Chapter 12

Modal Logic

Modal logic is the logic of necessity and possibility. The name "modal logic" stems from the fact that necessity and possibility have traditionally been regarded as *modes* of truth (or *ways* of being true). For example, some propositions, such as "All triangles have three sides," are traditionally regarded as *necessarily* true. Others, such as "Socrates was married to Xanthippe," are traditionally regarded as true *but not necessarily* true. We will explore these modes of truth in section 12.1.

Modal logic has many interesting applications, especially in regard to philosophical issues. For example, here is a modal version of Saint Anselm's famous ontological argument for the existence of God:

> If God exists, then God is a Supremely Perfect Being. If God is a Supremely Perfect Being, then it is impossible that God not exist. It is logically possible that God exists. So, God exists.

Is this argument valid? We will examine it closely in section 12.5. Modal logic has been developed in order to determine the validity of arguments such as this, whose forms make essential use of modal concepts.

Aristotle was the first philosopher to discuss the logical relationships between necessity and possibility. For example, in his *De Interpretatione*, Aristotle identified the following equivalences:[1]

> "It is impossible that ***p***" is logically equivalent to "It is necessary that not ***p***." (For example, "It is impossible that circular squares exist" is logically equivalent to "It is necessary that circular squares do not exist.")
>
> "It is not possible that not ***p***" is logically equivalent to "It is necessary that ***p***." (For example, "It is not possible that not every square is a rectangle" is equivalent to "Necessarily, every square is a rectangle.")

However, modal logic did not become highly developed until the 20th century. An American logician, C. I. Lewis (1883–1964), was dissatisfied with the so-called paradoxes of material implication. In symbols, they look like this:

$$p \rightarrow (q \rightarrow p)$$
$$\sim p \rightarrow (p \rightarrow q)$$

Both of these formulas are tautologies of statement logic. The first formula says (in effect) that a conditional is true if its consequent is true. The second formula says (in effect) that a conditional is true if its antecedent is false. These tautologies indicate that the material conditional is not very close to the ordinary English "if-then." Lewis wanted to capture the idea of a necessary connection between antecedent and consequent that frequently occurs in philosophical discourse—for example, "Necessarily, if I think, then I exist." In working his ideas out, Lewis made dramatic advances in modal logic. In this chapter, we will see how to develop a system of modal logic by adding to the system of statement logic developed in chapters 7 and 8.

12.1 Modal Concepts

Let us begin our exploration of the basic concepts of modal logic with an examination of the concept of a necessary truth. A **necessary truth** is one that cannot be false under any possible circumstances. Here are some standard examples:

1. Either trees exist or it is not the case that trees exist.
2. One plus one equals two.
3. All cats are cats.
4. All husbands are married.
5. If Sue is older than Tom, then Tom is not older than Sue.
6. Nothing is red all over and blue all over at the same time.
7. No prime minister is a prime number.[2]

These examples fall into certain significant categories. Item (1) is a tautology of statement logic. Recall that a tautology is a statement that is true in every row of its truth table.[3]

Item (2) is a mathematical truth. Since it does not seem possible for mathematical truths to be false, many philosophers regard all mathematical truths as necessary ones.

Items (3) and (4) are examples of what philosophers call analytic statements. Definitions of the term "analytic" vary, but it will be adequate for present purposes to say that an **analytic statement** is one that is either (a) true by virtue of its logical form or (b) transformable into a statement that is true by virtue of its logical form by replacing synonyms with synonyms.[4] Item (3) has the form "All *A* are *A*," and since it is not possible for statements of this form to be false, (3) is a necessary truth. As for (4), since the word "husbands" means "married men," we can transform (4) into "All married men are married." Thus, upon analyzing (4), we can see that it has the form "All *AB* are *A*." And because statements having this form cannot be false, (4) is necessary.

Items (5) through (7) are not so easily categorized. They do not seem to be analytic. A statement that is not analytic is said to be **synthetic**. But while (5), (6), and (7) are apparently synthetic, many philosophers believe that it is impossible for such statements to be false. In this view, (5), (6), and (7) are synthetic yet necessary. It must be admitted, however, that many philosophers are skeptical about the thesis that there are synthetic necessary truths. However, because this issue is very complicated, we cannot explore it here.

We noted previously that a necessary truth is one that cannot be false in any possible circumstances. What exactly is meant by a "possible circumstance"? We have been making use of this concept throughout this book, for we noted early on that if we can describe a *situation or circumstance* in which the conclusion of an argument is false while its premises are true, then we have demonstrated that the argument is invalid. For instance:

8. Someone is rich. So, Bill Gates (of Microsoft) is rich.

This argument is invalid, though both its premise and its conclusion are true. We can demonstrate the invalidity by describing a possible circumstance in which the premise is true while the conclusion is false. For example, here is a possible circumstance: "Bill Gates donates his entire fortune, every last penny, to charitable causes, freely choosing to live in poverty, but at least one other person who is currently rich remains so." In such a circumstance, the premise of (8) is true, but the conclusion is false.

A possible circumstance is *a way things could (or might) have been*. In ordinary life, we often make use of a distinction between *how things are* and *how they could (or might) have been*. For example, consider what happens when one makes a decision with far-reaching consequences. One may look back on the decision and realize that one's present circumstances could have been different because one's decision could have been different. Philosophers have attempted to analyze talk about "the way things could have been" in terms of the technical concept of possible worlds. So, before going further, we need to clarify the concept of a possible world.

A **possible world** is a *total way* things could have been. To get at this concept, let us begin with the actual world. As philosophers use this expression, the "actual world" is not merely the planet Earth, but the *complete situation* we find ourselves in—the entire universe (including all the stars and galaxies, subatomic particles, and their movements), every person, every object, and all events (past, present, and future). Moreover, as philosophers use the phrase, the "actual world" is one of many possible worlds. We can clarify the idea of possible worlds by considering some ways in which the actual world could have been different from what it is.

Let's start with a trivial case. Yesterday, January 29, 1998, I wore a blue tie all day. I might have chosen a different tie, say, a red one, but I didn't. So, the actual world includes the following circumstance or state of affairs: my wearing a blue tie all day on January 29, 1998. And we can say that there is a possible world that is exactly like the actual world except that on January 29, 1998, I wear a red tie instead of a blue one. Of course, that possible world is not the actual world, but it is a (total) way things could have been, a comprehensive situation.

Once you grasp the idea of a possible world, you can see that there are *lots* of them, for the actual world might have gone differently in all sorts of ways. For instance, it seems possible that some or even all of the persons who do exist might not have. Suppose my parents had never met, but instead had died of a childhood disease. Then, presumably, I would not exist. So, there is a possible world in which I do not exist. Consider another example: In the actual world, Bill Clinton was president of the United States in January 1998, but we can describe a possible world in which this is not so—for example, one in which Clinton resigned from office in December 1997. Likewise, we can conceive of a possible world in which airplanes were never invented or in which the Allies lost World War II. Perhaps we can even conceive of a possible world in which a large asteroid struck the earth in 1850, destroying the entire human race.

There are, in fact, infinitely many possible worlds. To see this, consider the case of Harvey, the sentimental mathematician. There is a possible world in which Harvey's favorite number is the number 1. In that world, Harvey is very fond of the number 1, thinks of it often, and frequently extols it to others. But, there is also a possible world in which Harvey's favorite number is the number 2, a possible world in which Harvey's favorite number is the number 3, and so on. Clearly, there are infinitely many possible worlds.

Because possible worlds are *total* ways things could have been, only one possible world can be actual. Think of it this way. A complete description of a possible world would be a list of statements that express all of what is true (and only what is true) in that world.[5] Now, take any two possible worlds (call them W_1 and W_2). Since these are different possible worlds, and a possible world is a comprehensive situation, there must be something true in one of these worlds that is not true in the other. For instance, suppose W_1 and W_2 are exactly alike, except that in W_1 you are presently wearing a purple hat while in W_2 you are not presently wearing a purple hat. Thus, if W_1 and W_2 are both actual, you are presently wearing a purple hat and yet not wearing a purple hat, which is a contradiction. Obviously, then, at most one of these worlds is actual.

Now that we have the concept of a possible world, we have an alternative way of defining "necessary truth." A **necessary truth** is one that is true in every possible world. And as we will see, this way of characterizing necessary truths has a number of advantages. But before proceeding further, we need to clarify the concept of a necessary truth by distinguishing it from some other concepts with which it is often confused.

First, a "necessary truth," as we have defined the term, is often said to be *logically* necessary as opposed to **physically necessary**. We can get at the concept of a physically necessary truth as follows. Many possible worlds have the same laws of nature (e.g., the law of gravity) as the actual world. Certain truths are true in all of these worlds. For example:

9. Each physical object is attracted to every other with a force varying as the product of the masses of the objects and inversely as the square of the distance between them. (Newton's law of gravitation)
10. Nothing travels faster than the speed of light.

Statements (9) and (10) are physically necessary truths. But because the laws of nature could have been different than they in fact are, physically necessary truths are not necessary in our sense. Remember, a logically necessary truth is one that is true in *every* possible world. And there are logically possible worlds in which physical objects behave in accordance with a slightly different law of gravity than they do in the actual world. There are also logically possible worlds in which something travels just a bit faster than the speed of light. Our interest is in *logically* necessary truths, which are true in every possible world, not in physically necessary truths, which are false in some possible worlds.

Second, a necessary truth is not the same thing as an **unalterable truth**. To illustrate:

11. John Wilkes Booth killed Abraham Lincoln.

Statement (11) seems to be unalterably true at this time. It cannot become false at this late date assuming that the past cannot be changed. But (11) is not a necessary truth, for there is surely a logically possible world in which Booth refrains from killing Lincoln. Booth did not act under logical necessity.

Third, the concept of a necessary truth is not identical to the concept of a **self-evident statement**. A statement is *self-evident* if one can know that it is true simply by grasping the concepts involved. Now, typical examples of self-evident statements do seem to be necessary. For instance:

12. No circles are squares.

But some necessary truths seem to be too complicated to be known in this way. Consider, for example, Goldbach's conjecture:

13. Every even number greater than 2 is equal to the sum of two prime numbers.

Since Goldbach's conjecture is a mathematical proposition, *either* it *or* its negation is presumably necessary. But neither Goldbach's conjecture nor its negation is obviously true. Moreover, neither Goldbach's conjecture nor its negation has been proved by mathematicians. So, neither Goldbach's conjecture nor its negation appears to be self-evident. Thus, we cannot equate necessary truths with self-evident truths.

Fourth, a necessary truth is not the same thing as an "**un-give-up-able**" **statement**, that is, a statement that *cannot be given up*. The American philosopher and logician Willard van Orman Quine (1908–) has decried the distinction between the necessary and the contingent. According to Quine, the truth is simply that there are some statements we would be very reluctant to give up (i.e., to stop believing) and others we would more readily give up. In other words, there are merely degrees of "un-give-up-ability," and there are no necessary truths at all. But consider the following argument:

> Clearly we cannot equate relative "ungiveupability" with necessity. . . . For a belief may be ungiveupable for reasons that have nothing to do with its truth, and are unrelated to the necessity of its truth. A belief may be a guiding belief for a person's life to such an extent that the person may be psychologically incapable of giving it up. Or I may find the belief that I exist ungiveupable, but its "ungiveupability" has nothing to do with the necessity of its truth.[6]

Thus, to say that a statement is un-give-up-able is to make a psychological claim, namely, that the person who believes it is psychologically incapable of ceasing to believe it. And this psychological concept is very different from the concept of a necessary truth, that is, the concept of a proposition that is true in every possible world.*

In thinking about necessary truths, we are led back to the distinction between statements and propositions, which we discussed briefly in section 4.1. We have said that a necessary truth cannot be false under any possible circumstances. But you may be thinking that linguistic meaning is changeable, and hence that there are possible worlds in which people speak the truth when they say, for example, "Some circles are squares." After all, there are possible worlds in which the word "circles" is synonymous with "rectangle" (as that word is currently used by English speakers). But this line of thinking is deeply flawed, because it confuses sentences with the propositions (truths or falsehoods) those sentences express. In ordinary usage, the English sentence "No circles are squares" expresses a necessary truth. But if the words composing the sentence

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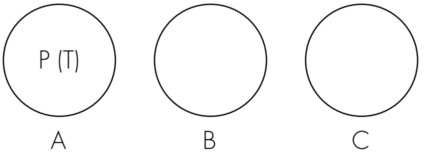
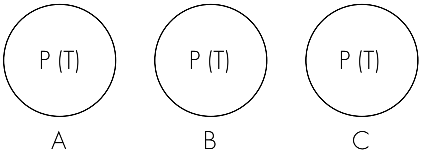
*Would it be better to interpret Quine as claiming that "Statement S is un-give-up-able" means "One cannot give S up *without doing something irrational*"? On this reading of Quine, "un-give-up-able" becomes an epistemological concept, that is, a concept regarding what is known or rationally believed. But even in this interpretation "un-give-up-able" still cannot be equated with "necessary," for it would surely be irrational for me to give up the statement "I am thinking" and yet "I am thinking" is not a necessary truth. For a closer look at these issues, see Kenneth Konyndyk, *Introductory Modal Logic* (Notre Dame, IN: University of Notre Dame Press, 1964), pp. 14–15.

were to change meaning, we would need to use a different sentence to express that same truth. So, to avoid confusion, we will make free use of the word "proposition" for the remainder of this chapter. A **proposition** is simply a truth or a falsehood. A given proposition may or may not be expressed in a sentence. And whereas a sentence (and hence a statement) belongs to a particular language, such as English or German, a proposition does not.

It may be helpful to depict possible worlds as circles for the sake of illustration. Let us label the actual world A and label two other possible worlds B and C. And, just for the sake of illustration, let us pretend that these three worlds are all of the possible worlds. (Of course, as we have already seen, there are in fact many more possible worlds.) The symbol *P(T)*, when placed in a circle, means that P is true in that world. For example:



This diagram says that P is true in A (the actual world). Similarly, the symbol *P(F)*, when placed in a circle, means that P is false in that world. Where the symbol *P(?)* appears, the question mark may be replaced with either a "T" or an "F" (as needed). Now, we can make a kind of picture of a necessary truth P as follows:

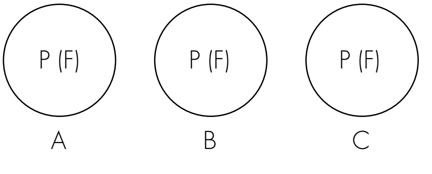


The diagram tells us that P is true in every possible world.

With the concept of a necessary truth in hand, we can readily characterize the other important modal concepts. For instance, a proposition is **impossible** *if and only if* it is necessarily false. Here is an example:

14. Some squares are triangles.

An impossible proposition is false in every possible world. We can depict an impossible proposition P as follows.

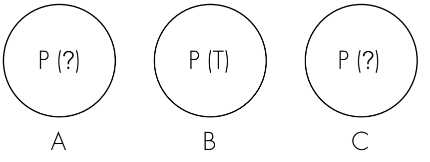


The diagram says that P is false in every possible world.

A proposition is **possible**, or **possibly true**, *if and only if* it is not necessarily false. By this definition, whatever is true is possibly true. Necessary truths also count as possibly true, since they are not necessarily false. In addition, many falsehoods are possibly true, such as this one:

15. Johannesburg, South Africa, is the most highly populated city in the world.

Possibly true propositions are true in *at least one* possible world. We can depict a generic possibly true proposition P as follows:



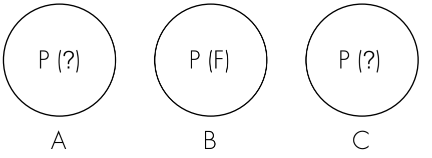
The diagram indicates that P is true in at least one possible world. Here, we made P true in world B, but of course, P is possibly true as long as P is true in *at least one* possible world, and that world need not be world B. So, our diagram is perhaps slightly misleading. The question marks may be replaced with either a "T" or an "F," depending on the proposition under consideration. For instance, if P is true in the actual world, then we replace the question mark with a "T" in world A.

A further clarification about possibly true propositions is needed. In saying that a proposition is *possibly* true, we are not saying that *for all we know* it is true. Philosophers say that a proposition is **epistemically possible** if it is not known to be false given the information currently available. (Epistemology is the branch of philosophy that concerns the theory of knowledge.) Consider again proposition (15). It is not epistemically possible, for we know it is false. Nevertheless, (15) is *logically* possible, for the following proposition is not a *necessary* falsehood: During the 20th century, Johannesburg grew steadily until it had a larger population than any other city in the world.

A proposition is **possibly false** *if and only if* it is not necessarily true. Possibly false propositions are false in *at least one* possible world. All false propositions (including necessarily false ones) are possibly false. But many true propositions are possibly false as well. For instance:

16. Pierre and Marie Curie discovered radium.

Although the Curies did discover radium, they might not have. For example, because their research was difficult and took a long time, they might easily have given up. We can depict a generic possibly false proposition P as follows:

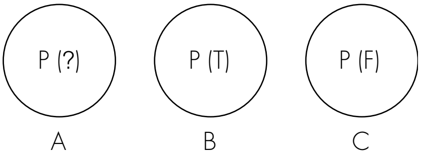


The diagram indicates that P is false in at least one world. Again, the question marks can be replaced by a "T" or an "F," depending on the proposition under consideration.

A proposition is **contingent** *if and only if* it is both possibly true and possibly false. For example, the following proposition is contingent:

17. I exist.

Since proposition (17) is actually true, it is possibly true. But it is also possibly false, for if a certain egg–sperm pair had never united, I presumably would not exist. And many things could have prevented that egg–sperm pair from uniting. If a proposition is contingent, then it is true in at least one possible world and false in at least one possible world. We can depict a generic contingent proposition P as follows:

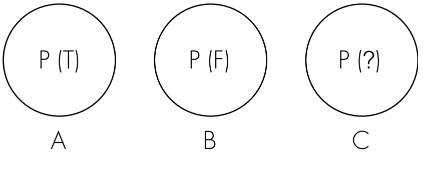


Note that a contingent proposition may or may not be true in the actual world.

A proposition is a **contingent truth** *if and only if* it is true but possibly false. In other words, a contingent truth is true in the actual world but false in at least one possible world. For instance:

18. Socrates died as a result of drinking hemlock.

Socrates did die as a result of drinking hemlock, but he might not have. For example, he might have escaped from prison, or he might simply have refused to drink the hemlock (in which case the Athenians would probably have executed him in some other way). We can depict a generic contingent truth P as follows:

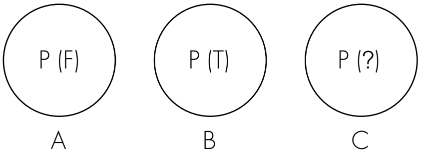


Note that a contingent truth is true in world A, the actual world.

Finally, a proposition is a **contingent falsehood** *if and only if* it is false but possibly true. For example:

19. Socrates died of smallpox.

A contingent falsehood is false in the actual world but true in at least one possible world. We can picture a generic contingent falsehood P as follows:



Note that a contingent falsehood is false in world A, the actual world.

Complete the following exercises to check your understanding of the basic modal concepts.

Exercise 12.1

Part A: True or False? Which of the following statements are true? Which are false?

* **1.** A *contingent truth* is defined as a proposition that is true in some possible worlds but false in others.

2. A proposition is *possibly true* if and only if it is true in at least one possible world.

3. The planet Saturn is an example of a *possible world*.

* **4.** The actual world is not a possible world.

5. A *necessary truth* is one that cannot be false under any possible circumstances.

6. Every necessary truth is possibly true.

* **7.** Every possibly true proposition is either a necessary truth or a contingent truth.

8. A proposition is *possibly true* if and only if it is true in the actual world.

9. A *contingent proposition* is true in at least one possible world and false in at least one possible world.

* **10.** A *necessary truth* is one that is true in every possible world.

11. If a proposition is *possibly true*, then it is not known to be false given the information currently available.

12. A proposition is *contingently false* if it is false in the actual world but true in at least one possible world.

* **13.** Every contingent proposition is false.

14. If P is a contingent truth, then there is a possible world in which its negation is true.

15. An *impossible proposition* is one that is necessarily false.

* **16.** A *necessarily false* proposition is one that is false in every possible world.

17. If a proposition is true in some possible worlds but false in others, then it must be a contingent truth.

18. If an argument is valid, then its conclusion is true in every possible world in which its premises are true.

* **19.** If an argument is invalid, then there is at least one possible world in which its premises are true while its conclusion is false.

20. An *analytic statement* is one that is true by virtue of its form.

Part B: Identifying Modalities Which of the following propositions are necessary? Contingent? Impossible? If a statement is contingent, also indicate its actual truth value if you can.

* **1.** Chairs exist.

2. All uncles are male.

3. Five plus five equals fifty-five.

* **4.** One plus one equals two *and* the Eiffel Tower is in France.

5. $\sim A \rightarrow (A \rightarrow B)$

6. Benjamin Franklin was the third president of the United States.

* **7.** The French artist Rodin once made a bronze sculpture that was a spherical cube.

8. Either grass is green or it is not the case that grass is green.

9. $A \rightarrow (A \rightarrow B)$

* **10.** Abraham Lincoln was never president of the United States.

11. $\sim(P \leftrightarrow P)$

12. John F. Kennedy was assassinated by Lee Harvey Oswald, but Kennedy did not die.

* **13.** Either two plus two equals four or two plus two equals five.

14. Two plus two equals twenty-two *and* there are some trees.

15. Either two equals three or two equals four.

* **16.** No human has ever set foot on Neptune.

17. Two plus three equals five *and* seven plus five equals twelve.

18. Some people enjoy rock music.

* **19.** There is at least one aunt who has neither a niece nor a nephew.

20. Alaska was the 49th state to join the United States.

12.2 The Modal Symbols

As noted previously, we can develop a system of modal logic by adding to the system of statement logic developed in chapters 7 and 8. We will use the symbol

"□" (called the "box" or "necessity operator") to stand for "necessarily." Thus, "□***p***" means "necessarily, ***p***." To illustrate:

20. Necessarily, all sisters are siblings. (S: All sisters are siblings)
    In symbols: □S

The box can be combined with the tilde. We can say that ***p*** is *not* a necessary truth in this way: ~□***p***. For instance:

21. It is not necessarily true that dogs exist. (D: Dogs exist)
    In symbols: ~□D

Notice that care must be taken with the placement of the tilde, for we symbolize "***p*** is a necessary falsehood" as □~***p***. For example:

22. It is necessarily false that two plus two equals five. (T: Two plus two equals five)
    In symbols: □~T

We will use the symbol "◇" (called the "diamond" or "possibility operator") to stand for "possibly." To illustrate:

23. Possibly, Sue is lying. (S: Sue is lying)
    In symbols: ◇S

We can say that ***p*** is logically *impossible* using the diamond and a tilde: ~◇***p***. For instance:

24. It is logically impossible that zero equals one. (Z: Zero equals one)
    In symbols: ~◇Z

Again, care must be taken regarding the placement of the tilde. Consider the following:

25. It is logically possible that Oswald did not kill Kennedy. (K: Oswald killed Kennedy)
    In symbols: ◇~K

Using our symbols we can write "***p*** is contingent" as ◇***p*** • ◇~***p***. In other words, to say that a proposition is contingent is to say that it is logically possible that it is true and logically possible that it is false. For example:

26. "Augusta is the capital of Maine" is a contingent statement. (A: Augusta is the capital of Maine)
    In symbols: ◇A • ◇~A

We can symbolize "***p*** is a contingent truth" as ***p*** • ◇~***p*** or as ***p*** • ~□***p***. For instance:

27. It is a contingent truth that Columbus is the capital of Ohio. (C: Columbus is the capital of Ohio)
    In symbols: C • ◇~C

We could also symbolize (27) as $C \bullet \sim\Box C$.

We can symbolize "*p* is a contingent falsehood" as $\sim p \bullet \Diamond p$. To illustrate:

28. It is contingently false that Dayton is the capital of Ohio. (D: Dayton is the capital of Ohio)
    In symbols: $\sim D \bullet \Diamond D$

We use parentheses to indicate the scope of the modal operators, just as we use parentheses to indicate the scope of the tilde (or the quantifiers in predicate logic). To see the need for this, compare the following statements:

29. Necessarily, both nine is an odd number and there are nine planets in our solar system. (N: Nine is a odd number; P: There are nine planets in our solar system)
    In symbols: $\Box(N \bullet P)$
30. Necessarily, nine is an odd number, and furthermore there are nine planets in our solar system.
    In symbols: $\Box N \bullet P$

Statements (29) and (30) are not logically equivalent. (29) says that the conjunction $N \bullet P$ is necessarily true. But a conjunction is necessary *if and only if* both its conjuncts are necessary. (If even one conjunct is not necessary, then there are worlds in which it is false, and in those worlds the conjunction itself is also false.) Moreover, it is not a necessary truth that there are nine planets in our solar system—there might have been more planets or fewer. So, (29) is false. (30), on the other hand, says that N ("Nine is an odd number") is necessary, but that P ("There are nine planets in our solar system") is true. Thus, (30) is a more cautious statement, and indeed a true one.

The main point here is that parentheses are needed to indicate the scope of the modal operators, and a shift in the placement of parentheses can change the meaning substantially. For example, just as $\sim(A \vee B)$ differs in meaning from $\sim A \vee B$ or from $\sim A \vee \sim B$, so $\Box(A \vee B)$ differs in meaning from $\Box A \vee B$ or from $\Box A \vee \Box B$. The general rule is that logical operators, such as the box and diamond, apply to the smallest immediately-following component that the punctuation permits. Thus, in the formula $\Diamond(P \rightarrow Q) \bullet R$, the diamond applies only to $P \rightarrow Q$; its scope does not extend to R.

It is important to understand that the modal operators are not truth-functional operators; that is, the meaning of the modal operators cannot be specified via truth tables. If the box and the diamond were truth-functional, then we could simply develop truth tables for them and set about evaluating arguments. Let's see what happens when we try to develop truth tables for the box and diamond.

p	$\Box p$
T	?
F	F

If p is false, then of course $\Box p$ is false. A proposition cannot be both false and necessarily true. But if we assume that p is true, we do not have enough information to determine that p is necessary. If p is merely a contingent truth, then $\Box p$ is false; but if p is a necessary truth, then $\Box p$ is true. So, the box is not truth-functional.

Now, let's consider the diamond. Is it truth-functional?

p	$\Diamond p$
T	T
F	?

If p is true, then of course p is possibly true. But suppose p is false. What is the truth value of $\Diamond p$? It may be either true or false, depending on the proposition in question. If p is impossible, then $\Diamond p$ is false; if p is contingently false, then $\Diamond p$ is true. So, the diamond is not truth-functional.

Truth tables give us the meaning of the tilde, dot, vee, arrow, and double-arrow. But as we have just seen, truth tables cannot give us the meaning for the box and the diamond. Following the work of Saul Kripke, who developed a theory of meaning (or semantics) for modal logic in the late 1950s, we will assign meaning to the box and diamond in terms of possible worlds. $\Box p$ is true in a given possible world *if and only if* p is true in every possible world. And $\Diamond p$ is true in a given possible world *if and only if* p is true in at least one possible world.

The following exercises will give you practice using the new symbols introduced in this section.

Exercise 12.2

Part A: Symbolizing Express the following statements in symbols, using G to abbreviate "God exists."

* **1.** It is logically possible that God exists.

2. Necessarily, God does not exist.

3. God exists, but "God exists" is not a necessary truth.

* **4.** It is impossible that God does not exist.

5. It is logically possible that God exists, but in fact God does not exist.

6. "God exists" is a contingent truth.

* **7.** "God exists" is a contingent proposition.

8. "God exists" is contingently false.

9. "God exists" is necessarily false.

* **10.** "God exists" is either necessary or impossible.

11. There is at least one possible world in which God exists.
12. God exists in every possible world.
* 13. There is no possible world in which God exists.
14. God exists in at least one possible world, but not in every possible world.
15. Either God exists in every possible world or God does not exist in the actual world.

Part B: More Symbolizing Express the following statements in symbols, using the schemes of abbreviation provided.

* 1. It is a contingent truth that Lincoln was the 16th president of the United States. (L: Lincoln was the 16th president of the U.S.)
2. "I exist" is a contingent proposition. (X: I exist)
3. It is necessarily false that Susan is a wife who is not married. (W: Susan is a wife; M: Susan is married)
* 4. George Washington was the first president of the United States, but it is logically possible that Benjamin Franklin was the first president of the United States. (G: George Washington was the first president of the United States; B: Benjamin Franklin was the first president of the United States)
5. It is impossible that if demons are invisible, then my wristwatch is inhabited by blue demons. (I: Demons are invisible; B: My wristwatch is inhabited by blue demons.)
6. It is a necessary truth that either it is raining or it is not raining. (R: It is raining)
* 7. Santa Claus does not exist, but it is logically possible that he does. (S: Santa Claus exists)
8. Either Goldbach's conjecture is necessarily true or its negation is necessarily true. (G: Goldbach's conjecture, that is, every even number greater than 2 is equal to the sum of two prime numbers)
9. Necessarily, if Jane is married, then Jane has a spouse. (M: Jane is married; S: Jane has a spouse)
* 10. It is a contingent truth that Rudolph is both red-nosed and a reindeer. (R: Rudolph is red-nosed; D: Rudolph is a reindeer)
11. Necessarily, both six and eight are divisible by two. (S: Six is divisible by two; E: Eight is divisible by two)
12. Possibly, Seattle is the capital of Washington, but as a matter of fact, Seattle is not the capital of Washington. (S: Seattle is the capital of Washington)
* 13. It is logically possible that if Kyle surfs on a tidal wave, then he surfs from California to Hawaii. (T: Kyle surfs on a tidal wave; C: Kyle surfs from California to Hawaii)
14. It is possible that Joan wins, but it is impossible that both Joan and Marsha win. (J: Joan wins; M: Marsha wins)

15. If it is necessary that five is odd, then it is necessary that six is even. (F: Five is odd; S: Six is even)

* 16. There is a possible world in which humans have ESP, but in the actual world, humans do not have ESP. (H: Humans have ESP)

17. One plus one equals two in every possible world, and one plus one equals eleven in no possible world. (O: One plus one equals two; E: One plus one equals eleven)

18. There is at least one possible world in which you win an Olympic gold medal and at least one possible world in which you do not win an Olympic gold medal. (Y: You win an Olympic gold medal)

* 19. There is a possible world in which every trapezoid is blue, but there is no possible world in which some trapezoid is a triangle. (B: Every trapezoid is blue; T: Some trapezoid is a triangle)

20. Although poverty exists in the actual world, it does not exist in every possible world. (P: Poverty exists)

12.3 Constructing Proofs

In this section, we introduce our first set of inference rules for modal logic.[7] Because we are building on the system of statement logic developed in chapter 8, all the rules of statement logic from chapter 8 may be used in proofs throughout this chapter. For reasons mentioned previously, however, we now assume that these rules apply to propositions (truths and falsehoods), as well as to statements (sentences having truth value).

If a proposition is necessarily true, then it is true. In other words, if a proposition is true in *every* possible world, then it is true in the actual world. This obvious principle gives us our first rule of modal logic, ***necesse ad esse*** (NE). Using the lowercase, italicized ***p*** to stand for any proposition, the rule may be stated as follows:

$$\begin{array}{l} \quad \Box p \\ \therefore\ p \end{array}$$

Literally translated, *necesse ad esse* means "from being necessary to being." Here is an English example:

31. Necessarily, no immaterial soul weighs 40 pounds. So, no immaterial soul weighs 40 pounds. (S: No immaterial soul weighs 40 pounds)

In symbols, the proof looks like this:

1. □S	∴ S
2. S	1, NE

Note that NE is an *implicational* rule, not an equivalence rule. Accordingly, it may not be applied to part of a line in a proof; rather, it must be applied to an entire line. For example, the following inference is *not* permitted:

1. ~□P
2. ~P   1, *incorrect use of NE*

This is like arguing, "It is not necessarily true that I exist. Therefore, I do not exist." Since the premise is true but the conclusion is false, this argument is invalid. (The premise says that I do not exist in *every* possible world or, in other words, that there is at least one possible world in which I do not exist. But the conclusion says that I do not exist in the actual world.)

If a proposition is actually true, then it is possibly true. In other words, if a proposition is true in the actual world, then it is true in at least one possible world. (Remember, the actual world is a possible world.) This obvious principle gives us our second rule of modal logic, ***esse ad posse*** (EP):

p
∴ ◇*p*

Literally translated, *esse ad posse* means "from being to being possible." Here is an English example:

32. Some bachelors are party animals. So, it is logically possible that some bachelors are party animals. (B: Some bachelors are party animals)

The proof runs as follows:

1. B   ∴ ◇B
2. ◇B   1, EP

Note that EP is an implicational rule. As such, it must be applied to whole lines in a proof, and not to parts of lines. For example, the following inference is not permitted:

1. C → M
2. ◇C → M   1, *incorrect use of EP*

This is like arguing, "If Demi Moore is a wife, then she is married. So, if it is *possible* that Demi Moore is a wife, then she is married." Again, the invalidity is obvious. The premise is true in every possible world. But the conclusion is false in those worlds in which Demi Moore is not married (although she could be a wife).

Our next inference rule is based on the principle that *every theorem of (nonmodal) statement logic is necessarily true*. A *theorem* is a statement that can be proved without any premises. Furthermore, every theorem of (nonmodal) statement logic is a tautology, and every such tautology is a theorem. (A *tautology* is a statement that is true in every row of its truth table.) Since it is impossible to assign truth values in such a way as to make tautologies or theorems false, they are true in every possible world. Our third inference rule, **theorem necessitation** (TN), may be formulated as follows:

p (derived without using any premises)
$\therefore \Box p$

Here is a proof that involves TN:

	∴ □[A → (~B ∨ A)]
1. A	Assume
2. A ∨ ~B	1, Add
3. ~B ∨ A	2, Com
4. A → (~B ∨ A)	1–3, CP
5. □[A → (~B ∨ A)]	4, TN

Note that we arrive at line (4) without using any premises. Here is another example of the use of TN:

1. □(P ∨ ~P) → □Q	∴ □Q
2. ~(P ∨ ~P)	Assume
3. ~P • ~~P	2, DeM
4. P ∨ ~P	2–3, RAA
5. □(P ∨ ~P)	4, TN
6. □Q	5, 1, MP

Here, we first prove the theorem, $P \vee \sim P$, without using any premises. Then we apply TN. Finally, we use line (5) together with the premise to obtain line (6), by MP.

Our fourth rule, **modal operator negation** (MN), is an equivalence rule. To understand this rule, consider the following pair of propositions:

33. Possibly, my shirt is green. (G: My shirt is green)
In symbols: ◇G

34. It is not necessarily true that my shirt is not green.
In symbols: ~□~G

Proposition (33) says that there is a possible world in which my shirt is green. That being so, the proposition "My shirt is not green" is not true in every possible world. In other words, if (33) is true, then (34) must be true. Similarly, if (34) is true, then (33) must be true, for if "My shirt is not green" is not true in every possible world, then "My shirt is green" is true in at least one possible world. Reflections of this sort should render the following equivalence rule intuitive:

$\Diamond p :: \sim\Box\sim p$

The four-dot symbol tells us that $\Diamond p$ implies $\sim\Box\sim p$ and also that $\sim\Box\sim p$ implies $\Diamond p$. This is the first form of modal operator negation (MN). MN comes in four forms. To understand the second form, consider the following pair of propositions:

35. Necessarily, red is a color. (R: Red is a color)
    In symbols: $\Box$R

36. It is not possible that red is not a color.
    In symbols: $\sim\Diamond\sim$R

Proposition (35) tells us that "Red is a color" is true in every possible world. (36) tells us that there is no possible world in which "Red is not a color" is true. It should be clear that each of these propositions implies the other. Our second form of MN is as follows:

$\Box p :: \sim\Diamond\sim p$

We can grasp the third form of MN by reflecting on the following pair of propositions:

37. It is not necessarily true that I exist. (X: I exist)
    In symbols: $\sim\Box$X

38. It is logically possible that I do not exist.
    In symbols: $\Diamond\sim$X

Proposition (37) tells us that "I exist" is not true in every possible world. And (38) tells us that there is at least one possible world in which "I do not exist" is true. Again, it should be obvious that each of these propositions implies the other. Stated formally, the third form of MN looks like this:

$\sim\Box p :: \Diamond\sim p$

It may be helpful to note that there is an analogy here between the behavior of the box and diamond and the behavior of universal and existential quantifiers. Recall that the QN rule lets us move from $\sim(x)Fx$ to $(\exists x)\sim$Fx, and vice versa. Similarly, MN lets us move from $\sim\Box$A to $\Diamond\sim$A, and vice versa.

The fourth form of MN can readily be understood by reflecting on the following pair of propositions.

39. Necessarily, it is not the case that circular squares exist (C: Circular squares exist)
    In symbols: $\Box\sim$C

40. It is not possible that circular squares exist.
In symbols: $\sim\Diamond C$

Proposition (39) tells us that "Circular squares do not exist" is true in every possible world. (40) tells us that "Circular squares exist" is not true in any possible world. Obviously, these propositions imply each other. And our last form of MN looks like this:

$\Box{\sim}p :: \sim\Diamond p$

Again, the box and diamond behave in a way analogous to universal and existential quantifiers. QN lets us move from $(x){\sim}Fx$ to $\sim(\exists x)Fx$, and vice versa. Similarly, MN lets us move from $\Box{\sim}A$ to $\sim\Diamond A$, and vice versa.

To sum up, the four forms of modal operator negation (MN) are as follows:

$\Diamond p :: {\sim}\Box{\sim}p$
$\Box p :: \sim\Diamond{\sim}p$
${\sim}\Box p :: \Diamond{\sim}p$
$\Box{\sim}p :: \sim\Diamond p$

When we employ any of these four rules in a proof, our annotation is simply MN. Since each form of MN is an equivalence rule, MN may be applied to parts of lines in a proof as well as to entire lines. Let's examine a proof that involves MN:

1. $\sim(\Diamond E \lor \Box F)$	
2. $\Box{\sim}E \to \sim\Diamond{\sim}G$	
3. $\Diamond{\sim}F \to {\sim}\Box{\sim}H$	$\therefore G \bullet \Diamond H$
4. $\sim\Diamond E \bullet {\sim}\Box F$	1, DeM
5. $\sim\Diamond E$	4, Simp
6. $\sim\Diamond E \to \sim\Diamond{\sim}G$	2, MN
7. $\sim\Diamond{\sim}G$	5, 6, MP
8. $\Box G$	7, MN
9. G	8, NE
10. ${\sim}\Box F$	4, Simp
11. $\Diamond{\sim}F$	10, MN
12. ${\sim}\Box{\sim}H$	3, 11, MP
13. $\Diamond H$	12, MN
14. $G \bullet \Diamond H$	9, 13, Conj

Note that MN is applied to part of a line (the antecedent of a conditional) in line (6).

Some of the rules of thumb we have used previously apply in modal logic. For example, it often helps to look at the conclusion first and then work back-

Summary of the First Set of Inference Rules

Necesse ad Esse **(NE)**

$\Box p$
$\therefore p$

Esse ad Posse **(EP)**

p
$\therefore \Diamond p$

Theorem Necessitation (TN)

p (derived without using any premises)
$\therefore \Box p$

Modal Operator Negation (MN)

$\Diamond p :: {\sim}\Box{\sim}p$
$\Box p :: {\sim}\Diamond{\sim}p$
${\sim}\Box p :: \Diamond{\sim}p$
$\Box{\sim}p :: {\sim}\Diamond p$

ward. Also, if it is possible to apply one of the rules of statement logic, it usually helps to do so. But the following rule of thumb is specific to modal logic:

> Rule of Thumb 1: When the tilde is combined with the box or diamond, it often helps to apply MN. Then use statement logic or other rules of modal logic.

Complete the following exercises to ensure your grasp of our first set of inference rules for modal logic.

Exercise 12.3

Part A: Annotating Annotate the following proofs.

* **1.** 1. $\Box$P $\therefore \Diamond$P
2. P
3. $\Diamond$P

2. 1. $\Box{\sim}$B • T $\therefore {\sim}\Diamond$B
2. $\Box{\sim}$B
3. ${\sim}\Diamond$B

3. 1. $\Box$(A • B) $\therefore \Diamond$B
2. A • B
3. B
4. $\Diamond$B

* **4.** 1. $\sim\Diamond\sim G \quad \therefore G$
2. $\Box G$
3. G

5. 1. $\Box\sim(B \leftrightarrow \sim B) \rightarrow A \quad \therefore \Diamond A$
2. $B \leftrightarrow \sim B$
3. $(B \rightarrow \sim B) \bullet (\sim B \rightarrow B)$
4. $B \rightarrow \sim B$
5. $\sim B \rightarrow B$
6. $\sim B \vee \sim B$
7. $\sim B$
8. $\sim\sim B \vee B$
9. $B \vee B$
10. B
11. $B \bullet \sim B$
12. $\sim(B \leftrightarrow \sim B)$
13. $\Box\sim(B \leftrightarrow \sim B)$
14. A
15. $\Diamond A$

6. 1. $\Diamond H \vee \sim\Box S$
2. $\Box\sim H \quad \therefore \Diamond\sim S$
3. $\sim\Diamond H$
4. $\sim\Box S$
5. $\Diamond\sim S$

* **7.** 1. $\Diamond\sim W \quad \therefore \sim\Box W$
2. $\sim\Box W$

8. 1. $\Box S \vee \Box P$
2. $\Diamond\sim S \quad \therefore \sim\Diamond\sim P$
3. $\sim\Box S$
4. $\Box P$
5. $\sim\Diamond\sim P$

9. 1. $\sim\Diamond E \quad \therefore \sim E$
2. $\Box\sim E$
3. $\sim E$

* **10.** $\therefore \Box[(A \bullet B) \rightarrow B]$
1. $A \bullet B$
2. B
3. $(A \bullet B) \rightarrow B$
4. $\Box[(A \bullet B) \rightarrow B]$

Part B: Correct or Incorrect? Which of the following inferences are permitted by one of our rules, and which are not? If an inference is permitted, indicate the rule that permits it. If it is not permitted, simply write "incorrect." (The question is whether one can move from the first statement to the second, in each instance, by a single application of one of our rules.)

* **1.** 1. $\Box N \rightarrow \Box E$
2. $N \rightarrow \Box E$

2. 1. $\Diamond L$
2. L

3. 1. $\sim\Diamond\sim K \rightarrow J$
2. $\Box K \rightarrow J$

* **4.** 1. $A \rightarrow B$
2. $\Diamond(A \rightarrow B)$

5. 1. C
2. $\Box C$

6. 1. $\Box(\Diamond D \rightarrow E)$
2. $\Diamond D \rightarrow E$

* **7.** 1. $F \rightarrow G$
2. $F \rightarrow \Diamond G$

8. 1. $P \rightarrow \Box Q$
2. $P \rightarrow Q$

9. 1. $\Box(H \bullet L)$
2. $\Box(H \bullet \Diamond L)$

* **10.** 1. $\sim\Box\sim R \leftrightarrow S$
2. $\Diamond R \leftrightarrow S$

11. 1. $T \vee \Diamond U$
2. $T \vee U$

12. 1. $W \vee X$
2. $W \vee \Diamond X$

* **13.** 1. $(\Diamond\sim Y \leftrightarrow \sim\Box\sim Z) \rightarrow \sim\Diamond V$
2. $(\Diamond\sim Y \leftrightarrow \Diamond Z) \rightarrow \sim\Diamond V$

14. 1. $\Box A \vee \Box B$
2. $A \vee B$

15. 1. $(\Diamond C \bullet \sim\Diamond\sim D) \rightarrow \Box E$
2. $(\Diamond C \bullet \Box D) \rightarrow \Box E$

Part C: Proofs Construct proofs to show that the following arguments are valid.

* **1.** $\Box C \bullet D, (\Diamond C \bullet D) \rightarrow S \therefore \Diamond S$

2. $\Box(P \rightarrow P) \rightarrow \Box Q \therefore Q$

3. $\Diamond J \vee \sim\Box K, \Diamond\sim L \rightarrow \sim\Diamond\sim K, \Box\sim J \therefore L$

* **4.** $\sim\Diamond M \bullet \Box N, \Box\sim M \rightarrow \sim\Diamond Q, \Diamond\sim N \rightarrow \sim\Box P,$
$(\Box\sim Q \vee \Diamond\sim P) \rightarrow \sim\Diamond R \therefore \sim R$

5. $\sim(\Box R \rightarrow \Box\sim S), \Diamond S \rightarrow T, \Diamond R \rightarrow U \therefore \Diamond(T \bullet U)$

6. $(\Box T \bullet \Diamond P) \vee (\Box T \bullet \Diamond\sim W), \Diamond S \rightarrow \sim\Box T, \Box W \therefore \Diamond\sim S \bullet \Diamond P$

* **7.** $\Diamond Z \vee (A \bullet \Box B), \Diamond Z \rightarrow \sim R, \Diamond\sim B \therefore \sim\Box R$

8. $\sim\Box(K \vee K), \sim\Box L \rightarrow \sim\Diamond\sim K, \Diamond P \rightarrow \Diamond\sim L \therefore \Diamond\sim P$

9. $\Box(A \rightarrow B), \Diamond\sim(\sim B \rightarrow \sim A) \vee \Box E, \Diamond E \rightarrow D \therefore \Diamond D$

* **10.** $(\Box M \vee \Box F) \bullet (\Box M \vee \sim\Box G), \Box M \rightarrow \sim\Diamond H, (\Box F \bullet \sim\Box G) \rightarrow \sim\Box J$
$\therefore \Diamond H \rightarrow \Diamond\sim J$

11. $\Box K \bullet (L \vee M), (\Box K \bullet L) \rightarrow \Box\sim N, (\Box K \bullet M) \rightarrow \Box\sim N,$
$\Diamond O \rightarrow \Diamond N \therefore \Diamond\sim O$

12. $\Diamond(P \bullet P), \Diamond\sim\sim P \rightarrow Q, \sim R \rightarrow \sim\Diamond Q \therefore \Diamond R$

* **13.** $\Box(A \vee B), \sim\Diamond\sim(B \vee A) \rightarrow \sim\Diamond C, \sim\Box\sim C \vee R \therefore \Diamond(R \vee S)$

14. $\sim(\Box F \rightarrow \sim\Diamond G), \sim\Diamond G \vee \Box H, \Diamond\sim F \vee \Box J \therefore \Diamond H \bullet \Box J$

15. $\Diamond C \rightarrow \sim\Box[\sim A \rightarrow (A \rightarrow B)] \therefore \sim C$

* **16.** $\Diamond Z \bullet (\Box Y \vee \sim\Diamond W), \Box\sim Z \vee \Diamond\sim Y, \Box\sim W \rightarrow \sim\Diamond U,$
$\sim\Box\sim U \vee \Box\sim T \therefore \Diamond\sim T$

17. $\Diamond\sim(S \rightarrow B), \sim\Box C \rightarrow \Box(\sim B \rightarrow \sim S), D \rightarrow \Diamond\sim C \therefore \Diamond\sim D$

18. $\Box\sim B \rightarrow (\Box\sim C \vee \sim\Diamond O), H \rightarrow \sim\Diamond B, (\sim\Diamond C \vee \Box\sim O) \rightarrow \sim\Diamond D,$
$\Box H \therefore D \rightarrow \Box E$

* **19.** $P \vee G, P \rightarrow \Box Z, G \rightarrow \sim\Diamond\sim Z, \Diamond\sim S \rightarrow \Diamond\sim Z \therefore \Box S$

20. $(\Box J \vee \Diamond K) \bullet (\Box J \vee \Diamond\sim L), \Box J \rightarrow \Diamond\sim M, (\Diamond K \bullet \sim\Box L) \rightarrow \sim\Box M,$
$\sim\Diamond N \vee \Box M \therefore \Diamond\sim N$

Part D: English Arguments Symbolize the following arguments and then prove them valid.

* **1.** Necessarily, it is false that free acts are coerced. Accordingly, it is impossible that free acts are coerced. (F: Free acts are coerced)

2. It is not logically possible that God does not exist. Therefore, God exists. (G: God exists)

3. Necessarily, invisible paintings do not exist. But either it is possible that invisible paintings exist, or it is necessary that all paintings are colored. Hence, all paintings are colored. (P: Invisible paintings exist; C: All paintings are colored)

* **4.** It's logically impossible that all paintings are forgeries. Consequently, not all paintings are forgeries. (F: All paintings are forgeries)

5. It is contingently false that vampires exist. So, it is possible that vampires exist. (V: Vampires exist)

6. Either it's impossible that humans have souls or it's necessary that moral agents have free will. But humans do have souls. It follows that moral agents have free will. (S: Humans have souls; M: Moral agents have free will)

* **7.** It is not possible that time travel occurs but it is necessary that time goes on. If it is possible that time is real and it is impossible that time travel occurs, then it is possible that time does not go on. So, time is not real. (O: Time travel occurs; G: Time goes on; R: Time is real)

8. It is contingently true that electrons exist. So, it is not necessarily true that electrons exist. (E: Electrons exist)

9. Either the soul is immortal or it is not necessarily true that if the good is real, then the soul is immortal only if the good is real. We may conclude that the soul is immortal. (S: The soul is immortal; G: the good is real)

10. If life is meaningless, then possibly happiness is both real and not real. It follows that life is not meaningless. (L: Life is meaningless; H: Happiness is real)

12.4 Modal Distribution and Strict Implication

We must now take a close look at inferences involving modal operators, the dot, the vee, and the arrow. We will first consider the behavior of the box and diamond as they relate to the dot and vee.

The **modal operator distribution equivalence** (MODE) rule comes in two forms. The first form governs the box and the dot. Using the lowercase, italicized letters ***p*** and ***q*** to stand for any proposition, the first form of MODE may be formulated as follows:

$$\Box(p \bullet q) :: (\Box p \bullet \Box q)$$

This makes explicit the principle that a conjunction is a necessary truth *if and only if* each of its conjuncts are necessary truths. For example, we can interchange the following statements because they are logically equivalent:

41. Necessarily, both two is even and three is odd.
42. Necessarily, two is even, and necessarily, three is odd.

Since MODE is an equivalence rule, it can be applied to parts of a line.

The second form of MODE governs the diamond and the vee:

$$\Diamond(p \vee q) :: (\Diamond p \vee \Diamond q)$$

This rule tells us that the diamond distributes over the vee. It makes explicit the logical equivalence between such statements as these:

43. It is logically possible that either Smith wins the election or Jones wins the election (or both).
44. Either it is logically possible that Smith wins the election or it is logically possible that Jones wins the election (or both).

The second form of MODE is also an equivalence rule.

Here is a short proof that involves the MODE rule:

1. $\Box(D \bullet E)$	
2. $\sim\Diamond\sim E \rightarrow \Diamond(K \vee L)$	$\therefore\ \Diamond K \vee \Diamond L$
3. $\Box D \bullet \Box E$	1, MODE
4. $\Box E$	3, Simp
5. $\sim\Diamond\sim E$	4, MN
6. $\Diamond(K \vee L)$	2, 5, MP
7. $\Diamond K \vee \Diamond L$	6, MODE

The **modal operator distribution implicational** (MODI) rule is, as the name suggests, an implicational rule. As such, MODI may be applied only to entire lines in a proof, and not to parts of lines. The first form of MODI concerns the box and the vee:

$\Box p \vee \Box q$
$\therefore\ \Box(p \vee q)$

This inference rule is based on the principle that a disjunction is a necessary truth if *at least one* of its disjuncts is a necessary truth. Here is an English example:

45. Either "Two plus two equals four" is necessary or "I am wearing a red shirt" is necessary. So, it is a necessary truth that either two plus two equals four or I am wearing a red shirt.

Warning: The first form of MODI does *not* let us move from $\Box(p \vee q)$ to $\Box p \vee \Box q$. The simplest way to see why is to consider the case in which q stands for $\sim p$, for instance, $\Box(A \vee \sim A)$, hence $\Box A \vee \Box \sim A$. It is easy to provide a counterexample to this argument form:

46. Necessarily, either Socrates died in 399 B.C.E. or Socrates did not die in 399 B.C.E. Therefore, either "Socrates died in 399 B.C.E." is a necessary truth or "Socrates did not die in 399 B.C.E." is a necessary truth.

Here, the premise is true. All propositions of the form "Either A or not A" are necessary truths. But the conclusion is false. Socrates might have died earlier or later than 399 B.C.E. So, "Socrates died in 399 B.C.E." is not a necessary truth. And since Socrates did in fact die in 399 B.C.E., it is clear that "Socrates did not die in 399 B.C.E." is not a *necessary* truth.

The second form of MODI governs the diamond and the dot:

$\Diamond(p \bullet q)$
$\therefore\ \Diamond p \bullet \Diamond q$

This rule is based on the principle that if a conjunction is possible, then each of its conjuncts is possible. Here is an English example:

47. It is logically possible that both Tom and Fred are lying. So, it is logically possible that Tom is lying and it is logically possible that Fred is lying.

Warning: The second form of MODI does not permit us to move from $\Diamond p \bullet \Diamond q$ to $\Diamond(p \bullet q)$. To see why, consider the following argument:

48. It is logically possible that Smith wins the race and logically possible that Jones wins the race. So, it is logically possible that Smith and Jones both win the race.

A tie is logically possible, but it is not logically possible that Smith and Jones both win the same race. Thus, arguments of this form can have a true premise and a false conclusion, and so the form is invalid.

Here is a proof involving the MODI rules:

1. $\Diamond(S \bullet R)$	
2. $\Diamond R \rightarrow (\sim\Diamond W \vee \Box Z)$	$\therefore\ \Box(\sim W \vee Z)$
3. $\Diamond S \bullet \Diamond R$	1, MODI
4. $\Diamond R$	3, Simp
5. $\sim\Diamond W \vee \Box Z$	2, 4, MP
6. $\Box\sim W \vee \Box Z$	5, MN
7. $\Box(\sim W \vee Z)$	6, MODI

Note that MN is applied in line (6) to set up an application of MODI.

Because confusion between the MODE and MODI rules readily leads to fallacies, it is important to grasp them clearly. So, let us pause here and summarize. Each form of MODE is an equivalence rule. The first form governs the box and the dot, and the second form governs the diamond and the vee:

$$\Box(p \bullet q) :: (\Box p \bullet \Box q) \qquad \text{MODE}$$
$$\Diamond(p \vee q) :: (\Diamond p \vee \Diamond q)$$

The first form tells us that the box can be moved from a conjunction to its conjuncts, and vice versa. The second form tells us that the diamond can be moved from a disjunction to its disjuncts, and vice versa.

Both forms of MODI, on the other hand, are implicational in nature. They govern the box and the vee, and the diamond and the dot.

$\Box p \vee \Box q$	$\Diamond(p \bullet q)$	MODI
$\therefore\ \Box(p \vee q)$	$\therefore\ \Diamond p \bullet \Diamond q$	

The form on the left tells us how the box and vee interrelate: The box can be moved from the disjuncts to the whole disjunction, *but not vice versa*. The form on the right tells us how the diamond and dot interrelate: The diamond can be moved from a conjunction to its conjuncts, *but not vice versa*.

Now that we have the MODE and MODI rules, we can add the following rules of thumb for modal logic:

Rule of Thumb 2: When the box is combined with the dot or the diamond is combined with the vee, it often helps to apply MODE.

Rule of Thumb 3: When the box is combined with the vee or the diamond is combined with the dot, it often helps to apply MODI.

As mentioned previously, C. I. Lewis wanted to construct a system of logic with a stronger conditional than the arrow. In particular, he sought a system in which "Necessarily, if *P*, then *Q*" was represented. We can construct this sort of conditional out of the arrow and the box, like this:

$\Box(P \to Q)$

It will also be convenient to have a special symbol for this stronger type of conditional, which we will call "strict implication" or "entailment." We will use the symbol "⇒" to indicate that *P strictly implies Q*. (Let us call this symbol "the modal arrow.") The expression ($P \Rightarrow Q$) may be read variously as "*P* strictly implies *Q*" or "*P* necessarily implies *Q*" or simply "*P* entails *Q*."*

Our first rule governing the modal arrow is called the law of **strict implication** (SI):

$\Box(\boldsymbol{p} \to \boldsymbol{q}) :: (\boldsymbol{p} \Rightarrow \boldsymbol{q})$

This rule is based on the definition of the modal arrow: "If ***p***, then ***q***" is necessary *if and only if* ***p*** entails ***q***. The utility of this equivalence rule will become apparent when we construct proofs involving necessarily true conditionals.

Now that we have two ways of symbolizing "if-then" statements (namely, the arrow and the modal arrow), the question naturally arises as to how we know when to translate the English "if-then" with an arrow and when to use the modal arrow. Of course, technical locutions such as "strictly implies," "necessarily implies," and "entails" call for the modal arrow. For example:

49. "Jill is an aunt" entails "Jill is a woman." (A: Jill is a aunt; W: Jill is woman)
    In symbols: $A \Rightarrow W$

And if we need to specify the arrow, we can use the locution "materially implies." To illustrate:

50. "Bob studies" materially implies "Bob passes." (S: Bob studies; P: Bob passes)
    In symbols: $S \to P$

Expressions of the form "Necessarily, if ***p***, then ***q***" can be translated in two ways. For example:

51. Necessarily, if Chris is a nephew, then Chris is male. (N: Chris is a nephew: M: Chris is male)

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*Among logicians, different notations are used to symbolize entailment. In systems in which material implication (our arrow) is symbolized with the horseshoe (⊃), the arrow is sometimes used to stand for entailment. The fishhook (⥽) is also used to symbolize entailment. I have chosen "⇒" as a natural addition to a system that uses the arrow for material implication.

In symbols: $N \Rightarrow M$. *Alternatively:* $\Box(N \rightarrow M)$.

Of course, in many actual cases, the explicit phrases mentioned here are not used, and so we are forced to rely on the context. In such cases, two general rules apply. First, give the author the benefit of the doubt. Thus, if using the arrow would render the argument invalid but using the modal arrow would render it valid, use the modal arrow. Always try to put the argument in its best possible light when translating into symbols. (This is an application of Principle 4 from chapter 3: Be fair and charitable in interpreting an argument.) Second, since using the modal arrow complicates the argument, do not use it if it is not needed to ensure validity. In general, if a relatively simple symbolization of an argument indicates that it has a valid form, then the original argument is valid *even if it contains complexities of form not represented in the symbolization*. For example, as we have seen previously, truth tables can be used to show many arguments valid even when the arguments contain statements, such as certain types of conditionals, whose meaning cannot be *fully* expressed via truth functional connectives.

Our next rule, modal **operator transfer** (OT), applies to strict implication but not to material implication. OT comes in two forms:

$p \Rightarrow q$	$p \Rightarrow q$
$\therefore\ \Box p \Rightarrow \Box q$	$\therefore\ \Diamond p \Rightarrow \Diamond q$

Note that both forms of OT are implicational rules, and so they must *not* be applied to parts of a line in a proof—only to entire lines. Here is an English example of each form of OT (assume that "if-then" expresses entailment in each case):

52. If eight is even, then eight is divisible by two (without remainder). So, if it is necessary that eight is even, then it is necessary that eight is divisible by two.
53. If Smith wins the race, then Jones does not win it. So, if it is possible that Smith wins the race, then it is possible that Jones does not win it.

The OT rule enables us to answer the following question: Suppose *A* entails *B*, and suppose *A* is a necessary truth. Does it follow that *B* is a necessary truth? The following proof reveals that the answer is yes.

1. $A \Rightarrow B$	
2. $\Box A$	$\therefore\ \Box B$
3. $\Box A \Rightarrow \Box B$	1, OT
4. $\Box(\Box A \rightarrow \Box B)$	3, SI
5. $\Box A \rightarrow \Box B$	4, NE
6. $\Box B$	2, 5, MP

A similar proof reveals that we can move from "*A* entails *B* and *A* is possible" to "*B* is possible":

1.	$A \Rightarrow B$	
2.	$\Diamond A$	$\therefore\ \Diamond B$
3.	$\Diamond A \Rightarrow \Diamond B$	1, OT
4.	$\Box(\Diamond A \rightarrow \Diamond B)$	3, SI
5.	$\Diamond A \rightarrow \Diamond B$	4, NE
6.	$\Diamond B$	2, 5, MP

Warning: Neither form of OT is valid for material implication. Consider the following argument form:

$$p \rightarrow q \ \therefore \Box p \rightarrow \Box q$$

This argument form will have a true premise and a false conclusion whenever p is a necessary truth and q is a contingent truth. In these circumstances, $p \rightarrow q$ is true, because both the antecedent and the consequent are true. However, $\Box p \rightarrow \Box q$ will be false because we have stipulated that p is a necessary truth, but since q is a contingent truth, it is not a necessary truth. In thinking about the following English example, bear in mind that we are assuming (for the sake of illustration) that "if-then" is here to be interpreted as *material* implication:

54. If all husbands were (are, and will be) married, then Socrates was married. So, if "All husbands were (are, and will be) married" is a necessary truth, then "Socrates was married" is a necessary truth.

The premise is equivalent to "Either not all husbands were (are, and will be) married or Socrates was married," which is true, since Socrates was married to Xanthippe. But the conclusion is false, since "Socrates was married" is not a necessary truth. (*Note:* If the premise of argument (54) is taken to express *strict* implication, then it is false. Although "All husbands are married" is true in every possible world, there are possible worlds in which Socrates remains a bachelor. So, it is not *necessarily* true that if all husbands are married, then Socrates is married.)

Similarly, the following argument form involving the material conditional is *invalid:*

$$p \rightarrow q \ \therefore \ \Diamond p \rightarrow \Diamond q$$

Suppose q is necessarily false and $\sim p$ is a contingent truth. Then the premise will be true, since it is equivalent to $\sim p \vee q$. But the conclusion will be false, since it is equivalent to $\sim \Diamond p \vee \Diamond q$. After all, $\sim \Diamond p$ is false, since $\sim p$ is by hypothesis a *contingent* truth (which implies $\Diamond p$); and $\Diamond q$ is false, since q is by hypothesis necessarily false, and hence impossible.

A number of inference rules for modal logic are highly analogous to inference rules in statement logic. For example, the following proof illustrates that an inference rule similar to *modus ponens* holds for strict implication:

1. $A \Rightarrow B$	
2. A	$\therefore B$
3. $\Box(A \rightarrow B)$	1, SI
4. $A \rightarrow B$	3, NE
5. B	2, 4, MP

Accordingly, let us add the following inference rule to our system, and call it **modal *modus ponens*** (MMP):

$$p \Rightarrow q$$
$$p$$
$$\therefore q$$

MMP will shorten many of our proofs.

A similar proof illustrates that an inference rule analogous to *modus tollens* holds for strict implication:

1. $A \Rightarrow B$	
2. $\sim B$	$\therefore \sim A$
3. $\Box(A \rightarrow B)$	1, SI
4. $A \rightarrow B$	3, NE
5. $\sim A$	2, 4, MT

So, let us also add **modal *modus tollens*** (MMT) to our system of rules:

$$p \Rightarrow q$$
$$\sim q$$
$$\therefore \sim p$$

Note that MMP and MMT are implicational rules. The following proof exemplifies the use of these rules:

1. $H \Rightarrow F$	
2. $\sim \Diamond F$	
3. $\sim H \Rightarrow G$	$\therefore G$
4. $\Box \sim F$	2, MN
5. $\sim F$	4, NE
6. $\sim H$	1, 5, MMT
7. G	3, 6, MMP

At this time, let us also add **modal hypothetical syllogism** (MHS) to our system of rules:

$p \Rightarrow q$
$q \Rightarrow r$
$\therefore\ p \Rightarrow r$

Note that MHS is an implicational rule.

Finally, let us add a rule of strict **logical equivalence** (LE), which governs mutual entailment (or strict biconditionals):

$p \Leftrightarrow q :: \Box(p \leftrightarrow q)$

Note that this is an equivalence rule, not an implicational rule. Here is a short proof involving typical applications of LE and MHS:

1. A ⇔ B	
2. B ⇒ ~C	∴ A ⇒ ~C
3. □(A ↔ B)	1, LE
4. □[(A → B) • (B → A)]	3, ME
5. □(A → B) • □(B → A)	4, MODE
6. □(A → B)	5, Simp
7. A ⇒ B	6, SI
8. A ⇒ ~C	7, 2, MHS

There is an important *amphiboly* (i.e., a double meaning due to a structural flaw) in some English conditionals. Our new symbols enable us to identify this amphiboly with precision. For example, consider this sentence:

55. If I think, then necessarily I exist. (T: I think; E: I exist)

It is not entirely clear which of the following symbolic statements translates (55):

56. $\Box(T \rightarrow E)$

57. $T \rightarrow \Box E$

Statement (56) says, "It is a necessary truth that *if* I think, *then* I exist." In other words, the whole conditional is a necessary truth. Medieval logicians called this *necessity of the consequence*, but using contemporary terminology, we might more naturally call it **necessity of the conditional**. Statement (57), on the other hand, says, "If I think, then 'I exist' is a necessary truth." Medieval logicians called this (for obvious reasons) **necessity of the consequent.** There is an important logical difference between (56) and (57), for (56) is obviously true, but (57) is false. (57) is false because its antecedent is true but its consequent is false. I do

Summary of the Second Set of Inference Rules

Modal Operator Distribution Equivalence Rule (MODE)

$\Box(p \cdot q) :: (\Box p \cdot \Box q)$
$\Diamond(p \vee q) :: (\Diamond p \vee \Diamond q)$

Modal Operator Distribution Implicational Rule (MODI)

$\Box p \vee \Box q$
$\therefore \Box(p \vee q)$

$\Diamond(p \cdot q)$
$\therefore \Diamond p \cdot \Diamond q$

Strict Implication (SI)

$\Box(p \rightarrow q) :: (p \Rightarrow q)$

Operator Transfer (OT)

$p \Rightarrow q$
$\therefore \Box p \Rightarrow \Box q$

$p \Rightarrow q$
$\therefore \Diamond p \Rightarrow \Diamond q$

Modal *Modus Ponens* (MMP)

$p \Rightarrow q$
p
$\therefore q$

Modal *Modus Tollens* (MMT)

$p \Rightarrow q$
$\sim q$
$\therefore \sim p$

Modal Hypothetical Syllogism (MHS)

$p \Rightarrow q$
$q \Rightarrow r$
$\therefore p \Rightarrow r$

Logical Equivalence (LE)

$(p \Leftrightarrow q) :: \Box(p \leftrightarrow q)$

in fact think (at least occasionally!), but since, as previously observed, I might not have existed at all, "I exist" is not a necessary truth.

Normally, it is best to translate an English conditional such as (55) as expressing necessity of the conditional. Thus, (56) is probably a better translation than (57). But sometimes it is hard to be sure what the original author intended. In *On Free Choice of the Will*, Saint Augustine considers an argument that could be paraphrased as follows:[8]

58. If God foreknows that I'll sin tomorrow, then necessarily I'll sin tomorrow. God foreknows that I'll sin tomorrow. So, necessarily I'll sin tomorrow. (G: God foreknows that I'll sin tomorrow; S: I'll sin tomorrow)

Superficially, this looks like a case of *modus ponens*. But is it really? That depends on how we translate the first premise:

59. $G \to \Box S, G \therefore \Box S$

60. $\Box(G \to S), G \therefore \Box S$

Argument (59) is an example of *modus ponens*, so it is valid. However, its first premise is false, or at least highly debatable, for it is natural to think that "I'll sin tomorrow" is a contingent proposition. (There is nothing *logically* impossible about resisting temptation for a day, however difficult it may be in fact.) And presumably, if God exists, then God knows lots of contingent propositions.

Argument (60), on the other hand, is invalid. Consider the following counterexample:

61. It is a necessary truth that if Socrates was a husband, then Socrates was married. Socrates was a husband. So, it is a necessary truth that Socrates was married.

The conclusion of this argument is false, for although Socrates was in fact married to Xanthippe, he might have chosen to remain a bachelor. But the premises of the argument are true. So, the argument is invalid. Thus, to analyze argument (58) fully and fairly, we have to consider two possible interpretations of the conditional premise. When we do, a subtle fallacy of amphiboly becomes apparent. In one way of reading the first premise, the argument is valid but the first premise is false (or at least very dubious). In the other way, the premises are true but the argument is invalid. Our symbolic tools give us a very clear way of identifying the problem.

The following exercises will test your understanding of the rules introduced in this section.

Exercise 12.4

Part A: Symbolizing Symbolize the following sentences using the schemes of abbreviation provided.

* **1.** "Joe is a man and Joe is not married" strictly implies "Joe is a bachelor." (J: Joe is a man; M: Joe is married; B: Joe is a bachelor)

2. Necessarily, if Linda is a niece, then she is female. (L: Linda is a niece; F: Linda is female)

3. "Chris is Bob's daughter" entails "Chris is not male." (D: Chris is Bob's daughter; M: Chris is male)
* 4. "Doug does not try" materially implies "Doug fails." (T: Doug tries; F: Doug fails)
5. "Erica is married" does not entail "Erica is happy." (M: Erica is married; H: Erica is happy)
6. "The Eiffel Tower is in Ohio" materially implies, but does not necessarily imply, "The Eiffel Tower is in France." (O: The Eiffel Tower is in Ohio; F: The Eiffel Tower is in France)
* 7. It is a necessary truth that if abortion is murder, then it is wrong. (M: Abortion is murder; W: Abortion is wrong)
8. "Killing innocent humans is always wrong" entails "Euthanasia is wrong." (K: Killing innocent humans is always wrong; E: Euthanasia is wrong)
9. It is necessarily false that some circles are triangles; however, some circles are purple. (T: Some circles are triangles; P: Some circles are purple)
10. Necessarily, if it is necessary that all humans are mortal and necessary that Socrates is human, then it is necessary that Socrates is mortal. (H: All humans are mortal; S: Socrates is human; M: Socrates is mortal)

Part B: Symbolizing and Evaluating Symbolize the following arguments. Which have valid forms? Which have invalid forms? *Note:* Two of the arguments have an amphibolous conditional premise, and hence these arguments can be interpreted as having two different forms. In these cases, identify both forms and indicate which of them is valid and which isn't.

* 1. Possibly Smith is guilty and possibly Jones is guilty. So, it is possible that both Smith and Jones are guilty. (S: Smith is guilty; J: Jones is guilty)
2. The proposition "Betty is an aunt" entails the proposition "Betty is female." Accordingly, the proposition "It is logically possible that Betty is an aunt" strictly implies the proposition "It is logically possible that Betty is female." (A: Betty is an aunt; F: Betty is female)
3. Necessarily, either Santa exists or he doesn't. Hence, either it is necessary that Santa exists or it is necessary that he doesn't exist. (S: Santa exists)
* 4. Either not all husbands are handsome or all wives are beautiful. Therefore, either it is not necessary that all husbands are handsome or it is necessary that all wives are beautiful. (H: All husbands are handsome; W: All wives are beautiful) [*Hint:* Apply MI.]
5. Either it is necessary that all bachelors own Fords or it is necessary that all bachelors own Porsches. Thus, it is necessary that either all bachelors own Fords or all bachelors own Porsches. (F: All bachelors own Fords; P: All bachelors own Porsches)
6. If I see that Al is stealing a TV, then necessarily Al is stealing a TV. I see that Al is stealing a TV. Therefore, it is necessarily true that Al is stealing a TV. (S: I see that Al is stealing a TV; A: Al is stealing a TV)

* **7.** It is logically possible that Fred and Sue are both arrogant. Hence, it is logically possible that Fred is arrogant and logically possible that Sue is arrogant. (F: Fred is arrogant; S: Sue is arrogant)

8. It is necessarily true that either Pat is a man or Pat is a woman. It follows that either it is necessary that Pat is a man or it is necessary that Pat is a woman. (M: Pat is a man; W: Pat is a woman)

9. "The Eiffel Tower is in Ohio" materially implies "The Eiffel Tower is in France." Thus, "It is possible that the Eiffel Tower is in Ohio" materially implies "It is possible that the Eiffel Tower is in France." (O: The Eiffel Tower is in Ohio; F: The Eiffel Tower is in France)

* **10.** If Socrates is sitting, then it must be the case that Socrates is sitting. Socrates is sitting. Therefore, it is necessarily true that Socrates is sitting. (S: Socrates is sitting)

11. Necessarily, either nothing is caused or every event has a cause. Therefore, either it is necessarily true that nothing is caused or it is necessarily true that every event has a cause. (N: Nothing is caused; E: Every event is caused)

12. Monism (i.e., the view that there is only one thing) is possibly true, and so is reincarnation (i.e., the view that when a person dies, his or her soul enters another body). Consequently, it is logically possible both that monism is true and that reincarnation is true. (M: Monism is true; R: Reincarnation is true)

13. Necessarily, if radical skeptics are right, then there is no external world. Radical skeptics are right. So, it is a necessary truth that there is no external world. (R: Radical skeptics are right; E: There is no external world)

14. Either it is necessary that God exists or it is necessary that the physical universe exists. Accordingly, it is necessarily true that either God exists or the physical universe exists. (G: God exists; P: The physical universe exists)

15. Possibly, both Hinduism and Zoroastrianism are true. Therefore, possibly Hinduism is true and possibly Zoroastrianism is true. (H: Hinduism is true; Z: Zoroastrianism is true)

Part C: Annotating

Annotate the following proofs.

* **1.**
1. $\Box(A \bullet \sim B)$ ∴ $\sim \Diamond B$
2. $\Box A \bullet \Box \sim B$
3. $\Box \sim B$
4. $\sim \Diamond B$

2.
1. $\Box C \vee \Box D$ ∴ $C \vee D$
2. $\Box(C \vee D)$
3. $C \vee D$

3.
1. $\Diamond E \vee \Diamond F$
2. $\Diamond(E \vee F) \rightarrow G$ ∴ $\Diamond G$
3. $\Diamond(E \vee F)$
4. G
5. $\Diamond G$

* **4.**
1. $\Diamond(H \bullet J)$
2. $\Diamond K \Rightarrow \sim \Diamond J$ ∴ $\Box \sim K$
3. $\Diamond H \bullet \Diamond J$
4. $\Diamond J$
5. $\sim\sim \Diamond J$
6. $\sim \Diamond K$
7. $\Box \sim K$

5.
1. $L \Rightarrow M$
2. $\Box \sim M$ ∴ $\Box \sim L$
3. $\sim \Diamond M$
4. $\Diamond L \Rightarrow \Diamond M$
5. $\sim \Diamond L$
6. $\Box \sim L$

6. 1. ◇A ⇒ B ∴ □~A ∨ B
2. □(◇A → B)
3. □(~◇A ∨ B)
4. ~◇A ∨ B
5. □~A ∨ B

* **7.** 1. N ⇔ O
2. □(N → P) ∴ O ⇒ P
3. N ⇒ P
4. □(N ↔ O)
5. □[(N → O) • (O → N)]
6. □(N → O) • □(O → N)
7. □(O → N)
8. O ⇒ N
9. O ⇒ P

8. 1. Q ⇒ R
2. ◇Q ∴ ◇R
3. ◇Q ⇒ ◇R
4. ◇R

9. 1. S ⇒ (T • U)
2. □S ∴ □U
3. □S ⇒ □(T • U)
4. □(T • U)
5. □T • □U
6. □U

* **10.** 1. ◇(W ∨ X)
2. ◇W ⇒ ◇Z
3. ◇X ⇒ ◇V ∴ ◇(Z ∨ V)
4. ◇W ∨ ◇X
5. □(◇W → ◇Z)
6. □(◇X → ◇V)
7. ◇W → ◇Z
8. ◇X → ◇V
9. ◇Z ∨ ◇V
10. ◇(Z ∨ V)

Part D: Correct or Incorrect? Which of the following inferences are permitted by one of our rules, and which are not? If an inference is permitted, indicate the rule that permits it. If it is not permitted, simply write "incorrect." (The question is whether one can move from the first statement to the second, in each instance, by a single application of one of our rules.)

* **1.** 1. ~A → ~B
2. ◇~A → ◇~B

2. 1. □C ⇒ □D
2. □(□C → □D)

3. 1. ◇(~E • F)
2. ◇~E • ◇F

* **4.** 1. □(G ∨ H)
2. □G ∨ □H

5. 1. ◇J • ◇K
2. ◇(J • K)

6. 1. ~L ⇒ ~M
2. □~L ⇒ □~M

* **7.** 1. ◇(~N ∨ P)
2. ◇~N ∨ ◇P

8. 1. Q ⇒ R
2. ◇Q ⇒ R

9. 1. □S ∨ □~T
2. □(S ∨ ~T)

* **10.** 1. ◇U ∨ ◇W
2. ◇(U ∨ W)

11. 1. ~Y ⇒ ~Z
2. ◇~Y ⇒ ◇~Z

12. 1. ~(A ⇒ B)
2. ~□(A → B)

* **13.** 1. □(~C • D)
2. □~C • □D

14. 1. E ⇔ F
2. (E ⇒ F) • (F ⇒ E)

15. 1. ~□(G → H)
2. ~(G ⇒ H)

Part E: Proofs Construct proofs to show that the following symbolic arguments are valid.

* **1.** A ⇒ B ∴ ~B ⇒ ~A

2. $C \Leftrightarrow D, D \therefore C$
3. $\Box F \therefore \Box(F \lor G)$
* 4. $\Box(E \rightarrow H), \sim\Box\sim E \therefore \Diamond H$
5. $S \Rightarrow T \therefore \sim\Diamond(S \bullet \sim T)$
6. $O \Leftrightarrow N, \Diamond O \therefore \Diamond N$
* 7. $\Diamond(L \lor M), \Box\sim L, M \Rightarrow \sim N \therefore \sim\Box N$
8. $\Diamond(\sim D \bullet \sim E), \Diamond A \Rightarrow \Box D \therefore \sim A$
9. $P \Leftrightarrow Q, Q \Rightarrow \sim S \therefore P \Rightarrow \sim S$
* 10. $A \Rightarrow E, \Box\sim E \therefore \Box\sim A$
11. $\Box(R \lor D), \Box\sim R \therefore \Box D$
12. $\Box B \bullet (E \lor \sim G), (\Box K \lor \Diamond H) \rightarrow \sim\Box B, \Diamond G \therefore \Diamond E \bullet \sim H$
* 13. $\Box[\sim R \lor (S \bullet T)], \Diamond R \therefore \Diamond S \bullet \Diamond T$
14. $\Diamond[S \lor (P \bullet R)], \Diamond Q \rightarrow \sim\Diamond S, \Diamond Q \rightarrow \sim\Diamond(P \bullet R) \therefore \sim Q$
15. $\Box[H \rightarrow (J \rightarrow K)], \Diamond J \therefore \Diamond(H \rightarrow K)$
* 16. $\Diamond[(L \lor M) \bullet (L \lor Q)], \Diamond P \Rightarrow \sim\Diamond Q, \Box\sim L \therefore \sim P$
17. $\sim\Box(P \bullet R), \Diamond\sim P \Rightarrow \Box Q, \Diamond\sim R \Rightarrow \Box Q \therefore Q$
18. $\sim\Diamond(S \lor T), (U \Rightarrow W) \rightarrow \Diamond S \therefore \Diamond\sim W$
* 19. $\Box[(E \bullet F) \rightarrow G], \Box E \therefore \sim G \Rightarrow \sim F$
20. $\Box(A \bullet B) \lor \Box(\sim A \bullet \sim B), (A \Leftrightarrow B) \rightarrow \sim\Diamond C \therefore \Diamond\sim C$

Part F: English Arguments Symbolize the following arguments, using the schemes of abbreviation provided. Then construct proofs to show that the arguments are valid.

* 1. Necessarily, if a first cause exists, then God exists. But it is not possible that a first cause does not exist. And hence, it is necessary that God exists. (F: A first cause exists; G: God exists)
2. "Contradictions are false" entails "Circular squares do not exist." And it is necessarily true that contradictions are false. Moreover, "Circular squares do not exist" entails "Some statements about what exists can be known independently of sensory experience." Therefore, necessarily, some statements about what exists can be known independently of sensory experience. (C: Contradictions are false; S: Circular squares exist; E: Some statements about what exists can be known independently of sensory experience)
3. Either two objects can have all properties in common or two physical objects cannot be in the same place at once. But necessarily, it is not the case that two objects can have all properties in common. Accordingly, two physical objects cannot be in the same place at once. (O: Two objects can have all properties in common; P: Two physical objects can be in the same place at once)

4. Either it is possible that every object has a cause of its existence or it is possible that at least one object exists uncaused. "It is possible that either every object has a cause of its existence or at least one object exists uncaused" strictly implies "Necessarily, it is not the case that some objects bring themselves into existence." Thus, it is impossible that some objects bring themselves into existence. (E: Every object has a cause of its existence; U: At least one object exists uncaused; B: Some objects bring themselves into existence)
5. Necessarily, if God is all-powerful and knowledge is power, then God is all-knowing. It is necessarily true that God is all-powerful. Therefore, the proposition "God is not all-knowing" entails the proposition "It is not the case that knowledge is power." (P: God is all-powerful; K: Knowledge is power; G: God is all-knowing)

12.5 Systems S4 and S5

C. I. Lewis described systems of modal logic in five stages. Each stage corresponds to one of the following principles:

1. If p is a theorem of statement logic, then $\Box p$.
2. If $\Box p$, then p.
3. If $\Box p$ and $p \Rightarrow q$, then $\Box q$.
4. If $\Box p$, then $\Box\Box p$.
5. If $\Diamond\Box p$, then $\Box p$.

The rules we have introduced so far reflect Principles 1, 2, and 3. Some philosophers accept *only* the first three principles. The modal system based on Principles 1–3 (and *not* including 4 and 5) is called system T. Accordingly, the system of modal logic developed in sections 12.3 and 12.4 is a variant of system T. The system of modal logic based on Principles 1–4 is called S4. And the system of modal logic based on all five principles is called S5. In this section, we will explore modal systems S4 and S5.

These various systems of modal logic are progressively stronger. For example, there are arguments that can be proved valid in system S4 that cannot be proved valid in system T. And there are arguments that can be proved valid in system S5 that cannot be proved valid in system S4. On the other hand, if an argument can be proved valid using the rules of system T, then it can be proved valid using the rules of system S4. And if an argument can be proved valid using the rules of system S4, then it can be proved valid using the rules of system S5.

System S4 gives us the following rules that govern repeated (or iterated) modal operators:

$\Box p :: \Box\Box p$

$\Diamond p :: \Diamond\Diamond p$

We will simply call these rules the **S4 rules**. (The annotation in a proof is "S4.") Note that these rules are equivalence rules, so they can be applied to parts of lines in a proof.

The S4 rules may seem initially puzzling. What do the iterated operators mean? For example, what does it mean to say that a proposition is *necessarily necessary*? And what does it mean to say that a proposition is *possibly possible*? Let us take a closer look at both rules.

First, consider $\Box p :: \Box\Box p$. Let's break this down into two separate inferences. The move from $\Box\Box p$ to $\Box p$ is actually redundant within our system. We can already make this move by NE. So, let's focus on the move from $\Box p$ to $\Box\Box p$. It helps to think about this in terms of possible worlds. Suppose $\Box p$ is true. Then p is true in every possible world. Could $\Box\Box p$ be false? Well, if $\Box\Box p$ is false, then $\Box p$ is false in at least one possible world. But if $\Box p$ is false in some possible world, then $\sim p$ is true in some possible world. So, it is not the case that p is true in every possible world, which contradicts our initial supposition that p is necessary. Therefore, when we think about the first S4 rule in terms of possible worlds, it seems clear that $\Box p$ implies $\Box\Box p$.*

Second, consider $\Diamond p :: \Diamond\Diamond p$. Again, break the inference rule down into two parts. The move from $\Diamond p$ to $\Diamond\Diamond p$ is redundant within our system, for we can already make this move by EP. So, let's focus on the move from $\Diamond\Diamond p$ to $\Diamond p$. Again, it helps to think in terms of possible worlds. Suppose $\Diamond\Diamond p$. Could $\Diamond p$ be false? If $\Diamond\Diamond p$, then there is at least one possible world (call it "W") in which p is possibly true. But if p is possibly true in W, then p is true in some possible world. But if $\Diamond p$ is false, then p is impossible, and hence p is false in every world. So, if we assume $\Diamond\Diamond p$, we are forced, on pain of contradiction, to accept $\Diamond p$. Thus, $\Diamond\Diamond p$ implies $\Diamond p$.

Now, consider the following English argument.

62. The proposition "Possibly, morality is relative to culture" entails the proposition "It is not necessarily necessary that torturing people for fun is wrong." But it is a necessary truth that torturing people for fun is wrong. Therefore, morality is not relative to culture. (M: Morality is relative to culture; T: Torturing people for fun is wrong)

Here is the symbolization and proof:

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*Here is one way of highlighting the philosophical issues that are at stake when the necessity operator is iterated. Some theologians claim that God created logic. This apparently implies that the correct logical rules could have been very different than they are—for example, that contradictions could have been true had God so ordained it. To reduce the philosophical alarm this view often produces, its proponents sometimes suggest that God made the law of noncontradiction a necessary truth in our world (the actual world), but God might have done otherwise. In other words, $\Box\sim(p \cdot \sim p)$, but $\sim\Box\Box\sim(p \cdot \sim p)$, which violates the S4 rules. This scenario is unacceptable if necessary truths are true in every possible world, for if p is necessary but God could have created a world in which p is not necessary, then $\Diamond\sim\Box p$. This means there is a possible world in which p is not necessary. But this could be so only if there is at least one world in which p is false. But then p isn't a necessary truth after all, contrary to our supposition.

1.	$\Diamond M \Rightarrow {\sim}\Box\Box T$	
2.	$\Box T$	$\therefore {\sim}M$
3.	$\Diamond M \Rightarrow {\sim}\Box T$	1, S4
4.	${\sim}{\sim}\Box T$	2, DN
5.	${\sim}\Diamond M$	3, 4, MMT
6.	$\Box{\sim}M$	5, MN
7.	${\sim}M$	6, NE

Our final set of rules belongs to Lewis's fifth system of modal logic, S5. We will simply call these rules the **S5 rules** (annotation: S5):

$\Diamond\Box p :: \Box p$

$\Diamond p :: \Box\Diamond p$

Once again, we have two rules, each of which is an equivalence rule. And once again, these rules are partly redundant within our system. For example, we can move from $\Box p$ to $\Diamond\Box p$ by EP; and we can move from $\Box\Diamond p$ to $\Diamond p$ by NE. But the move from $\Diamond\Box p$ to $\Box p$ is new and striking. If a proposition is *possibly* necessary, does it follow that it is necessary? The move from $\Diamond p$ to $\Box\Diamond p$ is new as well. If p is possible, does it follow that p is *necessarily* possible? A possible-worlds approach helps to explain why many logicians think the S5 rules are valid.

Consider the move from $\Diamond\Box p$ to $\Box p$. If $\Diamond\Box p$, then there is a possible world in which $\Box p$. For example, there is a possible world in which "No circles are squares" is necessary. But this means that there is a possible world (call it "W") in which "No circles are squares" is true *in every possible world*. But if in W "No circles are squares" is true in every possible word, then "No circles are squares" is necessary. In other words, if a proposition is necessary in one world, it is necessary in all.

Consider the move from $\Diamond p$ to $\Box\Diamond p$. This says that if a statement is logically possible, that is, true in at least one possible world, then it can't be impossible in some other world. Suppose $\Diamond p$ but ${\sim}\Box\Diamond p$ (for the sake of the argument). Now, ${\sim}\Box\Diamond p$ implies $\Diamond{\sim}\Diamond p$, by MN. But how could p be true in some possible world but impossible, that is, necessarily false, in some other possible world? If a statement is necessarily false in one possible world, then it is false in every possible world. It is reflections such as these that lead many philosophers to accept the S5 rules as valid.

Let us now use our new modal rules to consider a version of the famous ontological argument for the existence of God. The argument was first formulated by Saint Anselm (1033–1109), but we will consider a version that depends heavily on modal terms:

63. If God exists, then God is a Supremely Perfect Being. If God is a Supremely Perfect Being, then it is impossible that God not exist. It is logically possible

Summary of Rules for Systems S4 and S5	
S4 Rules	**S5 Rules**
$\Box p :: \Box\Box p$	$\Diamond\Box p :: \Box p$
$\Diamond p :: \Diamond\Diamond p$	$\Diamond p :: \Box\Diamond p$

that God exists. So, God exists. (G: God exists; S: God is a Supremely Perfect Being)

1. $G \Rightarrow S$	
2. $S \Rightarrow \sim\Diamond\sim G$	
3. $\Diamond G$	$\therefore$ G
4. $\Diamond G \Rightarrow \Diamond S$	1, OT
5. $S \Rightarrow \Box G$	2, MN
6. $\Diamond S \Rightarrow \Diamond\Box G$	5, OT
7. $\Diamond G \Rightarrow \Diamond\Box G$	4, 6, MHS
8. $\Diamond\Box G$	3, 7, MMP
9. $\Box G$	8, S5
10. G	9, NE

This argument is interesting because it reaches a philosophically dramatic conclusion on the basis of rather modest-looking premises. It seems reasonable to suppose that it is logically possible that God exists. (After all, we could say the same about unicorns or Santa Claus, couldn't we?) And if God does exist, then God must be a Supremely Perfect Being. After all, if there is no Supremely Perfect Being, then surely there is no entity that we would regard as God. So, if the argument is unsound, then surely either the second premise is false or at least one of the steps in the proof is invalid.

Philosophers who reject modal system S5 will point out that we made a crucial use of an S5 rule in line (9) of our proof. Others might reject the second premise. Why should we suppose that if God is a Supremely Perfect Being, then "God exists" is necessary? The defense might be that any being that could fail to exist would not be *supremely* perfect, because we could conceive of a more perfect being, namely, one that could not fail to exist.

On the other hand, if "A Supremely Perfect Being exists" is a necessary truth, then "A Supremely Perfect Being does not exist" is impossible. And many philosophers find this implausible. They claim that we must distinguish between *having* a concept (e.g., having the concept of a unicorn or of a Supremely Perfect Being) and *knowing* whether that concept applies to any real object. Furthermore, they claim we cannot tell *merely by examining a concept* whether it applies to anything in reality.

We cannot enter more deeply into these important metaphysical issues here. But our comments on the ontological argument illustrate how modal logic can be used to set philosophical issues up in a revealing fashion. The following exercises indicate how our modal tools can be used to frame several traditional philosophical problems with increased precision.

Exercise 12.5

Part A: Symbolizing Symbolize the following statements using the schemes of abbreviation provided.

* **1.** It is necessarily possible that Bozo is president. (B: Bozo is president)

2. "Possibly it is necessary that unicorns exist" strictly implies "necessarily unicorns exist." (U: Unicorns exist)

3. "Possibly, it is necessary that water is wet" materially implies, but does not strictly imply, "It is necessarily necessary that water is clear." (W: Water is wet; C: Water is clear)

* **4.** "The Great Pumpkin does not exist" is a contingent truth, and it is necessarily possible that the Great Pumpkin does exist. (G: The Great Pumpkin exists)

5. It is possibly possible that astrologers are wise, but it is not necessarily necessary that astrologers are wise. (A: Astrologers are wise)

6. Necessarily, either it is logically possible that miracles *can* occur or it is necessarily impossible that miracles occur. (M: Miracles occur)

* **7.** There is a possible world in which I am rich, but it is not possible that I am rich in every possible world. (R: I am rich)

8. Necessarily, if there is a possible world in which it is necessary that torturing innocent people is wrong, then there is no possible world in which torturing innocent people is not wrong. (T: Torturing innocent people is wrong)

9. "Ants are animals in every possible world" entails "In every possible world 'Ants are animals' is a necessary truth." (A: Ants are animals)

10. It is possibly necessary that if there is no possible world in which it is possible that green things are invisible, then it is necessarily necessary that green things are visible. (G: Green things are visible)

Part B: Annotating Annotate the following proofs.

* **1.**
1. $\Box\Box A \Rightarrow B$
2. $\Box A$ ∴ B
3. $\Box\Box A$
4. B

2.
1. $(\Diamond C \vee \Diamond\Diamond C) \Rightarrow D$ ∴ $\Box(\Diamond C \rightarrow D)$
2. $(\Diamond C \vee \Diamond C) \Rightarrow D$
3. $\Diamond C \Rightarrow D$
4. $\Box(\Diamond C \rightarrow D)$

3.
1. $\Box\Box E \Rightarrow \Box\Box F$
2. $\Diamond {\sim} F$ ∴ ${\sim}\Box E$
3. ${\sim}\Box F$
4. $\Box\Box E \Rightarrow \Box F$
5. ${\sim}\Box\Box E$
6. ${\sim}\Box E$

* **4.** 1. ◇G
2. □◇G ⇒ ◇□H ∴ H
3. □◇G
4. ◇□H
5. □H
6. H

5. 1. □~Q ⇒ □□~R
2. ◇R ∴ ◇Q
3. □~Q ⇒ □~◇R
4. □~Q ⇒ ~◇◇R
5. □~Q ⇒ ~◇R
6. ~~◇R
7. ~□~Q
8. ◇Q

6. 1. □□◇◇S ∴ ◇S
2. □◇◇S
3. □◇S
4. ◇S

* **7.** 1. ◇□J
2. ◇◇K ⇒ ~□J ∴ ~◇K
3. ◇◇K ⇒ ~◇□J
4. ~~◇□J
5. ~◇◇K
6. ~◇K

8. 1. L ⇒ M
2. M ⇒ □L
3. ◇L ∴ L
4. L ⇒ □L
5. ◇L ⇒ ◇□L
6. ◇□L
7. □L
8. L

9. 1. ◇□N ⇒ ◇◇P
2. ~◇P ∴ ~□N
3. ◇□N ⇒ ◇P
4. ~◇□N
5. ~□N

* **10.** 1. ◇□◇□T ∴ T
2. □◇□T
3. □□T
4. □T
5. T

Part C: Proofs Construct proofs to show that the following arguments are valid.

* **1.** □A, A ⇒ B ∴ □□B

2. ◇◇R ∴ □◇R

3. ◇□~H, ◇C ⇒ ◇H ∴ ~C

* **4.** □□(A • B) ∴ □A • □□B

5. ◇◇(P ∨ Q) ∴ ◇P ∨ ◇Q

6. □◇□◇F ∴ ◇F

* **7.** ◇□◇□~Q ∴ ~Q

8. ◇ ◇T ⇒ □◇P ∴ ~◇(◇T • ~◇P)

9. ~◇(A • ~B) ∴ ◇□A ⇒ ◇□B

* **10.** □□F ⇒ ◇□G ∴ ◇◇(~F ∨ □G)

11. □R, □□R ⇒ ◇~S ∴ ~□S

12. ◇□C ∴ ◇D ⇒ □□C

13. □□(H • ◇□E) ∴ □◇H • ◇E

14. □M ∨ ◇S ∴ □(M ∨ ◇S)

15. N ⇒ □N, ◇N ∴ N

16. $\Diamond J, \Box\Diamond J \Rightarrow \Diamond\Box K \therefore K$
17. $\Box\Diamond\Diamond(\sim F \vee \Diamond\sim G) \therefore \Diamond(\sim\Box F \vee \Box\Diamond\sim G)$
18. $\Diamond\Diamond(\sim L \vee \sim M) \therefore \sim(\Box L \bullet \Box M)$
19. $\Diamond E \Rightarrow \sim\Box\Box H, \Box H \therefore \sim E$
20. $\Diamond\sim S \Rightarrow \sim S, \Diamond S \therefore S$

Part D: English Arguments Symbolize the following arguments using the schemes of abbreviation provided. Then construct proofs to show that the arguments are valid.

* 1. Necessarily, if a supremely perfect island exists, then it is necessarily true that a supremely perfect island exists. Possibly, a supremely perfect island exists. Therefore, a supremely perfect island exists. (P: A supremely perfect island exists)

2. Necessarily, if God exists, God is nonphysical but omnipresent. Necessarily, if God is nonphysical, then it is impossible for God to be located anywhere. Necessarily, if God is omnipresent, God is located everywhere. Necessarily, if God is not located anywhere, then God is not located everywhere. Therefore, it is necessarily true that God does not exist. (G: God exists; N: God is nonphysical; O: God is omnipresent; L: God is located somewhere; E: God is located everywhere)

3. The proposition "It is possible that my soul can inhabit another body" strictly implies the proposition "It is not a necessary truth that I am my body." It is not necessary that my soul does not inhabit another body. The proposition "It is possible that I am not my body" entails the proposition "I am not my body." It follows that I am not my body. (B: I am identical with my body; S: My soul inhabits another body)

* 4. Necessarily, both Zeus and Yahweh are omnipotent. Necessarily, if it is necessary that Zeus is omnipotent, then Zeus can thwart Yahweh. Necessarily, if it is necessary that Yahweh is omnipotent, then Yahweh can thwart Zeus. However, "Zeus can thwart Yahweh" entails "It is not necessarily true that Yahweh thwarts Zeus." And necessarily, if it is possible that Yahweh does not thwart Zeus, then it is possible that Yahweh is not omnipotent. We must therefore conclude that Yahweh is not omnipotent. (Z: Zeus is omnipotent; Y: Yahweh is omnipotent; T: Zeus thwarts Yahweh; A: Yahweh thwarts Zeus)

5. Necessarily, it is possible that the word "bachelor" will change meaning over time. Necessarily, if it is possible that the word "bachelor" will change meaning over time, then it is not necessarily true that all bachelors are unmarried. Hence, it is possible that not all bachelors are unmarried. (W: The word "bachelor" will change meaning over time; A: All bachelors are unmarried)

6. The following conditional is logically possible: the proposition "There is a physical universe" entails the proposition "God exists." Therefore, "There is a physical universe" entails "God exists." (P: There is a physical universe; G: God exists)

7. If it is possibly possible that a powerful demon exists, then it is necessarily possible that I am deceived about the existence of an external world. Necessarily, it is possible that a powerful demon exists. If it is possible that I am deceived about the existence of an external world, then it is necessarily possible that I know nothing.

But the proposition "Possibly I know nothing" entails the proposition "I know nothing." Accordingly, I know nothing. (D: A powerful demon exists; E: I am deceived about the existence of an external world; K: I know nothing)

8. If a supremely perfect being created something, then a supremely perfect being created the best of all possible worlds. Possibly, it is necessary that a supremely perfect being created something. If a supremely perfect being created the best of all possible worlds, then the actual world is the best of all possible worlds. Therefore, it is not possible that the actual world is not the best of all possible worlds. (S: A supremely perfect being created something; B: A supremely perfect being created the best of all possible worlds; A: The actual world is the best of all possible worlds)

9. Possibly, it is necessary that if pointless suffering occurs, then God is not both omnipotent and perfectly good. Possibly, pointless suffering occurs. Therefore, necessarily either possibly God is not omnipotent or possibly God is not perfectly good. (S: Pointless suffering occurs; O: God is omnipotent; G: God is perfectly good)

10. Necessarily, using nuclear weapons involves indiscriminate killing if and only if it is wrong to use nuclear weapons. The proposition "It is possible that indiscriminate killing occurred at Hiroshima" strictly implies the proposition "It is possibly necessary that using nuclear weapons involves indiscriminate killing." Indiscriminate killing occurred at Hiroshima assuming that it is possible that any war *can* be won without killing a great many noncombatants. Possibly, any war *is* won without killing a great many noncombatants. Therefore, it is wrong to use nuclear weapons. (N: Using nuclear weapons involves indiscriminate killing; W: It is wrong to use nuclear weapons; H: Indiscriminate killing occurred at Hiroshima; K: Any war is won without killing a great many noncombatants)

Part E: Paradoxes of Strict Implication As noted previously, C. I. Lewis developed modal logic partly because he was troubled by the so-called paradoxes of material implication. However, the concept of strict implication has results that seem just as paradoxical to many. To understand these paradoxes, construct proofs to show that the following arguments are valid:

1. $\Box B \therefore A \Rightarrow B$

2. $\sim\Diamond A \therefore A \Rightarrow B$

Argument (1) tells us that *a necessary truth is strictly implied by any statement whatsoever.* It follows, for example, that:

3. "Trees exist" strictly implies "No circles are squares."

This is puzzling as the antecedent of (3) seems irrelevant to its consequent. Does it help to note that (3) is equivalent to (4), applying SI and MI?

4. Necessarily, either trees do not exist or no circles are squares.

Argument (2) tells us that *an impossible proposition entails any proposition whatsoever.* It follows, for example, that:

5. "There are circular squares" strictly implies "Santa exists."

Again the antecedent does not seem relevant to the consequent. Does it help to note that (5) is equivalent to (6), applying SI and MI?

6. Necessarily, either there are no circular squares or Santa exists.

Notes

1. The historical observations here regarding Aristotle and the one that follows regarding C. I. Lewis are gleaned from Kenneth Konyndyk, *Introductory Modal Logic* (Notre Dame, IN: University of Notre Dame Press, 1986), pp. 18–19, 26.
2. Example (7) is borrowed from Alvin Plantinga, *The Nature of Necessity* (Oxford: Clarendon Press, 1974), p. 2.
3. For a review of the concept of a tautology, see section 7.5.
4. My definition is adapted from Laurence BonJour, *In Defense of Pure Reason* (New York: Cambridge University Press, 1998), p. 32.
5. This is a note for the metaphysically alert reader. I realize that I am speaking somewhat loosely about statements here. We will take up the issue of statements and propositions momentarily.
6. Konyndyk, *Introductory Modal Logic*, p. 15.
7. The system of rules in this chapter is largely based on a system developed by Richard L. Purtill, *A Logical Introduction to Philosophy* (Englewood Cliffs, NJ: Prentice-Hall, 1989), pp. 171–176.
8. Saint Augustine, *On Free Choice of the Will*, trans. Anna S. Benjamin and L. H. Hackstaff (Indianapolis, IN: Bobbs-Merrill, 1964), pp. 90–93.