Having discussed about the formulation of linear programming problems and their solution using graphic and simplex methods, we now consider the subjects of duality and sensitivity analysis. Both of these have significant managerial implications. Their most important contribution is a thorough appreciation of economics of the problem. Shadow prices tell the manager what price would be reasonable for acquiring scarce resources, or which resources are in abundance and could possibly be diverted to other uses within the organisation. Sensitivity analysis lets the manager know if more effort should be given to estimating certain parameters of the problem, and when changes in those estimated parameters might produce a change in the basic nature of the decision.

**Duality**

For every LPP, there is another LPP that is related to it and is derived from it. The given problem is called the *primal* while the problem derived is called the *dual*. To write the dual of an LPP, the following conditions are to be satisfied:

1. The variables should all be non-negative.
2. The constraints should all be “≤” type, if the problem is of the maximisation type, and “≥” type if it is of the minimisation type.

If condition (1) is not satisfied so that some variable is given to be unrestricted in sign, then it is substituted by the difference of two non-negative variables. Thus, if \(x_3\) is given to be unrestricted, it may be replaced in the problem by \(x_4 - x_5\), where \(x_4, x_5 \geq 0\).

In respect of condition (2), if a constraint involves “≤” sign while the desired sign is “≥”, or vice versa, then it is multiplied by \(-1\). If a constraint involves an “=” sign, then it is replaced by a pair of inequalities in opposite directions. A constraint \(7x_1 + 9x_2 = 66\), for instance would be replaced by two constraints as \(7x_1 + 9x_2 \leq 66\) and \(7x_1 + 9x_2 \geq 66\). One of these would be reversed again as desired.

Once the two conditions are satisfied, the dual can be written by introducing dual variables and transposing the three matrices involved in the primal. The dual will have as many constraints as the number of primal variables, and will have as many variables as the number of constraints that the primal has. The rows and columns of the \(a_{ij}\) coefficients will be interchanged. It may be noted that for every unrestricted variable in the primal problem, a constraint in the dual is of the “=” form and vice-versa.

**Solution to Primal and Dual**

If the primal has an optimal solution, then the dual also would have optimal solution with identical objective function value. Not only this, the optimal values of the dual variables can be obtained from the \(\Delta_j\) values of the
slack/surplus variables in the optimal tableau. If the primal problem has unbounded solution, the dual will have infeasibility, and vice versa.

**Economic Interpretation of the Dual** The $\Delta_j$ values for the slack/surplus variables indicate the marginal profitability or shadow prices of the resources which they represent. The shadow price of a resource is the maximum price a manager would be prepared to pay for an additional unit of the resource. Note that only scarce resources have positive shadow prices. The surplus resources have no worth since any addition to them will cause no change in the profit.

**Sensitivity Analysis**

This deals with investigating the effect of changes in the objective function co-efficients ($c_j$’s), resource availability ($b_i$ values) and the LHS co-efficients ($a_{ij}$’s) of the constraints. The changes in the two-variable problems can be analysed graphically as well as in context of the simplex solution. However, for problems with three variables or more, graphic approach is ruled out.

In the graphic approach, to determine whether a change in the co-efficient of a variable in the objective function would lead to a change in the basis or not, it should be noted that a change in a co-efficient causes a change in the iso-profit line (for a maximisation problem). The current basis continues to be optimal as long as the current optimal solution happens to be the last point in the feasible region to make contact with the iso-profit lines while we move in the direction of increasing values of the objective function. If the current basis remains optimal, the values of the decision variables remain what they have been, but the optimal $Z$-value may change.

To determine whether a change in the right hand side value of a constraint would cause a change in the optimality of the current basis, we begin by considering the constraints that are binding for the current optimal solution. For a change in the RHS of a constraint, the current basis remains optimal as long as the point where the constraints are binding remains feasible. However, even if the current basis remains optimal, the values of the decision variables and the optimal value of the objective function might change.

In context of the simplex method solution, sensitivity analysis is performed as follows:

(a) *Changes in $c_j$ values:* For a variable included in the basis, divide the $\Delta_j$ row values by the corresponding row values in tableau for that variable. The range of $c_j$ for which the current optimal solution would not change is obtained by adjusting the value by the least positive and negative quotients. For a variable not included in the basis, any decrease in the co-efficient (profit) value or any increase by an amount less than the $\Delta_j$ (absolute) value shall cause no change in the solution.

(b) *Changes in $b_i$ values (Right hand side ranging:)* Divide the $b_i$ values in the optimal solution tableau by the $a_{ij}$ values of the slack variable of interest. The least positive and the least negative values are then subtracted from the $b_i$ value of the original problem. It gives the range of values for the given resource over which there shall be no change in its
shadow price. It means that any changes in the availability of resources within the limits established would increase/decrease profit at a rate of the shadow price.

(c) Changes in $a_{ij}$ values: They can also be analysed to determine their effect on the optimal solution. However, the analysis is relatively more complex.

The advisability of introduction of a new product in the existing set-up can also be determined. For this, we first obtain resource requirements and multiply them with the respective shadow prices. If the unit profit from the proposed product exceeds the sum of these multiplications, then it is desirable to add the new product.

**Multiple Parameter Changes: 100% Rule** When multiple changes take place in the coefficients in the objective function or in the RHS values of the constraints, then the 100 per cent Rule may be used to determine whether they would affect the current solution.