Chapter 2

Derivatives

Curriculum Expectations

Rate of Change

By the end of the course, students will:

3.2 verify the constant, constant multiple, sum, and difference rules graphically and numerically [e.g., by using the function $g(x) = kf(x)$ and comparing the graphs of $g'(x)$ and $kf'(x)$; by using a table of values to verify that $f'(x) + g'(x) = (f + g)'(x)$, given $f(x) = x$ and $g(x) = 3x$], and read and interpret proofs involving $\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$ of the constant, constant multiple, sum, and difference rules (student reproduction of the development of the general case is not required)

Technology Notes

The technology tools used in this chapter are primarily graphing calculators, specifically the TI-83+/84+ series and a computer algebra system (CAS), specifically the TI-89/89T series.

3.4 verify that the power rule applies to functions of the form $f(x) = x^n$, where n is a rational number [e.g., by comparing values of the slopes of tangents to the function $f(x) = x^{\frac{1}{2}}$ with values of the derivative function determined using the power rule], and verify algebraically the chain rule using monomial functions [e.g., by determining the same derivative for $f(x) = (5x^3)^{\frac{1}{3}}$ by using the chain rule and by differentiating the simplified form, $f(x) = 5^{\frac{1}{3}}x$ and the product rule using polynomial functions [e.g., by determining the same derivative for $f(x) = (3x + 2)(2x^2 - 1)$ by using the product rule and by differentiating the expanded form $f(x) = 6x^3 + 4x^2 - 3x - 2$]

3.5 solve problems, using the product and chain rules, involving the derivatives of polynomial functions, sinusoidal functions, exponential functions, rational functions [e.g., by expressing $f(x) = \frac{x^2 + 1}{x - 1}$ as the product $f(x) = (x^2 + 1)(x - 1)^{-1}$], radical functions [e.g., by expressing $f(x) = \sqrt{x^2 + 5}$ as the power $f(x) = (x^2 + 5)^{\frac{1}{2}}$, and other simple combinations of functions [e.g., $f(x) = x \sin x$, $f(x) = \frac{\sin x}{\cos x}$]*

*The emphasis of this expectation is on the application of the derivative rules and not on the simplification of resulting complex algebraic expressions.

Derivatives and Their Applications

By the end of the course, students will:

2.1 make connections between the concept of motion (i.e., displacement, velocity, acceleration) and the concept of the derivative in a variety of ways (e.g., verbally, numerically, graphically, algebraically)

2.2 make connections between the graphical or algebraic representations of derivatives and real-world applications (e.g., population and rates of population change, prices and inflation rates, volume and rates of flow, height and growth rates)

2.3 solve problems, using the derivative, that involve instantaneous rates of change, including problems arising from real-world applications (e.g., population growth, radioactive decay, temperature changes, hours of daylight, heights of tides), given the equation of a function*

*The emphasis of this expectation is on the application of the derivative rules and not on the simplification of resulting complex algebraic expressions.

Chapter 2 Planning Chart

Section Suggested Timing	Student Text Page(s)	Teacher's Resource Blackline Masters	Assessment	Tools
Chapter 2 Opener 10 min	69			
Prerequisite Skills 45–60 min	70–71	• BLM 2–1 Prerequisite Skills		
2.1 Derivative of a Polynomial Function 75 min	72–86	• BLM 2–2 Derivative Rules 1 • BLM 2–3 Section 2.1 Practice		• graphing calculator
Extension: Problem Solving with a Computer Algebra System 30 min	87			• computer algebra system (CAS)
2.2 The Product Rule 75 min	88–96	• BLM 2–4 Section 2.2 Practice		• grid paper • graphing calculator (optional)
2.3 Velocity, Acceleration, and Second Derivatives 70–150 min	97–110	• BLM 2–5 Section 2.3 Question 6 • BLM 2–6 Section 2.3 Question 7 • BLM 2–7 Section 2.3 Practice		• grid paper • graphing calculator • graphing software • Calculator-Based Ranger (CBR™) • calculator-to-CBR™ cable • ramp at least 3 m long • large ball, such as a basketball
2.4 The Chain Rule 75 min	111–119	• BLM 2–8 Section 2.4 Question 2 • BLM 2–9 Section 2.4 Practice	• BLM 2–10 Section 2.4 Achievement Check Rubric	• graphing calculator (optional)
2.5 Derivatives of Quotients 75 min	120–126	• BLM 2–11 Section 2.5 Practice		• graphing calculator (optional)
Extension: The Quotient Rule 30 min	127–129			• graphing calculator (optional)
2.6 Rate of Change Problems 75 min	130–141	• BLM 2–12 Section 2.6 Practice		
Chapter 2 Review 75 min	142–143	• BLM 2–2 Derivative Rules 1 • BLM 2–13 Derivative Rules 2 • BLM 2–14 Chapter 2 Review		• graphing calculator • computer algebra system (CAS) (optional)
Chapter 2 Problem Wrap-Up 75 min	143		• BLM 2–15 Chapter 2 Problem Wrap-Up Rubric	• grid paper • graphing calculator (optional) • computer algebra system (CAS) (optional) • computer with *The Geometer's Sketchpad*® (optional)
Chapter 2 Practice Test 75 min	144–145		• BLM 2–16 Chapter 2 Test	• graphing calculator (optional) • computer algebra system (CAS) (optional)
Chapter 2 Task: The Disappearing Lollipop 75 min	146		• BLM 2–17 Chapter 2 Task Rubric	• spherical lollipop on stick • measuring tape or string and centimetre ruler • timer with seconds hand • graphing software (optional)

Chapter 2 Blackline Masters Checklist

	BLM	Title	Purpose
Prerequisite Skills			
	BLM 2–1	Prerequisite Skills	Practice
2.1 Derivative of a Polynomial Function			
	BLM 2–2	Derivative Rules 1	Student Support
	BLM 2–3	Section 2.1 Practice	Practice
2.2 The Product Rule			
	BLM 2–4	Section 2.2 Practice	Practice
2.3 Velocity, Acceleration, and Second Derivatives			
	BLM 2–5	Section 2.3 Question 6	Student Support
	BLM 2–6	Section 2.3 Question 7	Student Support
	BLM 2–7	Section 2.3 Practice	Practice
2.4 The Chain Rule			
	BLM 2–8	Section 2.4 Question 2	Student Support
	BLM 2–9	Section 2.4 Practice	Practice
	BLM 2–10	Section 2.4 Achievement Check Rubric	Assessment
2.5 Derivatives of Quotients			
	BLM 2–11	Section 2.5 Practice	Practice
2.6 Rate of Change Problems			
	BLM 2–12	Section 2.6 Practice	Practice
Chapter 2 Review			
	BLM 2–2	Derivative Rules 1	Student Support
	BLM 2–13	Derivative Rules 2	Student Support
	BLM 2–14	Chapter 2 Review	Practice
Chapter 2 Chapter Problem Wrap-Up			
	BLM 2–15	Chapter 2 Problem Wrap-Up Rubric	Assessment
Chapter 2 Practice Test			
	BLM 2–16	Chapter 2 Test	Assessment
Chapter 2 Task: The Disappearing Lollipop			
	BLM 2–17	Chapter 2 Task Rubric	Assessment
	BLM 2–18	Chapter 2 Practice Masters Answers	Answers

Prerequisite Skills

Student Text Pages
70–71

Suggested Timing
45–60 min

Related Resources
- BLM 2–1 Prerequisite Skills

Assessment
You may wish to use **BLM 2–1 Prerequisite Skills** as a diagnostic assessment. Refer students to the Prerequisite Skills Appendix for examples and further practice of topics.

Differentiated Instruction
- Use an **anticipation guide** to introduce this chapter.

Chapter Problem

- The Chapter Problem is introduced on page 71. Have students discuss their understanding of the topic. The Chapter Problem is revisited in sections 2.1 (question 19), 2.2 (question 16), 2.4 (question 13), and 2.5 (question 12). These questions are designed to help students move toward the Chapter 2 Problem Wrap-Up on page 143. Alternatively, you may wish to assign the Chapter Problem questions and Chapter Problem Wrap-Up when students have completed the chapter, as part of a summative assessment.

2.1 Derivative of a Polynomial Function

Student Text Pages
72–86

Suggested Timing
75 min

Tools
- graphing calculator

Related Resources
- BLM 2–2 Derivative Rules 1
- BLM 2–3 Section 2.1 Practice

Teaching Suggestions

- Allow 35 to 45 min for students to complete the **Investigate** as they work in groups of 3 or 4. The Investigate will help students explore the simple derivative rules numerically and graphically.
- Note that students will be using the derivative **nDeriv(** on a graphing calculator for the first time in **Part B**. This function uses numerical methods to graph the derivative of a given function. Once graphed, you can access the Table of Values or determine the derivative for a given value of x as you can for any other function. Refer to the Technology Appendix, page 564.
- A computer algebra system (CAS) or *The Geometer's Sketchpad*® can be used to determine the equation of the derivative. Refer to the Technology Appendix, page 571.
- The **Derivative Rules** in Leibniz form are listed in the third column of the chart on page 76. Provide **BLM 2–2 Derivative Rules 1** to students who have difficulty with memory or writing.
- Refer to **Prerequisite Skills questions 2 to 4,** and **6** for this section.

- **Examples 1** and **2** illustrate how the power rule can be extended to include integral and rational exponents.
- For Example 2, graphing the numerical derivative followed by the derivative as calculated using a heavier line format provides a visual confirmation that the two are identical. Caution students that this does not constitute a proof.
- **Examples 4** and **6** represent the two types of tangent problems that are typically solved using derivatives.
- For **Example 5**, the tangent operation on a graphing calculator not only draws the tangent for visual confirmation of the solution but also provides the equation of the tangent. However, the equation will disappear as soon as another key is pressed. Ask students to write the equation, if needed. Refer to the Technology Appendix, page 565.

- As students consider the **Communicate Your Understanding** questions, draw out how the sum and difference rules can be combined and extended to polynomial functions with more than two terms. It is also important for students to understand the difference between *proving* a rule for the general case and *verifying* a rule using a specific example.
- Please note that although the proofs for some of the derivative rules are provided in this section or asked for in the questions, it is not necessary to have students memorize them or to include them in formal evaluation. Reasoning and Proving is, however, one of the Mathematical Processes. At this level of mathematics it may benefit students to have an understanding of how a formal proof is done and what the results represent.

- The purpose of **questions 6** and **8** is to have students understand that, at this point, the derivative rules they have learned only work when a quotient or product is first simplified into a sum and/or difference. They should conclude that additional rules are needed to differentiate more complex functions that involve quotients, products, and powers.
- The purpose of **question 10** is to have students discuss the meaning of the phrases *sum of derivatives* and *derivative of sums*.

- **Question 15** addresses all of the mathematical processes. In particular, students are asked to solve the problem, represent the function using graphing technology, reason their way through the scenario, select the appropriate tools to find specific answers, connect to material previously learned, reflect on their results, and communicate effectively their thoughts and ideas.
  - Students should be able to sketch an approximation of the arrow's path without technology.
  - Have students refer to Example 6 when looking at **part b**).
  - Remind students to use all the information they have when thinking through the problem.
  - Look for a clear explanation in **part e**).
  - If students do not have access to technology, an accurate hand drawn graph will help in confirming their answers for **parts a**) **to e**).
- For **questions 24** and **25**, point out that the solution steps to these questions are similar.
- For **question 27**, students should identify if the given input and output values correspond to $f(x)$ or $f'(x)$.
- For **question 28**, students should first identify if the given point is on the given curve.
- For **question 29**, students should identify that the given value is for y and x.
- For additional practice, provide **BLM 2–3 Section 2.1 Practice**.

Investigate Answers (pages 72–75)

	Original Function	Derivative Function
Constant Rule	$y = c$	$\frac{dy}{dx} = 0$
Power Rule	$y = x^n$	$\frac{dy}{dx} = nx^{n-1}$
Sum Rule	$y = f(x) + g(x)$	$\frac{dy}{dx} f'(x) + g'(x)$
Different Rule	$y = f(x) - g(x)$	$\frac{dy}{dx} f'(x) - g'(x)$
Constant Multiple Rule	$y = cf(x)$	$\frac{dy}{dx} = cf'(x)$

Part A

1. a) i) $y = 2$: The slope of the function at any point on its graph is 0.

ii) $y = -3$: The slope of the function at any point on its graph is 0.

iii) $y = 0.5$: The slope of the function at any point on its graph is 0.

b) Answers may vary. For example: The slope of the function $y = c$ would not be different for any value of $c \in \mathbb{R}$. The slope of the function at any point on the graph of a constant function is 0.

c) Answers may vary. For example: The derivative of a constant function $y = c$ for any $c \in \mathbb{R}$ is $\frac{dy}{dx} = 0$.

Part B

1. a) Graphs scales may vary. The graph consists of the line $y = x$ and the line $y = 1$.

b) The equation that corresponds to the graph of the derivative is $y = 1$.

c)

X	Y1	Y2
0	0	1
1	1	1
2	2	1
3	3	1
4	4	1
5	5	1
6	6	1

X=0

Answers may vary. For example: The y-values for the graph of the derivative function, **Y2** are all equal to 1. The graph of the equation found in part b), $y = 1$, would have the same y-values.

2. a) Graphs scales may vary. The graph consists of the parabola $y = x^2$ and the line $y = 2x$. The equation that corresponds to the graph of the derivative is $y = 2x$.

X	Y1	Y2
0	0	0
1	1	2
2	4	4
3	9	6
4	16	8
5	25	10
6	36	12

X=0

Answers may vary. For example: The y-values for the graph of the derivative function, **Y2**, are all equal to 2 multiplied by the x-values of the function $y = x^2$. The graph of the equation found, $y = 2x$, would have the same y-values.

b) Graphs scales may vary. The graph consists of the curve $y = x^3$ and the parabola $3x^2$. The equation that corresponds to the graph of the derivative is $y = 3x^2$.

X	Y1	Y2
0	0	1E-6
1	1	3
2	8	12
3	27	27
4	64	48
5	125	75
6	216	108

X=0

Answers may vary. For example: The y-values for the graph of the derivative function, **Y2**, are all equal to 3 multiplied by the square of the x-values of the function $y = x^3$. The graph of the equation found in, $y = 3x^2$, would have the same y-values.

c) Graphs scales may vary. The graph consists of the curve $y = x^4$ and the curve $y = 4x^3$. The equation that corresponds to the graph of the derivative is $y = 4x^3$.

X	Y1	Y2
0	0	0
1	1	4
2	16	32
3	81	108
4	256	256
5	625	500
6	1296	864

X=0

Answers may vary. For example: The y-values for the graph of the derivative function, **Y2**, are all equal to 4 multiplied by the cube of the x-values of the function $y = x^4$. The graph of the equation found in part b), $y = 4x^3$, would have the same y-values.

3.

$f(x)$	$f'(x)$
x	1
x^2	$2x$
x^3	$3x^2$
x^4	$4x^3$

4. a) Answers may vary. For example: Yes. There is a pattern in both the graphs and the derivatives in step 3. The exponent of the each of the original functions is the coefficient of the derivative. The exponent of the derivative function is one less than the original function. The derivative of a power, $f(x) = x^n$ is $f'(x) = nx^{n-1}$.

b) The derivative of $f(x) = x^5$ is $f'(x) = 5x^4$. Graphs scales may vary. The graph consists of the curve $y = x^5$ and the curve $y = 5x^4$ that looks like a parabola. The equation that corresponds to the graph of the derivative is $y = 5x^4$.

X	Y1	Y2
0	0	1E-12
1	1	5
2	32	80
3	243	405
4	1024	1280
5	3125	3125
6	7776	6480

X=0

Answers may vary. For example: The y-values for the graph of the derivative function, **Y2**, are all equal to 5 multiplied by the fourth power of the x-values of the function $y = x^5$. The graph of the equation found, $y = 5x^4$, would have the same y-values.

Part C

1. a) Graphs scales may vary.
   b) Answers may vary. For example: The graphs of **Y2**, **Y3**, and **Y4** are all straight lines and pass through the point (0, 0), as does the graph of **Y1**. Each of the graphs has been stretched vertically by a factor equal to the numerical coefficient of each function, which is equal to the slope of each of the lines. Also the lines become steeper as the value of the slope of each line increases.
   c) **Y1** $= x$, slope: 1; equation of the derivative: $\frac{dy}{dx} = 1$
      Y2 $= 2x$, slope: 2; equation of the derivative: $\frac{dy}{dx} = 2$
      Y3 $= 3x$, slope: 3; equation of the derivative: $\frac{dy}{dx} = 3$
      Y4 $= 4x$, slope: 4; equation of the derivative: $\frac{dy}{dx} = 4$
   d) Answers may vary. For example: The derivative of $y = cx$ for any constant $c \in \mathbb{R}$ is equal to the constant of the function multiplied by the derivative of the variable of the function.
2. a) Graph scales may vary.
   b) Answers may vary. For example: The graphs of **Y2**, **Y3**, and **Y4** are all parabolas and have the same vertex (0, 0), as does the graph of **Y1**. Each of the graphs has been stretched vertically by a factor equal to the coefficient of each of the functions.
   c) **Y1** $= x^2$, equation of the derivative: $\frac{dy}{dx} = 2x$
      Y2 $= 2x^2$, equation of the derivative: $\frac{dy}{dx} = 4x$
      Y3 $= 3x^2$, equation of the derivative: $\frac{dy}{dx} = 6x$
      Y4 $= 4x^2$, equation of the derivative: $\frac{dy}{dx} = 8x$
   d) Answers may vary. For example: The derivative of $y = cx^2$ for any constant $c \in \mathbb{R}$ is equal to the constant of the function multiplied by the derivative of the variable of the function.
3. Answers may vary. For example: If $f(x) = cx^n$, then $f'(x) = cnx^{n-1}$.

Part D

1. a) i) $f'(x) = 4$; $g'(x) = 7$; $f'(x) + g'(x) = 4 + 7 = 11$
      ii) $h(x) = 4x + 7x = 11x$; $h'(x) = 11$
   b) Answers may vary. For example: The results in part a) i) and ii) are the same. The predicted derivative is the sum of the derivatives of each term in the function $h(x) = x + x^2$.
   c) Answers may vary.
   d) Answers may vary. For example: In the table of values in Screen 6, the first column represents the x values for each of the functions. The second column represents the y-values for the derivative of the function $y = x^2$. The third column represents the y-values for the sum of the derivative of the function $y = x$ and the derivative of the function $y = x^3$.
   e) Answers may vary. For example: The relationship is that **Y1** + **Y2** = **Y3**.
2. Answers may vary. For example:
   a) The sum of the derivatives of each term in a function is equal to the derivative of the function.
   b) The thick line graph in Screen 5 is the sum of the two functions represented by the solid horizontal line and the dotted line. The thick line graph in Screen 5 is the same as the graph in Screen 8.
   c) The results confirm the prediction of the derivative made in step 1, part b).
3. Answers may vary. For example: The derivative of the sum of two functions is equal to the sum of the derivatives of the two functions.

Differentiated Instruction

- Use the **think-pair-share** strategy.

Common Errors

- Students differentiate simple rational functions, of the form $y = \frac{c}{x^n}$, where c is any constant, incorrectly.

Rx Remind students to express $y = \frac{c}{x^n}$ in the form $y = cx^{-n}$ before applying the power rule. Refer to **Prerequisite Skills, question 4**.

- Students differentiate simple rational functions, of the form $y = \sqrt[n]{x^m}$, incorrectly.

Rx Remind students to express $y = \sqrt[n]{x^m}$ in the form $y = x^{\frac{m}{n}}$ before applying the power rule. Refer to **Prerequisite Skills, question 3**.

4. Answers may vary. For example: Yes there is a similar rule for the derivative of the difference of two functions. The derivative of the difference of two functions is equal to the difference of the derivatives of the two functions. You could confirm your prediction by graphing the derivatives of each of the functions and the difference of the derivatives of the two functions, as well as the derivative of the difference of the two functions on the same viewing screen of a graphing calculator and determining that the rule for the derivative of the difference of two functions is correct.

Communicate Your Understanding Responses (page 83)

C1 The graph of a vertical line has an equation of the form $y = c$, where c is a constant. To find the derivative of the function, $y = c$, determine the slope of the tangent to the line. The tangent and the graph of $y = c$ are same. The slope of a horizontal line is zero. Therefore, the derivative of a constant is zero.

C2 If a polynomial function is the result of the addition and/or subtraction of terms you can use the sum and difference rules for differentiating polynomials. Differentiate each term separately and then add and subtract accordingly.

C3 By adding or subtracting three or more polynomial functions and simplifying, you can then apply the sum and differences rules to the simplified function.

For example,

$f(x) = x^2 + 3$, $g(x) = 2x^3 + 2x^2$, $h(x) = x^2$, and $k(x) = f(x) + g(x) - h(x)$.

Find $k'(x)$.

$k(x) = f(x) + g(x) - h(x)$
$= (x^2 + 3) + (2x^3 + 2x^2) - (x^2)$
$= 2x^3 + 2x^2 + 3$

$k'(x) = 6x^2 + 4x$

$k'(x) = f'(x) + g'(x) - h'(x)$
$= 2x + 6x^2 + 4x - 2x$
$= 6x^2 + 4x$

Also:

$k(x) = f(x) + g(x) - h(x)$

$k(x) = (x^2 + 3) + (2x^3 + 2x^2) - (x^2)$

$k'(x) = (2x + 0) + (6x^2 + 4x) - (2x)$

$k'(x) = 6x^2 + 4x$

The result is the same for all three methods.

C4 When you prove a derivative rule, you determine a rule that works for all functions. You can prove a derivative rule by using first principles. When you show that the derivative rules work for certain functions, it works for certain functions but might not work for all functions. You can show that the rules work by differentiating the functions using the derivative rules.

Mathematical Process Expectations

Process Expectation	Selected Questions
Problem Solving	10–12 15, 18, 24
Reasoning and Proving	9, 12–22, 27–30
Reflecting	7–12, 15, 18, 24
Selecting Tools and Computational Strategies	1–6, 8, 9, 11, 13–22, 24–27, 29
Connecting	3–18, 24
Representing	5, 6, 9, 15–18, 20–22, 24
Communicating	4, 7, 8, 10, 12, 15, 18, 24, 30

Ongoing Assessment

Achievement Check, question 26, on student text page 86

Achievement Check, question 26, student text page 86

Sample Solution

26. a) i) $p'(3) = 14$; at 3 h, the rate of growth will be 14 000 bacteria/h.

ii) $p'(8) = 4$; at 8 h, the rate of growth will be 4000 bacteria/h.

iii) $p'(13) = -6$; at 13 h, the rate of growth will be -6000 bacteria/h.

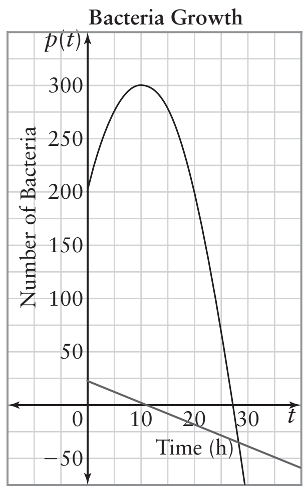
iv) $p'(18) = -16$; at 18 h, the rate of growth will be $-16\ 000$ bacteria/h.

b) Sometime between 8 h and 13 h, the population of the colony stops growing and begins to decline.

c) At $t = 8$, the equation of the tangent to $p(t)$ is $y = 4t + 264$, where y is the number of bacteria in thousands and t is the number of hours.

d) The population stops growing after 10 h. The population reaches a peak of 300 000 bacteria at this time.

e) The graph should not continue past the zero. The population increases from 0 to 10 h and decreases from 10 h to approximately 27.5 h when it dies out completely. The graph of the derivative is positive from 0 to 10 h and negative from 10 h on. Both graphs show that the rate of increase decreases until it is zero at 10 h and the rate of decrease is speeding up.



f) i) The population increases when $t \in [0,10)$.

ii) The population decreases when $t \in (10,27.32]$.

Student Text Page
87

Suggested Timing
30 min

Tools
- computer algebra system (CAS)

Problem Solving with a Computer Algebra System

Teaching Suggestions

- The length of time for this extension depends on whether students have previously used a CAS.
- You may wish to demonstrate how a CAS may be applied to **Example 6** in **section 2.1**.
- Students could use this technology to solve **questions 20 to 25** in **section 2.1**. Students could also create similar questions and then solve them using a CAS.

2.2 The Product Rule

Student Text Pages
88–96

Suggested Timing
75 min

Tools
- grid paper
- graphing calculator (optional)

Related Resources
- BLM 2–4 Section 2.2 Practice

Teaching Suggestions

- Allow 10 to 15 min for students to complete the **Investigate** individually or in pairs.
- By completing the Investigate, students should conclude that the method for determining the derivative of a sum or difference does *not* apply when determining the derivative of a product.
- Refer to **Prerequisite Skills, question 10** for this section.

- In each **Example**, the use of Leibniz notation helps identify the factor that is being differentiated. The use of colour for the derivative provides a good visual focus when teaching the examples.
- **Example 1** verifies the answers algebraically and using technology.
- **Example 3** illustrates how mathematical modelling is applied to determine a function that involves a product. The table in the solution of **part a**) is included to help students understand the thinking behind developing the equation. It does not have to be included when students solve similar questions.

- As students consider the **Communicate Your Understanding** questions, draw out that although the products in this section may be expanded and then differentiated using the methods of **section 2.1**, students are building rules that will be combined with the chain rule to determine derivatives of complex products that include powers. In these cases expanding first may not be efficient or even possible.

- Discuss with students the extent to which they are required to simplify their answers once the derivative is taken. Students could simply apply the product rule for **questions 1 to 3** and not simplify their answers. This will help them focus on learning and applying the rule.
- For question 3, the product rule should be used and students are not required to expand first.
- **Question 9** addresses all of the mathematical processes. In particular, students are asked to select the appropriate tools, represent the derivative in two different ways, reflect on the information they are discovering, connect to information previously learned, solve a problem, reason through the various parts of the question, and communicate effectively their thoughts and ideas.
  - Have students refer to **Example 3**.
  - Look for proper terminology in describing the results of **part a**).
  - Students should see the connection between when the derivative equals zero and the maximum value of the function.
  - Look for a sketch that is complete and correctly labelled.
- For **question 12**, it is possible for students to multiply the $2x^2$ and the first binomial to avoid using the product rule for three factors. Alternatively, you may have them consider $2x^2(x^2 + 2x)$ as the first factor or $f(x)$ and $(x - 1)$ as the second factor or $g(x)$.
- For **question 14**, have students write $-\frac{t}{18}$ as $-\frac{1}{18}t$.
- **Questions 17** and **19** are intended to give some insight into the chain rule, covered later in this chapter.
- For **question 18**, have students write the product fgh as $[fg]h$ or $f[gh]$. The use of Leibniz notation here may also benefit students.
- For additional practice, provide **BLM 2–4 Section 2.2 Practice**.

Differentiated Instruction

- Use a **graffiti strategy** to demonstrate understanding.

Common Errors

- Students mistake a coefficient as the first term or *f*(*x*) and then apply the product rule, as in **questions 14** and **15**.

R_x Point out that constants or coefficients are simply multiplied with the derivative of terms containing the main variable, which is usually *x*. In questions 14 and 15, the coefficients, 90 and 15, should not be considered as *f*(*x*) in the product rule.

- Students incorrectly differentiate a product by differentiating each part of the product.

R_x Refer students to the results of the **Investigate**. Have students write the definition of the product rule for $p(x) = f(x)g(x)$ as part of their derivative solution.

Investigate Answers (page 88)

1. a) $f'(x) = 3x^2$; $g(x) = 4x^3$
   b) $f'(x) \times g'(x) = 12x^5$
2. a) $f(x) \times g(x) = x^7$
   b) $\frac{d}{dx}[f(x) \times g(x)] = 7x^6$
3. Answers may vary. For example: Yes. The above result verifies that $f'(x) \times g'(x) \neq \frac{d}{dx}[f(x) \times g(x)]$ since $f'(x) \times g'(x) = 12x^5$ and $\frac{d}{dx}[f(x) \times g(x)] = 7x^6$
4. Answers may vary. For example:
   $$\begin{aligned} f(x)g'(x) + g(x)f'(x) &= (x^3)(4x^3) + (x^4)(3x^2) \\ &= 4x^6 + 3x^6 \\ &= 7x^6 \end{aligned}$$
   Yes, this expression will always work.

Communicate Your Understanding Responses (page 93)

C1 The product of the two derivatives of two functions is *not* equal to the derivative of the product of two functions. For example:

$$\begin{aligned} f(x) &= x + 2 \text{ and } g(x) = 3x^2 - 1 \\ f(x)g(x) &= (x + 2)(3x^2 - 1) \\ &= 3x^3 + 2x^2 - x - 2 \\ [f(x)g(x)]' &= 9x^2 + 4x - 1 \end{aligned}$$

$$\begin{aligned} f'(x) &= 1 \text{ and } g'(x) = 6x \\ f'(x)\, g'(x) &= 1(6x) \\ &= 6x \end{aligned}$$

C2 The product rule is a more efficient method of differentiating the product of two functions. Also, this rule can be used on functions that would take a long to expand and simplify, such as $y = (x^3 + 5x)^{15}(x^4 - 7x)^{20}$.

C3 You do not need to simplify the derivative before substituting the value of x because it is quicker to substitute the value into the derivative then simplify. The results will be the same using both methods.

C4 To differentiate $y = (2x - 5)^3$, express the cubed binomial as a product of a squared binomial and a binomial, $y = (2x - 5)^2(2x - 5)$. Then differentiate.

Mathematical Process Expectations

Process Expectation	Selected Questions
Problem Solving	9–11, 14–16, 18
Reasoning and Proving	9, 11, 14, 16, 17, 19, 21
Reflecting	1, 9, 10, 11, 14–18
Selecting Tools and Computational Strategies	1, 3–10, 12–14, 16, 17, 19–21
Connecting	1–11, 14–21
Representing	9–13, 16, 20
Communicating	9–11, 15, 16, 18

2.3

Velocity, Acceleration, and Second Derivatives

Student Text Pages

97–110

Suggested Timing

75–150 min

Tools

- grid paper
- graphing calculator
- graphing software
- Calculator-Based Ranger (CBR™)
- calculator-to-CBR™ cable
- ramp at least 3 m long
- large ball, such as a basketball

Related Resources

- BLM 2–5 Section 2.3 Question 6
- BLM 2–6 Section 2.3 Question 7
- BLM 2–7 Section 2.3 Practice

Teaching Suggestions

- Allow 5 min for **Investigate A**. Students may work individually or in pairs.
- For step 1 of Investigate A, it is interesting to note that taking the numerical derivative of a numerical derivative builds on errors introduced at the first step. However, the graphing calculator takes such small steps that it would take quite a few iterations for the error to become noticeable.
- Point out the notation for the second derivative found in the **Connections** box on page 102.
- Refer to **Prerequisite Skills, question 7** for this section.
- If you have not used a CBR™ before, take some time to practise with it before you begin **Investigate B**. The sonic sensor is very sensitive to aiming errors. It needs a large target to reflect from. If a graph shows any "artifacts", i.e., spurious spikes or other formations, the sensor is not receiving a reliable signal. Consider using a bigger target and ensure that you keep distances between 1 m and 3 m from the sensor.
- An additional note about the CBR™: the Ranger program derives velocity and acceleration data numerically from the distance and time data. You will find that the graphs become progressively less accurate.
- For Investigate B, allow 35 to 40 min. Students may work in groups of 4 or 5, depending on the number of CBR™s available. It may be necessary to have some groups set up their ramp in an alternative location such as the hallway or in the school concourse.
- Remind students to read the **Technology Tips**. You may wish to discuss these before they begin. Also review and discuss steps 1 to 4. A demonstration may be necessary if students are not familiar with using a CBR™.
- Group students who may have difficulty working with a CBR™ with those that are more familiar with this technology. Each member of the group should have a designated task and are expected to cooperate to complete the Investigate within the designated time.
- Remind them to answer **questions 5 to 8** and record their observations. Have each group compare and discuss their findings with another group or have one group present their findings for question 5, then a different group presents their findings for question 6, etc. up to question 8.

- For **Example 1**, students could use a graphing calculator to graph $f(x)$, $f'(x)$, and $f''(x)$ and then examine the corresponding table values in relation to the graphs shown on page 100. Remind students that *the y-values of the graph of a derivative are the slopes of the tangents to points on the original graph*. Therefore, when the slopes are positive on the original graph, the y-values are positive on the graph of the derivative. This relationship holds between the original function and the first derivative as well as the first derivative and the second derivative.
- Point out the difference between *speed* and *velocity*, as defined on page 101.
- In the solution for **Example 2, part d)**, remind students that there are two possible values for t, when solving $t^2 = 18.37$. In this case, the restriction on t makes the negative value inadmissible.
- In the solution for **Example 4**, have students draw or visualize the tangent lines drawn to the curve over the given intervals. Ask students if the slopes are positive, negative, increasing, or decreasing. Students may need to be reminded that when the slopes are positive and decreasing (as from A to B), the acceleration is negative and when the slopes are negative and increasing

(as from C to D), the acceleration is positive. Use the analogy of pedalling a bicycle along a hill that has the shape of the graph given in this Example.

- To provide additional challenge, ask students to consider if the motorcycle is speeding up or slowing down for the intervals in the chart.

- As students consider the **Communicate Your Understanding** questions, draw out the difference between *acceleration*, *deceleration*, *speeding up*, and *slowing down*.

- For students with visual acuity or motor dexterity difficulties, provide **BLM 2–5 Section 2.3 Question 6** and **BLM 2–6 Section 2.4 Question 7** for **questions 6** and 7.
- For **question 8**, ask students to visualize tangents along different points on each curve. They should ask themselves if the slopes of the tangents are positive or negative, increasing or decreasing. In particular, for **part c**), the slopes of the tangents are negative and decreasing (even though the tangents are becoming steeper along the curve from left to right).
- **Question 13** addresses several of the mathematical processes. In particular, students are asked to select the appropriate tools, to solve the problem posed, connect to the previous work covered, and reflect on the situation given.
  - Look for appropriate form in **part a**).
  - Look for calculations that are correct and clear.
  - Students should see the connections between velocity and acceleration.
- Students should analyse the graphs of the position, velocity, and acceleration functions for **question 16**. Point out to students that $s(t)$ represents the *position* or *location* of the particle at a particular time t and not the total displacement. Give an example of a line diagram to help them with their sketch in **part e**).
- For additional practice, provide **BLM 2–7 Section 2.3 Practice**.

Investigate Answers (page 97–99)

Investigate A: Method 1

1. a) $\frac{dy}{dx} = 3x^2$ **b)** $\frac{d^2y}{dx^2} = 6x$

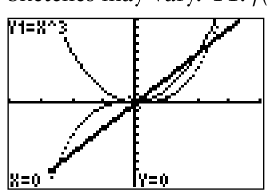
c) Answers may vary. For example: The result in part b) is a linear function. The original function is a cubic function.

d) Answers may vary. For example: It makes sense to call the result in part b) a second derivative, since it is found by differentiating the original function to get the first derivative and then differentiating the first derivative to get the second derivative.

Method 2

1. a) Sketches may vary. **Y1**: $f(x) = x^3$; **Y2**: $f(x) = 3x^2$; **Y3**: $f(x) = 6x$

b)



Answers may vary. For example: Yes, the graphs that were predicted are accurate.

c) Answers may vary. For example: The original function is a cubic function that goes through the point (0, 0). The first derivative of the cubic function is a quadratic function that has a vertex of (0, 0). The second derivative of the cubic function is a linear function that goes through the point (0, 0).

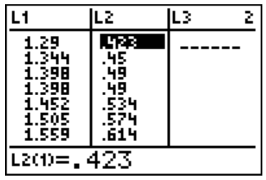
2. a) The equation of **Y2** is $y = 3x^2$. The equation of **Y3** is $y = 6x$.

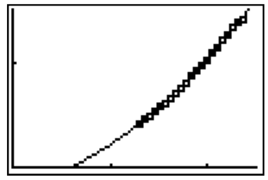
b) Answers may vary. For example: Yes. It is possible to differentiate a derivative. It makes sense to call **Y3** a second derivative since it is found by differentiating the original function to get the first derivative then differentiating the first derivative to get the second derivative.

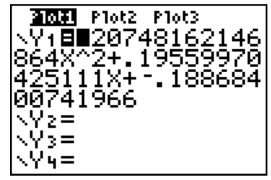
Investigate B

Answers may vary. Sample data for a ball rolling down the ramp is shown.

4.







The displacement-time graph models a quadratic function.

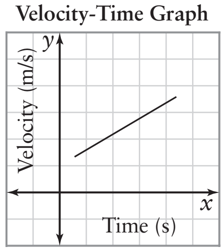
5. a) The ball was closest to the CBR™ at 1.29 s. This represents the point when the ball started to roll down the ramp. For a ball rolling up the ramp, the time may vary. This represents the point when the ball is closest to the CBR™ and will start to move down the ramp.

b) The displacement was increasing on the interval $t \in [1.29, 2.15]$, t in seconds. Answers may vary for the interval when the displacement was decreasing.

c) Answers may vary. For example: The ball was rolling away from the CBR™ during the interval when the displacement was increasing. The ball was rolling toward the CBR™ during the interval when the displacement was decreasing.

6. a) The sketch of the graph of the prediction for the corresponding velocity-time graph for a ball rolling down the ramp is shown.

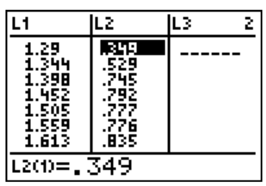
Velocity-Time Graph

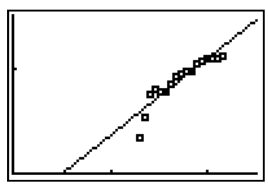


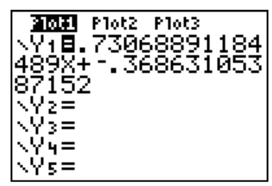
A graph for the corresponding velocity time graph for a ball rolling up the ramp with have the same shape but in the opposite direction.

Answers may vary. For example: The displacement-time graph is a quadratic function. Velocity is the first derivative of displacement. The velocity-time graph is predicted to be a linear function.

b)







Window variables: $x \in [0, 2.5]$, $y \in [0, 1.5]$. The velocity-time graph models a linear function.

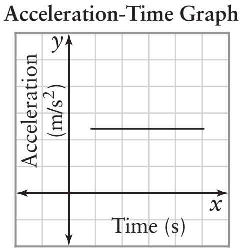
c) Answers may vary. For example: The time found in question 5 part a) represents the time when the ball started to roll down the ramp.

d) Answers may vary. For example: The intervals when the displacement was increasing is reflected on the velocity-time graph as the time when the velocity was increasing as it rolled down the ramp, and then the velocity was decreasing as the ball rolled to a stop.

e) Answers may vary. For example: The rate of change of distance is the velocity. The distance-time graph is a quadratic function and the velocity-time graph is a linear function. The rate of change or first derivative of a quadratic function is a linear function.

7. a) Sketch of the graph of the prediction for the corresponding acceleration-time graph for a ball rolling down a ramp is shown.

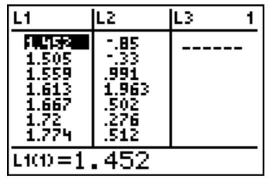
Acceleration-Time Graph

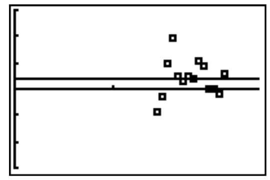


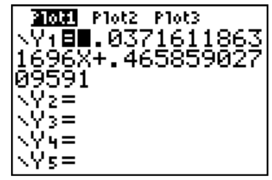
A sketch of a graph of the prediction for the corresponding acceleration-time graph for a ball rolling up a ramp will be the same shape and direction as the graph for a ball rolling down a ramp. The height of the graph above the *x*-axis may differ.

Answers may vary. For example: The velocity-time graph is predicted to be a linear function. Acceleration is the first derivative of displacement. The acceleration-time graph is predicted to be a constant function.

b)







The acceleration-time graph models a constant function.

c) Answers may vary. For example: The acceleration was positive when the ball was rolling down the ramp on the time interval $t \in [1.29, 2.258]$, t in seconds, and was negative when it started to slow down at the bottom of the ramp and stop.

Time intervals may vary for a ball rolling up a ramp but the acceleration will be positive going up the ramp and negative as it slows down as it approaches the CBR™.

d) Answers may vary. For example: Acceleration is the rate of change of velocity. The velocity is a linear function.

8. Answers may vary. For example: The rate of change of velocity is the acceleration. The velocity-time graph is a linear function and the acceleration-time graph is a constant function. The rate of change or first derivative of a linear function is a constant function.

Communicate Your Understanding Responses (page 106)

C1 Answers may vary. For example: An object is speeding up when it is accelerating. This occurs when the second derivative of the position function is positive. An object is slowing down when it is decelerating. This occurs when the second derivative of the position function is negative.

C2 Answers may vary. For example: When the position graph of an object is increasing, the object has positive velocity. When the position graph is decreasing, the object has negative velocity.

C3 Answers may vary. For example: Velocity and speed are both a measure of how fast an object is moving. The difference is that velocity has a direction while speed does not.

C4 Answers may vary. For example: $a(t)$ is one degree less than $v(t)$ which is one degree less than $s(t)$.

Mathematical Process Expectations

Process Expectation	Selected Questions
Problem Solving	1, 2, 13–19
Reasoning and Proving	5, 8, 9, 11, 14–19
Reflecting	5, 6, 8, 9, 12, 13, 15, 17
Selecting Tools and Computational Strategies	1–4, 6, 12–17
Connecting	3, 4, 6–13, 15, 17
Representing	7, 15, 16
Communicating	8, 9, 15

Differentiated Instruction

- Use a **Frayer model** to summarize the key concepts.
- Use a **journal entry**. Provide this question: "How are displacement, velocity, and acceleration related?"

Common Errors

- Students use the derivative to determine average velocity.

R_x Remind students that average velocity does not require the derivative but instantaneous velocity does. Average velocity is calculated over an interval of time and is found using the slope formula.

- Students interpret positive acceleration as meaning that the object is speeding up and negative acceleration as meaning that the object is slowing down.

R_x Have students refer to the chart solution for **Example 3** to understand that the sign of the *product of velocity and acceleration* determines if the object is speeding up or slowing down.

2.4 The Chain Rule

Student Text Pages
111–119

Suggested Timing
75 min

Tools
- graphing calculator (optional)

Related Resources
- BLM 2–8 Section 2.4 Question 2
- BLM 2–9 Section 2.4 Practice
- BLM 2–10 Section 2.4 Achievement Check Rubric

Teaching Suggestions

- Allow 10 min for students to complete the **Investigate**. They may work individually or in pairs.
- Refer to **Prerequisite Skills, questions 12 to 14** for this section.

- For **Example 3**, remind students to be prepared to combine the chain rule with other rules learned up to this point. As you present the example, discuss with students the extent to which you want them to simplify their answers.
- Students could also use *The Geometer's Sketchpad*® to check the equation of the derivative function, although it sometimes displays results in an unexpected format.
- **Example 4** involves mathematical modelling. Students may have some difficulty understanding how to develop the equation for $s(t)$ from the given information.

- As students consider the **Communicate Your Understanding** questions, draw out that the power of a function rule is a special form of the chain rule that applies to functions raised to a power. In later chapters, the chain rule will be used to differentiate other types of functions, such as trigonometric and logarithmic functions, which may or may not include powers.

- Provide **BLM 2–8 Section 2.4 Question 2** to save time in class when completing the table for **question 2**.
- Solutions to **questions 9 to 12** may be verified using technology. For question 12, students must combine the product and chain rules.
- **Question 15** addresses several of the mathematical processes. In particular, students are asked to connect the various representations, reflect on the cube, and then communicate effectively.
  - Look for a solution that is clear and complete.
  - Students should articulate their thoughts well.
- The purpose of **question 16** is to have students think about a method for differentiating quotients.
- For **question 18**, students must combine the product and chain rules.
- **Questions 19** and **20** allow students to focus on the definition of the chain rule and to decide what given information is required for the solution.
- For additional practice, provide **BLM 2–9 Section 2.4 Practice**.

Investigate Answers (page 111–112)

1. a) $f(x) = 2x$

b) $f'(x) = 2$

2. a) $g'(x) = \dfrac{1}{3x^{\frac{2}{3}}}$ or $g'(x) = \dfrac{1}{3}x^{-\frac{2}{3}}$

b) $g'[h(x)] = \dfrac{1}{3}(8x^3)^{-\frac{2}{3}}$

c) Answers may vary. For example: It is appropriate to refer to the expression $g'[h(x)]$ as a composite function since it is composed of the functions $g'(x)$ where the x value has been replaced by the function $h(x)$.

d) $h' = 24x^2$

DIFFERENTIATED INSTRUCTION

- Use the **think aloud** strategy.

COMMON ERRORS

- Students differentiate the outer and inner functions at the same time.

R_x Compare the chain rule to opening a wrapped gift box. It is necessary to first remove the wrapping paper (outer function) before opening the box to see what is inside (inner function). So the inner function does not change as the outer function is being differentiated.

- Students have difficulty combining the chain rule with the product rule.

R_x Point out to students that when the given function is of the form *f*(*x*)*g*(*x*), begin with the product rule and then apply the chain rule to differentiate *f*(*x*) and *g*(*x*) within the definition of the product rule.

3. a) $g'[h(x)] \times h'(x) = \frac{1}{3}(8x^3)^{-\frac{2}{3}} \times 24x^2$

b) 2

c) Answers may vary. For example: The derivative result from question 1, part b), is the same as the derivative result from question 3, part b).

4. a) $f(x) = (g \cdot h)(x)$; $f'(x) = g'[h(x)] \times h'(x)$

b) $f'(x) =$ derivative of the "outer function (inner function)" times the derivative of the "inner function"

c) $f'(x) = 2(2x^3 - 5)(6x^2)$

d) $f(x) = (2x^3 - 5)^2$
$f(x) = (2x^3 - 5)(2x^3 - 5)$
$f'(x) = (2x^3 - 5)(6x^2) + (2x^3 - 5)(6x^2)$
$f'(x) = 2(2x^3 - 5)(6x^2)$

Communicate Your Understanding Responses (page 117)

C1 Answers may vary. For example:
$y = (x^2 + 1)^2$
$= f(g(x))$
$y' = 2(x^2 + 1)(2x)$
$= f'(x)g'(x)$

C2 Answers may vary. For example: Yes, the product rule can be used to verify the chain rule by using a function with equal roots. For example: $y = (x + 1)(x + 1)$ and $y = (x + 1)^2$. The derivative of the first expression using the product rule is the same as the derivative of the second expression using the chain rule. The derivative using both methods is $\frac{dy}{dx} = 2(x + 1)$.

C3 Answers may vary. For example: No, the power of the function is just the chain rule with the inside function equal to the same function as the inside function in the chain rule.

C4 Answers may vary. For example: The multiplication operation creates the "chain" in the chain rule.

Mathematical Process Expectations

Process Expectation	Selected Questions
Problem Solving	10, 12, 13, 16
Reasoning and Proving	12, 20, 22
Reflecting	12, 14, 16, 17
Selecting Tools and Computational Strategies	4–12, 15–17, 21, 23
Connecting	1–3, 7–16, 18, 19, 22, 23
Representing	4, 5, 12, 15, 23
Communicating	12, 14, 16, 17

ONGOING ASSESSMENT

Achievement Check, question 17, on student text page 119

Achievement Check, question 17, student text page 119

This performance task is designed to assess the specific expectations covered in sections 2.1 to 2.4. Refer to **BLM 2–10 Section 2.4 Achievement Check Rubric** for evaluating student responses.

The following Math Process Expectations can be assessed.

Achievement Chart Category	Related Math Processes
Knowledge and Understanding	Selecting tools and computational strategies
Thinking	Problem solving Reasoning and proving Reflecting
Application	Selecting tools and computational strategies Connecting
Communication	Communicating Representing

Sample Solution

17. a) $p'(t) = \dfrac{105 + 44t}{\sqrt{210t + 44t^2}}$

$p'(2) \doteq 7.91$

The rate of growth after 2 years is approximately 8 squirrels per year.

b)
$$\begin{aligned} p(t) &= 60 \\ \sqrt{210t + 44t^2} &= 60 \\ 210t + 44t^2 &= 3600 \end{aligned}$$

Using the quadratic formula, $t = 6.9684$ years. The population will reach 60 squirrels in 7 years.

c)
$$\begin{aligned} p'(7) &= \frac{1}{4}\sqrt{753} \\ &\doteq 6.86 \end{aligned}$$

The rate of change is approximately 7 squirrels per year at that time.

d)
$$\begin{aligned} p'(t) &= 7 \\ \frac{105 + 44t}{\sqrt{210t + 44t^2}} &= 7 \\ (105 + 44t)^2 &= 49(210t + 44t^2) \\ 220t^2 + 1025t - 11\,025 &= 0 \end{aligned}$$

Using the quadratic formula, $t \doteq 5.084$. At approximately 5 years, the rate of change of the population is approximately 7 squirrels per year.

2.5 Derivatives of Quotients

Student Text Pages
120–126

Suggested Timing
75 min

Tools
- graphing calculator (optional)

Related Resources
- BLM 2–11 Section 2.5 Practice

Teaching Suggestions

- In this lesson, quotients are to be differentiated using the product rule. This requires expressing the quotient as a product by using a negative exponent for the denominator.
- Refer to **Prerequisite Skills**, **questions 5** and **6**.

- As students consider the **Communicate Your Understanding** questions, draw out that when the numerator of a quotient is a constant, the power of a function rule is a better tool for differentiating than the product rule. Have students discuss why the derivative of a quotient is *not* equal to the quotient of the derivatives.

- For **question 1**, the use of the power of a function rule is more appropriate whereas for **question 3**, the use of the product rule is more appropriate.
- Point out that although the derivatives in **question 5** need to be simplified, it is not necessary to do so for **question 6**.
- The function in **question 9** should be expressed as a quotient of powers.
- **Question 13** addresses all of the mathematical processes. Students are asked to reason, to solve the problem, represent the derivative, select the appropriate tools, reflect on the relationship between the values, connect, and communicate effectively.
  - Look for answers that are complete and clear.
  - Students should justify their answer to **part c**) in a variety of ways.
  - Students' interpretations should be clear and concise.
- The results of section 2.2, question 18 apply to question 13.
- Have students express the function in **question 14** as a power with a negative exponent.
- For **questions 16**, remind students to simplify the resulting composite functions before taking the derivative.
- For additional practice, provide **BLM 2–11 Section 2.5 Practice**.

Communicate Your Understanding Responses (page 124)

C1 Answers may vary. For example: The derivative of $y = \frac{1}{x^2 + 1}$ is $\frac{dy}{dx} = \frac{-2x}{(x^2 + 1)^2}$. The derivative of $y = \frac{x}{x^2 + 1}$ is $\frac{dy}{dx} = \frac{1 - 2x^2}{(x^2 + 1)^2}$. Both derivatives have the same denominator. Both derivatives have different denominators.

C2 Answers may vary. For example: $\frac{g(x) - g'(x)}{[g(x)]^2}$.

C3 Answers may vary. For example: This statement is false. The derivative of $y = \frac{x + 1}{x^2}$ is $\frac{dy}{dx} = \frac{-x^2 - 2x}{x^4}$. If determined according to the statement in question C3, the answer would be $\frac{1}{2x}$ which is obviously not the same result. The rule is if $q(x) = \frac{f(x)}{g(x)}$, then $q'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$.

C4 The functions $f(x)$ and $g(x)$ have to be differentiable functions otherwise you cannot differentiate $q(x)$. Also $g(x) \neq 0$ because division by zero is undefined in the real number system.

Differentiated Instruction

- Use these strategies: **think-pair-share** and **think aloud**.
- Use a **graphic organizer decision tree** to organize the derivative rules and list when to use them.

Common Errors

- Students have difficulty simplifying the resulting derivatives.

Rx When presenting **Examples 1** and **2**, be sure to point out how to simplify by common factoring. Remind students that common factoring involves division and so exponents are subtracted. Refer to **Prerequisite Skills, question 11**.

- Students incorrectly differentiate a quotient by differentiating the numerator and the denominator of the quotient separately.

Rx Refer students to **Examples 1** and **2**. Remind them to write each quotient as a product and then apply the product rule. Have students write out the definition of the product rule for $p(x) = f(x)g(x)$ as part of their derivative solution.

Mathematical Process Expectations

Process Expectation	Selected Questions
Problem Solving	10–13
Reasoning and Proving	7, 11, 13
Reflecting	10–13, 15
Selecting Tools and Computational Strategies	2, 4–9, 11, 13, 15, 16
Connecting	6–14
Representing	1, 3, 11, 15
Communicating	7, 11–13, 15

Extension

Student Text Page
127–129

Suggested Timing
30 min

Tools
- graphing calculator (optional)

The Quotient Rule

Teaching Suggestions

- Allow 10 to 15 min for students complete the **Investigate** as they work in small groups.
- Have students apply the quotient rule to **Examples 1** and **2** in **section 2.5**, and then compare their answers. Ask students to compare the methods of section 2.5 and the quotient rule. Have them discuss which method they prefer.
- Give students the option of using the product rule or the quotient rule to differentiate quotients.
- Discuss with students the extent to which you would like them to simplify their answers.
- Some students mix up the numerator when applying the quotient rule. Emphasize that the order of the derivative in the numerator matters and that it is necessary to first differentiate the numerator.
- When teaching the quotient rule, some students may find it helpful to call the numerator T for the "top" and the denominator B for the "bottom". Then the rule becomes *The bottom times the derivative of the top minus the top times the derivative of the bottom over the bottom squared*. The following memory aid may be helpful: Let $Q = \frac{T}{B}$ then $Q' = \frac{BT' - TB'}{B^2}$.

Investigate Answers (page 127–129)

Part A

1. $Q(x)g(x) = \frac{f(x)}{g(x)}g(x)$ $\qquad Q(x)g(x) = f(x)$

2. $Q'(x)g(x) + g'(x)Q(x) = f'(x)$

3. $Q'(x)g(x) = f'(x) - g'(x)Q(x)$

4. $Q'(x)g(x) = f'(x) - g'(x)\frac{f(x)}{g(x)}$

5. $Q'(x)g(x) = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)}$

6. $Q'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}$

7. $Q(x) = \frac{f(x)}{g(x)}$ $\qquad Q(x) = f(x)[g(x)]^{-1}$

$Q'(x) = f(x)(-1)[g(x)]^{-2}g'(x) + f'(x)[g(x)]^{-1}$

$Q'(x) = [g(x)]^{-2}[-f(x)g(x) + f'(x)g(x)]$

$Q'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

8. Answers may vary. For example: The results of step 6 and step 7 are the same. A quotient of two functions can be differentiated using the quotient rule or the product rule if the two functions are rearranged into a product of two functions.

Part B

1. Consider the function $Q(x) = \frac{x^3 - 4x^2}{3x^2 + x}$. Using the quotient rule to differentiate the function: $Q'(x) = \frac{(3x^2 + x)(3x^2 - 8x) - (x^3 - 4x^2)(6x + 1)}{(3x^2 + x)^2}$.

2. Using the product rule to differentiate the function:
$Q'(x) = \frac{(3x^2 + x)(3x^2 - 8x) - (x^3 - 4x^2)(6x + 1)}{(3x^2 + x)^2}$

3. **a)** The results for steps 1 and 2 are the same.
b) Answers may vary. For example: I prefer the method in step 1 because it has fewer parts to work with.

2.6 Rate of Change Problems

Student Text Pages
130–141

Suggested Timing
75 min

Related Resources
- BLM 2–12 Section 2.6 Practice

Teaching Suggestions

- Discuss the definitions of the business functions and their derivatives on pages 130 and 131. Ask students why the formulas for finding the revenue and profit functions make sense.

- In the solution for **Example 1, part a**) some students may require more details on how to determine equations 1 and 2.
- Refer students to the Technology Appendix for help finding the maxima, minima, and zeros of a function using a graphing calculator.

- As students consider the **Communicate Your Understanding** questions, draw out that the meaning of the word *marginal* and the relationship between actual cost and marginal cost.

- As students complete the practice questions, they must consider which derivative rule(s) are most appropriate for the given function.
- Have students refer to **section 2.3, Example 4** when completing **question 7**.
- **Question 8** addresses several of the mathematical processes. In particular, students are asked to represent the marginal cost, select appropriate tools to determine the marginal cost and marginal profit, connect to the work thus far, and communicate effectively.
  – Look for a marginal cost that is specific.
  – Students should connect the marginal cost to the selling price.
  – Explanations should be clear and concise.
- Students should create a composite of the two functions given in **question 13**.
- For **question 14**, students should simplify the derivative.
- Remind students that some of the variables that make up the functions in **questions 20** and **21** represent constants. Using Leibniz notation helps identify the main variable for the derivative.
- Question 21 provides an interesting application of derivatives. Some students may find the function a bit confusing, so identify the main variable. Point out to students that **part a**) gives the result that they must work towards when finding the derivative.
- For additional practice, provide **BLM 2–12 Section 2.6 Practice**.

Communicate Your Understanding Responses (page 137)

C1 Answers may vary. For example: *Marginal* refers to the rate of change of some quantity per item as a result of creating one extra item.

C2 Answers may vary. For example: The term *price function* is appropriate because the price function determines the demand for an item at a given price.

C3 Answers may vary. For example: Marginal revenue refers to the change in revenue per item if one more item is produced. If it is negative, then the revenue per item drops when more items are made. If it is positive then the revenue per item increases when more items are made.

C4 For large values of x, the marginal cost of when producing x items is approximately equal to the cost of producing one more item, the $(x + 1)$th item.

Differentiated Instruction

- Use the **what-so what double entry** strategy.
- Use a **jig saw** strategy to reinforce the key concepts.
- Use a **wordwall/information wall** to summarize the key concepts.

Common Errors

- Students have difficulty remembering each business function and their definitions.

R_x Students should copy the summaries on pages 130 and 131 into their notebooks. If students identify themselves as a business owner they will find that most of these functions will make sense and can easily be remembered.

- Students have difficulty reading, interpreting, and solving problems.

R_x Ask students to identify an Example that corresponds to the questions they find difficult. They should refer to the steps of the Example solution to solve the question.

Mathematical Process Expectations

Process Expectation	Selected Questions
Problem Solving	1–21
Reasoning and Proving	4, 6, 8–10, 15, 17, 18, 21
Reflecting	6, 8–10, 15–17
Selecting Tools and Computational Strategies	1–10, 15, 17, 21
Connecting	1–3, 5–10, 13–17
Representing	1–3, 8, 14, 17
Communicating	6–10, 13–18

Review

Student Text Pages
142–143

Suggested Timing
75 min

Tools
- graphing calculator
- computer algebra system (CAS)

Related Resources
- BLM 2–2 Derivatives Rules 1
- BLM 2–13 Derivatives Rules 2
- BLM 2–14 Chapter 2 Review

Study Guide

Use the following study guide to direct students who have difficulty with specific questions to appropriate examples to review.

Question	Section(s)	Refer to
1	2.1	Example 3 (pages 79–80)
2	2.1	Example 5 (page 81)
3	2.2	Example 1 (page 90)
4	2.2	Example 2 (page 91)
5	2.3	Example 1 (page 100)
6	2.3	Example 2 (pages 101–103)
7	2.4	Example 4 (page 116)
8	2.4	Example 2 (page 114)
9	2.5	Example 3 (page 122)
10	2.5	Example 1 (page 120) Example 2 (page 121)
11	2.5	Example 3 (page 122)
12	2.6	Example 1 (pages 131–133)
13	2.6	Example 4 (page 135)

Problem Wrap-Up

Student Text Page
143

Suggested Timing
75 min

Tools
- grid paper
- graphing calculator (optional)
- computer algebra system (CAS) (optional)
- computer with *The Geometer's Sketchpad®* (optional)

Related Resources
- BLM 2–15 Chapter 2 Problem Wrap-Up Rubric

Summative Assessment
- Use **BLM 2–15 Chapter 2 Problem Wrap-Up Rubric** to assess student achievement.

Using the Chapter Problem

- Students may work on the chapter problem individually or in pairs. Remind students to keep track of their solutions to the chapter problem questions. Discuss how the chapter problem questions relate to the Chapter Problem Wrap-Up. Students may apply technology to solve the Chapter Problem Wrap-Up.

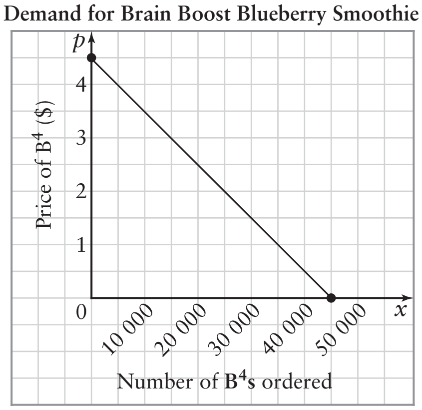
Level 3 Sample Response

a) Write the demand equation in the form $y = mx + b$.

$$p(x) = \frac{45\,000 - x}{10\,000} = \frac{45\,000}{10\,000} - \frac{1}{10\,000}x = 4.5 - 0.0001x$$

The x-intercept of this line is 45 000 and the y-intercept is 4.5. Plot these two points (45 000, 0) and (0, 4.5), or any other two points that satisfy the equation. Draw a line through the points for the following graph.

Demand for Brain Boost Blueberry Smoothie



b) These quantities may be easily read from the grid of the graph. Find the point on the curve that corresponds to $p = \$0.50$ along the vertical axis. Determine the value on the horizontal axis that corresponds to this point. It is 40 000. Therefore, when the price is \$0.50 the number of $\mathbf{B}^4$ orders is 40 000. Similarly, when the price is \$3 the number of $\mathbf{B}^4$ orders is 15 000.

c) These quantities cannot be easily read from the grid of the graph so use the equation for accuracy. Substitute each price into $p(x) = 4.5 - 0.0001x$ and solve for x.

i)
$$\begin{aligned} 2.75 &= 4.5 - 0.0001x \\ -1.75 &= -0.0001x \\ x &= 17\,500 \end{aligned}$$

When the price is \$2.75, the number of $\mathbf{B}^4$ orders is 17 500.

ii)
$$\begin{aligned} 3.90 &= 4.5 - 0.0001x \\ -0.6 &= -0.0001x \\ x &= 6000 \end{aligned}$$

When the price is \$3.90, the number of $\mathbf{B}^4$ orders is 6000.

d)
$$\begin{aligned} R(x) &= xp(x) \\ &= x(4.5 - 0.0001x) \\ &= 4.5x - 0.0001x^2 \end{aligned}$$

$$R'(x) = 4.5 - 0.0002x$$

$$\begin{aligned} R'(20\,000) &= 4.5 - 0.0002(20\,000) \\ &= 0.5 \end{aligned}$$

When 20 000 $\mathbf{B}^4$s are sold per year the rate of increase in revenue is \$0.50 for each additional order.

e) $P(x) = R(x) - C(x)$
$= 4.5x - 0.0001x^2 - (10\ 000 + 0.75x)$
$= -0.0001x^2 + 3.75x - 10\ 000$

$P(15\ 000) = -0.0001(15\ 000)^2 + 3.75(15\ 000) - 10\ 000$
$= 23\ 750$

$P(30\ 000) = -0.0001(30\ 000)^2 + 3.75(30\ 000) - 10\ 000$
$= 12\ 500$

The profit on the sale of 15 000 $\mathbf{B}^4$s is greater than the profit on the sale of 30 000 $\mathbf{B}^4$s by $11 250.

$P'(x) = -0.0002x + 3.75$

$P'(15\ 000) = -0.0002(15\ 000) + 3.75$
$= 0.75$

$P'(30\ 000) = -0.0002(30\ 000) + 3.75$
$= -2.25$

When 15 000 $\mathbf{B}^4$s are ordered, the profit is increasing at a rate of $0.75 per additional order but when 30 000 $\mathbf{B}^4$s are ordered, the profit is decreasing at a rate of $2.25. This implies that the cost of producing 30 000 $\mathbf{B}^4$s is greater than the cost of producing 15 000 $\mathbf{B}^4$s.

Level 3 Notes

- Student's graph is fairly accurate and properly labelled.
- Student provides solutions to most parts of the question.
- Solutions may contain very minor errors.
- Student demonstrates a fairly good understanding of how to determine the marginal revenue, profit, and marginal profit functions.
- Student demonstrates understanding of how to interpret most of the results that correspond to the given values.
- Student demonstrates a fairly good understanding of problem solving techniques.
- Student provides a fairly organized solution and clear justification for most responses.

What Distinguishes Level 2

- Student's graph is partially accurate and is not fully or properly labelled.
- Student provides solutions to some parts of the question.
- Solutions may contain significant errors.
- Student demonstrates some understanding of how to determine the marginal revenue, profit, and marginal profit functions.
- Student demonstrates some understanding of how to interpret the results for some of the given values.
- Student demonstrates some understanding of problem solving techniques.
- Student provides a disorganized solution and some justification for responses.

What Distinguishes Level 4

- Student's graph is accurate and properly labelled.
- Student provides solutions to all parts of the question.
- Solutions do not contain any errors.
- Student demonstrates a high degree of understanding of how to determine the marginal revenue, profit, and marginal profit functions.
- Student demonstrates a clear understanding of how to interpret the results for each given value.
- Student demonstrates a clear understanding of problem solving techniques.
- Student provides a highly organized solution and clear justification for responses.

Practice Test

Student Text Pages
144–145

Suggested Timing
75 min

Tools
- graphing calculator (optional)
- computer algebra system (CAS) (optional)

Related Resources
- BLM 2–16 Chapter 2 Test

Summative Assessment
- You may wish to use **BLM 2–16 Chapter 2 Test** as a summative assessment.

Study Guide

Use the following study guide to direct students who have difficulty with specific questions to appropriate examples to review.

Question	Section(s)	Refer to
1	2.1 2.2 2.5	Derivative Rules (page 76) Key Concepts (page 93) Key Concepts (page 124)
2	2.3	Example 3 (pages 103–105)
3	2.5 Extension	Example 1 (page 120) Example 1 (page 128)
4	2.3	Example 1 (page 100)
5 a)	2.4	Investigate (pages 111–112)
5 b)	2.2	Example 1 (page 90)
6	2.1 2.2 2.4 2.4 2.5	Example 3 (pages 79–80) Example 1 (page 90) Example 1 (pages 112–113) Example 3 (page 115) Example 2 (page 121)
7	2.3	Example 2 (pages 101–103)
8	2.5	Example 3 (page 122)
9	2.1	Example 6 (page 82)
10	2.3	Example 4 (page 105)
11	2.6	Example 1 (page 131–133)
12	2.5	Example 4 (page 123)
13	2.6	Example 1 (page 131–133)
14	2.5	Example 4 (page 123)

Can students do each of the following?

- apply the constant rule, power rule, constant multiple rule, sum and difference rules, product rule, and chain rule to differentiate a given function
- differentiate a quotient
- determine the instantaneous rate of change at a given point on a curve using derivatives
- determine a point or points on a curve at which a given rate of change occurs
- connect and interpret the graphical and algebraic representations of a derivative
- solve problems involving velocity and acceleration
- apply derivative rules to solve a variety of rate problems

Task

Student Text Page
146

Suggested Timing
70 min

Tools
- spherical lollipop on stick
- measuring tape or string and centimetre ruler
- timer with seconds hand
- graphing software (optional)

Related Resources
- BLM 2–17 Chapter 2 Task Rubric

Ongoing Assessment
- Use **BLM 2–17 Task Rubric** to assess student achievement.

The Disappearing Lollipop

Teaching Suggestions

- Read the instructions over with the class. Organize the class into groups of 2 or 3. The task could be assigned as an in-class assignment or as an independent assignment to be completed outside of class. You may wish to provide water and paper towels to clean the measuring tape after each measurement if being performed during class time.

- Student responses are being assessed for the level of mathematical understanding they represent. As you assess each response, consider the following questions:
  - Has the student provided the original radius of the lollipop?
  - Did the student comprehend the given information and instructions?
  - Has the student chosen the appropriate tools and computational strategies?
  - Has the student provided clearly communicated responses to **parts e) to g)**?
  - Did the student determine the rates of change of height using the first derivative of the function?

Level 3 Sample Response

Answers may vary according to the type of lollipop used and the accuracy of the measurements.

a) Original radius is 2 cm.

b)

Time (s)	0	30	60	90	120	150	180	210	240	270	300
Radius of Lollipop (cm)	2	1.93	1.87	1.80	1.72	1.64	1.54	1.43	1.31	1.15	0.94

c) By creating a scatter plot and examining the graph, I can see the relation is not linear. Try quadratic or cubic models for the data.
Quadratic regression: $r = -0.000\,007t^2 - 0.001\,082t + 1.978\,17$
The model would be quadratic if the surface area is decreasing at a steady rate.
Cubic regression: $r = -0.000\,000\,028t^3 + 0.000\,005t^2 - 0.0025t + 2.005\,74$
The model would be cubic if the volume is decreasing at a steady rate.
The rate of change of the radius with respect to time would be:
Using the quadratic regression: $\frac{dr}{dt} = -0.000\,014t - 0.001\,082$
Using the cubic regression: $\frac{dr}{dt} = -0.000\,000\,084t^2 + 0.000\,01t - 0.0025$

d) $V(r) = \frac{4}{3}\pi r^3$
Using the quadratic equation: $V(t) = -1.4 \times 10^{-15}t^6 - 6.7 \times 10^{-13}t^5 + 1.1 \times 10^{-9}t^4 + 3.7 \times 10^{-7}t^3 - 0.0003t^2 - 0.05t + 32.4$
Using the cubic equation:
$V(t) = -9.2 \times 10^{-23}t^9 + 4.9 \times 10^{-20}t^8 - 3.3 \times 10^{-17}t^7 + 2.9 \times 10^{-14}t^6 - 1 \times 10^{-11}t^5 + 4.6 \times 10^{-9}t^4 - 0.000\,002t^3 + 0.000\,41t^2 - 0.126t + 33.8$
These equations are found by substituting the regression equations into the volume equation. They were simplified with the help of a computer algebra system.

e) The radius would be 1 cm at half its original value.
$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$
Using the quadratic regression model: $\frac{dV}{dt} = 4\pi\,(1)(-0.001\,096)$
The rate of change in the volume is approximately $-0.013\,773$ cm^3/s.

Using the cubic regression model: $\frac{dV}{dt} = 4\pi\,(1)(-0.002\ 49)$, the rate of change in the volume is approximately $-0.312\ 91$ cm^3/s.

f) It would be unusual to find that the rate of change of volume with respect to time to be a constant as this is a very crude scientific experiment. It relies on the person consuming the lollipop uniformly for 30-s intervals and also measuring the radius of a sphere. Both of these things are difficult to do with a high degree of accuracy.

g) The quadratic model shows that the lollipop would be completely consumed in approximately 460 s. The cubic model shows total consumption in 401 s. Both of these values were found using graphing software.

h) The rate of change of the volume of the lollipop compared with the surface area would be: $\frac{V'}{SA} = \frac{4\pi r^2 \frac{dr}{dt}}{4\pi r^2}$. Therefore the proportion is $\frac{dr}{dt}$, which is not a constant. Therefore, the experiment does not confirm the hypothesis.

i) When the initial radius is multiplied by a factor of k, and the radius equation is solved, the time to consume the lollipop becomes approximately $\sqrt{k}$ times as many seconds. This was found by multiplying the initial radius by different factors and then analysing the results.

Level 3 Notes

- Student makes an attempt at a high degree of accuracy in the measurements.
- Student finds an appropriate model using scatter plots and regression equations.
- Student demonstrates an understanding of composite functions when expressing the volume as a function of time.
- Student demonstrates an understanding of the chain rule when finding the rate of change of volume with respect to time, including units.
- Student provides reasonable arguments as to whether or not the rate of change of volume would be a constant.
- Student extrapolates from a graph or finds the root of the model for the time when the lollipop would be totally consumed.
- Student provides a reasonable explanation for confirmation (or not) of the hypothesis with some justification.
- Student makes an attempt to change the initial radius and see how that affects the consumption time.

What Distinguishes Level 2

- Student shows some accuracy in the measurements.
- Student needs assistance in finding an appropriate model using scatter plots and regression equations.
- Student uses composite functions when expressing the volume as a function of time with some errors.
- Student uses the chain rule when finding the rate of change of volume with respect to time with some errors.
- Student states whether or not the rate of change of volume would be a constant without justification.
- Student needs assistance when extrapolating from a graph or finding the root of the model for the time when the lollipop would be totally consumed.
- Student provides confirmation (or not) of the hypothesis without justification.
- Student makes no attempt to change the initial radius and see how that affects the consumption time.

What Distinguishes Level 4

- Student shows a high degree of accuracy in the measurements.
- Student finds an appropriate model using scatter plots and regression equations with explanations and justification for the model used.
- Student correctly uses composite functions when expressing the volume as a function of time with some algebraic manipulation.
- Student correctly uses the chain rule when finding the rate of change of volume with respect to time, including correct units.
- Student provides reasonable arguments as to whether or not the rate of change of volume would be a constant.
- Student accurately extrapolates from a graph or finds the root of the model for the time when the lollipop would be totally consumed.
- Student provides a reasonable explanation for confirmation (or not) of the hypothesis with some graphical or algebraic justification.
- Student changes the initial radius to see how that affects the consumption time and giving a general conjecture.