CHAPTER 3

The Standard Atmosphere

Sometimes gentle, sometimes capricious, sometimes awful, never the same for two moments together; almost human in its passions, almost spiritual in its tenderness, almost divine in its infinity.

John Ruskin, The Sky

Aerospace vehicles can be divided into two basic categories: atmospheric vehicles such as airplanes and helicopters, which always fly within the sensible atmosphere, and space vehicles such as satellites, the Apollo lunar vehicle, and deep-space probes, which operate outside the sensible atmosphere. However, space vehicles do encounter the earth's atmosphere during their blastoffs from the earth's surface and again during their reentries and recoveries after completion of their missions. If the vehicle is a planetary probe, then it may encounter the atmospheres of Venus, Mars, Jupiter, etc. Therefore, during the design and performance of any aerospace vehicle, the properties of the atmosphere must be taken into account.

The earth's atmosphere is a dynamically changing system, constantly in a state of flux. The pressure and temperature of the atmosphere depend on altitude, location on the globe (longitude and latitude), time of day, season, and even solar sunspot activity. To take all these variations into account when considering the design and performance of flight vehicles is impractical. Therefore, a *standard atmosphere* is defined in order to relate flight tests, wind tunnel results, and general airplane design and performance to a common reference. The standard

PREVIEW BOX

Before you jump into a strange water pond or dive into an unfamiliar swimming pool, there are a few things you might like to know. How cold is the water? How clean is it? How deep is the water? These are things that might influence your swimming performance in the water, or even your decision to go swimming at all. Similarly, before we can study the performance of a flight vehicle speeding through the air, we need to know something about the properties of the air itself. Consider an airplane flying in the atmosphere, or a space vehicle blasting through the atmosphere on its way up to space, or a vehicle coming back from space through the atmosphere. In all these cases, the performance of the flight vehicle is going to be dictated in part by the properties of the atmosphere—the temperature, density, and pressure of the atmosphere.

What are the properties of the atmosphere? We know they change with altitude, but how do they change? How do we find out? These are important questions, and they are addressed in this chapter. Before you can go any further in your study of flight vehicles, you need to know about the atmosphere. Here is the story—please read on.

atmosphere gives mean values of pressure, temperature, density, and other properties as functions of altitude; these values are obtained from experimental balloon and sounding-rocket measurements combined with a mathematical model of the atmosphere. To a reasonable degree, the standard atmosphere reflects average atmospheric conditions, but this is not its main importance. Rather, its main function is to provide tables of common reference conditions that can be used in an organized fashion by aerospace engineers everywhere. The purpose of this chapter is to give you some feeling for what the standard atmosphere is all about and how it can be used for aerospace vehicle analyses.

We might pose the rather glib question: *Just what is the standard atmosphere?* A rather glib answer is: *The tables in Apps. A and B at the end of this book.* Take a look at these two appendixes. They tabulate the temperature, pressure, and density for different altitudes. Appendix A is in SI units, and App. B is in English engineering units. Where do these numbers come from? Were they simply pulled out of thin air by somebody in the distant past? Absolutely not. The numbers in these tables were obtained on a rational, scientific basis. One purpose of this chapter is to develop this rational basis. Another purpose is to show you how to use these tables.

The road map for this chapter is given in Fig. 3.1. We first run down the left side of the road map, establishing some definitions and an equation from basic physics (the hydrostatic equation) that are necessary tools for constructing the numbers in the standard atmosphere tables. Then we move to the right side of the road map and discuss how the numbers in the tables are actually obtained. We go through the construction of the standard atmosphere in detail. Finally, we define some terms that are derived from the numbers in the tables—the pressure, density, and temperature altitudes—which are in almost everyday use in aeronautics.

Finally, we note that the details of this chapter are focused on the determination of the standard atmosphere for earth. The tables in Apps. A and B are for the

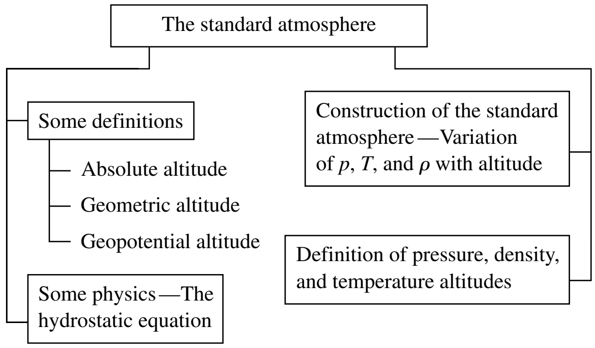

Figure 3.1 Road map for Chap. 3.

earth's atmosphere. However, the physical principles and techniques discussed in this chapter are also applicable to constructing model atmospheres for other planets, such as Venus, Mars, and Jupiter. So the applicability of this chapter reaches far beyond the earth.

It should be mentioned that several different standard atmospheres exist, compiled by different agencies at different times, each using slightly different experimental data in the models. For all practical purposes, the differences are insignificant below 30 km (100,000 ft), which is the domain of contemporary airplanes. A standard atmosphere in common use is the 1959 ARDC model atmosphere. (ARDC stands for the U.S. Air Force's previous Air Research and Development Command, which is now the Air Force Research Laboratory.) The atmospheric tables used in this book are taken from the 1959 ARDC model atmosphere.

3.1 DEFINITION OF ALTITUDE

Intuitively, we all know the meaning of altitude. We think of it as the distance above the ground. But like so many other general terms, it must be more precisely defined for quantitative use in engineering. In fact, in the following sections we define and use six different altitudes: absolute, geometric, geopotential, pressure, temperature, and density altitudes.

First, imagine that we are at Daytona Beach, Florida, where the ground is at sea level. If we could fly straight up in a helicopter and drop a tape measure to the ground, the measurement on the tape would be, by definition, the geometric altitude h_G, that is, the geometric height above sea level.

Now, if we bored a hole through the ground to the center of the earth and extended our tape measure until it hit the center, then the measurement on the tape would be, by definition, the *absolute altitude* h_a. If r is the radius of the earth, then $h_a = h_G + r$. This is illustrated in Fig. 3.2.

The absolute altitude is important, especially for space flight, because the local acceleration of gravity g varies with h_a. From Newton's law of gravitation,

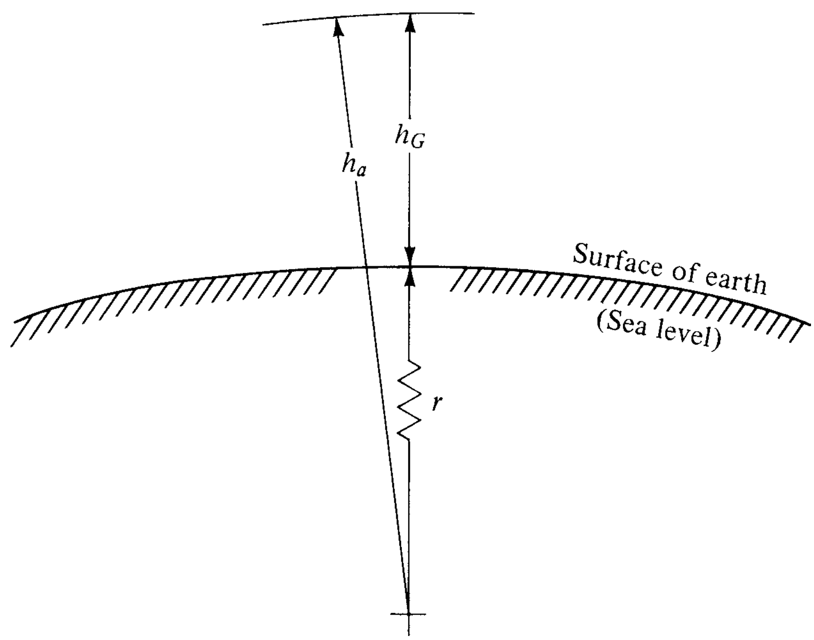

Figure 3.2 Definition of altitude.

g varies inversely as the square of the distance from the center of the earth. By letting g_0 be the gravitational acceleration at *sea level,* the local gravitational acceleration g at a given absolute altitude h_a is

$$g = g_0 \left(\frac{r}{h_a}\right)^2 = g_0 \left(\frac{r}{r + h_G}\right)^2 \tag{3.1}$$

The variation of g with altitude must be taken into account when you are dealing with mathematical models of the atmosphere, as discussed in the following sections.

3.2 HYDROSTATIC EQUATION

We will now begin to piece together a model that will allow us to calculate variations of p, ρ, and T as functions of altitude. The foundation of this model is the hydrostatic equation, which is nothing more than a force balance on an element of fluid at rest. Consider the small stationary fluid element of air shown in Fig. 3.3. We take for convenience an element with rectangular faces, where the top and bottom faces have sides of unit length and the side faces have an infinitesimally small height dh_G. On the bottom face, the pressure p is felt, which gives rise to an upward force of $p \times 1 \times 1$ exerted on the fluid element. The top face is slightly higher in altitude (by the distance dh_G), and because pressure varies with altitude, the pressure on the top face will be slightly different from that on the bottom face, differing by the infinitesimally small value dp. Hence,

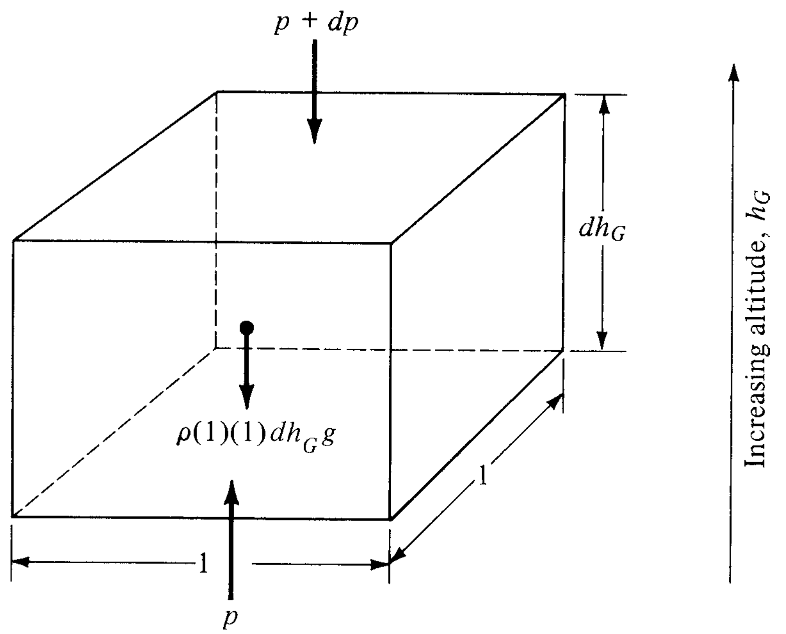

Figure 3.3 Force diagram for the hydrostatic equation.

on the top face, the pressure $p + dp$ is felt. It gives rise to a downward force of $(p + dp)(1)(1)$ on the fluid element. Moreover, the volume of the fluid element is $(1)(1)\, dh_G = dh_G$, and since ρ is the mass per unit volume, then the mass of the fluid element is simply $\rho(1)(1)dh_G = \rho\, dh_G$. If the local acceleration of gravity is g, then the weight of the fluid element is $g\rho\, dh_G$, as shown in Fig. 3.3. The three forces shown in Fig. 3.3, pressure forces on the top and bottom, and the weight must balance because the fluid element is not moving. Hence,

$$p = p + dp + \rho g\, dh_G$$

Thus,

$$dp = -\rho g\, dh_G \tag{3.2}$$

Equation (3.2) is the *hydrostatic equation* and applies to any fluid of density ρ, for example, water in the ocean as well as air in the atmosphere.

Strictly speaking, Eq. (3.2) is a differential equation; that is, it relates an infinitesimally small change in pressure dp to a corresponding infinitesimally small change in altitude dh_G, where in the language of differential calculus, dp and dh_G are differentials. Also note that g is a variable in Eq. (3.2); g depends on h_G as given by Eq. (3.1).

To be made useful, Eq. (3.2) should be integrated to give what we want, namely, the variation of pressure with altitude $p = p(h_G)$. To simplify the integration, we make the *assumption* that g is constant throughout the atmosphere, equal to its value at sea level g_0. This is something of a historical convention in aeronautics. Hence, we can write Eq. (3.2) as

$$dp = -\rho g_0 \, dh \tag{3.3}$$

However, to make Eqs. (3.2) and (3.3) numerically identical, the altitude h in Eq. (3.3) must be slightly different from h_G in Eq. (3.2), to compensate for the fact that g is slightly different from g_0. Suddenly, we have defined a new altitude h, which is called the *geopotential altitude* and which differs from the geometric altitude. To better understand the concept of geopotential altitude, consider a given geometric altitude, h_G, where the value of pressure is p. Let us now increase the geometric altitude by an infinitesimal amount, dh_G, such that the new geometric altitude is $h_G + dh_G$. At this new altitude, the pressure is $p + dp$, where the value of dp is given by Eq. (3.2). Let us now put this *same value* of dp in Eq. (3.3). Dividing Eq. (3.3) by (3.2), we have

$$1 = \left(\frac{g_0}{g}\right)\left(\frac{dh}{dh_G}\right)$$

Clearly, since g_0 and g are different, then dh and dh_G must be different; that is, the numerical values of dh and dh_G that correspond to the *same* change in pressure, dp, are different. As a consequence, the numerical values of h and h_G that correspond to the same actual physical location in the atmosphere are different values.

For the practical mind, geopotential altitude is a "fictitious" altitude, defined by Eq. (3.3) for ease of future calculations. However, many standard atmosphere tables quote their results in terms of geopotential altitude, and care must be taken to make the distinction. Again, geopotential altitude can be thought of as that fictitious altitude that is physically compatible with the assumption of $g = \text{const} = g_0$.

3.3 RELATION BETWEEN GEOPOTENTIAL AND GEOMETRIC ALTITUDES

We still seek the variation of p with geometric altitude $p = p(h_G)$. However, our calculations using Eq. (3.3) will give, instead, $p = p(h)$. Therefore, we need to relate h to h_G, as follows. Dividing Eq. (3.3) by (3.2), we obtain

$$1 = \frac{g_0}{g}\frac{dh}{dh_G}$$

or

$$dh = \frac{g}{g_0} dh_G \tag{3.4}$$

We substitute Eq. (3.1) into (3.4):

$$dh = \frac{r^2}{(r + h_G)^2} dh_G \tag{3.5}$$

By convention, we set both h and h_G equal to zero at sea level. Now, consider a given point in the atmosphere. This point is at a certain geometric altitude h_G, and associated with it is a certain value of h (different from h_G). Integrating Eq. (3.5) between sea level and the given point, we have

$$\int_0^h dh = \int_0^{h_G} \frac{r^2}{(r+h_G)^2} dh_G = r^2 \int_0^{h_G} \frac{dh_G}{(r+h_G)^2}$$

$$h = r^2 \left(\frac{-1}{r+h_G} \right)_0^{h_G} = r^2 \left(\frac{-1}{r+h_G} + \frac{1}{r} \right) = r^2 \left(\frac{-r+r+h_G}{(r+h_G)r} \right)$$

Thus,

$$\boxed{h = \frac{r}{r+h_G} h_G} \tag{3.6}$$

where h is geopotential altitude and h_G is geometric altitude. This is the desired relation between the two altitudes. When we obtain relations such as $p = p(h)$, we can use Eq. (3.6) to subsequently relate p to h_G.

A quick calculation using Eq. (3.6) shows that there is little difference between h and h_G for low altitudes. For such a case, $h_G \ll r$, $r/(r+h_G) \approx 1$; hence, $h \approx h_G$. Putting in numbers, $r = 6.356766 \times 10^6$ m (at a altitude of 45°), and if $h_G = 7$ km (about 23,000 ft), then the corresponding value of h is, from Eq. (3.6), $h = 6.9923$ km, about 0.1 of 1 percent difference! Only at altitudes above 65 km (213,000 ft) does the difference exceed 1 percent. (Note that 65 km is an altitude at which aerodynamic heating of NASA's Space Shuttle becomes important during reentry into the earth's atmosphere from space.)

3.4 DEFINITION OF THE STANDARD ATMOSPHERE

We are now in a position to obtain p, T, and ρ as functions of h for the standard atmosphere. The keystone of the standard atmosphere is a *defined* variation of T with altitude, based on experimental evidence. This variation is shown in Fig. 3.4. Note that it consists of a series of straight lines, some vertical (called the constant-temperature, or *isothermal,* regions) and others inclined (called the *gradient* regions). Given $T = T(h)$ as *defined* by Fig. 3.4, then $p = p(h)$ and $\rho = \rho(h)$ follow from the laws of physics, as shown in the following.

First, consider again Eq. (3.3):

$$dp = -\rho g_0 \, dh$$

Divide by the equation of state, Eq. (2.3):

$$\frac{dp}{p} = -\frac{\rho g_0 \, dh}{\rho RT} = -\frac{g_0}{RT} dh \tag{3.7}$$

Consider first the isothermal (constant-temperature) layers of the standard atmosphere, as given by the vertical lines in Fig. 3.4 and sketched in Fig. 3.5. The temperature, pressure, and density at the base of the isothermal layer shown in

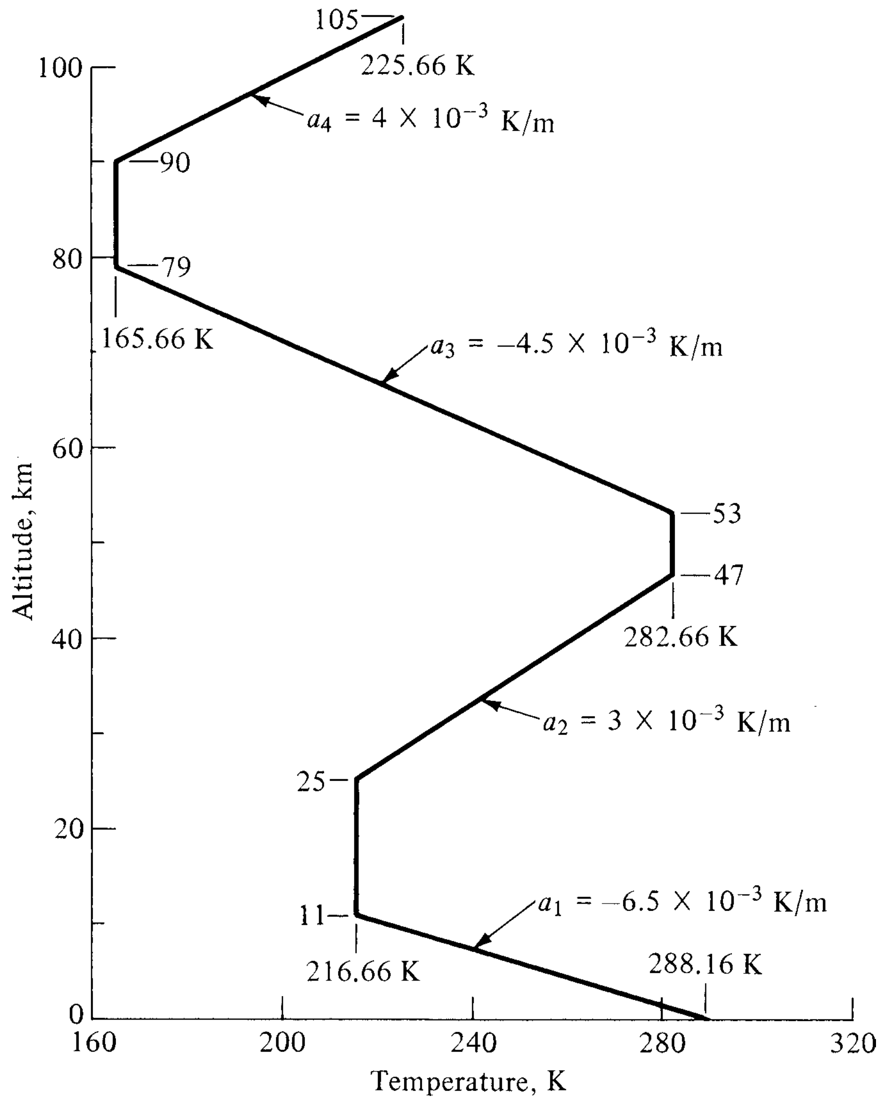

Figure 3.4 Temperature distribution in the standard atmosphere.

Fig. 3.5 are T_1, p_1, and ρ_1, respectively. The base is located at a given geopotential altitude h_1. Now consider a given point in the isothermal layer above the base, where the altitude is h. The pressure p at h can be obtained by integrating Eq. (3.7) between h_1 and h.

$$\int_{p_1}^{p} \frac{dp}{p} = -\frac{g_0}{RT} \int_{h_1}^{h} dh \tag{3.8}$$

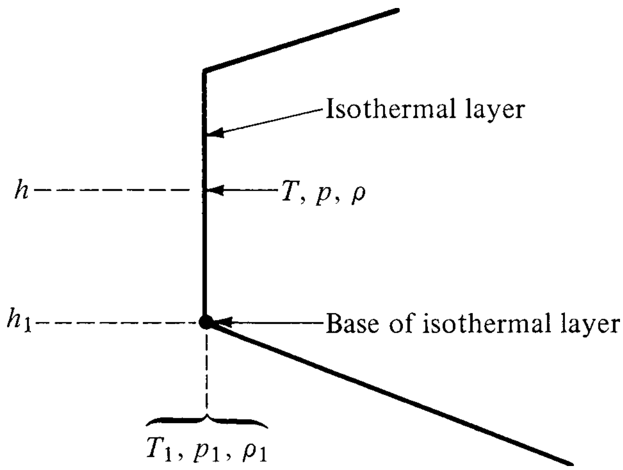

Figure 3.5 Isothermal layer.

Note that g_0, R, and T are constants that can be taken outside the integral. (This clearly demonstrates the simplification obtained by assuming that $g = g_0 =$ const, and therefore dealing with geopotential altitude h in the analysis.) Performing the integration in Eq. (3.8), we obtain

$$\ln \frac{p}{p_1} = -\frac{g_0}{RT}(h - h_1)$$

or

$$\boxed{\frac{p}{p_1} = e^{-[g_0/(RT)](h-h_1)}} \tag{3.9}$$

From the equation of state,

$$\frac{p}{p_1} = \frac{\rho T}{\rho_1 T_1} = \frac{\rho}{\rho_1}$$

Thus,

$$\boxed{\frac{\rho}{\rho_1} = e^{-[g_0/(RT)](h-h_1)}} \tag{3.10}$$

Equations (3.9) and (3.10) give the variation of p and ρ versus geopotential altitude for the isothermal layers of the standard atmosphere.

Considering the gradient layers, as sketched in Fig. 3.6, we find the temperature variation is linear and is geometrically given as

$$\frac{T - T_1}{h - h_1} = \frac{dT}{dh} \equiv a$$

where a is a *specified* constant for each layer obtained from the defined temperature variation in Fig. 3.4. The value of a is sometimes called the *lapse rate* for

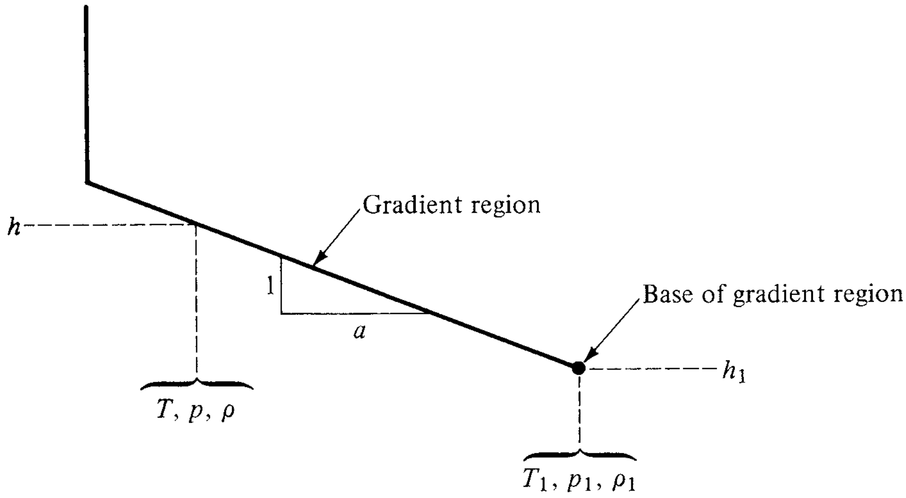

Figure 3.6 Gradient layer.

the gradient layers.

$$\boxed{a \equiv \frac{dT}{dh}}$$

Thus,

$$dh = \frac{1}{a} dT$$

We substitute this result into Eq. (3.7):

$$\frac{dp}{p} = -\frac{g_0}{aR} \frac{dT}{T} \tag{3.11}$$

Integrating between the base of the gradient layer (shown in Fig. 3.6) and some point at altitude h, also in the gradient layer, Eq. (3.11) yields

$$\int_{p_1}^{p} \frac{dp}{p} = -\frac{g_0}{aR} \int_{T_1}^{T} \frac{dT}{T}$$

$$\ln \frac{p}{p_1} = -\frac{g_0}{aR} \ln \frac{T}{T_1}$$

Thus,

$$\boxed{\frac{p}{p_1} = \left(\frac{T}{T_1}\right)^{-g_0/(aR)}} \tag{3.12}$$

From the equation of state,

$$\frac{p}{p_1} = \frac{\rho T}{\rho_1 T_1}$$

Hence, Eq. (3.12) becomes

$$\frac{\rho T}{\rho_1 T_1} = \left(\frac{T}{T_1}\right)^{-g_0/(aR)}$$

$$\frac{\rho}{\rho_1} = \left(\frac{T}{T_1}\right)^{-[g_0/(aR)]-1}$$

or

$$\boxed{\frac{\rho}{\rho_1} = \left(\frac{T}{T_1}\right)^{-\{[g_0/(aR)]+1\}}} \tag{3.13}$$

Recall that the variation of T is linear with altitude and is given the specified relation

$$\boxed{T = T_1 + a(h - h_1)} \tag{3.14}$$

Equation (3.14) gives $T = T(h)$ for the gradient layers; when it is plugged into Eq. (3.12), we obtain $p = p(h)$; similarly from Eq. (3.13) we obtain $\rho = \rho(h)$.

Now we can see how the standard atmosphere is pieced together. Looking at Fig. 3.4, start at sea level ($h = 0$), where standard sea level values of pressure, density, and temperature—p_s, ρ_s, and T_s, respectively—are

$$\boxed{\begin{aligned} p_s &= 1.01325 \times 10^5 \text{ N/m}^2 = 2116.2 \text{ lb/ft}^2 \\ \rho_s &= 1.2250 \text{ kg/m}^3 = 0.002377 \text{ slug/ft}^3 \\ T_s &= 288.16 \text{ K} = 518.69^\circ\text{R} \end{aligned}}$$

These are the base values for the first gradient region. Use Eq. (3.14) to obtain values of T as a function of h until $T = 216.66$ K, which occurs at $h = 11.0$ km. With these values of T, use Eqs. (3.12) and (3.13) to obtain the corresponding values of p and ρ in the first gradient layer. Next, starting at $h = 11.0$ km as the base of the first isothermal region (see Fig. 3.4), use Eqs. (3.9) and (3.10) to calculate values of p and ρ versus h, until $h = 25$ km, which is the base of the next gradient region. In this manner, with Fig. 3.4 and Eqs. (3.9), (3.10), and (3.12) to (3.14), a table of values for the standard atmosphere can be constructed.

Such a table is given in App. A for SI units and App. B for English engineering units. Look at these tables carefully and become familiar with them. They *are* the standard atmosphere. The first column gives the geometric altitude, and the second column gives the corresponding geopotential altitude obtained from Eq. (3.6). The third through fifth columns give the corresponding standard values

DESIGN BOX

The first step in the design process of a new aircraft is the determination of a set of *specifications,* or *requirements,* for the new vehicle. These specifications may include such performance aspects as a stipulated maximum velocity at a given altitude or a stipulated maximum rate of climb at a given altitude. These performance parameters depend on the aerodynamic characteristics of the vehicle, such as lift and drag. In turn, the lift and drag depend on the properties of the atmosphere. When the specifications dictate certain performance at a given altitude, this altitude is taken to be the standard altitude in the tables. Therefore, in the preliminary design of an airplane, the designer uses the standard atmosphere tables to define the pressure, temperature, and density at the given altitude. In this fashion, many calculations made during the preliminary design of an airplane contain information from the standard altitude tables.

of temperature, pressure, and density, respectively, for each altitude, obtained from the previous discussion.

We emphasize again that the standard atmosphere is a reference atmosphere only and certainly does not predict the actual atmospheric properties that may exist at a given time and place. For example, App. A says that at an altitude (geometric) of 3 km, $p = 0.70121 \times 10^5\ \text{N/m}^2$, $T = 268.67\ \text{K}$, and $\rho = 0.90926\ \text{kg/m}^3$. In reality, situated where you are, if you could right now levitate yourself to 3 km above sea level, you would most likely feel a p, T, and ρ different from the values obtained from App. A. The standard atmosphere allows us only to reduce test data and calculations to a convenient, agreed-upon reference, as will be seen in subsequent sections of this book.

Comment: Geometric and Geopotential Altitudes Revisited We now can appreciate better the meaning and significance of the geometric altitude, h_G, and the geopotential altitude, h. The variation of the properties in the standard atmosphere are calculated from Eqs. (3.9) to (3.14). These equations are derived using the simplifying assumption of a constant value of the acceleration of gravity equal to its value at sea level; that is, $g = \text{constant} = g_0$. Consequently, the altitude that appears in these equations is, by definition, the geopotential altitude, h. Examine these equations again—you see g_0 and h appearing in these equations, not g and h_G. The simplification obtained by assuming a constant value of g is the *sole reason* for defining the geopotential altitude. This is the only use of geopotential altitude we will make in this book—for the calculation of the numbers that appear in Apps. A and B. Moreover, since h and h_G are related via Eq. (3.6), we can always obtain the geometric altitude, h_G, that corresponds to a specified value of geopotential altitude, h. The geometric altitude, h_G, is the actual height above sea level and therefore is more practical. That is why the first column in Apps. A and B is h_G, and the entries are in even intervals of h_G. The second column gives the corresponding values of h, and these are the values used to generate the corresponding numbers for p, ρ, and T via Eqs. (3.9) to (3.14).

In the subsequent chapters in this book, any dealings with altitude involving the use of the standard atmosphere tables in Apps. A and B will be couched in terms of the geometric altitude, h_G. For example, if reference is made to a "standard altitude" of 5 km, it means a geometric altitude of $h_G = 5$ km. Now that we have seen how the standard atmosphere tables are generated, after the present chapter we will have no reason to deal anymore with geopotential altitude.

Hopefully, you now have a better understanding of the statement made at the end of Sec. 3.2 that geopotential altitude is simply a "fictitious" altitude, defined by Eq. (3.3) for the single purpose of simplifying the subsequent derivations.

EXAMPLE 3.1

Calculate the standard atmosphere values of T, p, and ρ at a geopotential altitude of 14 km.

■ **Solution**

Remember that T is a *defined* variation for the standard atmosphere. Hence, we can immediately refer to Fig. 3.4 and find that at $h = 14$ km,

$$T = 216.66 \text{ K}$$

To obtain p and ρ, we must use Eqs. (3.9) to (3.14), piecing together the different regions from sea level up to the given altitude with which we are concerned. Beginning at sea level, the first region (from Fig. 3.4) is a gradient region from $h = 0$ to $h = 11.0$ km. The lapse rate is

$$a = \frac{dT}{dh} = \frac{216.66 - 288.16}{11.0 - 0} = -6.5 \text{ K/km}$$

or

$$a = -0.0065 \text{ K/m}$$

Therefore, using Eqs. (3.12) and (3.13), which are for a gradient region and where the base of the region is sea level (hence $p_1 = 1.01 \times 10^5$ N/m^2 and $\rho_1 = 1.23$ kg/m^3), we find at $h = 11.0$ km

$$p = p_1 \left(\frac{T}{T_1}\right)^{-g_0/(aR)} = (1.01 \times 10^5) \left(\frac{216.66}{288.16}\right)^{-9.8/[-0.0065(287)]}$$

where $g_0 = 9.8$ m/s^2 in SI units. Hence, p (at $h = 11.0$ km) $= 2.26 \times 10^4$ N/m^2.

$$\rho = \rho_1 \left(\frac{T}{T_1}\right)^{-[g_0/(aR)+1]}$$

$$= (1.23) \left(\frac{216.66}{288.16}\right)^{-\{9.8/[-0.0065(287)]+1\}}$$

$$= 0.367 \text{ kg/m}^3 \qquad \text{at } h = 11.0 \text{ km}$$

The above values of p and ρ now form the *base* values for the first isothermal region (see Fig. 3.4). The equations for the isothermal region are Eqs. (3.9) and (3.10), where now $p_1 = 2.26 \times 10^4$ N/m^2 and $\rho_1 = 0.367$ kg/m^3. For $h = 14$ km, $h - h_1 = 14 - 11 = 3$ km $= 3000$ m. From Eq. (3.9),

$$\boxed{\begin{aligned} &p = p_1 e^{-[g_0/(RT)](h-h_1)} = (2.26 \times 10^4) e^{-[9.8/287(216.66)](3000)} \\ &p = 1.41 \times 10^4 \text{ N/m}^2 \end{aligned}}$$

From Eq. (3.10),

$$\frac{\rho}{\rho_1} = \frac{p}{p_1}$$

Hence,

$$\rho = \rho_1 \frac{p}{p_1} = 0.367 \frac{1.41 \times 10^4}{2.26 \times 10^4} = \boxed{0.23 \text{ kg/m}^3}$$

These values check, within roundoff error, with the values given in App. A. *Note:* This example demonstrates how the numbers in Apps. A and B are obtained!

3.5 PRESSURE, TEMPERATURE, AND DENSITY ALTITUDES

With the tables of Apps. A and B in hand, we can now define three new "altitudes"—pressure, temperature, and density altitudes. This is best done by example. Imagine that you are in an airplane flying at some real, geometric altitude. The value of your actual altitude is immaterial for this discussion. However, at this altitude, you measure the actual outside air pressure to be 6.16×10^4 N/m^2. From App. A, you find that the standard altitude that corresponds to a pressure of 6.16×10^4 N/m^2 is 4 km. Therefore, by *definition,* you say that you are flying at a *pressure altitude* of 4 km. Simultaneously, you measure the actual outside air temperature to be 265.4 K. From App. A, you find that the standard altitude that corresponds to a temperature of 265.4 K is 3.5 km. Therefore, by definition, you say that you are flying at a *temperature altitude* of 3.5 km. Thus, you are simultaneously flying at a pressure altitude of 4 km and a temperature altitude of 3.5 km while your actual geometric altitude is yet a different value. The definition of *density altitude* is made in the same vein. These quantities—pressure, temperature, and density altitudes—are just convenient numbers that, via App. A or B, are related to the actual p, T, and ρ for the actual altitude at which you are flying.

EXAMPLE 3.2

If an airplane is flying at an altitude where the actual pressure and temperature are 4.72×10^4 N/m^2 and 255.7 K, respectively, what are the pressure, temperature, and density altitudes?

■ Solution

For the pressure altitude, look in App. A for the standard altitude value corresponding to $p = 4.72 \times 10^4$ N/m^2. This is 6000 m. Hence,

$$\boxed{\text{Pressure altitude} = 6000\,\text{m} = 6\,\text{km}}$$

For the temperature altitude, look in App. A for the standard altitude value corresponding to $T = 255.7$ K. This is 5000 m. Hence,

$$\boxed{\text{Temperature altitude} = 5000\,\text{m} = 5\,\text{km}}$$

For the density altitude, we must first calculate ρ from the equation of state:

$$\rho = \frac{p}{RT} = \frac{4.72 \times 10^4}{287(255.7)} = 0.643\ \text{kg/m}^3$$

Looking in App. A and interpolating between 6.2 and 6.3 km, we find that the standard altitude value corresponding to $\rho = 0.643$ kg/m^3 is about 6.240 m. Hence,

$$\boxed{\text{Density altitude} = 6240\,\text{m} = 6.24\,\text{km}}$$

Note that temperature altitude is not a unique value. The answer for temperature altitude could equally well be 5.0, 38.2, or 59.5 km because of the multivalued nature of the altitude-versus-temperature function. In this section, only the lowest value of temperature altitude is used.

EXAMPLE 3.3

The flight test data for a given airplane refer to a level-flight maximum-velocity run made at an altitude that simultaneously corresponded to a pressure altitude of 30,000 ft and density altitude of 28,500 ft. Calculate the temperature of the air at the altitude at which the airplane was flying for the test.

■ Solution

From App. B:
For pressure altitude = 30,000 ft:

$$p = 629.66\ \text{lb/ft}^2$$

For density altitude = 28,500 ft:

$$\rho = 0.9408 \times 10^{-3}\ \text{slug/ft}^3$$

These are the values of p and ρ that simultaneously existed at the altitude at which the airplane was flying. Therefore, from the equation of state,

$$T = \frac{p}{\rho R} = \frac{629.66}{(0.94082 \times 10^{-3})(1716)} = \boxed{390^\circ\text{R}}$$

EXAMPLE 3.4

Consider an airplane flying at some real, geometric altitude. The outside (ambient) pressure and temperature are 5.3×10^4 N/m^2 and 253 K, respectively. Calculate the pressure and density altitudes at which this airplane is flying.

■ Solution

Consider the ambient pressure of 5.3×10^4 N/m^2. In App. A, there is not a precise entry for this number. It lies between the value $p_1 = 5.331 \times 10^4$ N/m^2 at altitude $h_{G,1} = 5100$ m and $p_2 = 5.2621 \times 10^4$ N/m^2 at altitude $h_{G,2} = 5200$ m. We have at least two choices. We could simply use the nearest entry in the table, which is for an altitude $h_{G,2} = 5100$ m, and say that the answer for pressure altitude is 5100 m. This is acceptable if we are making only approximate calculations. However, if we need greater accuracy, we can *interpolate* between entries. Using linear interpolation, the value of h_G corresponding to $p = 5.3 \times 10^4$ N/m^2 is

$$h_G = h_{G,1} + (h_{G,2} - h_{G,1})\left(\frac{p_1 - p}{p_1 - p_2}\right)$$

$$h_G = 5100 + (5200 - 5100)\left(\frac{5.331 - 5.3}{5.331 - 5.2621}\right)$$

$$= 5100 + 100(0.4662) = 5146.6 \text{ m}$$

The pressure altitude at which the airplane is flying is 5146.6 m. (Note that in this example and in Examples 3.2 and 3.3, we are interpreting the word *altitude* in the tables to be the geometric altitude h_G rather than the geopotential altitude h. This is for convenience, because h_G is tabulated in round numbers, in contrast to the column for h. Again, at the altitudes for conventional flight, the difference between h_G and h is not significant.)

To obtain the density altitude, calculate the density from the equation of state.

$$\rho = \frac{p}{RT} = \frac{5.3 \times 10^4}{(287)(253)} = 0.72992 \text{ kg/m}^3$$

Once again we note that this value of ρ falls between two entries in the table. It falls between $h_{G,1} = 5000$ m where $\rho_1 = 0.73643$ kg/m^3 and $h_{G,2} = 5100$ m where $\rho_2 = 0.72851$ kg/m^3. (Note that these subscripts denote different lines in the table from those used in the first part of this example. It is good never to become a slave to subscripts and symbols. Just always keep in mind the significance of what you are doing.) We could take the nearest entry, which is for an altitude $h_G = 5100$ m, and say that the answer for the density altitude is 5100 m. However, for greater accuracy, let us linearly interpolate between the two entries.

$$h_G = h_{G,1} + (h_{G,2} - h_{G,1})\left(\frac{\rho_1 - \rho}{\rho_1 - \rho_2}\right)$$

$$= 5000 + (5100 - 5000)\left(\frac{0.73643 - 0.72992}{0.73643 - 0.72851}\right)$$

$$= 5000 + 100(0.82197) = 5082.2 \text{ m}$$

The density altitude at which the airplane is flying is 5082.2 m.

3.6 HISTORICAL NOTE: THE STANDARD ATMOSPHERE

With the advent of ballooning in 1783 (see Chap. 1), people suddenly became interested in acquiring a greater understanding of the properties of the atmosphere above ground level. However, a compelling reason for such knowledge did not arise until the coming of heavier-than-air flight in the 20th century. As we shall see in subsequent chapters, the flight performance of aircraft is dependent upon such properties as the pressure and density of the air. Thus, a knowledge of these properties, or at least some agreed-upon standard for worldwide reference, is absolutely necessary for intelligent aeronautical engineering.

The situation in 1915 was summarized by C. F. Marvin, Chief of the U.S. Weather Bureau and chairman of an NACA subcommittee to investigate and report upon the existing status of atmospheric data and knowledge. In his "Preliminary Report on the Problem of the Atmosphere in Relation to Aeronautics," NACA Report No. 4, 1915, Marvin writes:

> The Weather Bureau is already in possession of an immense amount of data concerning atmospheric conditions, including wind movements at the earth's surface. This information is no doubt of distinct value to aeronautical operations, but it needs to be collected and put in form to meet the requirements of aviation.

The following four years saw such efforts to collect and organize atmospheric data for use by aeronautical engineers. In 1920, the Frenchman A. Toussaint, director of the Aerodynamic Laboratory at Saint-Cyr-l'Ecole, France, suggested the following formula for the temperature decrease with height:

$$T = 15 - 0.0065h$$

where T is in degrees Celsius and h is the geopotential altitude in meters. Toussaint's formula was formally adopted by France and Italy with the Draft of Inter-Allied Agreement on Law Adopted for the Decrease of Temperature with Increase of Altitude, issued by the Ministere de la Guerre, Aeronautique Militaire, Section Technique, in March 1920. One year later, England followed suit. The United States was close behind. Since Marvin's report in 1915, the U.S. Weather Bureau had compiled measurements of the temperature distribution and found Toussaint's formula to be a reasonable representation of the observed mean annual values. Therefore, at its executive committee meeting of December 17, 1921, NACA adopted Toussaint's formula for airplane performance testing, with the statement: "The subcommittee on aerodynamics recommends that for the sake of uniform practice in different countries that Toussaint's formula be adopted in determining the standard atmosphere up to 10 km (33,000 ft). . . ."

Much of the technical data base that supported Toussaint's formula was reported in NACA Report No. 147, "Standard Atmosphere," by Willis Ray Gregg in

1922. Based on free-flight tests at McCook Field in Dayton, Ohio, and at Langley Field in Hampton, Virginia, and on the other flights at Washington, District of Columbia, as well as artillery data from Aberdeen, Maryland, and Dahlgren, Virginia, and sounding-balloon observations at Fort Omaha, Nebraska, and St. Louis, Missouri, Gregg was able to compile a table of mean annual atmospheric properties. An example of his results follows:

Altitude, m	Mean Annual Temperature in United States, K	Temperature from Toussaint's Formula, K
0	284.5	288
1,000	281.0	281.5
2,000	277.0	275.0
5,000	260.0	255.5
10,000	228.5	223.0

Clearly, Toussaint's formula provided a simple and reasonable representation of the mean annual results in the United States. This was the primary message in Gregg's report in 1922. However, the report neither gave extensive tables nor attempted to provide a document for engineering use.

Thus, it fell to Walter S. Diehl (who later became a well-known aerodynamicist and airplane designer as a captain in the Naval Bureau of Aeronautics) to provide the first practical tables for a standard atmosphere for aeronautical use. In 1925, in NACA Report No. TR 218, entitled (again) "Standard Atmosphere," Diehl presented extensive tables of standard atmospheric properties in both metric and English units. The tables were in increments of 50 m up to an altitude of 10 km and then in increments of 100 m up to 20 km. In English units, the tables were in increments of 100 ft up to 32,000 ft and then in increments of 200 ft up to a maximum altitude of 65,000 ft. Considering the aircraft of that day (see Fig. 1.31), these tables were certainly sufficient. Moreover, starting from Toussaint's formula for T up to 10,769 m, then assuming $T = \text{const} = -55°\text{C}$ above 10,769 m, Diehl obtained p and ρ in precisely the same fashion as described in the previous sections of this chapter.

The 1940s saw the beginning of serious rocket flights, with the German V-2 and the initiation of sounding rockets. Moreover, airplanes were flying higher than ever. Then, with the advent of intercontinental ballistic missiles in the 1950s and space flight in the 1960s, altitudes began to be quoted in terms of hundreds of miles rather than feet. Therefore, new tables of the standard atmosphere were created, mainly extending the old tables to higher altitudes. Popular among the various tables is the ARDC 1959 Standard Atmosphere, which is used in this book and is given in Apps. A and B. For all practical purposes, the old and new tables agree for altitudes of greatest interest. Indeed, it is interesting to compare values, as shown in the following:

Altitude, m	*T* from Diehl, 1925, K	*T* from ARDC, 1959, K
0	288	288.16
1,000	281.5	281.66
2,000	275.0	275.16
5,000	255.5	255.69
10,000	223.0	223.26
10,800	218.0	218.03
11,100	218.0	216.66
20,000	218.0	216.66

So Diehl's standard atmosphere from 1925, at least up to 20 km, is just as good as the values today.

3.7 Summary

Some of the major ideas of this chapter are listed as follows.

1. The standard atmosphere is defined in order to relate flight tests, wind tunnel results, and general airplane design and performance to a common reference.
2. The definitions of the standard atmospheric properties are based on a given temperature variation with altitude, representing a mean of experimental data. In turn, the pressure and density variations with altitude are obtained from this empirical temperature variation by using the laws of physics. One of these laws is the hydrostatic equation:
$$dp = -\rho g\, dh_G \tag{3.2}$$
3. In the isothermal regions of the standard atmosphere, the pressure and density variations are given by
$$\frac{p}{p_1} = \frac{\rho}{\rho_1} = e^{-[g_0/(RT)](h-h_1)} \qquad \text{(3.9) and (3.10)}$$
4. In the gradient regions of the standard atmosphere, the pressure and density variations are given by, respectively,
$$\frac{p}{p_1} = \left(\frac{T}{T_1}\right)^{-g_0/(aR)} \tag{3.12}$$
$$\frac{\rho}{\rho_1} = \left(\frac{T}{T_1}\right)^{-\{[(g_0/(aR)]+1\}} \tag{3.13}$$
where $T = T_1 + a(h - h_1)$ and a is the given lapse rate.
5. The pressure altitude is that altitude in the standard atmosphere that corresponds to the actual ambient pressure encountered in flight or laboratory experiments. For

example, if the ambient pressure of a flow, no matter where it is or what it is doing, is 393.12 lb/ft^2, the flow is said to correspond to a pressure altitude of 40,000 ft (see App. B). The same idea can be used to define density and temperature altitudes.

Bibliography

Minzner, R. A., K. S. W. Champion, and H. L. Pond: *The ARDC Model Atmosphere, 1959,* Air Force Cambridge Research Center Report No. TR-59-267, U.S. Air Force, Bedford, MA, 1959.

Problems

3.1 At 12 km in the standard atmosphere, the pressure, density, and temperature are 1.9399×10^4 N/m^2, 3.1194×10^{-1} kg/m^3, and 216.66 K, respectively. Using these values, calculate the standard atmospheric values of pressure, density, and temperature at an altitude of 18 km, and check with the standard altitude tables.

3.2 Consider an airplane flying at some real altitude. The outside pressure and temperature are 2.65×10^4 N/m^2 and 220 K, respectively. What are the pressure and density altitudes?

3.3 During a flight test of a new airplane, the pilot radios to the ground that she is in level flight at a standard altitude of 35,000 ft. What is the ambient air pressure far ahead of the airplane?

3.4 Consider an airplane flying at a pressure altitude of 33,500 ft and a density altitude of 32,000 ft. Calculate the outside air temperature.

3.5 At what value of the geometric altitude is the difference $h - h_G$ equal to 2 percent of the geopotential altitude, h?

3.6 Using Toussaint's formula, calculate the pressure at a geopotential altitude of 5 km.

3.7 The atmosphere of Jupiter is essentially made up of hydrogen, H_2. For H_2, the specific gas constant is 4157 J/(kg)(K). The acceleration of gravity of Jupiter is 24.9 m/s^2. Assuming an isothermal atmosphere with a temperature of 150 K and assuming that Jupiter has a definable surface, calculate the altitude above that surface where the pressure is one-half the surface pressure.

3.8 An F-15 supersonic fighter aircraft is in a rapid climb. At the instant it passes through a standard altitude of 25,000 ft, its time rate of change of altitude is 500 ft/s, which by definition is the *rate of climb,* discussed in Chap. 6. Corresponding to this rate of climb at 25,000 ft is a time rate of change of ambient pressure. Calculate this rate of change of pressure in units of pounds per square foot per second.

3.9 Assume that you are ascending in an elevator at sea level. Your eardrums are very sensitive to minute changes in pressure. In this case, you are feeling a 1-percent decrease in pressure per minute. Calculate the upward speed of the elevator in meters per minute.

3.10 Consider an airplane flying at an altitude where the pressure and temperature are 530 lb/ft^2 and 390°R, respectively. Calculate the pressure and density altitudes at which the airplane is flying.

3.11 Consider a large rectangular-shaped tank of water open to the atmosphere, 10 ft deep, with walls of length 30 ft each. When the tank is filled to the top with water, calculate the force (in tons) exerted on the side of each wall in contact with the water. The tank is located at sea level. (*Note:* The specific weight of water is 62.4 lb_f/ft^3, and 1 ton = 2000 lb_f.) (*Hint:* Use the hydrostatic equation.)

3.12 A discussion of the entry of a space vehicle into the earth's atmosphere after it has completed its mission in space is given in Chap. 8. An approximate analysis of the vehicle motion and aerodynamic heating during atmospheric entry assumes an approximate atmospheric model called the "exponential atmosphere," where the air density variation with altitude is assumed to be

$$\frac{\rho}{\rho_0} = e^{-g_0 h/(RT)}$$

where ρ_0 is the sea-level density and h is the altitude measured above sea level. This equation is only an approximation for the density variation with altitude throughout the whole atmosphere, but its simple form makes it very useful for approximate analyses. Using this equation, calculate the density at an altitude of 45 km. Compare your result with the actual value of density from the standard altitude tables. In the preceding equation, assume $T = 240$ K (a reasonable representation for the value of the temperature between sea level and 45 km, which you can see by scanning down the standard atmosphere table).