_0, in this simple case, namely:

$$c_0 = \frac{\pi \max(uS_0 - K, 0) + (1 - \pi)\max(ds_0 - k, 0)}{1 + r_f}$$

With N periods to expiration, the risk-neutral expected value of the expiration value of the call, discounted at the risk-free rate, again generates the current no-arbitrage value of the call. Now, however, there are $N + 1$ possible final call values, each determined by the number of up moves, 0, 1, . . . , N. There is only one path for N up moves, and the risk-neutral probability of arriving there is π^N. The value of the option with a strike price of K at this point is $\max(0, u^N S_0 - K)$. For $N - 1$ up moves, the value of the option is $\max(0, u^{N-1}S_0 - K)$, which multiplies the risk-neutral probability of $\pi^{N-1}(1 - \pi)$. However, there are N such paths, one for each of the N dates at which the single down move can occur. For $N - 2$ up moves, each path to $\max(0, u^{N-2}d^2S_0 - K)$ has a risk-neutral probability of $\pi^{N-2}(1 - \pi)^2$. There are $N(N - 1)/2$ such paths.

In general, for j up moves, $j = 0, \ldots, N$, each path has a risk-neutral probability of $\pi^j(1 - \pi)^{N-j}$, and there are $N!/[j!(N - j)!]$ such paths to the associated value of $\max(0, u^j d^{N-j} S_0 - K)$.[13] Therefore the 'expected' future value of a European call option, where the expectation uses the risk-neutral probabilities to weight the outcome, is

$$\sum_{j=0}^{N} \frac{N!}{j!(N - j)!} \pi^j (1 - \pi)^{N-j} \max(0, u^j d^{N-j} S_0 - K)$$

[13] The expression $n!$ means $n(n - 1)(n - 2) \ldots \times 3 \times 2 \times 1$, with the special case of $0!$ being equal to 1.

This expression, discounted at the risk-free rate of r_f per binomial period, gives the value of the call option.

Results

Result 8.7

(*The binomial formula*.) The value of a European call option with a strike price of K and N periods to expiration on an expiration with no dividends to expiration and a current value of S_0 is

$$c_0 = \frac{1}{(1+r_f)^N} \sum_{j=0}^{N} \frac{N!}{j!(N-j)!} \pi^j (1-\pi)^{N-j} \max(0, u^j d^{N-j} S_0 - K) \quad (8.2)$$

where

- r_f = risk-free return per period
- π = risk-neutral probability of an up move
- u = ratio of the share price to the prior share price, given that the up state has occurred over a binomial step
- d = ratio of the share price to the prior share price, given that the down state has occurred over a binomial step.

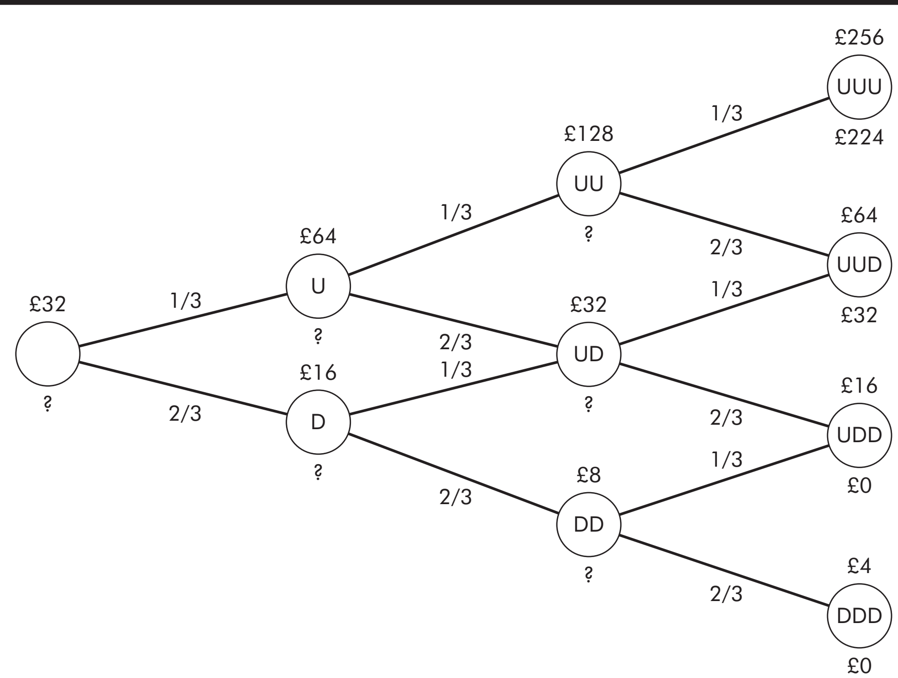
Example 8.5 applies this formula numerically.

Example 8.5

Valuing a European Call Option with the Binomial Formula

Use equation (8.2) to find the value of a three-month at-the-money call option on Smith Group plc trading at £32 a share. To keep the computations simple, assume that $r_f = 0$, $u = 2$, $d = 0.5$, and the number of periods (computed as months), $N = 3$.

Exhibit 8.7 Binomial Tree Diagram for the Value of Smith Group Equity (Above Node) and the Smith Group Call Option (Below Node)



Answer: Exhibit 8.7 illustrates the tree diagram for Smith Group's equity and call option. The shares can have a final value (seen on the right-hand side of the diagram) of £256 (three up moves to node UUU), £64 (two up moves to node UUD), £16 (one up move to node UDD) or £4 (0 up moves to node DDD). The corresponding values for the option are £224 = $u^3S_0 > K$, £32 = $u^2dS_0 > K$, £0 and £0. Since $\pi = 1/3 = (1 + r_f - d)/(u - d)$ and $(1 - \pi) = 2/3$ for these values of u, d and r_f, the expected future value of the option using risk-neutral probabilities is

$$\left(\frac{1}{3}\right)^3(£256 - £32) + 3\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)(£64 - £32) + 3\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^2(£0) + \left(\frac{2}{3}\right)^3(£0) = £15.41$$

Since the discount rate is 0, £15.41 is also the value of the call option.

8.5 Binomial Valuation of American Options

The last section derived a formula for valuing European calls.[14] This section illustrates how to use the binomial model to value options that may be prematurely exercised. These include American puts and American calls on dividend-paying equities.

American Puts

The procedure for modelling American option values with the binomial approach is similar to that for European options. As with European options, work backwards from the right-hand side of the tree diagram. At each node in the tree diagram, look at the two future values of the option, and use risk-neutral discounting to determine the value of the option at that node. In contrast to European options, however, this value is only the value of the option at that node, provided that the investor holds on to it for one more period. If the investor exercises the option at the node, and the underlying asset is worth S at the node, then the value captured is $S - K$ for a call and $K - S$ for a put, rather than the discounted risk-neutral expectation of the values at the two future nodes, as was the case for the European option.

This suggests a way to value the option, assuming that it will be optimally exercised. Working backwards, at each node, compare (1) the value from early exercise of the option with (2) the value of waiting one more period and achieving one of two values. The value to be placed at that decision node is the larger of the exercise value and the value of waiting. Example 8.6 illustrates this procedure for an American put.

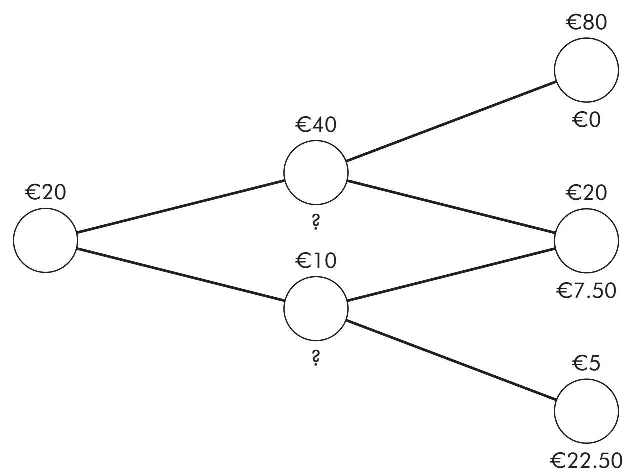
Note that, even with the possibility of early exercise, it is still possible to track the put option in Example 8.6 with a combination of BBVA equity and a risk-free asset. However, the tracking portfolio here is different from that for a European put option with comparable features. In tracking the American put, the major difference is due to what happens at node D in Exhibit 8.8, when early exercise is optimal. To track the option value, do not solve for a portfolio of the equity and the risk-free security that has a value of €3.00 if an up move occurs and €12.00 if a down move occurs, but solve for the portfolio that has a value of €3.00 if an up move occurs and €17.50 if a down move occurs. This still amounts to solving two equations with two unknowns. With the European option, the variation between the up and down state is –€9.00. With the American option in Example 8.6, the variation is –€14.50. Since the variation in the share price between nodes U and D is €30 (= €40 – €10), –0.3 (= –€9/€30) of a share perfectly tracks a European option, and –0.483333 (= –€14.50/€30) of a share perfectly tracks an American option. The associated risk-free investments solve

$$-0.3(€40) + 1.25x = €3 \text{ or } x = €12 \text{ for the European put}$$

$$-0.483333(€40) + 1.25x = €3 \text{ or } x = €17.867 \text{ for the American put}$$

[14] As suggested earlier, American calls on equities that pay no dividends until expiration can be valued as European calls with the method described in the last section.

Exhibit 8.8 Binomial Tree Diagram for the Price of BBVA Equity (Above Node) and an American Put Option (Below Node) with K = €27.50 and T = 2



Example 8.6

Valuing an American Put

Assume that, in each period, the non-dividend-paying equity of BBVA, a major Spanish bank, can either double or halve in value: that is, $u = 2$, $d = 0.5$. If the initial share price is €20 per share and the risk-free rate is 25 per cent per period (this is incredibly high, but we need it to make our example work!), what is the value of an American put expiring two periods from now with a strike price of €27.50?

Answer: Exhibit 8.8 outlines the path of BBVA's equity and the option. At each node in the diagram, the risk-neutral probabilities solve

$$\frac{2\pi + 0.5(1 - \pi)}{1.25} = 1$$

Thus $\pi = 0.5$ throughout the problem.

In the final period, at the far right-hand side of the diagram, the equity is worth €80, €20 or €5. From node U, where the share price is €40, the price can move to €80 (node UU) with an associated put value of €0, or it can move to €20 (node UD), which gives a put value of €7.50. There is no value from early exercise at node U, because the €40 share price exceeds the €27.50 strike price. This means that the put option at node U is worth the discounted expected value of the €0 and €7.50 outcomes, or

$$\frac{0\pi + €7.50(1 - \pi)}{1.25} = €3.00$$

From node D, where the share price is €10 a share, early exercise leads to a put value of €17.50. Compare this with the value from not exercising and waiting one more period, which leads to share prices of €20 (node UD) and €5 (node DD), and respective put values of €7.50 and €22.50. The value of these two outcomes is

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$$\frac{€7.50\pi + €22.50(1-\pi)}{1.25} = €12.00$$

This is less than the value from early exercise. Therefore the optimal exercise policy is to exercise when the share price hits €10 at node D.

From the leftmost node, there are two possible subsequent values for the equity. If the share price rises to €40 a share (node U), the put, worth €3, should not be exercised early. If the share price declines to €10 a share (node D), the put is worth €17.50, achieved by early exercise. If the market sets a price for the American put, believing it will be optimally exercised, the market value at this point would be €17.50. To value the put at the leftmost node, weight the €3.00 and €17.50 outcomes by the risk-neutral probabilities and discount at the risk-free rate. This value is

$$\frac{€3.00\pi + €17.50(1-\pi)}{1.25} = €8.20$$

Early exercise leads to a €7.50 put at this node, which is inferior. Thus the value of the put is €8.20, and it pays to wait another period before possibly exercising.

This also implies that the European put is worth –0.3(€20) + €12 = €6, which is €2.20 less than the American put. One can also compute this €2.20 difference by multiplying $1 - \pi$ (which is 0.5) by the €5.50 difference between the non-exercise value and the actual value of the American put at node D, and discounting back one period at a 25 per cent rate.

Valuing American Options on Dividend-Paying Equities

An equity that pays dividends has two values at the node representing the ex-dividend date: (1) the **cum-dividend value** of the equity, which is the value of the equity prior to the ex-dividend date; and (2) the **ex-dividend value**, the share price after the ex-dividend date, which is lower by the amount of the dividend, assuming no taxes. At the ex-dividend date, arbitrage forces dictate that the share price should drop by the amount of the present value of the declared dividend (which is negligibly less than the amount of the dividend, since a cheque is generally mailed a few weeks after the ex-dividend date. Our analysis ignores this small amount of discounting).[15]

It never pays to exercise an American put just before the ex-dividend date. For example, if the dividend is £5 per share, a put that is about to be exercised is worth £5 more just after the ex-dividend date than it was prior to it. By contrast, it makes sense to exercise an American call just before the ex-dividend date, if one chooses to exercise prematurely at all. If the call is in the money both before and after the ex-date of a £5 dividend, the exercise value of the call is £5 higher before the ex-dividend date than after it.

The assumption that dividends are riskless creates a problem if an investor is not careful. For example, it may be impossible to have a risk-free dividend if the ex-date is many periods in the future and large numbers of down moves occur. In taking this 'bad path' along the binomial tree, an investor might find that a riskless dividend results in a *negative* ex-dividend value for the equity – which is impossible. There are two ways to model the dividend process that avoid such problems. One approach, which works but is difficult to implement, assumes that the size of the dividend depends on the path that the share price takes. After all, if an equity declines substantially in value, the company may reduce or suspend the dividend. This requires the ability to model the dividend accurately along all paths a share price might take. Such a dividend would be a risky cash flow, because the path the share price will follow is unknown in advance.

15 If investors know that the share price will drop by less than the dividend amount, buying the equity just before it goes ex-dividend and selling just after it goes ex-dividend means a loss equal to the drop in the share price, which is more than offset by the dividend received. If the share price drops by more than the dividend, selling short the equity just before it goes ex-dividend and buying it back just after is also an arbitrage opportunity. With taxes, share prices may fall by less than the amount of the dividend. For more detail, see Chapter 15.

Example 8.7

Valuing an American Call Option on a Dividend-Paying Equity

Assume that BBVA pays a dividend, and that the values above each node in Exhibit 8.8 represent the price process for BBVA equity stripped of its rights to a risk-free dividend of €6.25 paid at nodes U and D (which is assumed to be the only dividend prior to expiration). (1) Describe the tree diagram for the actual share price, assuming a risk-free rate of 25 per cent, and (2) value an American call option expiring in the final period with a strike price of €20.

Answer: (1) At the expiration date, on the far right of Exhibit 8.8, the actual share price and the ex-dividend share price are the same: the dividend has already been paid! At the intermediate period, each of the two nodes has two values for the equity. Ex-dividend, the share prices are €40 and €10 at nodes U and D, respectively, and the corresponding cum-dividend values are €46.25 and €16.25, derived by adding the €6.25 dividend to the two ex-dividend share prices. Since the present value of the €6.25 dividend is €5.00 one period earlier, the actual share price at the initial date is

$$€25 = €20 + \frac{€6.25}{€1.25}$$

(2) The value of the option at the intermediate period requires a comparison of its exercise value with its value from waiting until expiration. Exercising just before the ex-dividend date generates €46.25 – €20.00 = €26.25 at the U node. The value from not exercising is the node U value of the two subsequent option expiration values, €60 (= €80 – €20) at node UU and €0 at node UD. Example 8.6 found that the two risk-neutral probabilities are each 0.5. Hence this value is

$$\frac{€60\pi + €0(1-\pi)}{1.25} = €24$$

Since €24.00 is less than the value of €26.25 obtained by exercising at node U, early exercise just prior to the ex-dividend instant is optimal. At the D node, the option is worth 0 since it is out of the money (cum-dividend) at node D and is not in the money for either of the two stock values at the nodes UD and DD at expiration. The initial value of the option is therefore

$$\frac{€26.25\pi + €0(1-\pi)}{1.25} = €10.50$$

A second approach is to ignore the dividend and model the path taken by the value of the equity stripped of its dividend rights between the initial date of valuation and the expiration date of the option. For the binomial process, start out with a price $S_0^* = S_0$ – PV(dividends to expiration). Then select a constant u and d to trace out the binomial tree at all dates t for S_t^*. To obtain the tree diagram for the actual value of the equity, S_t, add back the present value of the risk less dividend(s). With this method (see Example 8.7), an investor never has to worry about the value of the underlying equity being less than the dividend.

As in the case of put valuation (see Example 8.6), the American call option in Example 8.7 is worth more than a comparable European call option. If the option in this example had been a European option, it would have been worth €9.60 = [(0.5)€24 + (0.5)€0]/1.25. This is smaller than the American option value, because the right of premature exercise is used at node U and therefore has value.

Any suboptimal exercise policy lowers the value of the premature exercise option and transfers wealth from the buyer of the American option to the seller. This issue often arises in corporations, which are well known to exercise the American option implicit in callable bonds that they issue at a much later date than is optimal. Such suboptimal exercise transfers wealth from the corporation's equity holders to the holders of the callable bonds.

8.6 Black–Scholes Valuation

Up to this point, we have valued derivatives using **discrete models** of share prices, which consider only a finite number of future outcomes for the share price, and only a finite number of points in time. We now turn our attention to **continuous-time models**. These models allow for an infinite number of share price outcomes, and they can describe the distribution of equity and option prices at any point in time.

Black–Scholes Formula

Chapter 7 noted that if time is divided into large numbers of short periods, a binomial process for share prices can approximate the continuous-time lognormal process for these prices. Here, particular values are selected for u and d that are related to the standard deviation of the stock return, which also is known as the equity's **volatility**. The limiting case where the time periods are infinitesimally small is one where the binomial formula developed in the last section converges to an equation known as the **Black–Scholes formula**.

The Black–Scholes formula provides no-arbitrage prices for European call options and American call options on underlying securities with no cash dividends until expiration (because such options should have the same values as European call options with the same features). The Black–Scholes formula is also easily extended to price European puts (see exercise 8.2). Finally, the formula reasonably approximates the values of more complex options. For example, to price a call option on an equity that pays dividends, one can calculate the Black–Scholes values of options expiring at each of the ex-dividend dates and those expiring at the true expiration date. The largest of these option values is a quick and often accurate approximation of the value obtained from more sophisticated option pricing models. This largest value is known as the **pseudo-American value** of the call option.

Result 8.8

(*The Black–Scholes formula.*) If an equity that pays no dividends before the expiration of an option has a return that is lognormally distributed, can be continuously traded in frictionless markets, and has a constant variance, then, for a constant risk-free rate,[16] the value of a European call option on that equity with a strike price of K and T years to expiration is given by

$$c_0 = S_0N(d_1) - \mathrm{PV}(K)N(d_1 - \sigma\sqrt{T}) \tag{8.3}$$

where

$$d_1 = \frac{\ln[S_0/\mathrm{PV}(K)]}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}$$

The Greek letter σ is the annualized standard deviation of the natural logarithm of the stock return, ln( ) represents the natural logarithm, and $N(z)$ is the probability that a normally distributed variable with a mean of zero and variance of 1 is less than z.

Results

[16] The formula also holds if the variance and risk-free rate can change in a predictable way. In the former case, use the average volatility (the square root of the variance) over the life of the option. In the latter case, use the yield associated with the zero-coupon bond maturing at the expiration date of the option. If the risk-free rate or the volatility changes in an unpredictable way, and is not perfectly correlated with the share price, no risk-free hedge between the equity and the option exists. Additional securities (such as long- and short-term bonds or other options on the same equity) may be needed to generate this hedge, and the Black–Scholes formula would have to be modified accordingly. However, for most short-term options, the modification to the option's value due to unpredictable changes in interest rates is a negligible one.

Example 8.8 illustrates how to use a normal distribution table to compute Black–Scholes values.

The difference between the strike price and the underlying price is £0.04. Therefore the estimate of £0.21 for the value of the call option reflects the time value of the BP option and the probability that the price in the future will be above £6.20.

Example 8.8

Computing Black–Scholes Warrant Values for Chrysler

Use the normal distribution table (Table A.5 in Appendix A) at the end of the book (or a spreadsheet function such as NORM.S.DIST in Microsoft Excel 2010) to calculate the fair market value of a three-month BP in-the-money option that is listed on Euronext.liffe. For the purposes of calculation, we shall need to provide hypothetical estimates for some of the variables in this example. However, as much as possible, we shall utilize real data. Assume that: (1) the price of BP shares is £6.16 today; (2) BP's historical annualized share price return volatility, σ, is 16 per cent per year (which is 0.16 in decimal form); (3) the risk-free rate is 4.78 per cent per year (annually compounded); (4) the options are American options, but no dividends are expected to be paid over the life of the option (it is thus possible to value the options as European options); (5) the options expire in exactly three months; and (6) the strike price is £6.20.

Answer: The present value of the strike price is $£6.20/1.0478^{0.25} = £6.1280$. The volatility times the square root of time to maturity is the product of 0.16 and the square root of 0.25, or 0.08. The Black–Scholes equation says that the value of the call is therefore

$$c_0 = £6.16N(d_1) - £6.1280N(d_1 - 0.08)$$

where

$$d_1 = \frac{\ln(£6.16/£6.1280)}{0.08} + \frac{1}{2}(0.08) = 0.1050$$

The normal distribution values for d_1 (= 0.1050) and $d_1 - 0.08$ (= 0.0250) in Table A.5 in Appendix A indicate that the probability of a standard normal variable being less than 0.1050 is 0.5418, and the probability of it being less than 0.0250 is 0.5100. Thus the no-arbitrage call value is

$$£6.16(0.5418) - £6.1280(0.5100) = £0.21$$

Dividends and the Black–Scholes Model

It is especially important with continuous-time modelling to model the path of the share price when stripped of its dividend rights, if dividends are paid before the expiration date of the option. For example, the Black–Scholes model assumes that the logarithm of the return on the underlying equity is normally distributed. This means that, in any finite amount of time, the share price may be close to zero, albeit with low probability. Subtracting a finite dividend from this low value would result in a negative ex-dividend share price. Hence, unless one is willing to describe the dividend for each of an infinite number of share price outcomes in this type of continuous price model, it is important to model the process for the equity stripped of its right to the dividend. This was illustrated with the binomial model earlier, but it is more imperative here.[17]

[17] A numerical example illustrating the Black–Scholes valuation of a European option on a dividend-paying equity appears later in this chapter. In terms of the above formula, one merely substitutes the value of the equity stripped of the present value of the dividends to expiration for S_0 in equation (8.3) to arrive at a correct answer. This is simply the current share price less the risk-free discounted value of the dividend payment.

8.7 Estimating Volatility

The only parameter that requires estimation in the Black–Scholes model is the volatility, σ. This volatility estimate also may be of use in estimating u and d in a binomial model (see Chapter 7).

There are a number of ways to estimate σ, assuming it is constant. One method is to use historical data, as shown in Exhibit 8.9. We now analyse this issue.

Exhibit 8.9 Computation of the Volatility Estimate for the Black–Scholes Model Using Historical Return Data on BP

Date	Price (pence)	Return	Gross return	ln(Gross return)	Date	Price (pence)	Return	Gross return	ln(Gross return)
Jan 06	576.26	9.77	90.23	–0.10	Jul 08	462.26	–10.59	110.59	0.10
Feb 06	541.96	–5.95	105.95	0.06	Aug 08	468.69	1.39	98.61	–0.01
Mar 06	568.63	4.92	95.08	–0.05	Sep 08	411.29	–12.25	112.25	0.12
Apr 06	581.96	2.34	97.66	–0.02	Oct 08	449.63	9.32	90.68	–0.10
May 06	545.14	–6.33	106.33	0.06	Nov 08	466.91	3.84	96.16	–0.04
Jun 06	549.94	0.88	99.12	–0.01	Dec 08	466.25	–0.14	100.14	0.00
Jul 06	562.58	2.30	97.70	–0.02	Jan 09	438.33	–5.99	105.99	0.06
Aug 06	529.18	–5.94	105.94	0.06	Feb 09	397.33	–9.35	109.35	0.09
Sep 06	515.89	–2.51	102.51	0.02	Mar 09	417.94	5.19	94.81	–0.05
Oct 06	516.77	0.17	99.83	0.00	Apr 09	427.69	2.33	97.67	–0.02
Nov 06	508.80	–1.54	101.54	0.02	May 09	465.56	8.85	91.15	–0.09
Dec 06	503.04	–1.13	101.13	0.01	Jun 09	432.86	–7.02	107.02	0.07
Jan 07	473.78	–5.82	105.82	0.06	Jul 09	452.99	4.65	95.35	–0.05
Feb 07	462.70	–2.34	102.34	0.02	Aug 09	498.32	10.01	89.99	–0.11
Mar 07	489.30	5.75	94.25	–0.06	Sep 09	519.31	4.21	95.79	–0.04
Apr 07	501.26	2.44	97.56	–0.02	Oct 09	536.07	3.23	96.77	–0.03
May 07	500.38	–0.18	100.18	0.00	Nov 09	551.23	2.83	97.17	–0.03
Jun 07	534.50	6.82	93.18	–0.07	Dec 09	575.59	4.42	95.58	–0.05
Jul 07	511.90	–4.23	104.23	0.04	Jan 10	563.03	–2.18	102.18	0.02
Aug 07	494.17	–3.46	103.46	0.03	Feb 10	568.70	1.01	98.99	–0.01
Sep 07	503.04	1.79	98.21	–0.02	Mar 10	612.62	7.72	92.28	–0.08
Oct 07	554.00	10.13	89.87	–0.11	Apr 10	565.55	–7.68	107.68	0.07
Nov 07	522.98	–5.60	105.60	0.05	May 10	494.80	–12.51	112.51	0.12
Dec 07	545.14	4.24	95.76	–0.04	Jun 10	319.36	–35.46	135.46	0.30
Jan 08	471.57	–13.50	113.50	0.13	Jul 10	405.95	27.11	72.89	–0.32
Feb 08	483.98	2.63	97.37	–0.03	Aug 10	380.60	–6.24	106.24	0.06
Mar 08	453.84	–6.23	106.23	0.06	Sep 10	435.05	14.31	85.69	–0.15
Apr 08	541.59	19.34	80.66	–0.21	Oct 10	425.80	–2.13	102.13	0.02
May 08	538.93	–0.49	100.49	0.00	Nov 10	425.95	0.04	99.96	0.00
Jun 08	517.00	–4.07	104.07	0.04	Dec 10	465.55	9.30	90.70	–0.10

Using Historical Data

The appropriate volatility computation for the σ in the Black–Scholes model is based on the volatility of instantaneous returns.

1. Obtain historical returns for the equity the option is written on. The second column of Exhibit 8.9 ('Price') represents the closing price on BP plc ordinary shares at the end of each month, and 'Return' reports the monthly percentage returns of BP from January 2006 to December 2010.
2. Convert the returns to gross returns (100 per cent plus the rate of return in percentage form, 1 plus the return in decimal form), as shown in the 'Gross return' column of Exhibit 8.9.
3. Take the natural logarithm of the decimal version of the gross return: thus, before taking the log, divide by 100 if the gross return is in percentage form.
4. Compute the unbiased sample standard deviation of the logged return series, and annualize it by multiplying it by the square root of the ratio of 365 to the number of days in the return interval (for example, for monthly returns multiply by the square root of 12, and for quarterly returns multiply by the square root of 4).

Using Spreadsheets to Compute the Volatility

Spreadsheet standard deviation functions typically provide the unbiased estimate of the standard deviation.[18] Remember to annualize the standard deviation obtained from the spreadsheet, because the spreadsheet does not know whether the returns were taken weekly, monthly, daily, and so on. In Exhibit 8.9, which reports monthly returns, this adjustment amounts to multiplying the output from the spreadsheet by 3.4641, which is the square root of 12.

Frequency Choice

Exhibit 8.9 uses monthly data to estimate the volatility of BP for the Black–Scholes model. Statistical theory suggests that one should use returns that are sampled more frequently to obtain more precise volatility estimates; our preference is weekly data. The use of daily data may be inferior, because the bid–ask spread tends to make volatility estimates overstate the true volatility of returns.

Improving the Volatility Estimate

Procedures similar to those designed to improve beta estimation for the Capital Asset Pricing Model can improve the volatility estimate. Consider the spectrum of historical estimates of σ for a large number of securities. Those securities with the highest (lowest) estimated volatilities from historical data are more likely to have overestimates (underestimates) of the true volatility because of sampling error. This information can be used to improve volatility estimates. In particular, an improved volatility estimate can be derived by taking a weighting of the average estimated volatility over a large group of securities and the historical volatility estimate for a single security.

The Implied Volatility Approach

An alternative approach for estimating volatility in a security is to look at other options on the same security. If market values for the options exist, there is a unique **implied volatility** that makes the Black–Scholes model consistent with the market price for a particular option.

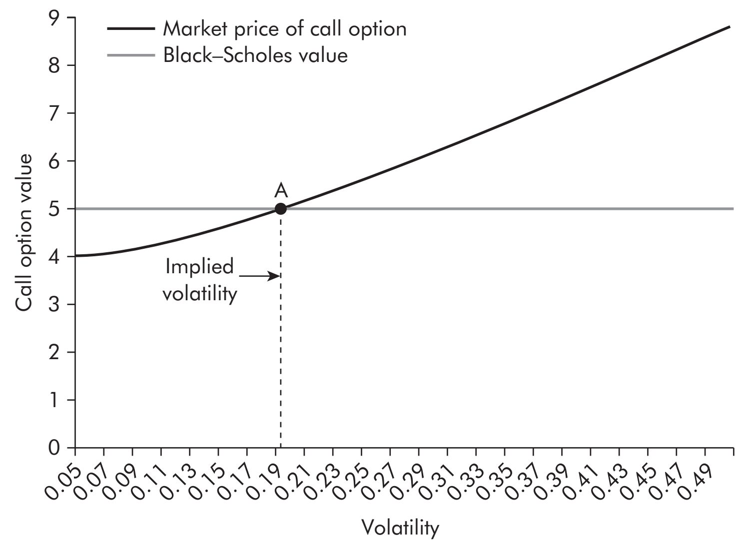
Exhibit 8.10 illustrates this concept. The σ at point A, the intersection of the horizontal line (representing the £5 market price of the call option) and the upward-sloping line (representing the Black–Scholes value), is the implied volatility. In this graph, it is about 19 per cent.

Averaging the implied volatilities of other options on the same security is a common approach to obtaining the volatility necessary for obtaining the Black–Scholes valuation of an option. Since the implied volatilities of the other options are obtained from the Black–Scholes model, this method implicitly assumes that the other options are priced correctly by that model.

Implied volatility is a concept that is commonly used in options markets, particularly the over-the-counter markets. For example, price quotes are often given to sophisticated customers in terms of implied

[18] For example, STDEV.S in Excel 2010.

Exhibit 8.10 The Value of a Call Option as a Function of Its Volatility



volatilities, because these volatilities change much less frequently than the equity and option prices. A customer or a trader can consider an implied volatility quote and know that tomorrow the same quote is likely to be valid, even though the price of the option will be different because the price of the underlying asset has changed. Implied volatility quotes help the investor who is comparison-shopping among dealers for the best option price.

If options of the same maturity but different strike prices have different implied volatilities, arbitrage is possible. This arbitrage requires purchase of the low implied volatility option and writing of the high implied volatility option. The next section explores the arbitrage of mispriced options in more depth.

8.8 Black–Scholes Price Sensitivity to Share Price, Volatility, Interest Rates and Expiration Time

This section develops some intuition for the Black–Scholes model by examining whether the call option value in the formula, equation (8.3), increases or decreases as various parameters in the formula change. These parameters are the current share price S_0, the share price return volatility σ, the risk-free interest rate r_f, and the time to maturity T.

Delta: The Sensitivity to Share Price Changes

The Greek letter delta (Δ) is commonly used in mathematics to represent a change in something. In finance, **delta** is the change in the value of the derivative security with respect to movements in the share price, holding everything else constant. The delta of the option is the derivative of the option's price with respect to the share price: $\partial c_0/\partial S_0$ for a call, $\partial p_0/\partial S_0$ for a put. The derivative of the right-hand side of the Black–Scholes formula, equation (8.3), with respect to S_0, the delta of the call option, is $N(d_1)$ (see exercise 8.4).

Delta as the Number of Shares of Equity in a Tracking Portfolio

Delta has many uses. One is in the formation of a tracking portfolio. Delta can be viewed as the number of shares of equity needed in the tracking portfolio. Let x be the number of shares in the tracking portfolio. For a given change in s_0, ds_0, the change in the tracking portfolio is $x ds_0$. Hence, unless x equals $\partial c_0/\partial s_0$ for a call or $\partial p_0/\partial s_0$ for a put, $x ds_0$ will not be the same as the change in the call or put value, and the tracking of the option pay-off with a portfolio of the underlying equity and a risk-free bond will not be perfect.

Because $N(d_1)$ is a probability, the number of shares needed to track the option lies between zero and one. In addition, as time elapses and the share price changes, d_1 changes, implying that the number of shares in the tracking portfolio needs to change continuously. Thus the tracking of the option using the Black–Scholes model, like that for the binomial model, requires dynamically changing the quantities of the equity and risk-free bond in the tracking portfolio. However, these changes require no additional financing.

Delta and Arbitrage Strategies

If the market prices of options differ from their theoretical prices, it is possible to design an arbitrage. Once set up, the arbitrage is self-financing until the arbitrage position is closed out.

As always, arbitrage requires the formation of a tracking portfolio using the underlying asset and a risk-free security. One goes long in the tracking portfolio and short in the option, or vice versa, to achieve arbitrage (see exercise 8.3).

In the design of the arbitrage, the ratio of the underlying asset position to the option position must be the negative of the partial derivative of the *theoretical* option price with respect to the price of the underlying security. For a call option that is priced by the Black–Scholes formula, this partial derivative is $N(d_1)$, the option's delta.

Delta and the Interpretation of the Black–Scholes Formula

Viewing $N(d_1)$ as the number of shares of equity in the tracking portfolio lends a nice interpretation to the Black–Scholes call option formula. The first part of the Black–Scholes formula in equation (8.3), $S_0N(d_1)$, is the cost of the shares needed in the tracking portfolio. The second term, $\text{PV}(K)N(d_1 - \sigma\sqrt{T})$, represents the amount of cash borrowed at the risk-free rate. The difference between the two terms is the cost of the tracking portfolio. Hence the Black–Scholes formula is simply an arbitrage relation. The left-hand side of the equation, c_0, is the value of the option. The right-hand side represents the market price of the tracking portfolio.

An examination of the tracking portfolio reveals that call options on equity are equivalent to leveraged positions in equity. When you focus on the capital market line, the more leverage you have, the greater the beta, standard deviation and expected return, as pointed out in Chapter 5. For this reason, call options per unit of cash invested are always riskier than the underlying equity per unit of cash invested.

Black–Scholes Option Values and Equity Volatility

One can use the Black–Scholes formula, equation (8.3), to show that an option's value is increasing in σ, the volatility of the underlying equity. (Refer again to the Black–Scholes call value in Exhibit 8.10, which has a positive slope, and see exercise 8.5.)

Results

Result 8.9

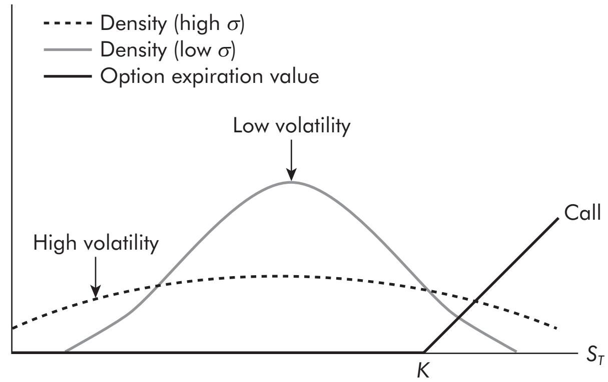
As the volatility of the share price increases, the values of both put and call options written on the equity increase.

Result 8.9 is a general property of options, holding true for calls and puts and both American and European options. It has a number of implications for the financial behaviour of both corporate finance executives and portfolio managers, and is a source of equity holder–debt holder conflicts.[19]

The intuition for this result is that increased volatility spreads the distribution of the future share price, fattening up both tails of the distribution, as shown in Exhibit 8.11. Good things are happening for a call (put) option when the right (left) tail of the distribution is more likely to occur. It is not good when the left (right) tail of the distribution is more likely to occur, but exercise of the option does *not* take place in this region of outcomes, so it is not so bad either. After a certain point, a fatter tail on the left (right) of the distribution can hurt the investor much less than a fat tail on the right (left) of the distribution can help. The worst that can happen is that an option expires worthless.

[19] These conflicts are discussed in Chapter 16.

Exhibit 8.11 Effect of Increasing Volatility



Option Values and Time to Option Expiration

European calls on equities that pay no dividends are more valuable the longer the time to expiration, other things being equal, for two reasons. First, the terminal share price is more uncertain, the longer the time to expiration. Uncertainty, described in our discussion of σ, makes options more valuable. Moreover, given the same strike price, the longer the time to maturity, the lower is the present value of the strike price paid by the holder of a call option.

In contrast, the discounting of the strike price makes European puts less valuable. Thus the combination of the effects of uncertainty and discounting leaves an ambiguous result. European puts can increase or decrease in value, the longer the time to expiration. American puts and calls, incidentally, have an unambiguous sensitivity to expiration time. These options have all the rights of their shorter-maturity cousins in addition to the rights of exercise for an even longer period. Hence the longer the time to expiration, the more valuable are American call and put options, even for dividend-paying equities.

Option Values and the Risk-Free Interest Rate

The Black–Scholes call option value is increasing in the interest rate r (see exercise 8.6) because the payment of K for the share of equity whose call option is in the money costs less in today's money (that is, it has a lower discounted value) when the interest rate is higher. The opposite is true for a European put. Since one receives cash, puts are less valuable when the interest rate is higher. As noted earlier in this chapter, the timing of equity delivery is irrelevant as long as there are no dividends. Only the present value of the cash payment upon option exercise and whether this cash is paid out (call) or received (put) determine the effect of interest rates on the option value. Once again, this is a general result that holds for European and American options, and works the same in the binomial model and the Black–Scholes model.

A Summary of the Effects of the Parameter Changes

Exhibit 8.12 summarizes the results in this section, and includes the impact on the value of a long forward contract (equivalent to a call less a put). These exercises hold constant all the other factors determining option value, except the one in the relevant row.

An increase in strike price K or an increase in the dividend paid, although not listed in the exhibit, will have the opposite effect of an increase in the share price. It also is interesting to observe what has not been included in Exhibit 8.12 because it has no effect. In particular, risk aversion and the expected growth rate of the share price have no effect on option values, which are determined solely by the no-arbitrage relation between the option and its tracking portfolio (as Chapter 7 emphasized). When tracking is impossible (for example, executive share options), option values are not tied to tracking portfolios. In such cases, considerations such as risk aversion or share price growth rates may play a role in option valuation.

Exhibit 8.12 Determinants of Current Option and Forward Prices: Effect of a Parameter Increase

Parameter increased	Long forward	American call	American put	European call	European put
S_0	↑	↑	↓	↑	↓
T	↑	↑	↑	↑	Ambiguous
σ	No effect	↑	↑	↑	↑
r_f	↑	↑	↓	↑	↓

8.9 Valuing Options on More Complex Assets

Options exist on many assets. For example, corporations use currency options to hedge their foreign currency exposure. They also use swap options, which are valued in exactly the same manner as bond options, to hedge interest rate risk associated with a callable bond issued in conjunction with interest rate swaps. Option markets exist for semiconductors, agricultural commodities, precious metals and oil. There are even options on futures contracts for a host of assets underlying the futures contracts.

These options trade on many organized exchanges and over the counter. For example, currency options are traded on Euronext.liffe, but large currency option transactions tend to be over the counter, with a bank as one of the parties. Clearly, an understanding of how to value some of these options is important for many finance practitioners.

The Forward Price Version of the Black–Scholes Model

To value options on more complex assets such as currencies, it is important to recognize that the Black–Scholes model is a special case of a more general model having arguments that depend on (zero-cost) forward prices instead of spot prices. Once you know how to compute the forward price of an underlying asset, it is possible to determine the Black–Scholes value for a European call on the underlying asset.

To transform the Black–Scholes formula into a more general formula that uses forward prices, substitute the no-arbitrage relation from Chapter 7, $S_0 = F_0/(1 + r_f)^T$, where F_0 is the forward price of an underlying asset in a forward contract maturing at the option expiration date, into the original Black–Scholes formula, equation (8.3). The Black–Scholes equation can then be rewritten as

$$c_0 = \mathrm{PV}[F_0N(d_1) - KN(d_1 - \sigma\sqrt{T})$$

where

$$d_1 = \frac{\ln[F_0/K]}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}$$

and where:

PV is the risk-free discounted value of the expression inside the brackets.

This equation looks a bit simpler than the original Black–Scholes formula.

Investors and financial analysts can apply this simpler, more general version of the Black–Scholes formula to value European call options on currencies, bonds,[20] dividend-paying equities and commodities.

[20] The Black–Scholes formula assumes that the volatility of the bond price is constant over the life of the option. Since bond volatilities diminish at the approach of the bond's maturity, the formula provides only a decent approximation to the fair market value of an option that is relatively short-lived compared with the maturity of the bond.

Computing Forward Prices from Spot Prices

In applying the forward price version of the Black–Scholes formula, it is critical to use the appropriate forward price for the underlying asset. The appropriate forward price F_0 represents the price one agrees to today, but pays at the option expiration date, to acquire one unit of the underlying asset at that expiration date.

The following are a few rules of thumb for calculating forward prices for various underlying assets.

- *Foreign currency*. Multiply the present spot foreign currency rate, expressed as *home currency/foreign currency* (for example, R/$ for a South African firm and £/€ for a British firm) by the *present value* of a riskless unit of foreign currency paid at the forward maturity date (where the discounting is at the foreign riskless interest rate). Multiply this value by the *future value* (at the maturity date) of one unit of currency paid today: that is, multiply by $(1 + r_f)^T$, where T represents the years to maturity and r_f is the domestic riskless interest rate. Chapter 7 describes forward currency rate computations in detail when currencies are expressed as *foreign currency/home currency*.
- *Riskless coupon bond*. Find the present value of the bond when stripped of its coupon rights until maturity: that is, current bond price less PV(coupons). Multiply this value by the future value (at the maturity date) of one unit of currency paid today. (Depending on how the bond price is quoted, an adjustment to add accrued interest to the quoted price of the bond to obtain the full price may also need to be made, as described in Chapter 2.)
- *Stock with dividend payments*. Find the present value of the equity when stripped of its dividend rights until maturity: that is, current share price less PV(dividends). Multiply this value by the future value (at the maturity date) of one unit of currency paid today.
- *Commodity*. Add the present value of the storage costs until maturity to the current price of the commodity. Subtract the present value of the benefits, known as the *convenience yield*, associated with holding an inventory of the commodity to maturity.[21] Multiply this value by the future value, at the maturity date, of one unit of currency paid today.

Note that all forward prices multiply some adjusted market value of the underlying investment by $(1 + r_f)^T$, where r_f is the riskless rate of interest. Multiplying by one plus the risk-free rate of interest accounts for the cost of holding the investment until maturity. The cost is the lost interest on the money spent to acquire the investment. For example, with an equity that pays no dividends, the present value of having the equity in one's possession today as opposed to some future date is the same. However, the longer the investor can postpone paying a pre-specified amount for the equity, the better off he or she is. The difference between the forward price and the share price thus reflects interest to the party with the short position in the forward contract, who is essentially holding the equity for the benefit of the party holding the long position in the forward contract.

The difference between the forward price and the share price also depends on the benefits and any costs of having the investment in one's possession until forward maturity. With equity, the benefit is the payment of dividends; with bonds, the payment of coupons; with foreign currency, the interest earned in a foreign bank when that currency is deposited; with commodities, the convenience of having an inventory of the commodity less the cost of storage.

Applications of the Forward Price Version of the Black–Scholes Formula

Example 8.9 demonstrates the use of the generalized Black–Scholes formula to value European call options having these more complex underlying assets.

American Options

The forward price version of the Black–Scholes model also has implications for American options. For example, the reason why American call options on equities that pay no dividends sell for the same price as comparable European call options is that the risk-free discounted value of the forward price at expiration is never less than the current share price, no matter how much time has elapsed since the purchase

[21] See Chapter 22 for a more detailed discussion of convenience yields.

of the option. If a discrete dividend is about to be paid and the equity is relatively close to expiration, the current cum-dividend price of the equity exceeds the discounted value of the forward price, and premature exercise may be worth while. If one can be certain that this will not be the case, waiting is worth while. One can generalize this result as follows.

Example 8.9

Pricing Securities with the Forward Price Version of the Black–Scholes Model

The UK sterling risk-free rate is assumed to be 6 per cent per year, and all σs are assumed to be 25 per cent per year. Use the forward price version of the Black–Scholes model to compute the value of a European call option to purchase one year from now.

a €1 at a strike price of £0.75. The current spot exchange rate is £0.70/€ and the one-year risk-free rate is 4 per cent in France.

b A 30-year bond with an 8 per cent semi-annual coupon at a strike price of £100 (full price). The bond is currently selling at a full price of £102 (which includes accrued interest) per £100 of face value, and has two scheduled coupons of £4 before option expiration, to be paid six months and one year from now. (At the forward maturity date, the second coupon payment has just been made.)

c A FTSE 100 contract with a strike price of £6,800. The FTSE 100 has a current price of £6,609 and a present value of next year's dividends of £165.

d A barrel of oil with a strike price of £46.57 ($95 a barrel at an exchange rate of $2.04/£). A barrel of oil currently sells for £43.62 ($89 a barrel). The present value of next year's storage costs is £1, and the present value of the convenience of having a barrel of oil available over the next year (for example, if there are long gas lines owing to an oil embargo and extremely bad weather in the Gulf region) is £1.

Answer: The forward prices are, respectively:

a $\frac{£0.70(1.06)}{1.04} = £0.7134$

b $£102(1.06) - £4\sqrt{1.06} - £4 = £100.0017$

c $(£6{,}609 - £165)(1.06) = £6{,}830.64$

d $(£43.62 + £1 - £1)(1.06) = £46.24$

Plugging these values into the forward price version of the Black–Scholes model yields:

a $$c_0 = \frac{£0.7134N(d_1) - 0.75N(d_1 - 0.25)}{1.06}$$

where

$$d_1 = \frac{\ln(0.7134/0.75)}{0.25} + 0.125 = -0.751$$

Thus c_0 is approximately £0.0528.

b $$c_0 = \frac{£100.0017N(d_1) - £100N(d_1 - 0.25)}{1.06}$$

where

$$d_1 = \frac{\ln(100.0017/100)}{0.25} + 0.125 = 0.13$$

Thus c_0 is approximately £9.39.

c $c_0 = \dfrac{£6{,}830.64N(d_1) - £6{,}800N(d_1 - 0.25)}{1.06}$

where

$$d_1 = \frac{\ln(6{,}830.64/6{,}800)}{0.25} + 0.125 = 0.1430$$

Thus c_0 is approximately £654.14.

d $c_0 = \dfrac{£46.24N(d_1) - £46.57N(d_1 - 0.25)}{1.06}$

where

$$d_1 = \frac{\ln(46.24/46.57)}{0.25} + 0.125 = 0.966$$

Thus c_0 is approximately £4.20 (or \$8.57 a barrel).

Result 8.10

An American call (put) option should not be prematurely exercised if the value of the forward price of the underlying asset at expiration, discounted back to the present at the risk-free rate, either equals or exceeds (is less than) the current price of the underlying asset. As a consequence, if one is certain that over the life of the option this will be the case, American and European options should sell for the same price if there is no arbitrage.

Results

There are several implications of Result 8.10. With call options on the FTSE 100 where dividends of different equities pay off on different days so that the overall dividend stream resembles a continuous flow, American and European call options should sell for the same price if the risk-free rate to expiration exceeds the dividend yield. Also, with bonds where the risk-free rate to option expiration is greater (less) than the coupon yield of the risk-free bond, American and European call (put) options should sell for the same amount as their European counterparts if the option strike price is adjusted for accrued interest (as it usually is, unlike in Example 8.9) so that the coupon stream on the bond is like a continuous flow. This suggests that when the term structure of interest rates (see Chapter 10) is steeply upward (downward) sloping, puts (calls) of the European and American varieties are likely to have the same value.

American Call and Put Currency Options

Result 8.10 also has implications for American currency options on both calls and puts. These implications are given in Result 8.11.

Result 8.11

If the domestic interest rate is greater (less) than the foreign interest rate, the American option to buy (sell) domestic currency in exchange for foreign currency should sell for the same price as the European option to do the same.

Results

8.10 Empirical Biases in the Black–Scholes Formula

The Black–Scholes model is particularly impressive in the general thrust of its implications about option pricing. Option prices tend to be higher in environments with high interest rates and high volatility. Moreover, you find remarkable similarities when comparing the values of many options, even American options, with the results given by the Black–Scholes model.

However, after a close look at the prices in the newspaper, it is easy to see that the Black–Scholes formula tends to underestimate the market values of some kinds of option and overestimate others. MacBeth and Merville (1979) used daily closing prices to study actively traded options on six equities in 1976. Their technique examined the implied volatilities of various options on the same securities, and found these volatilities to be inversely related to the strike prices of the options. This meant that the Black–Scholes value, on average, was too high for deep out-of-the-money calls and too low for in-the-money calls. These biases grew larger, the further the option was from expiration.

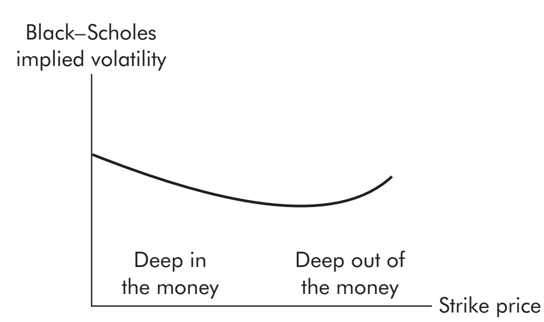
A similar but more rigorous set of tests was conducted by Rubinstein (1985). Using transaction data, he first paired options with similar characteristics. For example, a pairing might consist of two call options on the same equity with two different strike prices, but the same expiration date, that traded within a small interval of time (for example, 15 minutes). With these pairings, 50 per cent of the members of the pair with low strike prices should have higher implied volatilities than their counterparts with high strike prices. Rubinstein also used pairings in which the only difference was the time to expiration. He then evaluated the pairings and found that shorter times to expiration led to higher implied volatility for out-of-the-money calls. This would suggest that the Black–Scholes formula tends to underestimate the values of call options close to expiration relative to call options with longer times to expiration.

Rubinstein also found a strike price bias that was dependent on the period examined. From August 1976 to October 1977, lower strike prices meant higher implied volatility. In this period, the Black–Scholes model underestimated the values of in-the-money call options and overestimated the values of out-of-the-money call options, which is consistent with the findings of MacBeth and Merville. However, for the period from October 1977 to August 1978, Rubinstein found the opposite result, except for out-of-the-money call options that were close to expiration.

Although these biases were highly statistically significant, Rubinstein concluded that they had little economic significance. In general, Rubinstein found that the biases in the Black–Scholes model were of the order of a 2 per cent deviation from Black–Scholes pricing. Hence a fairly typical €2 Black–Scholes price would coincide with a €2.04 market price. *This means that the Black–Scholes model, despite its biases, is still a fairly accurate estimator of the actual prices found in options markets*.

Since these studies were completed, many traders have referred to what is known as the **smile effect** (see Exhibit 8.13). If one plots the implied volatility of an option against its strike price, the graph of implied volatility looks like a 'smile'. This formation suggests that the Black–Scholes model underprices both deep in-the-money and deep out-of-the-money options relative to near at-the-money options.

Exhibit 8.13 Implied Volatility versus Strike Price



8.11 Summary and Conclusions

This chapter developed the put–call parity formula, relating the prices of European calls to those of European puts, and used it to generate insights into minimum call values, premature exercise policy for American calls, and the relative valuation of American and European calls. The put–call parity formula also provided insights into corporate securities and portfolio insurance.

This chapter also applied the results on derivative securities valuation from Chapter 7 to price options, using two approaches: the binomial approach and the Black–Scholes approach. The results on European call pricing with these two approaches can be extended to European puts. The put–call parity formula provides a method for translating the pricing results with these models into pricing results for European puts.

The pricing of both American calls on dividend-paying equities and American puts cannot be derived from European call pricing formulae, because it is sometimes optimal to exercise these securities prematurely. This chapter used the binomial method to show how to price these more complicated types of option.

This chapter also discussed various issues relating to the implementation of the theory, including: (1) the estimation of volatility and its relation to the concept of an implied volatility; (2) extending the pricing formulae to complex underlying securities; and (3) the known empirical biases in option pricing formulae. Despite a few biases in the Black–Scholes option pricing formula, it appears that the formulae work reasonably well when properly implemented.

Key Concepts

Result 8.1: (*The put–call parity formula*.) If no dividends are paid to holders of the underlying equity prior to expiration, then, assuming no arbitrage,

$$c_0 - p_0 = S_0 - \text{PV}(K)$$

That is, a long position in a call and a short position in a put sells for the current share price less the strike price discounted at the risk-free rate.

Result 8.2: It never pays to exercise an American call option prematurely on an equity that pays no dividends before expiration.

Result 8.3: An investor does not capture the full value of an American call option by exercising between ex-dividend or (in the case of a bond option) ex-coupon dates.

Result 8.4: If the underlying equity pays no dividends before expiration, then the no-arbitrage values of American and European call options with the same features are the same.

Result 8.5: (*Put–call parity formula generalized*.) $c_0 - p_0 = S_0 - \text{PV}(K) - \text{PV}(\text{div})$. The difference between the no-arbitrage values of a European call and a European put with the same features is the current share price less the sum of the present value of the strike price and the present value of all dividends to expiration.

Result 8.6: It is possible to view equity as a call option on the assets of the firm and to view risky corporate debt as riskless debt worth PV(K) plus a short position in a put option on the assets of the firm ($-p_0$) with a strike price of K.

Result 8.7: (*The binomial formula*.) The value of a European call option with a strike price of K and N periods to expiration on an equity with no dividends to expiration and a current value of S_0 is

$$c_0 = \frac{1}{(1+r_f)^N}\sum_{j=0}^{N}\frac{N!}{j!(N-j)!}\pi^j(1-\pi)^{N-j}\max(0, u^j d^{N-j}S_0 - K)$$

where

r_f = risk-free return per period

π = risk-neutral probability of an up move

u = ratio of the share price to the prior share price, given that the up state has occurred over a binomial step

d = ratio of the share price to the prior share price, given that the down state has occurred over a binomial step.

Result 8.8: (*The Black–Scholes formula*.) If an equity that pays no dividends before the expiration of an option has a return that is lognormally distributed, can be continuously traded in frictionless markets, and has a constant variance, then, for a constant risk-free rate, the value of a European call option on that equity with a strike price of K and T years to expiration is given by

$$c_0 = S_0 N(d_1) - \text{PV}(K)N(d_1 - \sigma\sqrt{T})$$

where

$$d_1 = \frac{\ln[S_0/\text{PV}(K)]}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}$$

The Greek letter σ is the annualized standard deviation of the natural logarithm of the stock return, ln( ) represents the natural logarithm, and $N(z)$ is the probability that a normally distributed variable with a mean of zero and variance of 1 is less than z.

Result 8.9: As the volatility of the share price increases, the values of both put and call options written on the equity increase.

Result 8.10: An American call (put) option should not be prematurely exercised if the value of the forward price of the underlying asset at expiration, discounted back to the present at the risk-free rate, either equals or exceeds (is less than) the current price of the underlying asset. As a consequence, if one is certain that over the life of the option this will be the case, American and European options should sell for the same price if there is no arbitrage.

Result 8.11: If the domestic interest rate is greater (less) than the foreign interest rate, the American option to buy (sell) domestic currency in exchange for foreign currency should sell for the same price as the European option to do the same.

Key Terms

Exercises

8.1 You hold an American call option with a £30 strike price on an equity that sells at £35. The option sells for £5 one year before expiration. Compare the cash flows at expiration from: (1) exercising the option now, and putting the £5 proceeds in a bank account until the expiration date; and (2) holding on to the option until expiration, selling short the equity, and placing the £35 you receive into the same bank account.

8.2 Combine the Black–Scholes formula with the put–call parity formula to derive the Black–Scholes formula for European puts.

8.3 HSBC Holdings equity has a volatility of $\sigma = 0.25$ and a price of £9.25 a share. A European call option on HSBC stock with a strike price of £10 and an expiration time of one year has a price of £1. Using the Black–Scholes model, describe how you would construct an arbitrage portfolio, assuming that the present value of the strike price is £9.43. Would the arbitrage portfolio increase or decrease its position in HSBC if shortly thereafter the share price of HSBC rose to £9.30 a share?

8.4 Take the partial derivative of the Black–Scholes value of a call option with respect to the underlying security's price, S_0. Show that this derivative is positive and equal to $N(d_1)$. *Hint*: first show that $S_0N'(d_1) - \text{PV}(K)N'(d_1 - \sigma\sqrt{T})$ equals zero by using the fact that the derivative of N with respect to d_1, $N'(d_1)$, equals $1/\sqrt{2\pi}\,[\exp^{(-0.5d_1^2)}]$.

8.5 Take the partial derivative of the Black–Scholes value of a call option with respect to the volatility parameter. Show that this derivative is positive and equal to $S_0\sqrt{T}N'(d_1)$.

8.6 If $\text{PV}(K) = K/[(1 + r)^T]$, take the partial derivative of the Black–Scholes value of a call option with respect to the interest rate r_f. Show that this derivative is positive and equal to $T \times \text{PV}(K)$ $N(d_1 - \sigma\sqrt{T})/(1 + r_f)$.

8.7 Suppose you observe a European call option on an asset that is priced at less than the value of $S_0 - \text{PV}(K) - \text{PV}(\text{div})$. What type of transaction should you execute to achieve arbitrage? (Be specific with respect to amounts, and avoid using puts in this arbitrage.)

8.8 Consider a position of two purchased calls (BASF, three months, K = €96) and one written put (BASF, three months, K = €96). What position in BASF equity will show the same sensitivity to price changes in BASF equity as the option position described above? Express your answer algebraically as a function of d_1 from the Black–Scholes model.

8.9 The present price of an equity share of Strategy AB is €50. The equity follows a binomial process where each period the share price either goes up 10 per cent or down 10 per cent. Compute the fair market value of an American put option on Strategy AB equity with a strike price of €50 and two periods to expiration. Assume Strategy AB pays no dividends over the next two periods. The risk-free rate is 2 per cent per period.

8.10 Steady plc has a share value of £50. At-the-money American call options on Steady plc with nine months to expiration are trading at £3. Sure plc also has a share value of £50. At-the-money American call options on Sure plc with nine months to expiration are trading at £3. Suddenly, a merger is announced. Each share in both corporations is exchanged for one share in the combined corporation, 'Sure & Steady'. After the merger, options formerly on one share of either Sure plc or Steady plc were converted to options on one share of Sure & Steady. The only change is the difference in the underlying asset. Analyse the likely impact of the merger on the values of the two options before and after the merger. Extend this analysis to the effect of mergers on the equity of firms with debt financing.

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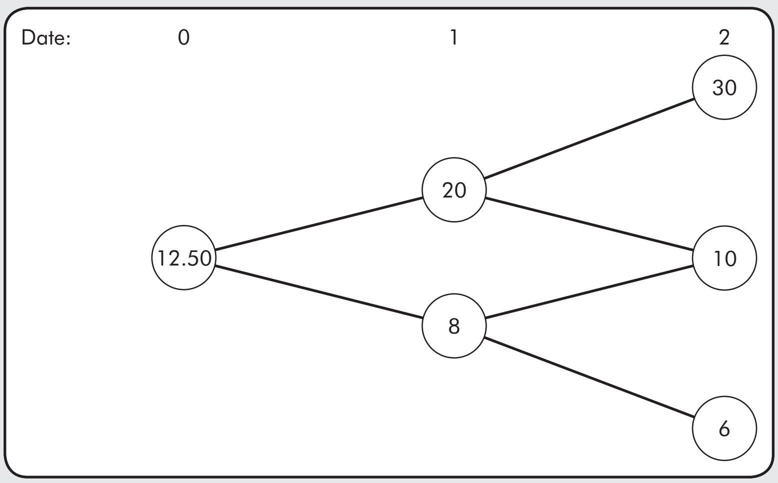
8.11 FSA is a privately held firm. As an analyst trying to determine the value of FSA's ordinary equity and bonds, you have estimated the market value of the firm's assets to be €1 million and the standard deviation of the asset return to be 0.3. The debt of FSA, which consists of zero-coupon bank loans, will come due one year from now at its face value of €1 million. Assuming that the risk-free rate is 5 per cent, use the Black–Scholes model to estimate the value of the firm's equity and debt.

8.12 Describe what happens to the amount of equity held in the tracking portfolio for a call (put) as the share price goes up (down). *Hint*: prove this by looking at delta.

8.13 Callable bonds appear to have market values that are determined as though the issuing corporation optimally exercises the call option implicit in the bond. You know, however, that these options tend to get exercised past the optimal point. Write up a non-technical presentation for your boss, the portfolio manager, explaining why arbitrage exists, and how to take advantage of it with this investment opportunity.

8.14 The following tree diagram outlines the share price of a company over the next two periods:



The risk-free rate is 12 per cent from date 0 to date 1, and 15 per cent from date 1 to date 2. A European call on this equity (1) expires in period 2, and (2) has a strike price of £8.

a Calculate the risk-neutral probabilities implied by the binomial tree.
b Calculate the pay-offs of the call option at each of three nodes at date 2.
c Compute the value of the call at date 0.

8.15 A non-dividend-paying equity has a current price of £30 and a volatility of 20 per cent per year.

a Use the Black–Scholes equation to value a European call option on the equity above with a strike price that has a present value of £28 and time to maturity of three months.
b Without performing calculations, state whether this price would be higher if the call were American. Why?
c Suppose the equity pays dividends. Are otherwise identical American and European options likely to have the same value? Why?

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Practical Insights for Part II

Allocating Capital for Real Investment

- Mean-variance analysis can help determine the risk implications of product mixes, mergers and acquisitions, and carve-outs. This requires thinking about the mix of real assets as a portfolio. (Section 4.6)
- Theories to value real assets identify the types of risk that determine discount rates. Most valuation problems will use either the CAPM or APT, which identify market risk and factor risk, respectively, as the relevant risk attributes. (Sections 5.8, 6.10)
- An investment's covariance with other investments is a more important determinant of its discount rate than is the variance of the investment's return. (Section 5.7)
- The CAPM and the APT both suggest that the rate of return required to induce investors to hold an investment is determined by how the investment's return covaries with well-diversified portfolios. However, existing evidence suggests that most of the well-diversified portfolios that have traditionally been used, in either a single-factor or a multiple-factor implementation, do a poor job of explaining the historical returns of ordinary equities. Although multifactor models do better than single-factor models, all model implementations have difficulty (to varying degrees) explaining the historical returns of investments with extreme size, market-to-book ratios and momentum. These shortcomings need to be accounted for when allocating capital to real investments that fit into these anomalous categories. (Sections 5.11, 6.12)

Financing the Firm

- When issuing debt or equity, the CAPM and APT can provide guidelines about whether the issue is priced fairly. (Sections 5.8, 6.10)
- Because equity can be viewed as a call option on the assets of the firm when there is risky debtfinancing, the equity of firms with debt is riskier than the equity of firms with no debt. (Sections 8.3, 8.8)
- Derivative valuation theory can be used to value risky debt and equity in relation to one another. (Section 8.3)

Knowing Whether and How to Hedge Risk

- The fair market values, not the actual market values, determine appropriate ratios for hedging. These are usually computed from the valuation models for derivatives. (Section 8.8)
- Portfolio mathematics can enable the investor to understand the risk attributes of any mix of real assets, financial assets and liabilities. (Section 4.6)
- Forward currency rates can be inferred from domestic and foreign interest rates. (Section 7.2)

Allocating Funds for Financial Investments

- Portfolios generally dominate individual securities as desirable investment positions. (Section 5.2)
- Per unit of cash invested, leveraged positions are riskier than unleveraged positions. (Section 4.7)
- There is a unique optimal risky portfolio when a risk-free asset exists. The task of an investor is to identify this portfolio. (Section 5.4)
- Mean-variance analysis is frequently used as a tool for allocating funds between broad-based portfolios. Because of estimation problems, mean-variance analysis is difficult to use for determining allocations between individual securities. (Section 5.6)

- If the CAPM is true, the optimal portfolio to hold is a broad-based market index. (Section 5.8)
- If the APT is true, the optimal portfolio to hold is a weighted average of the factor portfolios. (Section 6.10)
- Since derivatives are priced relative to other investments, opinions about cash flows do not matter when determining their values. With perfect tracking possible here, mastery of the theories is essential if one wants to earn arbitrage profits from these investments. (Section 7.3)
- Apparent arbitrage profits, if they exist, must arise from market frictions. Hence, to obtain arbitrage profits from derivative investments, one must be more clever than competitors at overcoming the frictions that allow apparent arbitrage to exist. (Section 7.6)
- Derivatives can be used to insure a portfolio's value. (Section 8.3)
- The somewhat disappointing empirical evidence for the CAPM and APT may imply an opportunity for portfolio managers to beat aggregate market indices and other benchmarks they are measured against. (Sections 5.8, 6.10, 6.13)
- Per unit of investment, call options are riskier than the underlying asset. (Section 8.8)

Executive Perspective

Myron S. Scholes

For large financial institutions, financial models are critical to their continuing success. Since they are liability managers as well as asset managers, models are crucial in pricing and evaluating investment choices, and in managing the risk of their positions. Indeed, financial models, similar to those developed in Part II of this text, are in everyday use in these firms.

The mean-variance model, developed in Chapters 4 and 5, is one example of a model that we use in our activities. We use it and stress management technology to optimize the expected returns on our portfolio subject to risk, concentration and liquidity constraints. The mean-variance approach has influenced financial institutions in determining risk limits and measuring the sensitivity of their profit and loss to systematic exposures.

The risk–expected return models presented in Part II, such as the CAPM and the APT, represent another set of useful tools for money management and portfolio optimization. These models have profoundly affected the way investment funds are managed, and the way individuals invest and assess performance. For example, passively managed funds, which generally buy and hold a proxy for the market portfolio, have grown dramatically, accounting for more than 20 per cent of institutional investment. This has occurred, in part, because of academic writings on the CAPM and, in part, because performance evaluation using these models has shown that professional money managers as a group do not systematically outperform these alternative investment strategies. Investment banks use both debt and equity factor models – extremely important tools – to determine appropriate hedges to mitigate factor risks. For example, my former employer, Salomon Brothers, uses factor models to determine the appropriate hedges for its equity and debt positions.

All this pales, of course, with the impact of derivatives valuation models, starting with the Black–Scholes option-pricing model that I developed with Fischer Black in the early 1970s. Using the option-pricing technology, investment banks have been able to produce products that customers want. An entire field called financial engineering has emerged in recent years to support these developments. Investment banks use option-pricing technology to price sophisticated contracts and to determine the appropriate hedges to mitigate the underlying risks of producing these contracts. Without the option-pricing technology, presented in Chapters 7 and 8, the global financial picture would be far different. In the old world, banks were underwriters, matching those who wanted to buy with those who wanted to offer coarse contracts such as loans, bonds and equities. Derivatives have reduced the costs to provide financial services and products that are more finely tuned to the needs of investors and corporations around the world.

Mr Scholes is currently a partner in Oak Hill Capital Management, L.P., and Chairman of Oak Hill Platinum Partners, L.P., located in Menlo Park, CA, and Rye Brook, NY, respectively. He is also the Frank E. Buck Professor of Finance Emeritus, Stanford University Graduate School of Business and a recipient of the 1997 Nobel Prize in Economics.