4.3 Space of Boolean Functions

It is useful to consider Boolean functions as a working example for the concept of linear separability and elementary classifier design.

An n-dimensional Boolean function is a map from a domain space that comprises 2^n possible combinations of the n variables into the set $\{0, 1\}$. Geometrically, it is instructive to think of these 2^n domain points as corners of an n-dimensional hypercube. For example, in two dimensions, the set of possible inputs is $\{00, 01, 10, 11\}$. We denote precisely these four corners as the Boolean 2-cube: $\mathbb{B}^2 = \{0, 1\} \times \{0, 1\} = \{0, 1\}^2 = \{00, 01, 10, 11\}$. These four points form the corners of the unit square $\mathbb{I}^2 = [0, 1] \times [0, 1] = [0, 1]^2$ in $\mathbb{R}^2$. Similarly, in three dimensions there are eight input combinations that comprise the Boolean 3-cube $\mathbb{B}^3$.

$[0, 1] \times [0, 1]$ represents the Cartesian product of the two closed unit intervals. In general, $\mathbb{I}^n$ is sometimes referred to as the *fuzzy* n–cube [323]. We will have opportunity to revisit this in Chapter 14.

In $\mathbb{B}^n$, there are 2^n points corresponding to each of the 2^n possible combinations of n variables. In a *Boolean function*, each of these domain points maps to either 0 or 1, $f : \mathbb{B}^n \to \{0, 1\}$. For the present discussion, we denote the set of points that map to 0 as $\mathcal{X}_0$, and the set of those that map to 1 as $\mathcal{X}_1$. Put another way, the sets $\mathcal{X}_0, \mathcal{X}_1$ comprise points that are pre-images of range points 0, 1: $\mathcal{X}_0 = f^{-1}(0)$ and $\mathcal{X}_1 = f^{-1}(1)$. Each unique assignment of 0-1 values to the 2^n possible inputs in n-dimensions represents a Boolean function. Therefore, in n-dimensions, there are 2^{2^n} such unique assignments that can be made, and thus 2^{2^n} possible functions.

If there exists a hyperplane that separates $\mathcal{X}_0$ from $\mathcal{X}_1$, then they are linearly separable. The hyperplane should not pass through any of the points of $\mathbb{B}^n$ since that would result in an ambiguous classification for the point in question. In terms of Boolean functions we therefore have the following alternative but equivalent definition for linear separability.

Definition 4.3.1

A Boolean function $f(x_1, \ldots, x_n) : \mathbb{B}^n \rightarrow \{0, 1\}$, is linearly separable if there exists a hyperplane Π in $\mathbb{R}^n$ that strictly separates $\mathcal{X}_0$ from $\mathcal{X}_1$, and $\Pi \cap \{0, 1\}^n = \phi$.

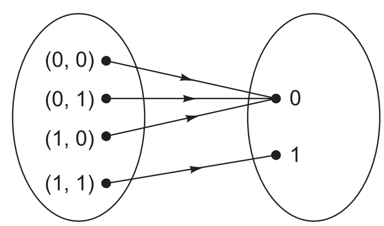
Of the 2^{2^n} possible Boolean functions, some are linearly separable and some are not. Examples of functions that are not linearly separable are the exclusive OR and the exclusive NOR functions.

4.3.1 Example: Two Dimensional Logical AND Function

Consider the two dimensional logical AND function. Assuming domain Boolean variables x_1 and x_2, Fig. 4.4(a) shows the truth table of the function. Figure 4.5(b) depicts the function $f_\wedge(x_1, x_2) : \mathbb{B}^2 \rightarrow \{0, 1\}$ as a map from a domain of 2^2 points ($n = 2$) which comprise $\mathbb{B}^2$, into the range set $\{0, 1\}$. For $f_\wedge(\cdot)$, $\mathcal{X}_0 = f_\wedge^{-1}(0) = \{(0, 0), (0, 1), (1, 0)\}$ and $\mathcal{X}_1 = f_\wedge^{-1}(1) = \{(1, 1)\}$.

x_1	x_2	$f_\wedge$
0	0	0
0	1	0
1	0	0
1	1	1

(a)



(b)

Fig. 4.5 The Boolean AND function $f_\wedge : \mathbb{B}^2 \rightarrow \{0, 1\}$

The AND function is linearly separable because the convex hulls $C_0 = C(\mathcal{X}_0)$ (the shaded triangle with corners at $(0, 0)$, $(0, 1)$, and $(1, 0)$ in Fig. 4.5) and $C_1 = C(\mathcal{X}_1)$ (which is the singleton set $\{(1, 1)\}$), are disjoint.

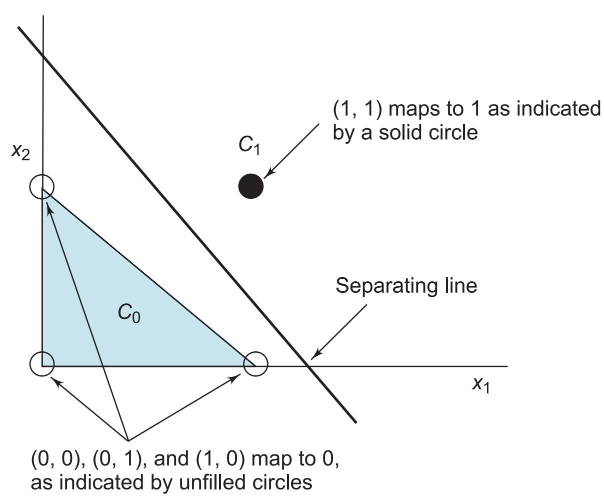


Fig. 4.6 The geometry of the Boolean AND function

It is convenient and instructive to view the problem geometrically as shown in Fig. 4.6. Here the four domain points comprising $\mathbb{B}^2$ have been marked on the real plane $\mathbb{R}^2$, with unfilled circles corresponding to points that map to a 0 and a solid circle corresponding to points that map to a 1. Referring further to Fig. 4.6, note that the convex hulls $C_0 = C(\mathcal{X}_0)$ (the shaded triangle with corners at (0,0), (0,1), and (1,0)) and $C_1 = C(\mathcal{X}_1)$ (the singleton set $\{(1, 1)\}$), are disjoint. Clearly, one can draw many straight lines that separate the solid circle from the unfilled circles. Figure 4.5 shows one such possibility, and $f_\wedge(\cdot)$ is therefore a linearly separable function.