Chapter 6

Geometric Vectors

In physics, the effects of a variety of forces acting in a given situation must be considered. For example, according to Newton's second law of motion, force = mass × acceleration. If all forces act in the same direction, this is a very simple rule. However, real life can be much more complicated. The fall of a skydiver is affected by the force of gravity and the force of air resistance. A ship's course is affected by the speed and direction of the water's current and the wind. The design of a tall building must take into consideration both the forces of earthquakes from below the surface and the velocity of the wind at high elevations. Problems like these can be solved using a mathematical model that involves vectors.

In this chapter, you will investigate vectors, which are quantities with both magnitude and direction.

By the end of this chapter, you will

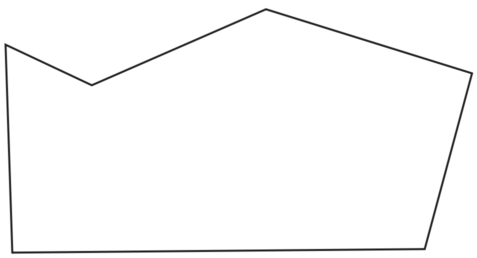
- recognize a vector as a quantity with both magnitude and direction, and identify, gather, and interpret information about real-world applications of vectors
- represent a vector in two-space geometrically as a directed line segment, with directions expressed in different ways (e.g., 320°, N40°W), and algebraically (e.g., using Cartesian coordinates; using polar coordinates), and recognize vectors with the same magnitude and direction but different positions as equal vectors
- perform the operations of addition, subtraction, and scalar multiplication on vectors represented as directed line segments in two-space, and on vectors represented in Cartesian form in two-space and three-space
- determine, through investigation with and without technology, some properties (e.g., commutative, associative, and distributive properties) of the operations of addition, subtraction, and scalar multiplication of vectors
- solve problems involving the addition, subtraction, and scalar multiplication of vectors, including problems arising from real-world applications

Prerequisite Skills

Round lengths and angles to the nearest tenth, if necessary.

Scale Drawings

1. Measure each side and angle, and sketch the polygon using the scale 3 cm represents 1 cm.



2. Decide on an appropriate scale for drawing a scale diagram of each line segment on an eighth of a sheet of letter paper. Draw each scale diagram.

a) 200 km **b)** 50 m

c) 120 cm **d)** 4000 km

Transformations of Angles

3. The initial arm of each angle is on the positive x-axis. Draw each angle. Then, find the measure of the smaller angle between the positive y-axis and the terminal arm.

a) an angle in standard position measuring 50°

b) terminal arm 10° below the positive x-axis

c) terminal arm in the third quadrant and 20° from the negative y-axis

d) an angle in standard position measuring 340°

4. Find the measure of the smaller angle between the positive y-axis and the terminal arm of each angle after a reflection in the origin.

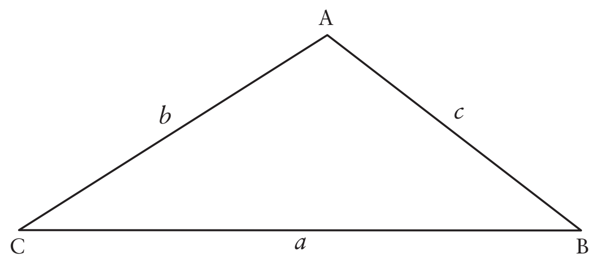
a) an angle in standard position measuring 30°

b) terminal arm 170° clockwise from the positive y-axis

c) terminal arm in the fourth quadrant and 25° from the negative y-axis

d) terminal arm in the second quadrant and 60° from the positive y-axis

Sine and Cosine Laws



Sine Law

$$\frac{\sin(\angle A)}{a} = \frac{\sin(\angle B)}{b} = \frac{\sin(\angle C)}{c}$$

or

$$\frac{a}{\sin(\angle A)} = \frac{b}{\sin(\angle B)} = \frac{c}{\sin(\angle C)}$$

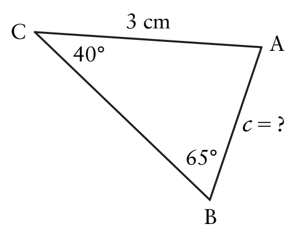
Cosine Law

$$a^2 = b^2 + c^2 - 2bc\cos(\angle A)$$

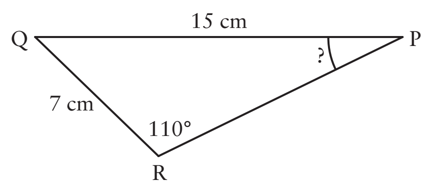
$$b^2 = a^2 + c^2 - 2ac\cos(\angle B)$$

$$c^2 = a^2 + b^2 - 2ab\cos(\angle C)$$

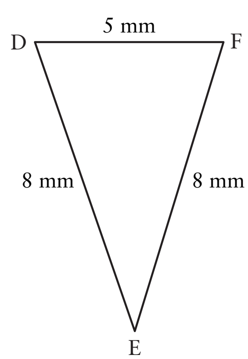
5. a) Use the sine law to find the length of side c.



b) Use the sine law to find the measure of ∠P.



c) Use the cosine law to find the measure of ∠E.

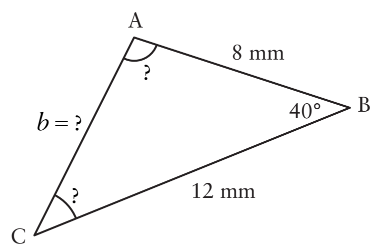


6. a) Given $\triangle$ABC with $b = 10$ cm, $c = 12$ cm, and $\angle$A $= 25°$, use the cosine law to calculate the length of side a.

   b) Given $\triangle$PQR with $p = 7$ m, $q = 6$ m, and $r = 9$ m, use the cosine law to calculate the measure of $\angle$P.

   c) Given $\triangle$DEF with $e = 6.9$ km, $f = 4.0$ km, and $\angle$E $= 120°$, use the sine law to calculate the measure of $\angle$F.

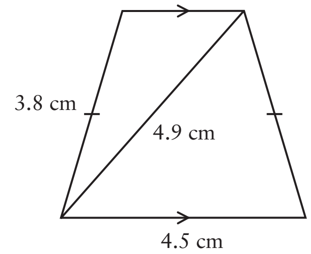
7. Given $\triangle$ABC, find the length of side b and the measures of $\angle$A and $\angle$C.



8. Solve $\triangle$PQR, with $\angle$P $= 40°$, $\angle$Q $= 30°$, and PR $= 10$ cm.

9. A tower that is 200 m tall is leaning to one side. From a certain point on the ground on that side, the angle of elevation to the top of the tower is 70°. From a point 55 m closer to the tower, the angle of elevation is 85°. Determine the acute angle from the horizontal at which the tower is leaning, to one decimal place.

10. Find the interior angles of the isosceles trapezoid, to the nearest tenth of a degree.



Number Properties

11. The properties in the table can be used to simplify expressions, where a, b, and $c \in \mathbb{R}$. Explain each property in your own words. Give a numeric example for each property.

Property	Addition	Multiplication
Commutative	$a + b = b + a$	$a \times b = b \times a$
Associative	$(a + b) + c = a + (b + c)$	$(a \times b) \times c = a \times (b \times c)$
Distributive	$a(b + c) = ab + ac$	

12 Explain how the properties in the table in question 11 can be used to simplify each expression, where a, b, $c \in \mathbb{R}$.

   a) $a + 4 + (-a)$

   b) $3(b + 10)$

   c) $c \times (-4) \times a$

   d) $(a + 2)(a - 2)$

13. Write an expression for which, in order to simplify, you would have to use at least three properties from the table in question 11. Trade expressions with a classmate and simplify. Explain each step.

CHAPTER PROBLEM

In TV shows such as Star Trek Enterprise, you often see the flight officer discussing the Galactic coordinates of the flight vector with the commanding officer. This chapter problem will investigate various situations where vectors are used in aeronautics. The engineers who design an airplane must consider the effects of forces caused by air resistance. Pilots need to consider velocity when flying an airplane. Air traffic controllers must consider the velocity and displacement of aircraft so they do not interfere with each other.

6.1 Introduction to Vectors

Not all physical qualities can be expressed by magnitude alone. Gravitational pull has magnitude, but it also has downward direction. A pilot needs to set both the speed and direction of flight. Police at an accident scene need to consider the momentum of cars of different masses travelling in different directions.

A **scalar** is a quantity that describes magnitude or size only (with or without units). It does not include direction.

A **vector** is a quantity that has both magnitude and direction.

Scalars	Examples	Vectors	Examples
numbers	$1, 3.2, -5, \sqrt{2}$		
temperature	-5°C, 72°F		
area	24 m^2, 15 cm^2		
distance	1 cm, 5.3 km	**displacement**	1 cm at an angle of 30°, 5.3 km north
speed	10 m/s, 80 km/h	**velocity**	10 m/s upward, 80 km/h west
mass	0.5 g, 23 kg	**force**	10 N downward, 35 N to the left

Example 1 Vector or Scalar?

State whether each of the following is an example of a vector or a scalar quantity.

a) a car travelling at 50 km/h to the east

b) a child pulling a wagon with a force of 100 N at 30° to the horizontal

c) a man's mass of 88 kg

d) a woman skiing at a speed of 25 km/h

e) a parachutist falling at 20 km/h downward

f) acceleration due to gravity on Earth of 9.8 m/s^2 downward

g) the number 5

h) your weight on a bathroom scale

Solution

a) The magnitude is 50 km/h, and the direction is east. This is a vector.

b) The magnitude is 100 N, and the direction is 30° to the horizontal. This is a vector.

c) The magnitude is 88 kg, but there is no direction. This is a scalar.

d) The magnitude is 25 km/h, but no direction is given. This is a scalar.

e) The magnitude is 20 km/h, and the direction is downward. This is a vector.

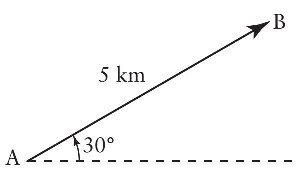
f) The magnitude is 9.8 m/s^2, and the direction is downward. This is a vector.

g) The number 5 has magnitude only, so it is a scalar. It does not matter that it has no units.

h) A scale uses the downward acceleration of gravity to calculate your weight. So, weight on a scale is a vector. Weight is sometimes used as a synonym for force. Your weight, in newtons, is your mass, in kilograms, multiplied by the acceleration due to gravity, which is 9.8 m/s^2 downward on Earth. Although your mass remains constant, your weight would be different on another planet because gravity is different on other planets.

A vector can be represented in several ways:

- In words, for example, as 5 km at an angle of 30° to the horizontal
- In a diagram, as a **geometric vector**, which is a representation of a vector using an arrow diagram, or directed line segment, that shows both magnitude (or size) and direction. The length of the arrow represents, and is proportional to, the vector's magnitude.



- In symbols, using the endpoints of the arrow: $\overrightarrow{AB}$

  Point A is the starting or initial point of the vector (also known as the "tail").

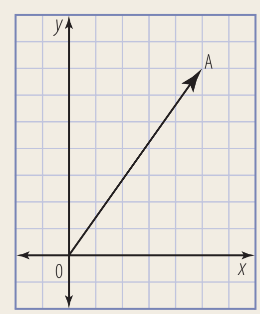
  Point B is the end or terminal point of the vector (also known as the "tip" or "head").

- In symbols, using a single letter: $\vec{v}$

The **magnitude**, or size, of a vector is designated using absolute value brackets. The magnitude of vector $\overrightarrow{AB}$ or $\vec{v}$ is written as $|\overrightarrow{AB}|$ or $|\vec{v}|$.

CONNECTIONS

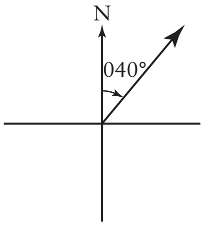
A position vector is a vector whose tail is at the origin, O, of a Cartesian coordinate system. For example, $\overrightarrow{OA}$ is a position vector. It describes the position of the point A relative to the origin. You will make extensive use of this concept in Chapters 7 and 8.



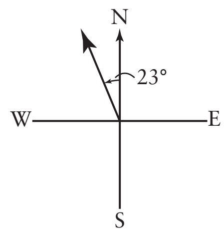
A vector's direction can be expressed using several different methods. In the diagram of $\overrightarrow{AB}$, it is expressed as an angle, moving counterclockwise with respect to a horizontal line. In navigation, vector directions are expressed as bearings.

A **true bearing** (or **azimuth bearing**) is a compass measurement where the angle is measured from north in a clockwise direction.

True bearings are expressed as three-digit numbers, including leading zeros. Thus, north is a bearing of 000°, east is 090°, south is 180°, and west is 270°. For example, a bearing of 040° is an angle of 40° in a clockwise direction from due north. For simplicity, we will use the word *bearing* to refer to a true bearing.



Directions can also be expressed using a **quadrant bearing**, which is a measurement between 0° and 90° east or west of the north-south line. The quadrant bearing N23°W is shown in the diagram.

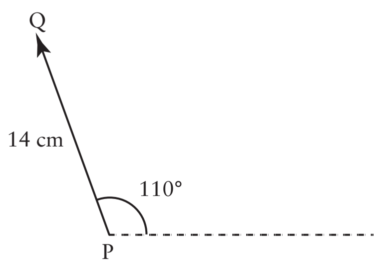


A quadrant bearing always has three components: the direction it is measured from (north in this case), the angle (23°), and the direction toward which it is measured (west). The quadrant bearing N23°W is read as 23° west of north, whereas S20°E is read as 20° east of south. All quadrant bearings are referenced from north or south, not from west or east.

Example 2 Describe Vectors

Describe each vector in words.

a)



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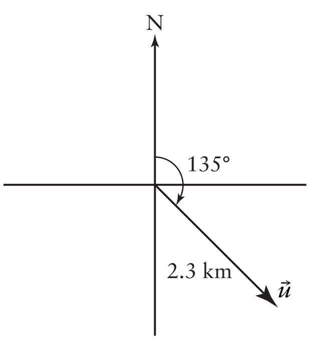
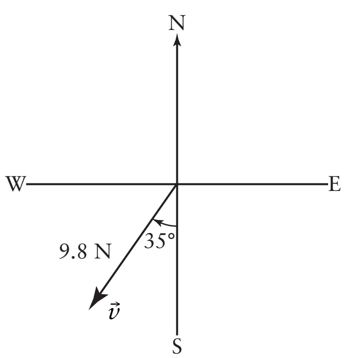


c)



Solution

a) 14 cm at 110° to the horizontal

b) 2.3 km at a true bearing of 135°

c) 9.8 N at a quadrant bearing of S35°W

Example 3 Draw Bearings

Draw a geometric vector with each bearing. Show the scale that you used on each diagram.

a) $\vec{v} = 2$ km at a bearing of 020°

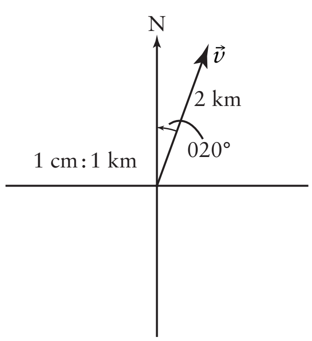
b) $\vec{v} = 4$ km at a bearing of 295°

c) $\vec{u} = 30$ km/h at a quadrant bearing of N40°E

d) $\vec{u} = 40$ km/h at a quadrant bearing of S70°W

Solution

a)



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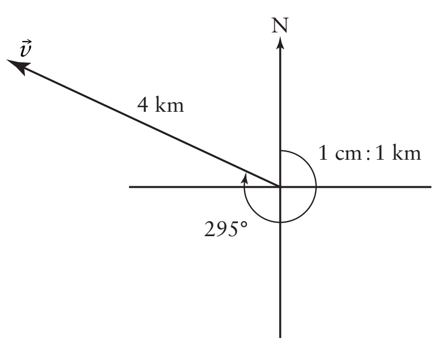
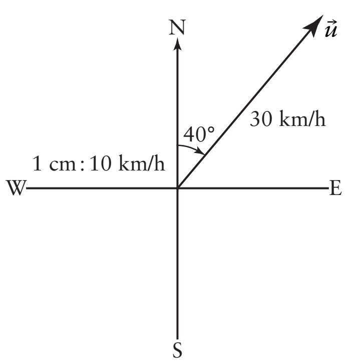
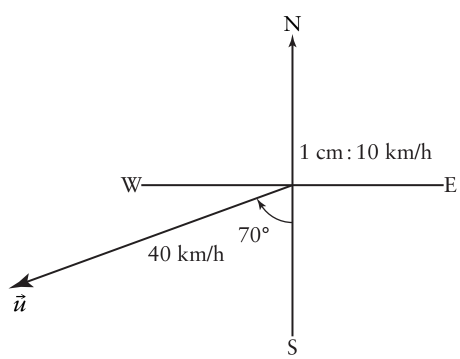


c)



d)



CONNECTIONS

When a compass is installed on an aircraft, it will often give false readings due to magnetic fields generated by radios and other aircraft components. The compass must be tested, and a "compass correction card" is attached to the instrument panel. This process is known as "swinging the compass." True bearings are used universally in aviation, rather than quadrant bearings. When planning a flight, the pilot draws the route and then determines the true track. The local magnetic variation is applied to the true track to obtain the magnetic track. Since the pilot steers the airplane using a magnetic compass, the pilot needs the magnetic track.

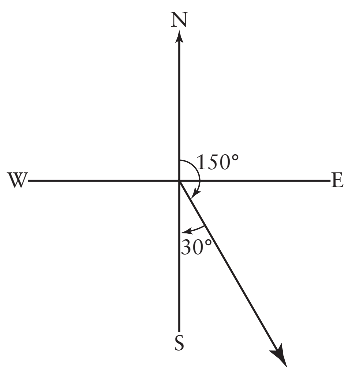
Example 4 Convert Between True Bearings and Quadrant Bearings

a) Write the true bearing 150° as a quadrant bearing.

b) Write the quadrant bearing N50°W as a true bearing.

Solution

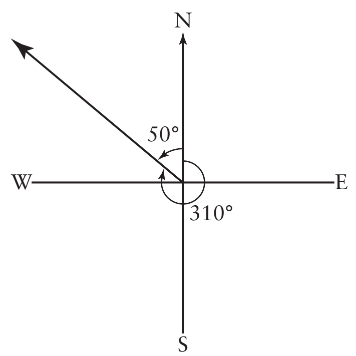
a)



$180° - 150° = 30°$

A bearing of 150° is equivalent to a quadrant bearing of S30°E.

b)

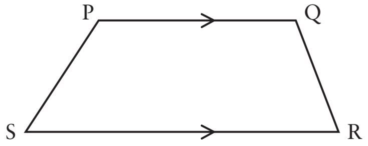


$360° - 50° = 310°$

A quadrant bearing of N50°W is equivalent to a true bearing of 310°.

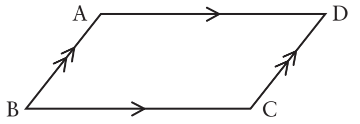
Parallel vectors have the same or opposite direction, but not necessarily the same magnitude.

In trapezoid PQRS, $\overrightarrow{PQ} \parallel \overrightarrow{RS}$ and $\overrightarrow{PQ} \parallel \overrightarrow{SR}$.



Equivalent vectors have the same magnitude and the same direction. The location of the vectors does not matter.

Opposite vectors have the same magnitude but opposite direction. Again, the location of the vectors does not matter. The opposite of a vector $\overrightarrow{AB}$ is written as $-\overrightarrow{AB}$.

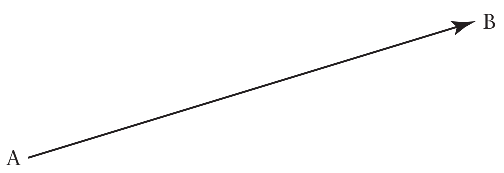


Consider parallelogram ABCD. The table on the next page shows the pairs of equivalent and opposite vectors.

	Vectors	Vector Equation
Equivalent	$\overrightarrow{AB}$ and $\overrightarrow{DC}$	$\overrightarrow{AB} = \overrightarrow{DC}$
	$\overrightarrow{BA}$ and $\overrightarrow{CD}$	$\overrightarrow{BA} = \overrightarrow{CD}$
	$\overrightarrow{AD}$ and $\overrightarrow{BC}$	$\overrightarrow{AD} = \overrightarrow{BC}$
	$\overrightarrow{DA}$ and $\overrightarrow{CB}$	$\overrightarrow{DA} = \overrightarrow{CB}$
Opposite	$\overrightarrow{AB}$ and $\overrightarrow{CD}$	$\overrightarrow{AB} = -\overrightarrow{CD}$
	$\overrightarrow{BA}$ and $\overrightarrow{DC}$	$\overrightarrow{BA} = -\overrightarrow{DC}$
	$\overrightarrow{AD}$ and $\overrightarrow{CB}$	$\overrightarrow{AD} = -\overrightarrow{CB}$
	$\overrightarrow{DA}$ and $\overrightarrow{BC}$	$\overrightarrow{DA} = -\overrightarrow{BC}$

Example 5 Equivalent and Opposite Vectors

a) Draw a vector equivalent to $\overrightarrow{AB}$, labelled $\overrightarrow{EF}$.



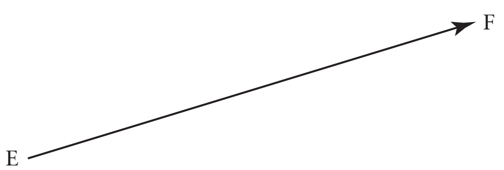
b) Write an expression to show how $\overrightarrow{AB}$ and $\overrightarrow{EF}$ are related.

c) Draw a vector opposite to $\overrightarrow{AB}$, labelled $\overrightarrow{GH}$.

d) Write an expression to show how $\overrightarrow{AB}$ and $\overrightarrow{GH}$ are related.

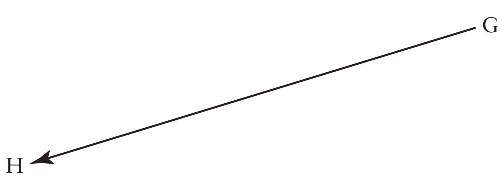
Solution

a)



b) $\overrightarrow{AB} = \overrightarrow{EF}$, because they have the same direction and magnitude.

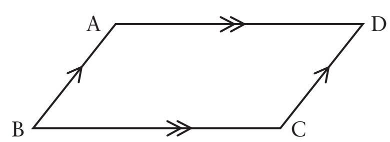
c)



d) $\overrightarrow{AB} = -\overrightarrow{GH}$, because they have the same magnitude but opposite directions.

KEY CONCEPTS

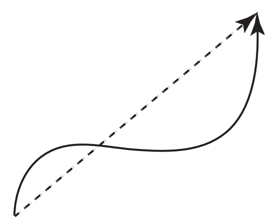
- A vector is a quantity that has both magnitude and direction.
- A scalar is a quantity that describes only magnitude.
- A vector can be represented in words, in a diagram, or in symbols.
- A true bearing (or bearing) is a directed compass measurement, beginning at north and rotating clockwise.
- A quadrant bearing is a compass measurement east or west of the north-south line.
- Equivalent vectors are equal in magnitude and direction. In parallelogram ABCD, $\overrightarrow{AB} = \overrightarrow{DC}$.
- Opposite vectors are equal in magnitude but opposite in direction. In parallelogram ABCD, $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are opposite vectors, and $\overrightarrow{AB} = -\overrightarrow{CD}$.



Communicate Your Understanding

C1 Friction causes an ice skater to slow down. Explain why friction is considered a vector.

C2 The curved arrow shows the path of a cyclist. Which represents the displacement, the curved arrow or the dotted arrow? Explain.



A Practise

1. For which of the following situations would a vector be a suitable mathematical model? Provide a reason for your decision.

a) A boat is travelling at 35 km/h east.

b) A boat is travelling at 10 knots.

c) A line segment of length 6 cm is drawn at 30° to the horizontal.

d) A racecar goes around an oval track at 220 km/h.

e) A baby's mass is 2.9 kg.

f) A box is pushed 10 m across the floor.

g) A chair has a weight of 50 N.

h) A cup of coffee has a temperature of 90°C.

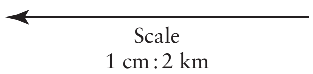
i) A pulley system uses a force of 1000 N to lift a container.

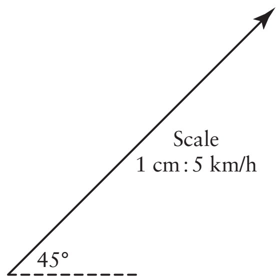
2. State three examples of vectors and three examples of scalars that are different from those in question 1.

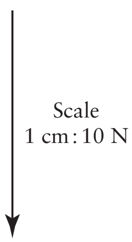
3. Copy and complete the table. Explain your answers.

Quantity	Vector or Scalar?
$\vec{v}$	
$\lvert\vec{v}\rvert$	
6	
$-\overrightarrow{CD}$	
$-\lvert\overrightarrow{AB}\rvert$	
π	
$-\sqrt{7}$	

4. Describe the magnitude and direction of each vector. Describe each vector in words and in symbols.

a) 

b) 

c) 

5. Convert each true bearing to its equivalent quadrant bearing.

a) 070° **b)** 180° **c)** 300°
d) 140° **e)** 210° **f)** 024°

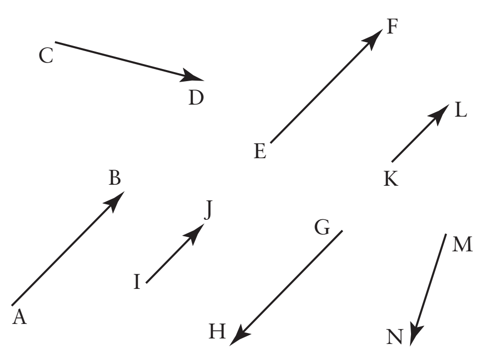
6. Convert each quadrant bearing to its equivalent true bearing.

a) N35°E **b)** N70°W **c)** S10°W
d) S52°E **e)** S18°E **f)** N87°W

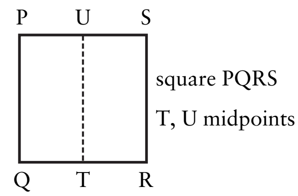
7. **a)** Which vectors are parallel to vector $\overrightarrow{AB}$?

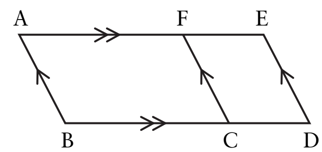
b) Which vectors are equivalent to vector $\overrightarrow{AB}$?

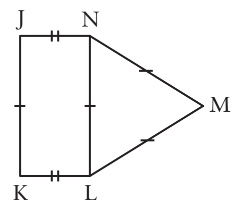
c) Which vectors are opposite to vector $\overrightarrow{AB}$?



8. Name all the equivalent vectors in each diagram.

a) 

b) 

c) 

9. State the opposite of each vector.

a) 200 km east

b) 500 N upward

c) 25 km/h on a bearing of 060°

d) 150 km/h on a quadrant bearing of S50°W

e) $\overrightarrow{AB}$

f) $\vec{v}$

10. Describe a vector that is parallel to each vector in parts a) to d) of question 9.

11. Use an appropriate scale to draw each vector. Label the magnitude, direction, and scale.

a) displacement of 40 m east

b) velocity of 100 km/h at a bearing of 035°

c) force of 5000 N upward

d) acceleration of 10 m/s^2 downward

e) velocity of 50 km/h at a quadrant bearing of S20°E

f) displacement of 2000 miles on a bearing of 250°

g) force of 600 N at 15° to the horizontal

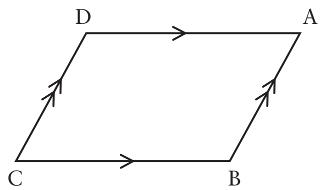
h) two forces of 500 N at 30° to each other

B Connect and Apply

12. The tread on a car's tires is worn down. Which is the most likely cause: distance, speed, displacement, or velocity? Explain.

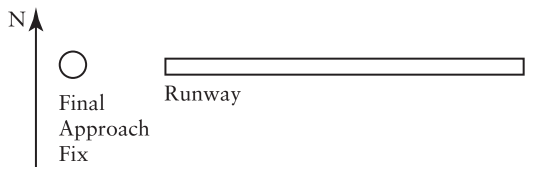
13. Given parallelogram ABCD, what is the relationship between

a) $\overrightarrow{AB}$ and $\overrightarrow{DC}$? **b)** $\overrightarrow{BC}$ and $\overrightarrow{DA}$?



Justify your response.

14. Chapter Problem Air traffic control (ATC) will often assign a pilot a velocity to fly, known as an approach vector, such that the aircraft arrives over a point known as the final approach fix (FAF) at a particular time. From this fix, the pilot turns toward the runway for landing. Suppose that an aircraft is 60 km west and 25 km north of the FAF shown. ATC would like the aircraft to be over the FAF in 10 min. Determine the approach vector to be assigned.



C Extend and Challenge

15. The standard unit of measurement of force is the newton (N). It is the force needed to accelerate a mass of 1 kg at 1 m/s^2. On Earth's surface, a mass of 1 kg requires a force of 9.8 N to counteract the acceleration due to gravity of 9.8 m/s^2 downward. Multiplying the mass by this acceleration gives the weight. On the Moon, the acceleration due to gravity is 1.63 m/s^2 downward.

Reasoning and Proving / Representing / **Selecting Tools** / **Problem Solving** / **Connecting** / **Reflecting** / Communicating

a) A person has a mass of 70 kg. What would this person weigh on Earth? on the Moon?

b) A truck has a mass of 2000 kg. What would it weigh on Earth? on the Moon?

c) When a certain object is floating in water on Earth, 75% of it is submerged. If water were found on the Moon, and the same object was floating in it, how much of it would be submerged?

16. Prove or disprove each statement.

a) If $\vec{a} = \vec{b}$, then $|\vec{a}| = |\vec{b}|$.

b) If $|\vec{a}| = |\vec{b}|$, then $\vec{a} = \vec{b}$.

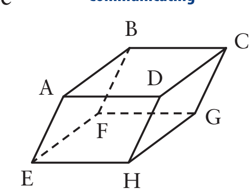
17. The diagram below is a parallelepiped.

Reasoning and Proving / **Representing** / Selecting Tools / Problem Solving / **Connecting** / Reflecting / **Communicating**

a) State one equivalent vector and one opposite vector for each of the following.

i) $\overrightarrow{AB}$ **ii)** $\overrightarrow{ED}$

iii) $\overrightarrow{BD}$ **iv)** $\overrightarrow{FB}$



b) Does $\overrightarrow{AG} = \overrightarrow{CE}$? Explain.

CONNECTIONS

A parallelepiped is a solid whose six faces are parallelograms.

18. Math Contest Which expression is equivalent to the zero vector?

a) $\overrightarrow{QB} + \overrightarrow{YW} + \overrightarrow{BY}$ **b)** $\overrightarrow{CK} - \overrightarrow{KJ} - \overrightarrow{JC}$

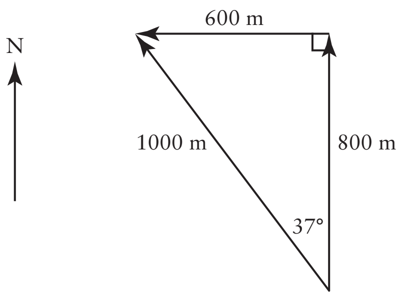
c) $\overrightarrow{EU} - \overrightarrow{EP} - \overrightarrow{PU}$ **d)** $\overrightarrow{KJ} - \overrightarrow{KC} - \overrightarrow{JC}$

19. Math Contest The centroid of a triangle is where the three medians of a triangle meet. $\triangle$DEF has vertices D(1, 3) and E(6, 1) and centroid at C(3, 4). Determine the coordinates of point F.

20. Math Contest Quadrilateral ABCD has vertices at A(13, 9), B(14, 2), C(7, 1), and D(5, 5). Show that ABCD is cyclic; that is, all four points lie on a circle.

6.2 Addition and Subtraction of Vectors

When you add two or more vectors, you are finding a single vector, called the **resultant**, that has the same effect as the original vectors applied one after the other. For example, if you walk north 800 m and then west 600 m, the result is the same as if you walked N37°W for 1000 m. The resultant vector is often represented by the symbol $\vec{R}$.

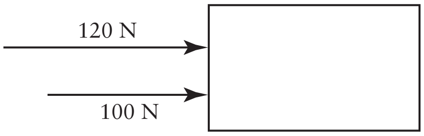


For another example, in a tug-of-war, four people pull to the left with forces of 100 N, 87 N, 95 N, and 102 N. The total force is 100 N + 87 N + 95 N + 102 N = 384 N to the left.

Investigate How can you add vectors?

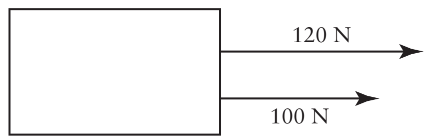
1. Consider two students moving an audiovisual cart in a classroom.

a) If they both push the cart, one with a force of 100 N, and the other with a force of 120 N, in the same direction, describe the magnitude and direction of the total force.



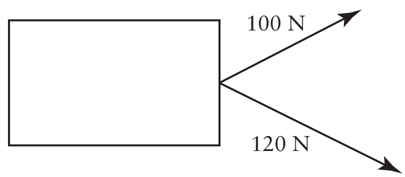
b) If one student pushes the cart and one student pulls the cart in the same direction, using the same forces as in part a), describe the magnitude and direction of the total force.

c) If both students pull the cart in the same direction, using the same forces as in part a), describe the magnitude and direction of the total force.



d) **Reflect** Describe vector addition for parallel forces.

2. If the students each pull the cart, using the same forces as in step 1a), but in different directions, would the cart move as fast as if they were both pulling in the same direction? Explain.



3. Suppose you walk 200 m to the southeast and then walk 300 m to the northeast.

a) Draw a scale diagram to illustrate this situation.

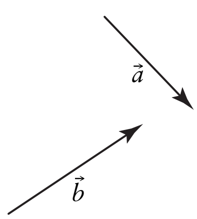
b) How could you determine how far you are from your starting point? In other words, how could you determine your displacement?

c) **Reflect** Explain how this situation is an example of vector addition.

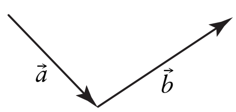
4. **Reflect** If you are given two vectors, $\vec{a}$ and $\vec{b}$, how would you find the resultant $\vec{a} + \vec{b}$?

Vector Addition

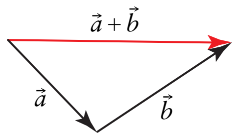
Consider two vectors, $\vec{a}$ and $\vec{b}$.



Think of $\vec{a} + \vec{b}$ as $\vec{a}$ followed by $\vec{b}$. Translate $\vec{b}$ so that the tail of $\vec{b}$ touches the head of $\vec{a}$.



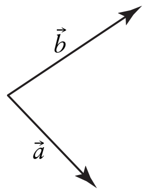
Find the sum by drawing and measuring from the tail of $\vec{a}$ to the head of $\vec{b}$. This new vector is the resultant $\vec{a}+\vec{b}$.



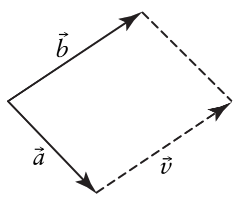
This is called the head-to-tail (or triangle) method.

Another method can be used when the vectors are tail to tail. Consider the same two vectors as above, $\vec{a}$ and $\vec{b}$.

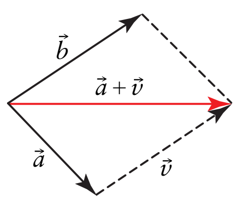
Translate $\vec{b}$ so that the tail of $\vec{b}$ touches the tail of $\vec{a}$.



Complete the parallelogram that has $\vec{a}$ and $\vec{b}$ as two of its sides.



Because of the properties of parallelograms, $\vec{b}$ and $\vec{v}$ are equivalent vectors. Thus, $\vec{a}+\vec{b}=\vec{a}+\vec{v}$. Use the head-to-tail method above to find $\vec{a}+\vec{v}$.



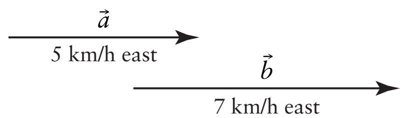
The resultant $\vec{a}+\vec{b}$ is the indicated diagonal of the parallelogram. This is called the tail-to-tail (or parallelogram) method.

CONNECTIONS

Go to *www.mcgrawhill.ca/links/calculus12* and follow the links to **6.2**. Download the file **Vector Addition.gsp**, an applet for adding vectors.

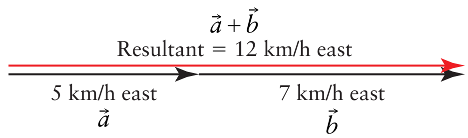
Adding Parallel Vectors

Vectors $\vec{a}$ and $\vec{b}$ are parallel and have the same direction.



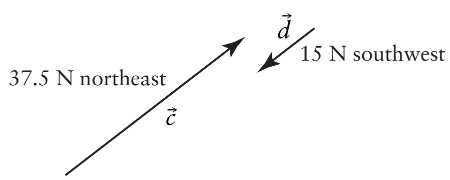
To find $\vec{a} + \vec{b}$, place the tail of $\vec{b}$ at the head of $\vec{a}$.

The resultant is the vector from the tail of the first vector, $\vec{a}$, to the tip of the second vector, $\vec{b}$.

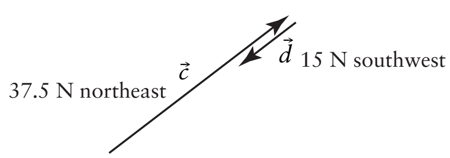


From the diagram, you can see that to add parallel vectors having the same direction, add their magnitudes. The resultant has the same direction as the original vectors.

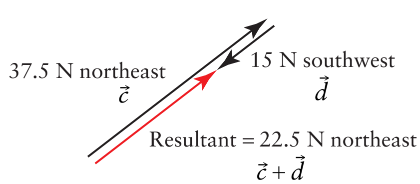
Vectors $\vec{c}$ and $\vec{d}$ are parallel, but have **opposite** directions.



To find $\vec{c} + \vec{d}$, place the tail of $\vec{d}$ at the head of $\vec{c}$.



The resultant is the vector from the tail of $\vec{c}$ to the head of $\vec{d}$.



From the diagram, the magnitude of the resultant is equal to the magnitude of $\vec{c}$ minus the magnitude of $\vec{d}$. The direction of the resultant is the same as the direction of $\vec{c}$, or northeast.

In general, for parallel vectors $\vec{a}$ and $\vec{b}$ having opposite directions, and their resultant, $\vec{R}$,

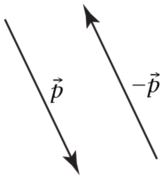
- If $|\vec{a}| > |\vec{b}|$, then $|\vec{R}| = |\vec{a}| - |\vec{b}|$, and $\vec{R}$ has the same direction as $\vec{a}$.
- If $|\vec{b}| > |\vec{a}|$, then $|\vec{R}| = |\vec{b}| - |\vec{a}|$, and $\vec{R}$ has the same direction as $\vec{b}$.

Subtracting Vectors

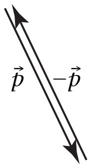
The relationship between addition and subtraction with vectors is similar to the relationship between addition and subtraction with scalars. To subtract $\vec{u} - \vec{v}$, add the opposite of $\vec{v}$ to $\vec{u}$. In other words, $\vec{u} - \vec{v}$ is equivalent to $\vec{u} + (-\vec{v})$, or $\vec{u}$ followed by $-\vec{v}$.

Adding Opposite Vectors and the Zero Vector

When you add two opposite integers, the result is zero. A similar result occurs when you add two opposite vectors. Consider vector $\vec{p}$ and its opposite vector, $-\vec{p}$. They have the same magnitude, but opposite directions.



To add these vectors, place them head to tail.

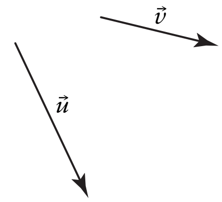


The vector from the tail of $\vec{p}$ to the tip of $-\vec{p}$ has no magnitude. This is the **zero vector**, which is written as $\vec{0}$. It has no specific direction.

Example 1 Add and Subtract Vectors

a) Find $\vec{u} + \vec{v}$.

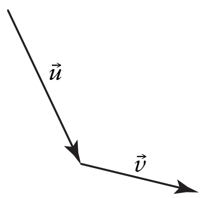
b) Find $\vec{u} - \vec{v}$.



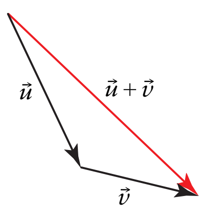
Solution

a) Method 1: ***Use the Head-to-Tail Method***

Translate $\vec{v}$ so that its tail is at the head of $\vec{u}$.

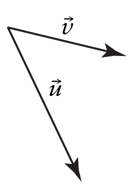


Draw the resultant $\vec{u} + \vec{v}$ from the tail of $\vec{u}$ to the head of $\vec{v}$.

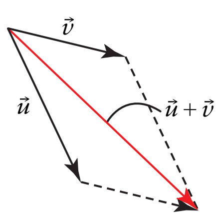


Method 2: ***Use the Parallelogram Method***

Translate $\vec{v}$ so that $\vec{u}$ and $\vec{v}$ are tail to tail.



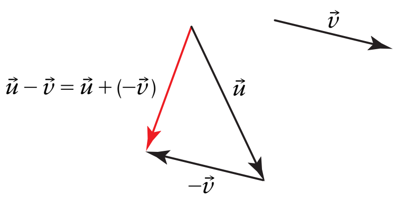
Construct a parallelogram using vectors equivalent to $\vec{u}$ and $\vec{v}$.



The resultant $\vec{u} + \vec{v}$ is the indicated diagonal of the parallelogram.

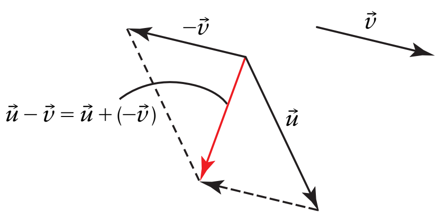
b) **Method 1:** ***Use the Head-to-Tail Method***

Draw the opposite of $\vec{v}$, $-\vec{v}$. Then, translate $-\vec{v}$ so its tail is at the head of $\vec{u}$, and add $\vec{u} + (-\vec{v})$.



Method 2: ***Use the Parallelogram Method***

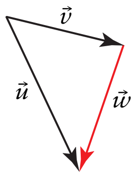
Draw the opposite of $\vec{v}$ and place $\vec{u}$ and $\vec{v}$ tail to tail. Complete the parallelogram, and draw the resultant $\vec{u} + (-\vec{v})$.



The resultant is the indicated diagonal of the parallelogram.

Method 3: ***Use the Tail-to-Tail Method***

Translate $\vec{v}$ so that $\vec{u}$ and $\vec{v}$ are tail to tail and draw a vector from the head of $\vec{v}$ to the head of $\vec{u}$. Call this vector $\vec{w}$.

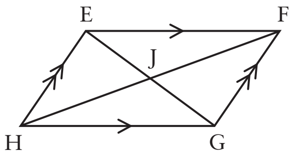


$$\begin{aligned}\vec{w} &= -\vec{v} + \vec{u}\\ &= \vec{u} - \vec{v}\end{aligned}$$

$\vec{u} - \vec{v}$ can be drawn from the head of $\vec{v}$ to the head of $\vec{u}$ when $\vec{u}$ and $\vec{v}$ are placed tail to tail.

Example 2 Vectors in Parallelograms

Consider parallelogram EFGH with diagonals EG and FH that intersect at J.



a) Express each vector as the sum of two other vectors in two ways.

i) $\overrightarrow{HF}$ **ii)** $\overrightarrow{FH}$ **iii)** $\overrightarrow{GJ}$

b) Express each vector as the difference of two other vectors in two ways.

i) $\overrightarrow{HF}$ **ii)** $\overrightarrow{FH}$ **iii)** $\overrightarrow{GJ}$

Solution

There are several possibilities for each vector. Two examples are shown for each.

a) **i)** $\overrightarrow{HF} = \overrightarrow{HE} + \overrightarrow{EF}$ — Use the head-to-tail method.

or

$\overrightarrow{HF} = \overrightarrow{HE} + \overrightarrow{HG}$ — Use the parallelogram method.

ii) $\overrightarrow{FH} = \overrightarrow{FE} + \overrightarrow{EH}$

or

$\overrightarrow{FH} = \overrightarrow{FE} + \overrightarrow{FG}$

iii) $\overrightarrow{GJ} = \overrightarrow{GH} + \overrightarrow{HJ}$

or

$\overrightarrow{GJ} = \overrightarrow{GF} + \overrightarrow{FJ}$

b) **i)** $\overrightarrow{HF} = \overrightarrow{HE} + \overrightarrow{EF}$ — Express $\overrightarrow{HF}$ as a sum, and then convert to subtraction.

$= \overrightarrow{HE} - \overrightarrow{FE}$

or

$\overrightarrow{HF} = \overrightarrow{GF} - \overrightarrow{GH}$ — Subtract from the head of $\overrightarrow{GH}$ to the head of $\overrightarrow{GF}$.

ii) $\overrightarrow{FH} = \overrightarrow{FE} + \overrightarrow{EH}$

$= \overrightarrow{FE} - \overrightarrow{HE}$

or

$\overrightarrow{FH} = \overrightarrow{GH} - \overrightarrow{GF}$

iii) $\overrightarrow{GJ} = \overrightarrow{GH} + \overrightarrow{HJ}$

$= \overrightarrow{GH} - \overrightarrow{JH}$

or

$\overrightarrow{GJ} = \overrightarrow{FJ} - \overrightarrow{FG}$

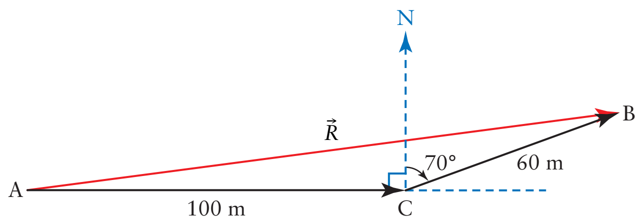
Example 3 Solve a Bearing Problem

In an orienteering race, you walk 100 m due east and then walk N70°E for 60 m. How far are you from your starting position, and at what bearing?

Solution

Method 1: ***Use Paper and Pencil***

Draw a diagram with scale 1 cm : 20 m. Draw a 5-cm arrow to represent the vector 100 m due east. Use a protractor to draw a 3-cm arrow at N70°E with its tail at the head of the first vector. Label the triangle as shown.



Measure the length of the resultant, $\vec{R} = \overrightarrow{AC} + \overrightarrow{CB}$, and then measure $\angle A$.

$|\vec{R}| = 15.8$ cm, which represents 158 m, and $\angle A = 7°$.

$\angle A$ is relative to east. Convert to a quadrant bearing: $90° - 7° = 83°$.

You have travelled about 158 m from your starting position, at a quadrant bearing of about N83°E.

Method 2: ***Use Trigonometry***

Use the cosine law to find the magnitude of $\vec{R}$. From the diagram in Method 1, $\angle ACB = 90° + 70° = 160°$.

$$|\vec{R}|^2 = |\overrightarrow{AC}|^2 + |\overrightarrow{BC}|^2 - 2|\overrightarrow{AC}||\overrightarrow{BC}|\cos(\angle ACB)$$
$$= 100^2 + 60^2 - 2(100)(60)\cos 160°$$
$$|\vec{R}| \doteq 157.7$$

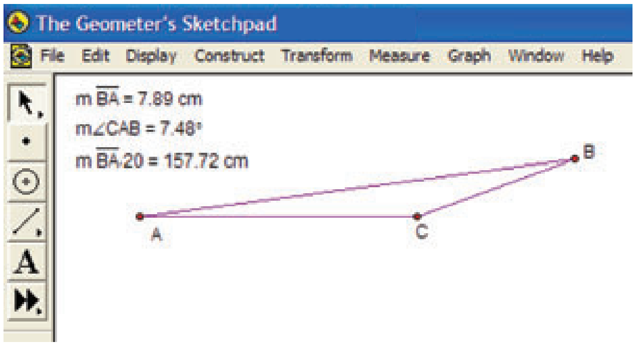
Use the sine law to find $\angle BAC$.

$$\frac{\sin\angle BAC}{|\overrightarrow{BC}|} = \frac{\sin\angle BCA}{|\vec{R}|}$$
$$\sin\angle BAC = \frac{|\overrightarrow{BC}|\sin\angle BCA}{|\vec{R}|}$$
$$= \frac{60\sin 160°}{157.7}$$
$$\angle BAC = \sin^{-1}\left(\frac{60\sin 160°}{157.7}\right)$$
$$\doteq 7.5°$$

$\angle BAC$ is relative to east. Convert to a quadrant bearing: $90° - 7.5° = 82.5°$.

You have travelled about 158 m from your starting position, at a quadrant bearing of about N83°E.

Method 3: *Use* The Geometer's Sketchpad®



1. Choose a scale.
2. Turn on automatic labelling of points. Draw point A.
3. From the **Transform** menu, choose **Translate,** and ensure that **Polar** is selected. Translate A 100 m at an angle of 0° using the scale. Label the translated point C.
4. Translate C 50 m at an angle of 20°. Label the translated point B.
5. Join points to complete the diagram.
6. Measure the magnitude and direction of the resultant.

Investigate Properties of vector addition and subtraction

Method 1: *Use Paper and Pencil*

A: Commutative Property for Vector Addition

Does order matter when you are adding or subtracting two vectors?

1. **a)** Draw any vectors $\vec{u}$ and $\vec{v}$. Translate $\vec{v}$ so that its tail is at the head of $\vec{u}$. Find $\vec{u}+\vec{v}$.

   b) Now translate $\vec{u}$ so that its tail is at the head of $\vec{v}$. Find $\vec{v}+\vec{u}$.

   c) Measure the magnitude and direction of $\vec{u}+\vec{v}$ and of $\vec{v}+\vec{u}$ using a ruler and a protractor. What do you notice?
2. Repeat step 1 for several different pairs of vectors.
3. Draw $\vec{u}-\vec{v}$ and also $\vec{v}-\vec{u}$. How do these two resultants compare?
4. **Reflect** The commutative property for vector addition says that for any vectors $\vec{u}$ and $\vec{v}$, $\vec{u}+\vec{v}=\vec{v}+\vec{u}$. How do your results from steps 1 and 2 demonstrate that this property is true?
5. **Reflect** What can you say about a commutative property for vector subtraction?

B: Associative Property for Vector Addition

How do you add three vectors at a time? What is the meaning of $\vec{p}+\vec{q}+\vec{r}$?

1. a) Draw any vectors $\vec{p}$, $\vec{q}$, and $\vec{r}$. Translate vectors to add $\vec{q}+\vec{r}$, and then add $\vec{p}$.
   b) Now translate vectors to add $\vec{p}+\vec{q}$, and then add $\vec{r}$.
   c) Measure the magnitude and direction of $\vec{p}+(\vec{q}+\vec{r})$ and $(\vec{p}+\vec{q})+\vec{r}$ using a ruler and a protractor. What do you notice?
2. Repeat step 1 for several different groups of three vectors.
3. Does the associative property work for vector subtraction? Is either of the following true? Explain.
   $\vec{p}-\vec{q}-\vec{r}=(\vec{p}-\vec{q})-\vec{r}$ or $\vec{p}-\vec{q}-\vec{r}=\vec{p}-(\vec{q}-\vec{r})$
4. **Reflect** The associative property for vector addition says that for any vectors $\vec{p}$, $\vec{q}$, and $\vec{r}$, $\vec{p}+(\vec{q}+\vec{r})=(\vec{p}-\vec{q})+\vec{r}$. How do your results from steps 1 and 2 demonstrate that this property is true?

Method 2: *Use* The Geometer's Sketchpad®

Tools

- computer with *The Geometer's Sketchpad®*

A: Commutative Property for Vector Addition

1. Go to *www.mcgrawhill.ca/links/calculus12* and follow the links to **6.2**. Download the file **Commutative Addition of Vectors.gsp**. Open the sketch.
2. Look at the black vector. Explain why this vector represents $\vec{u}+\vec{v}$.
3. Explain why the black vector also represents $\vec{v}+\vec{u}$.
4. Does this relation depend on the magnitude and direction of $\vec{u}$ or $\vec{v}$? Drag point A. What happens? Then, drag point B. What happens?
5. **Reflect** What conjecture can you make about the relation between $\vec{u}+\vec{v}$ and $\vec{v}+\vec{u}$? What would happen if the vector operation were subtraction rather than addition?

B: Associative Property for Vector Addition

1. Go to *www.mcgrawhill.ca/links/calculus12* and follow the links to **6.2**. Download the file **Associative Addition of Vectors.gsp**. Open the sketch.
2. Look at the magenta vector. Explain why this vector represents $(\vec{u}+\vec{v})+\vec{w}$.
3. Explain why the magenta vector also represents $\vec{u}+(\vec{v}+\vec{w})$.
4. Does this relation depend on the magnitude and direction of $\vec{u}$, $\vec{v}$, or $\vec{w}$? Drag point A. What happens? Then, drag point B. Note what happens. Finally, drag point C. What happens?
5. **Reflect** What conjecture can you make about the relation between $(\vec{u}+\vec{v})+\vec{w}$ and $\vec{u}+(\vec{v}+\vec{w})$? Which of these expressions can be used to add three vectors at a time?

Just as with integer addition, there is an **identity property** for vector addition. It says that for any vector $\vec{u}$, $\vec{u}+\vec{0}=\vec{u}=\vec{0}+\vec{u}$. This parallels the identity property for scalar addition of integers.

Example 4 Simplify Vector Expressions

Simplify each expression.

a) $(\vec{u}+\vec{v})-\vec{u}$

b) $[(\vec{p}+\vec{q})-\vec{p}]-\vec{q}$

Solution

a)
$$\begin{aligned}(\vec{u}+\vec{v})-\vec{u} &= (\vec{v}+\vec{u})-\vec{u} && \text{Commutative property}\\ &= (\vec{v}+\vec{u})+(-\vec{u}) && \text{Add the opposite.}\\ &= \vec{v}+(\vec{u}+(-\vec{u})) && \text{Associative property}\\ &= \vec{v}+\vec{0} && \text{Opposites add to the zero vector.}\\ &= \vec{v} && \text{Identity property}\end{aligned}$$

b)
$$\begin{aligned}[(\vec{p}+\vec{q})-\vec{p}]-\vec{q} &= [(\vec{q}+\vec{p})-\vec{p}]-\vec{q}\\ &= [(\vec{q}+\vec{p})+(-\vec{p})]-\vec{q}\\ &= (\vec{q}+[\vec{p}+(-\vec{p})])-\vec{q}\\ &= (\vec{q}+\vec{0})-\vec{q}\\ &= \vec{q}-\vec{q}\\ &= \vec{q}+(-\vec{q})\\ &= \vec{0}\end{aligned}$$

KEY CONCEPTS

- Vectors in different locations are equivalent if they have the same magnitude and direction. This allows us to construct diagrams for the addition and subtraction of vectors.
- Think of adding vectors as applying one vector after the other.
- You can add two vectors using the head-to-tail (triangle) method or the parallelogram method.
- If two vectors, $\vec{u}$ and $\vec{v}$, are parallel and in the same direction, then $|\vec{u}+\vec{v}|=|\vec{u}|+|\vec{v}|$, and $\vec{u}+\vec{v}$ is in the same direction as $\vec{u}$ and $\vec{v}$.
- If $\vec{u}$ and $\vec{v}$ have opposite directions and $|\vec{u}|>|\vec{v}|$, then $|\vec{u}+\vec{v}|=|\vec{u}|-|\vec{v}|$ and $\vec{u}+\vec{v}$ is in the same direction as $\vec{u}$.
- Subtract vectors by adding the opposite: $\vec{u}-\vec{v}=\vec{u}+(-\vec{v})$.
- The zero vector, $\vec{0}$, is defined as having zero magnitude and no specific direction. It is the resultant of adding two opposite vectors.
- For any vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$:
  $\vec{u}+\vec{v}=\vec{v}+\vec{u}$ (commutative property)
  $(\vec{u}+\vec{v})+\vec{w}=\vec{u}+(\vec{v}+\vec{w})$ (associative property)
  $\vec{v}+\vec{0}=\vec{v}=\vec{0}+\vec{v}$ (identity property)
- Simplifying vector expressions involving addition and subtraction is similar to simplifying expressions involving integers.

Communicate Your Understanding

C1 Two non-parallel vectors have magnitudes of 5 km/h and 9 km/h. Can the sum of the vectors have a magnitude of 14 km/h? Explain.

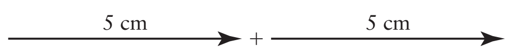
C2 Suppose you are given the resultant and one vector in the addition of two vectors. How would you find the other vector?

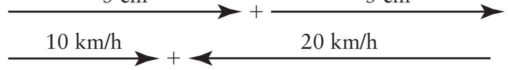
C3 Example 3 described three methods of solving a bearing problem: pencil and paper, trigonometry, and geometry software. Which method is the most accurate? Explain.

C4 Suppose you and a friend run to school using different routes. You run 2 km north and then 1 km west. Your friend runs 1 km west and then 2 km north. How is this an illustration of the commutative property of vector addition?

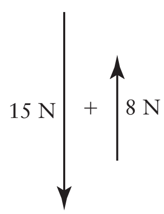
A Practise

1. Draw a diagram to illustrate each vector sum or difference.

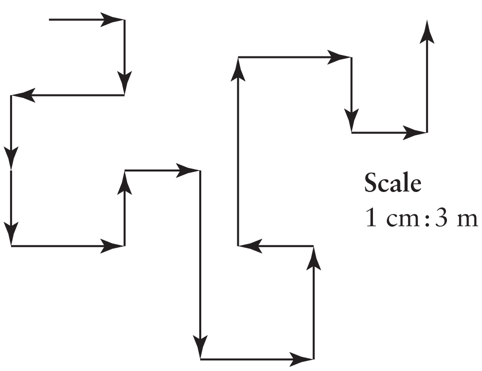
a) 

b) 

c) 

d) 

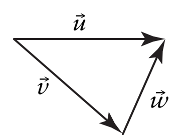
2. The diagram represents the path of an obstacle course.

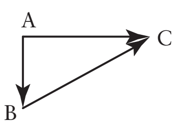


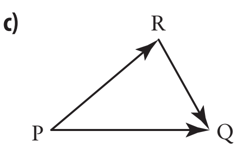
a) Determine the distance travelled and the displacement.

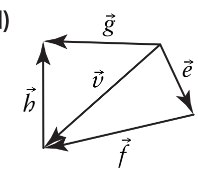
b) Are the distance and the displacement the same or different? Explain.

3. Express the shortest vector in each diagram as the sum or difference of two other vectors.

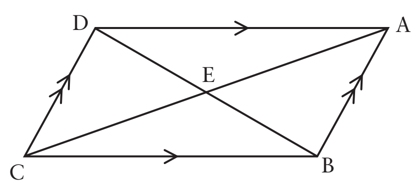
a) 

b) 

c) 

d) 

4. ABCD is a parallelogram, and E is the intersection point of the diagonals AC and BD. Name a single vector equivalent to each expression.



a) $\overrightarrow{AE} + \overrightarrow{EB}$

b) $\overrightarrow{BC} + \overrightarrow{BA}$

c) $\overrightarrow{AE} + \overrightarrow{AE}$

d) $\overrightarrow{AD} + \overrightarrow{AB}$

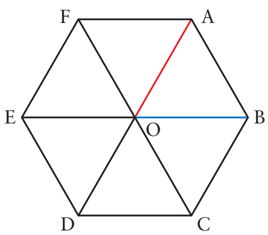
e) $\overrightarrow{BA} + \overrightarrow{AE} + \overrightarrow{ED} + \overrightarrow{DC}$

f) $\overrightarrow{AB} - \overrightarrow{DB}$

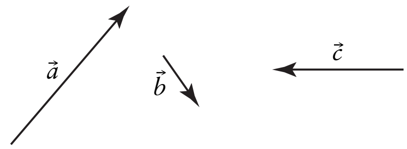
g) $\overrightarrow{AB} - \overrightarrow{CB} - \overrightarrow{DC}$

h) $\overrightarrow{AE} - \overrightarrow{EB} - \overrightarrow{BC}$

5. ABCDEF is a regular hexagon, and O is its centre. Let $\vec{a}=\overrightarrow{OA}$ and $\vec{b}=\overrightarrow{OB}$. Write $\overrightarrow{AB}$, $\overrightarrow{OC}$, $\overrightarrow{CO}$, and $\overrightarrow{AE}$ in terms of $\vec{a}$ and $\vec{b}$.



6. Use the following set of vectors to draw a diagram of each expression.



a) $\vec{a}+\vec{b}+\vec{c}$

b) $\vec{a}+\vec{b}-\vec{c}$

c) $\vec{a}-\vec{b}-\vec{c}$

B Connect and Apply

7. Niki, Jeanette, and Allen are playing a three-way tug-of-war. Three ropes of equal length are tied together. Niki pulls with a force of 210 N, Jeanette pulls with a force of 190 N, and Allen pulls with a force of 200 N. The angles between the ropes are equal.

   a) Draw a scale diagram showing the forces on the knot.

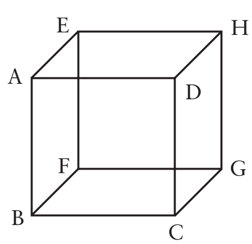
   b) Determine the magnitude and direction of the resultant force on the knot.

8. $\triangle$ABC is an equilateral triangle, with O its centroid.

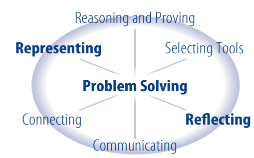
   a) Show that $\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}=\vec{0}$.

   b) Is it also true that $\overrightarrow{AO}+\overrightarrow{BO}+\overrightarrow{CO}=\vec{0}$? Justify your answer.

9. The diagram shows a cube. Let $\vec{u}=\overrightarrow{AB}$, $\vec{v}=\overrightarrow{AD}$, and $\vec{w}=\overrightarrow{AE}$. Express vectors $\overrightarrow{AH}$, $\overrightarrow{DG}$, $\overrightarrow{AG}$, $\overrightarrow{CE}$, and $\overrightarrow{BH}$ in terms of vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$.

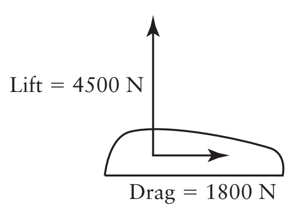


10. **Chapter Problem** The force of the air moving past an airplane's wing can be broken down into two forces: lift and drag. Use an appropriate scale drawing to approximate the resultant force acting on the airplane wing in the diagram. Go to *www.mcgrawhill.ca/links/calculus12* and follow the links to **6.2**. Download the file **Vector Addition.gsp** and use it to check your measurements.



CONNECTIONS

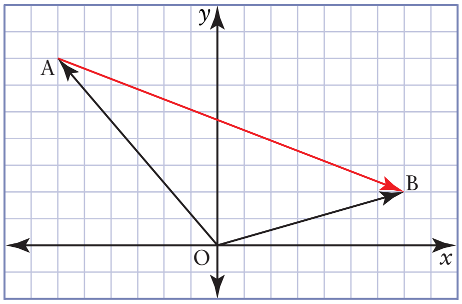
Lift is the lifting force that is perpendicular to the direction of travel. Drag is the force that is parallel and opposite to the direction of travel. Every part of an airplane contributes to its lift and drag. An airplane wing also uses flaps to increase or decrease lift and drag.



CONNECTIONS

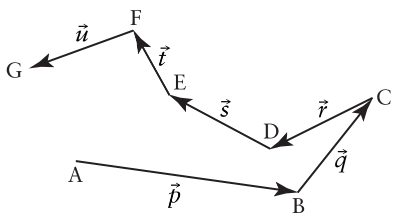
NASA has experimented with "lifting bodies," where all of the lift comes from the airplane's fuselage; the craft has no wings. Go to *www.mcgrawhill.ca/links/calculus12* and follow the links to find out more about lifting bodies .

11. **a)** Let A and B be any two points on a plane, and let O be the origin. Prove that $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$.

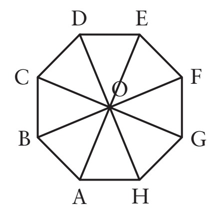


b) Let X be any other point on the plane. Prove that $\overrightarrow{AB} = \overrightarrow{XB} - \overrightarrow{XA}$.

12. Show that $(\vec{r} + \vec{u}) + (\vec{t} + \vec{p}) + (\vec{q} + \vec{s}) = \overrightarrow{AG}$.



13. Prove that the sum of the vectors from the vertices to the centre of a regular octagon is the zero vector.



Achievement Check

14. In a soccer match, the goalkeeper stands on the midpoint of her goal line. She kicks the ball 25 m at an angle of 35° to the goal line. Her teammate takes the pass and kicks it 40 m farther, parallel to the sideline.

a) Draw a scale diagram illustrating the vectors and resultant displacement.

b) What is the resultant displacement of the ball?

c) If the field is 110 m long, how far must the next player kick the ball to take a good shot at the centre of the goal, and in approximately which direction?

C Extend and Challenge

15. When would each expression be true? Support your answer with diagrams.

a) $|\vec{u} + \vec{v}| > |\vec{u} - \vec{v}|$

b) $|\vec{u} + \vec{v}| < |\vec{u} - \vec{v}|$

c) $|\vec{u} + \vec{v}| = |\vec{u} - \vec{v}|$

16. ABCD is a parallelogram, with P, Q, R, and S the midpoints of AB, BC, CD, and DA, respectively. Use vector methods to prove that PQRS is a parallelogram.

17. Prove that the statement $|\vec{u} + \vec{v}| \leq |\vec{u}| + |\vec{v}|$ is true for all vectors.

18. **Math Contest** If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are non-zero vectors with $\vec{a} = \vec{b} + \vec{c}$, then which statement is true?

a) $\vec{a}$ and $\vec{b}$ are collinear or $\vec{a}$ and $\vec{c}$ are collinear.

b) $|\vec{a}|$ is larger than $|\vec{b}|$ and $|\vec{a}|$ is larger than $|\vec{c}|$.

c) $|\vec{a}|$ is larger than $|\vec{b}| + |\vec{c}|$.

d) None of the above.

19. **Math Contest** Show that the functions $y = x^3 + 24x$ and $y = -x^3 + 12x^2 + 16$ intersect at exactly one point.

6.3 Multiplying a Vector by a Scalar

A variety of operations can be performed with vectors. One of these operations is multiplication by a scalar. What happens when you double your speed? What about doubling your velocity? If a car part has half the mass of another car part, is the force needed to install it halved? In this section, you will learn how multiplying by a number, or scalar, affects a vector quantity.

Scalar Multiplication

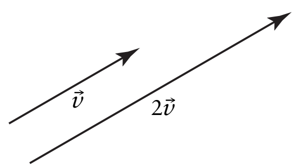
Given a vector $\vec{v}$ and a scalar k, where $k \in \mathbb{R}$, the **scalar multiple** of k and $\vec{v}$, $k\vec{v}$, is a vector $|k|$ times as long as $\vec{v}$. Its magnitude is $|k||\vec{v}|$.

If $k > 0$, then $k\vec{v}$ has the same direction as $\vec{v}$.

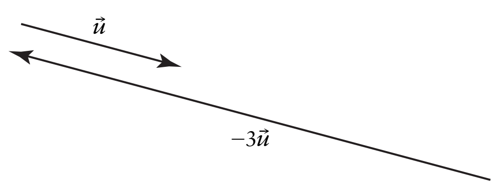
If $k < 0$, then $k\vec{v}$ has the opposite direction to $\vec{v}$.

A vector and its scalar multiple are parallel.

Vector $2\vec{v}$ has double the magnitude of vector $\vec{v}$ and has the same direction.



Vector $-3\vec{u}$ has triple the magnitude of vector $\vec{u}$, but has the opposite direction.

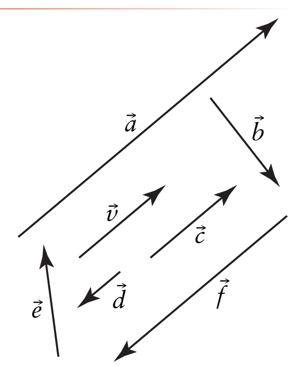


Example 1 Scalar Multiples

a) Which of these vectors are scalar multiples of vector $\vec{v}$? Explain.

b) Find the scalar k for each scalar multiple in part a).

c) For those vectors that are not scalar multiples of vector $\vec{v}$, explain why they are not.



Solution

a) $\vec{a}$, $\vec{c}$, $\vec{d}$, and $\vec{f}$ are all scalar multiples of $\vec{v}$, because they are all parallel to $\vec{v}$.

b) $\vec{a}$ is three times as long as $\vec{v}$ and has the same direction as $\vec{v}$, so $k = 3$.

$\vec{c}$ has the same length as $\vec{v}$ and the same direction as $\vec{v}$, so $k = 1$.

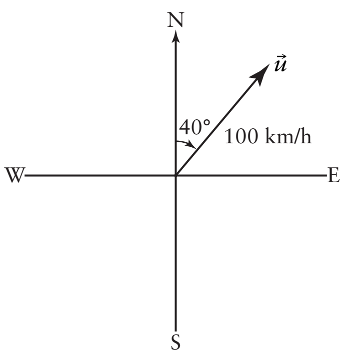
$\vec{d}$ is half as long as $\vec{v}$ and has the opposite direction to $\vec{v}$, so $k = -\frac{1}{2}$.

$\vec{f}$ is twice as long as $\vec{v}$ and has the opposite direction to $\vec{v}$, so $k = -2$.

c) $\vec{b}$ and $\vec{e}$ are not parallel to $\vec{v}$, so they are not scalar multiples of $\vec{v}$.

Example 2 Represent Scalar Multiplication

Consider vector $\vec{u}$ with magnitude $|\vec{u}| = 100$ km/h, at a quadrant bearing of N40°E.



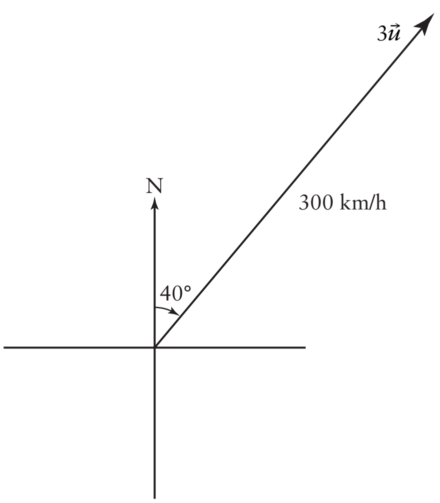
Draw a vector to represent each scalar multiplication. Describe, in words, the resulting vector.

a) $3\vec{u}$ **b)** $0.5\vec{u}$ **c)** $-2\vec{u}$

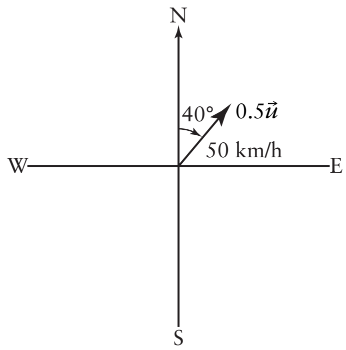
Solution

a) Draw an arrow three times as long as $\vec{u}$, in the same direction as $\vec{u}$.

The velocity is 300 km/h at a quadrant bearing of N40°E.

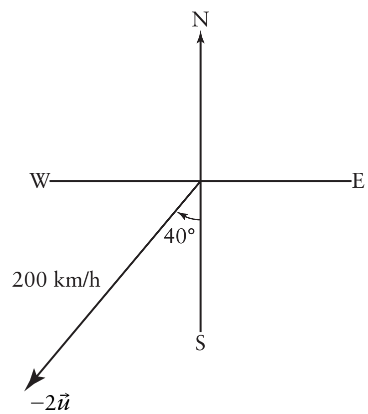


b) Draw an arrow half as long as $\vec{u}$, and in the same direction as $\vec{u}$.



The velocity is 50 km/h at a quadrant bearing of N40°E.

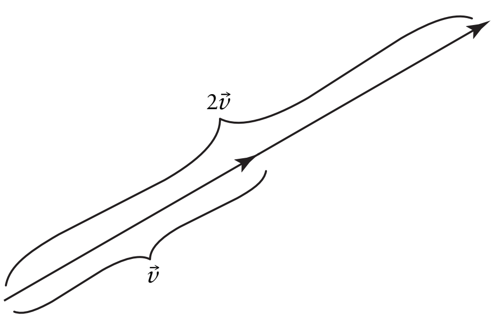
c) Draw an arrow twice as long as $\vec{u}$, but in the opposite direction to $\vec{u}$.

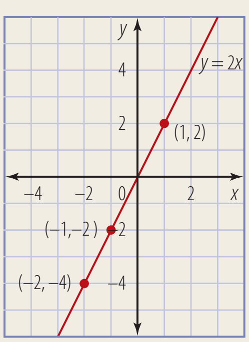


The velocity is 200 km/h at a quadrant bearing of S40°W.

Collinear Vectors

Vectors are **collinear** if they lie on a straight line when arranged tail to tail. The vectors are also scalar multiples of each other, which means that they are parallel.



> **CONNECTIONS**
>
> Three or more points are collinear if they lie on the same line.
>
> 
>
>
> The points (–2, –4), (–1, –2), and (1, 2) are collinear because they all lie on the line $y = 2x$.

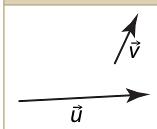
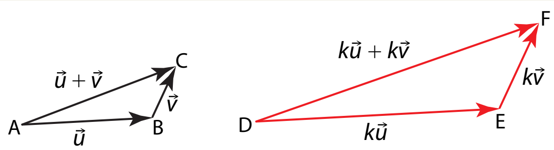
The following statements about non-zero vectors $\vec{u}$ and $\vec{v}$ are equivalent:

- $\vec{u}$ and $\vec{v}$ are scalar multiples of each other; that is, $\vec{u} = k\vec{v}$ for $k \in \mathbb{R}$.
- $\vec{u}$ and $\vec{v}$ are collinear.
- $\vec{u}$ and $\vec{v}$ are parallel.

Investigate — The distributive property

Method 1: ***Use Paper and Pencil***

The following is a sketch of the proof of the distributive property for vectors. Copy the table, and then explain each step. The first explanation is filled in for you.

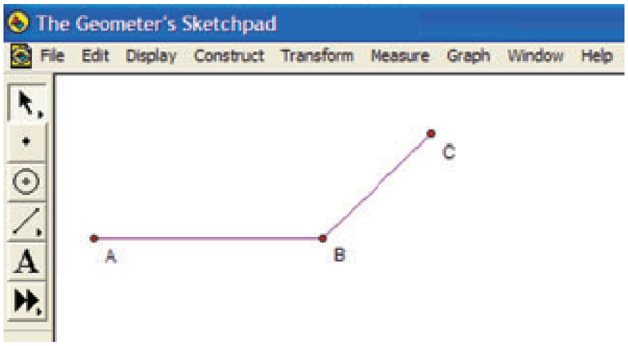
	Explanation
	Construct vector $\vec{u}$ and vector $\vec{v}$.
	
	
	
The two triangles are similar.	
$\frac{DF}{AC} = \frac{EF}{BC}$ $\frac{\lvert k\vec{u} + k\vec{v} \rvert}{\lvert \vec{u} + \vec{v} \rvert} = \frac{\lvert k\vec{v} \rvert}{\lvert \vec{v} \rvert}$	
$\frac{\lvert k\vec{u} + k\vec{v} \rvert}{\lvert \vec{u} + \vec{v} \rvert} = \frac{\lvert k \rvert \lvert \vec{v} \rvert}{\lvert \vec{v} \rvert}$ $\frac{\lvert k\vec{u} + k\vec{v} \rvert}{\lvert \vec{u} + \vec{v} \rvert} = \lvert k \rvert$	
$\lvert k\vec{u} + k\vec{v} \rvert = \lvert k \rvert \lvert \vec{u} + \vec{v} \rvert$ $\lvert k\vec{u} + k\vec{v} \rvert = \lvert k(\vec{u} + \vec{v}) \rvert$	
The magnitude of $k\vec{u} + k\vec{v}$ is the same as the magnitude of $k(\vec{u} + \vec{v})$, and $k\vec{u} + k\vec{v}$ and $k(\vec{u} + \vec{v})$ are collinear and have the same direction.	
Thus, $k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v}$, and the distributive property for vectors holds.	

Tools

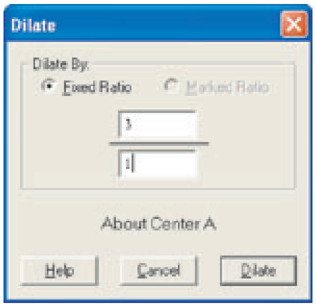
- computer with *The Geometer's Sketchpad*®

Method 2: *Use* The Geometer's Sketchpad®

1. Draw a line segment AB and a second line segment BC.



2. Mark point A as centre.
3. Using the **Transform** menu, dilate both segments and points A, B, and C by a 3 : 1 factor.



4. Label the image points B′ and C′.
5. Draw a segment from point A to the image point C′ of the dilation.
6. Draw a segment from point A to point C.
7. Which segment represents the vector sum $\overrightarrow{AB} + \overrightarrow{BC}$?
8. Write $\overrightarrow{AB'}$ in terms of $\overrightarrow{AB}$ and $\overrightarrow{B'C'}$ in terms of $\overrightarrow{BC}$.
9. Which segment represents the vector sum $3\overrightarrow{AB} + 3\overrightarrow{BC}$?
10. Measure the ratio of $\overrightarrow{AC'}$ to $\overrightarrow{AC}$.
11. Is $\overrightarrow{AC'}$ parallel to $\overrightarrow{AC}$?
12. Write $\overrightarrow{AC'}$ in terms of $\overrightarrow{AC}$.
13. Does the distributive property for scalar multiplication of vectors hold in this case? That is, does $3(\overrightarrow{AB} + \overrightarrow{BC}) = 3\overrightarrow{AB} + 3\overrightarrow{BC}$?
14. Repeat steps 1 to 13 for different dilation factors, k. Replace the number 3 in steps 3, 9, and 13 with your choice of k.
15. Try a negative dilation factor. Describe your results.
16. **Reflect** How does this investigation demonstrate that the distributive property for scalar multiplication of vectors holds?

Vector Properties for Scalar Multiplication

Distributive property: For any scalar $k \in \mathbb{R}$ and any vectors $\vec{u}$ and $\vec{v}$, $k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v}$.

Associative property: For any scalars a and $b \in \mathbb{R}$ and any vector $\vec{v}$, $(ab)\vec{v} = a(b\vec{v})$.

Identity property: For any vector $\vec{v}$, $1\vec{v} = \vec{v}$.

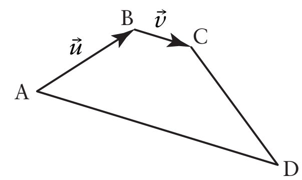
Linear Combinations of Vectors

Given two vectors $\vec{u}$ and $\vec{v}$ and scalars s and $t \in \mathbb{R}$, the quantity $s\vec{u} + t\vec{v}$ is called a **linear combination** of vectors $\vec{u}$ and $\vec{v}$.

For example, $2\vec{u} - 7\vec{v}$ is a linear combination of vectors $\vec{u}$ and $\vec{v}$, where $s = 2$ and $t = -7$.

Example 3 Linear Combinations of Vectors

In trapezoid ABCD, BC || AD and AD = 3BC. Let $\overrightarrow{AB} = \vec{u}$ and $\overrightarrow{BC} = \vec{v}$. Express $\overrightarrow{AD}$, $\overrightarrow{BD}$, and $\overrightarrow{CD}$ as linear combinations of $\vec{u}$ and $\vec{v}$.



Solution

$\overrightarrow{AD} = 3\vec{v}$

$\overrightarrow{BD} = \overrightarrow{BA} + \overrightarrow{AD}$

$= -\vec{u} + 3\vec{v}$

$\overrightarrow{CD} = \overrightarrow{CB} + \overrightarrow{BA} + \overrightarrow{AD}$

$= -\vec{v} + (-\vec{u}) + 3\vec{v}$

$= (-\vec{u}) + (-\vec{v}) + 3\vec{v}$ Commutative property

$= -\vec{u} + 2\vec{v}$ Associative property

KEY CONCEPTS

- When you multiply a vector by a scalar, the magnitude is multiplied by the scalar and the vectors are parallel. The direction remains unchanged if the scalar is positive, and becomes opposite if the scalar is negative.
- For any vectors $\vec{u}$ and $\vec{v}$ and scalars $k, m \in \mathbb{R}$:

  $k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v}$ (distributive property)

  $k(m\vec{u}) = (km)\vec{u}$ (associative property)

  $1\vec{u} = \vec{u}$ (identity property)
- Linear combinations of vectors can be formed by adding scalar multiples of two or more vectors.

Communicate Your Understanding

C1 Why does the direction not change when you multiply a vector by a positive scalar? Explain.

C2 Explain how the vectors $\vec{u}$, $5\vec{u}$, and $-5\vec{u}$ are related.

C3 Explain why these three sentences are equivalent.

- $\vec{u}$ and $\vec{v}$ are scalar multiples of each other.
- $\vec{u}$ and $\vec{v}$ are collinear.
- $\vec{u}$ and $\vec{v}$ are parallel.

A Practise

1. Let $|\vec{v}| = 500$ km/h, at a quadrant bearing of S30°E. Draw a scale diagram illustrating each related vector.

a) $2\vec{v}$ **b)** $0.4\vec{v}$

c) $-3\vec{v}$ **d)** $-5\vec{v}$

2. Simplify each of the following algebraically.

a) $\vec{u} + \vec{u} + \vec{u}$

b) $2\vec{u} - 3\vec{v} - 3\vec{u} + \vec{v}$

c) $3(\vec{u} + \vec{v}) - 3(\vec{u} - \vec{v})$

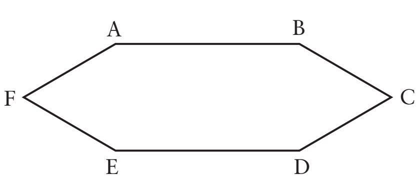
d) $3\vec{u} + 2\vec{v} - 2(\vec{v} - \vec{u}) + (-3\vec{v})$

e) $-(\vec{u} + \vec{v}) - 4(\vec{u} - 2\vec{v})$

f) $2(\vec{u} + \vec{v}) - 2(\vec{u} + \vec{v})$

3. Draw a vector diagram to illustrate each combination of vectors in question 2.

4. In hexagon ABCDEF, opposite sides are parallel and equal, and $\overrightarrow{FC} = 2\overrightarrow{AB}$. Let $\overrightarrow{AB} = \vec{u}$ and $\overrightarrow{FA} = \vec{v}$.

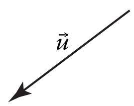


Express each vector in terms of $\vec{u}$ and $\vec{v}$. Simplify.

a) $\overrightarrow{CF}$ **b)** $\overrightarrow{FB}$ **c)** $\overrightarrow{FD}$

d) $\overrightarrow{CA}$ **e)** $\overrightarrow{EB}$ **f)** $\overrightarrow{BE}$

5. Copy vector $\vec{u}$.

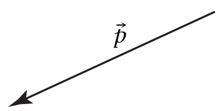


Show geometrically that $(k + m)\vec{u} = k\vec{u} + m\vec{u}$ for

a) $k = 2$ and $m = 3$ **b)** $k = 5$ and $m = -1$

c) $k = -4$ and $m = 2$ **d)** $k = -3$ and $m = -2$

6. Copy vector $\vec{p}$.



Show geometrically that $(ab)\vec{p} = a(b\vec{p})$ for

a) $a = 0.5$ and $b = 4$

b) $a = 3$ and $b = -2$

c) $a = -6$ and $b = \frac{1}{3}$

d) $a = -2$ and $b = -5$

7. Draw vectors $\vec{u}$ and $\vec{v}$ at right angles to each other with $|\vec{u}| = 3$ cm and $|\vec{v}| = 4.5$ cm. Then, draw the following linear combinations of $\vec{u}$ and $\vec{v}$.

a) $\vec{u} + \vec{v}$ **b)** $2\vec{u} - \vec{v}$

c) $0.5\vec{u} + 2\vec{v}$ **d)** $\vec{v} - \vec{u}$

8. What is the magnitude of $\frac{1}{|\vec{v}|}\vec{v}$ for any vector $\vec{v}$?

B Connect and Apply

9. Five people push a disabled car along a road, each pushing with a force of 350 N straight ahead. Explain and illustrate how the concept of scalar multiplication of a vector can be applied to this context.

10. Newton's second law of motion states that the force of gravity, $\vec{F}_g$, in newtons, is equal to the mass, m, in kilograms, times the acceleration due to gravity, $\vec{g}$, in metres per square second, or $\vec{F}_g = m \times \vec{g}$. On Earth's surface, acceleration due to gravity is 9.8 m/s^2 downward. On the Moon, acceleration due to gravity is 1.63 m/s^2 downward.

a) Write a vector equation for the force of gravity on Earth.

b) What is the force of gravity, in newtons, on Earth, on a 60-kg person? This is known as the weight of the person.

c) Write a vector equation for the force of gravity on the Moon.

d) What is the weight, on the Moon, of a 60-kg person?

11. PQRS is a parallelogram with A and B the midpoints of PQ and SP respectively. If $\vec{u} = \overrightarrow{QA}$ and $\vec{v} = \overrightarrow{PB}$, express each of the following vectors in terms of $\vec{u}$ and $\vec{v}$.

a) $\overrightarrow{PS}$ **b)** $\overrightarrow{AP}$ **c)** $\overrightarrow{RS}$

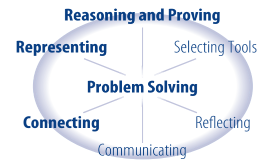
d) $\overrightarrow{AB}$ **e)** $\overrightarrow{QS}$ **f)** $\overrightarrow{AS}$

g) $\overrightarrow{BR}$ **h)** $\overrightarrow{PR}$ **i)** $\overrightarrow{RP}$

12. Show that the definitions of vector addition and scalar multiplication are consistent by drawing an example to show that $\vec{u} + \vec{u} + \vec{u} + \vec{u} = 4\vec{u}$.

13. If $\vec{u}$ is a vector and k is a scalar, is it possible that $\vec{u} = k\vec{u}$? Under what conditions can this be true?

14. An airplane takes off at 130 km/h at 15° above the horizontal. Provide an example to indicate scalar multiplication by 3.



15. Describe a scenario that may be represented by each scalar multiplication.

a) $3\vec{v}$, given $|\vec{v}| = 10$ N

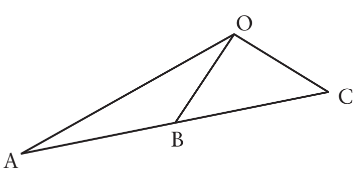
b) $2\vec{u}$, given $|\vec{u}| = 40$ km/h

c) $\frac{1}{2}\vec{w}$, given $|\vec{w}| = 9.8$ m/s^2

d) $10\vec{a}$, given $|\vec{a}| = 100$ km

16. Show geometrically that $(-1)k\vec{u} = k(-\vec{u})$.

17. Three points, A, B, and C, are collinear, such that B is the midpoint of AC. Let O be any non-collinear point. Prove that $\overrightarrow{OA} + \overrightarrow{OC} = 2\overrightarrow{OB}$.



18. From the given information, what can you say about the vectors $\vec{u}$ and $\vec{v}$ in each case?

a) $2\vec{u} = 3\vec{v}$

b) $\vec{u} - \vec{v} = \vec{0}$

c) $3(\vec{u} + \vec{v}) - 2(\vec{u} - \vec{v}) = \vec{u} + 5\vec{v}$

19. Which of the following are true statements? Give your reasoning.

a) $5\overrightarrow{AA} = -2\overrightarrow{AA}$

b) $-3\overrightarrow{AB} = 3(-\overrightarrow{BA})$

c) $-2(3\overrightarrow{BA}) = 6\overrightarrow{AB}$

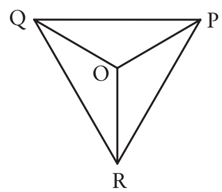
d) $2\overrightarrow{AB}$ and $-3\overrightarrow{BA}$ are collinear

e) $2\overrightarrow{AB}$ and $-3\overrightarrow{BA}$ have the same direction

C Extend and Challenge

20. If $\overrightarrow{OA} + \overrightarrow{OC} = 2\overrightarrow{OB}$, prove that A, B, and C are collinear and B is the midpoint of AC. This is the converse of the result in question 14.

21. $\triangle PQR$ is an equilateral triangle, and O is the centroid of the triangle. Let $\vec{u} = \overrightarrow{PQ}$ and $\vec{v} = \overrightarrow{PR}$. Express $\overrightarrow{PO}$, $\overrightarrow{QO}$, and $\overrightarrow{RO}$ in terms of $\vec{u}$ and $\vec{v}$.



22. Vector $\overrightarrow{AB}$ has endpoints A(4, 3) and B(−5, 1). Determine the coordinates of point C if
    a) $\overrightarrow{AC} = 5\overrightarrow{AB}$
    b) $\overrightarrow{AC} = -2\overrightarrow{AB}$
    Explain your strategy.

23. Given two perpendicular vectors $\vec{u}$ and $\vec{v}$, simplify $|\vec{u} + \vec{v}|^2 + |\vec{u} - \vec{v}|^2$. Illustrate the results geometrically.

24. The diagonals of quadrilateral ABCD bisect each other. Use vectors to prove that ABCD is a parallelogram.

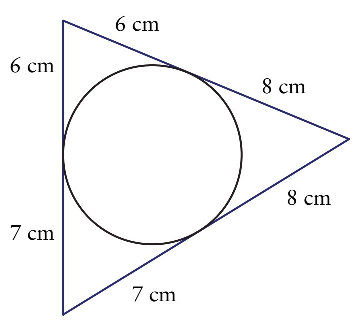
25. **a)** Prove that the diagonals of a cube intersect at a common point.
    b) Show that this point also bisects the line segment joining the midpoints of any two opposite edges of the cube.

26. If $\overrightarrow{OA} = \frac{1}{3}\overrightarrow{OB} + \frac{2}{3}\overrightarrow{OC}$, then prove that A, B, and C are collinear and that A divides the segment BC in the ratio of 2 : 1.

27. ABCDEF is a hexagon with opposite sides equal and parallel. Choose the midpoints of two pairs of opposite sides. Prove that the quadrilateral formed by these four midpoints is a parallelogram.

28. **Math Contest** The angle between two non-zero vectors $\vec{a}$ and $\vec{b}$ is 30° and $|\vec{a}| > |\vec{b}|$. Which statement is true?
    a) $|\vec{a}| + |\vec{b}| < |\vec{a} + \vec{b}|$
    b) $|\vec{a} - \vec{b}| < |\vec{a} + \vec{b}|$
    c) $|\vec{a} - \vec{b}| > |\vec{a} + \vec{b}|$
    d) None of the above

29. **Math Contest** A circle is inscribed in a triangle, as shown. Determine the radius of the circle.



CAREER CONNECTION

Tanica completed a 4-year bachelor of science in chemical engineering at Queen's University. She works for a company that designs and manufactures environmentally friendly cleaning products. In her job, Tanica and her team are involved in the development, safety testing, and environmental assessment of new cleaning products. During each of these phases, she monitors the rates of change of many types of chemical reactions. Tanica then analyses the results to produce a final product that is safe for the environment.

6.4 Applications of Vector Addition

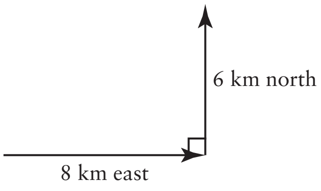
Ships and aircraft frequently need to steer around dangerous weather. A pilot must consider the direction and speed of the wind when making flight plans. Heavy objects are often lifted by more than one chain hanging from a horizontal beam. Velocity directions are often expressed in terms of a compass quadrant. Vector situations such as these can be modelled using triangles. Knowing the net effect of a number of forces determines the motion of an object. The single force—the resultant—has the same effect as all the forces acting together. You will use vector addition, the Pythagorean theorem, and trigonometry to solve vector problems involving oblique triangles.

Two vectors that are perpendicular to each other and add together to give a vector $\vec{v}$ are called the **rectangular vector components** of $\vec{v}$.

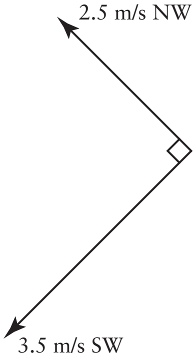
Example 1 Resultants of Rectangular Components

Draw the resultant for each set of rectangular components. Then, calculate the magnitude and direction, relative to the horizontal vector, of the resultant.

a) A sailboat's destination is 8 km east and 6 km north.



b) In a numerical model of the Bay of Fundy, the velocity of the water is given using rectangular vector components of 2.5 m/s northwest and 3.5 m/s southwest.

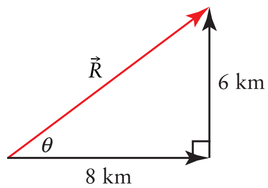


CONNECTIONS

You will reverse this process and find the rectangular components of a vector in Section 6.5.

Solution

a) Keep the vectors head to tail. The resultant, $\vec{R}$, is the hypotenuse of the right triangle.



Use the Pythagorean theorem to calculate the magnitude of the resultant.

$$|\vec{R}|^2 = 6^2 + 8^2$$
$$|\vec{R}| = 10$$

Calculate the angle the sailboat makes with the direction east.

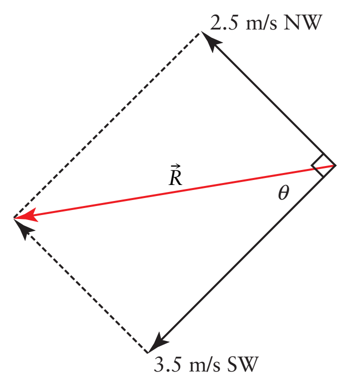
$$\theta = \tan^{-1}\left(\frac{6}{8}\right)$$
$$\doteq 36.9°$$

Since this situation involves navigation, calculate the direction as a bearing.

$90° - 36.9° = 53.1°$

The resultant displacement is 10 km at a bearing of about 053.1°.

b) Use the parallelogram method to draw the resultant.



Use the Pythagorean theorem to calculate the magnitude of the resultant.

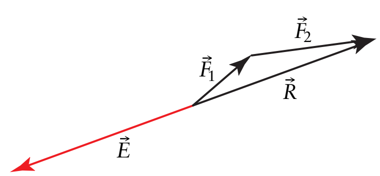
$$|\vec{R}|^2 = 3.5^2 + 2.5^2$$
$$|\vec{R}| \doteq 4.3$$

Find the value of θ.

$$\theta = \tan^{-1}\left(\frac{2.5}{3.5}\right)$$
$$\doteq 35.5°$$

Since the 3.5-m/s vector is 45° below the horizontal, $\vec{R}$ is 45° − 35.5° or 9.5° below the horizontal. This is 90° − 9.5° or 80.5° west of south. The water velocity is about 4.3 m/s S80.5°W.

An **equilibrant vector** is one that balances another vector or a combination of vectors. It is equal in magnitude but opposite in direction to the resultant vector. If the equilibrant is added to a given system of vectors, the sum of all vectors, including the equilibrant, is $\vec{0}$.



Example 2 Find a Resultant and Its Equilibrant

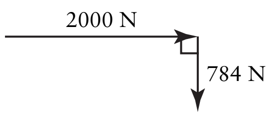
A clown with mass 80 kg is shot out of a cannon with a horizontal force of 2000 N. The vertical force is the acceleration due to gravity, which is 9.8 m/s^2, times the mass of the clown.

a) Find the magnitude and direction of the resultant force on the clown.

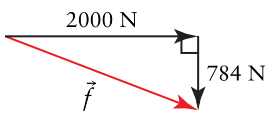
b) Find the magnitude and direction of the equilibrant force on the clown.

Solution

a) Draw a diagram of the situation. The vertical force is 9.8×80 or 784 N downward.



To find the resultant force, $\vec{f}$, add the vectors.



Since the forces are perpendicular, use the Pythagorean theorem to find the magnitude of the resultant, $|\vec{f}|$.

$$|\vec{f}|^2 = 2000^2 + 784^2$$
$$= 4\,614\,656$$
$$|\vec{f}| \doteq 2148$$

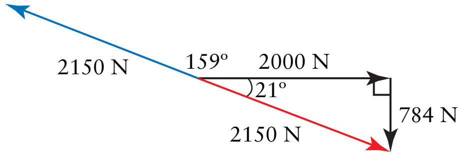
The magnitude of the resultant is about 2150 N.

To find the direction of the resultant force, use trigonometry. Let θ represent the angle of $\vec{f}$ to the horizontal.

$$\tan\theta = \frac{784}{2000}$$
$$\theta = \tan^{-1}\frac{784}{2000}$$
$$\doteq 21°$$

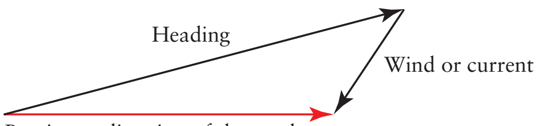
The resultant has a magnitude of about 2150 N and a direction of 21° below the horizontal.

b) Draw the equilibrant on the diagram.

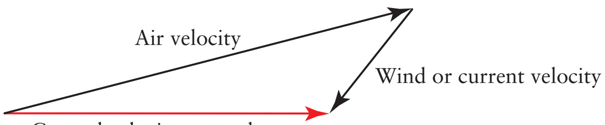


The equilibrant has magnitude of about 2150 N and a direction of 159° counterclockwise from the horizontal.

A **heading** is the direction in which a vessel is steered to overcome other forces, such as wind or current, with the intended resultant direction being the bearing.



Ground velocity is the velocity of an object relative to the ground. It is the resultant, or bearing velocity, when the heading velocity, or **air velocity**, and the effects of wind or current are added.

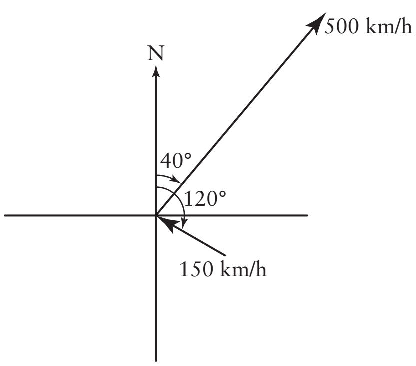


Example 3 Solve a Flight Problem

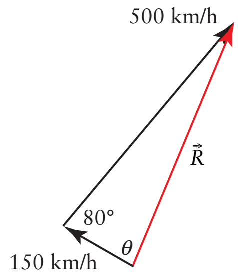
An airplane is flying at an airspeed of 500 km/h, on a heading of 040°. A 150-km/h wind is blowing from a bearing of 120°. Determine the ground velocity of the airplane and the direction of flight.

Solution

Draw a diagram illustrating the velocities and resultant vector. Use a compass quadrant graph.



Redraw the diagram, showing the resultant.



Let $\vec{R}$ be the resultant ground velocity of the airplane.

Let θ be the angle the resultant makes with the wind direction.

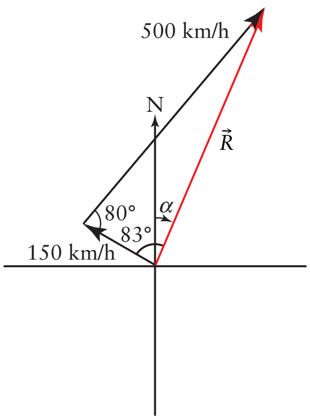
Use the cosine law to solve for $|\vec{R}|$.

$$|\vec{R}|^2 = 150^2 + 500^2 - 2(150)(500)\cos 80°$$
$$|\vec{R}| \doteq 496.440$$

Use the sine law to calculate θ.

$$\frac{\sin\theta}{500} = \frac{\sin 80°}{496.440}$$
$$\theta \doteq 82.689$$

To determine the bearing, translate the resultant so its tail is at the origin of the compass quadrant.



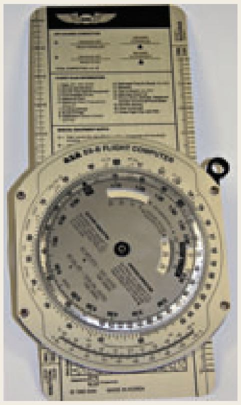
The head of the original wind vector is 60° off 180°. So, the head of the translated wind vector is 60° off 360°.

$$\alpha = 82.689° - 60°$$
$$= 22.689°$$

The airplane is flying at a ground velocity of about 496 km/h on a bearing of about 023°.

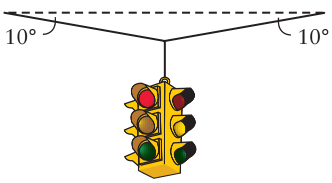
CONNECTIONS

The E6B Flight Computer, also known as a "whiz wheel," is used in flight training to calculate fuel burn, wind correction, time in the air, and groundspeed.



Example 4 Solve a Tension Problem

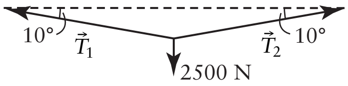
A traffic light at an intersection is hanging from two wires of equal length making angles of 10° below the horizontal. The traffic light weighs 2500 N. What are the tensions in the wires?



Solution

Draw a vector diagram so that the two tension vectors are keeping the traffic light at equilibrium.

Let $\vec{T_1}$ and $\vec{T_2}$ represent the two equal-magnitude tensions.

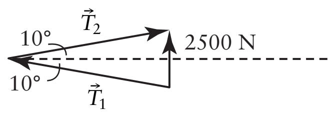


The 2500-N downward force of the traffic light is the equilibrant vector. Find the resultant of the tensions in the wires.

CONNECTIONS

Tension is the equilibrant force in a rope or chain keeping an object in place.

Redraw the diagram to show the sum of the tension vectors with a resultant of 2500 N upward.



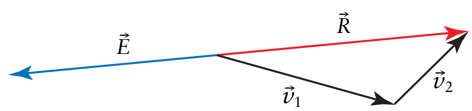
The equal angles at the base of the isosceles triangle each measure $(180° - 20°) \div 2 = 80°$.

$$\frac{|\vec{T}_1|}{\sin 80°} = \frac{2500}{\sin 20°}$$
$$|\vec{T}_1| = \frac{2500 \sin 80°}{\sin 20°}$$
$$|\vec{T}_1| \doteq 7198$$

The wires each have a tension of about 7200 N at 10° below the horizontal.

KEY CONCEPTS

- Two vectors that are perpendicular to each other and add together to give a vector $\vec{v}$ are called the rectangular vector components of $\vec{v}$.
- When two vectors act on an object, you can use vector addition, the Pythagorean theorem, and trigonometry to find the resultant.
- An equilibrant vector is the opposite of the resultant.



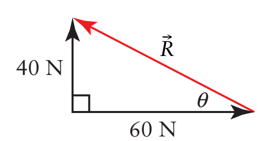
- Directions of resultants can be expressed as angles relative to one of the given vectors, or they can be expressed as bearings.

CONNECTIONS

Rectangular components are used to analyse voltage and current relationships in electronic circuits, such as those found in televisions and computers.

Communicate Your Understanding

C1 Rolly determined the resultant of the vector diagram as shown. Describe the error in his thinking.



$$|\vec{R}|^2 = 40^2 + 60^2$$
$$|\vec{R}| \doteq 72.1 \text{ N}$$

$$\theta = \tan^{-1}\left(\frac{40}{60}\right)$$
$$\theta \doteq 33.7°$$

The resultant force is 72.1 N at 33.7° vertically from the left.

C2 Explain the difference between the following two statements, and how it affects the vector diagram.

- A 15-km/h wind is blowing from a bearing of 030°.
- A 15-km/h wind is blowing at a bearing of 030°.

C3 Describe how you would draw a vector diagram to illustrate the following statement. Include the resultant and equilibrant.
"Forces of 100 N and 130 N are applied at 25° to each other."

C4 Describe the differences between airspeed, wind speed, and groundspeed when solving vector problems associated with airplane flight.

A Practise

1. Determine the resultant of each vector sum.

a) 34 km/h east and then 15 km/h north

b) 100 m/s south and then 50 m/s west

c) 45 km/h vertically and then 75 km/h horizontally

d) 3.6 m/s horizontally and then 2.3 m/s vertically

e) 10 N at 045° and then 8 N at 068°

f) 1200 N at 120° and then 1100 N at 300°

g) 300 m east and then 400 m northeast

h) 15 m/s^2 at 80° above the horizontal and then gravitational acceleration of 9.8 m/s^2

2. An airplane is flying at 550 km/h on a heading of 080°. The wind is blowing at 60 km/h from a bearing of 120°.

a) Draw a vector diagram of this situation.

b) Find the ground velocity of the airplane.

3. A boat with forward velocity of 14 m/s is travelling across a river directly toward the opposite shore. At the same time, a current of 5 m/s carries the boat down the river.

a) What is the velocity of the boat relative to the shore?

b) Find the direction of the boat's motion relative to the shore.

c) Go to *www.mcgrawhill.ca/links/calculus12* and follow the links to **6.4**. Download the file **Vector Addition.gsp** and use it to check your answers to parts a) and b).

4. A box weighing 450 N is hanging from two chains attached to an overhead beam at angles of 70° and 78° to the horizontal.

a) Draw a vector diagram of this situation.

b) Determine the tensions in the chains.

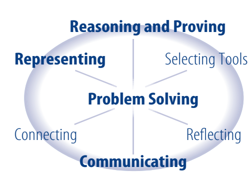
B Connect and Apply

5. Nancy is a pilot in Canada's North. She needs to deliver emergency supplies to a location that is 500 km away. Nancy has set the aircraft's velocity to 270 km/h on a northbound heading. The wind velocity is 45 km/h from the east.

a) Determine the resultant ground velocity of the aircraft.

b) Will Nancy be able to make the delivery within 2 h, at this ground velocity? Justify your response.

6. A golfer hits a golf ball with an initial velocity of 140 km/h due north. A crosswind blows at 25 km/h from the west.



a) Draw a vector diagram of this situation.

b) Find the resultant velocity of the golf ball immediately after it is hit.

7. A rocket is fired at a velocity with initial horizontal component 510 m/s and vertical component 755 m/s.

a) Draw a vector diagram of this situation.

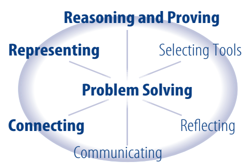
b) What is the ground velocity of the rocket?

8. A small aircraft, on a heading of 225°, is cruising at 150 km/h. It is encountering a wind blowing from a bearing of 315° at 35 km/h.

a) Draw a vector diagram of this situation.

b) Determine the aircraft's ground velocity.

9. A cruise ship's captain sets the ship's velocity to be 26 knots at a heading of 080°. The current is flowing toward a bearing of 153° at a speed of 8 knots.



a) Draw a vector diagram of this situation.

b) What is the ground velocity of the cruise ship?

10. Two astronauts use their jet packs to manoeuvre a part for the space station into position. The first astronaut applies a force of 750 N horizontally, while the second astronaut applies a force of 800 N vertically, relative to Earth.

a) Determine the resultant force on the part.

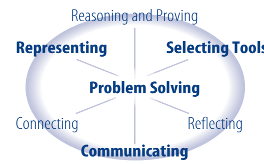
b) What would be the effect of doubling the vertical force?

c) What would be the effect of doubling both forces?

11. Emily and Clare kick a soccer ball at the same time. Emily kicks it with a force of 120 N at an angle of 60° and Clare kicks it with a force of 200 N at an angle of 120°. The angles are measured from a line between the centres of the two goals. Calculate the magnitude and direction of the resultant force.

12. A reproduction of a Group of Seven painting weighs 50 N and hangs from a wire placed on a hook so that the two segments of the wire are of equal length, and the angle separating them is 100°. Determine the tension in each segment of the wire.

13. **Chapter Problem**



Tanner is a pilot hired to transport a tranquilized bear 200 km due south to an animal preserve. The tranquilizer lasts for only 1 h, 45 min, so the airplane must reach its destination in 1.5 h or less. The wind is blowing from the west at 35 km/h. Tanner intends to fly the airplane at 180 km/h.

a) Draw a vector diagram of this situation.

b) Determine the heading Tanner needs to set in order to arrive within the allotted time.

c) Explain your choice of strategies.

14. Andrea flies planes that drop water on forest fires. A forest fire is 500 km away, at a bearing of 230°. A 72-km/h wind is blowing from a bearing of 182°.

a) Determine the heading that Andrea should set, if the airplane will be flying at

i) 230 km/h **ii)** 300 km/h

b) Explain the difference in results.

15. In a collision, a car with a momentum of 18 000 kg·m/s strikes another car whose momentum is 15 000 kg·m/s. The angle between their directions of travel is 32°.

a) Draw a vector diagram of this situation.

CONNECTIONS

Momentum is the product of mass and velocity.

b) Determine the resultant momentum of the cars upon impact. Round your answer to the nearest thousand.

16. Two forces act on an object at an angle of 40° to each other. One force has a magnitude of 200 N, and the resultant has a magnitude of 600 N.

a) Determine the angle the resultant makes with the 200-N force.

b) Determine the magnitude of the second force.

17. A car is stopped on a hill that is inclined at 5°. The brakes apply a force of 2124 N parallel to the road. A force of 9920 N, perpendicular to the surface of the road, keeps the car from sinking into the ground.

a) Explain why the rectangular components are not vertical and horizontal.

b) The weight of the car is the sum of the forces. What is the weight of the car?

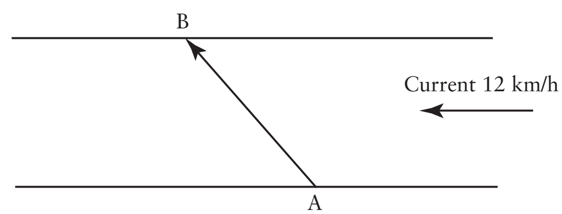
18. A jet's take-off velocity has a horizontal component of 228.3 km/h and a vertical component of 74.1 km/h. What is the jet's displacement from the end of the airstrip after 3 min?

19. While on a search and rescue mission for a boat lost at sea, a helicopter leaves its pad and travels 75 km at N20°E, turns and travels 43 km at S70°E, turns again and travels 50 km at S24°W, and makes a final turn and travels 18 km at N18°W, where the boat is found. What is the displacement of the boat from the helicopter pad?

20. A supply boat needs to cross a river from point A to point B, as shown in the diagram. Point B is 1.5 km downstream. The boat can travel at a speed of 20 km/h relative to the water. The current is flowing at 12 km/h. The width of the river to 500 m.

a) Determine the heading the captain should set to cross the river to point B.

b) Determine the heading the captain must set to return to point A.



Achievement Check

21. A barge is heading directly across a river at 3.5 km/h, while you are walking on the barge at 4 km/h toward the opposite shore. The current is flowing downstream at 1.4 km/h.

a) What is your actual velocity relative to the shore?

b) What would your velocity be relative to the shore if you were walking toward the shore you just left? Interpret the results of your calculations.

C Extend and Challenge

22. Three mutually perpendicular forces (in three dimensions) are applied to an object: $|\vec{F}_x| = 35$ N, $|\vec{F}_y| = 45$ N, and $|\vec{F}_z| = 25$ N. Determine the magnitude of the resultant force.

23. A ball is thrown horizontally at 15 m/s from the top of a cliff. The acceleration due to gravity is 9.8 m/s^2 downward. Ignore the effects of air resistance.

a) Develop a vector model to determine the velocity of the ball after t seconds.

b) What is the velocity of the ball after 2 s?

24. Flight-training manuals simplify the forces acting on an aircraft in flight to thrust (forward horizontally), drag (backward horizontally), lift (vertically upward), and weight (vertically downward). Find the resultant force acting on an airplane with a mass of 600 kg when its engine produces 12 000 N of thrust, its wings produce 8000 N of lift, and its total drag is 9000 N.

25. While starting up, a wheel 9.0 cm in diameter is rotating at 200 revolutions per minute (rpm) and has an angular acceleration of 240 rpm per minute. Determine the acceleration of a given point on the wheel.

26. A party planner has suspended a 100-N crate filled with balloons from two equally long ropes, each making an angle of 20° with the horizontal, and attached at a common point on the crate.

a) What is the tension in each rope?

b) If one rope were 1.5 times as long as the other rope, how would this affect the vector diagram? Assume the distance between the ropes and the total length of rope is the same as in part a).

27. A vector joins the points (3, 5) and (−2, 7). Determine the magnitude and direction of this vector.

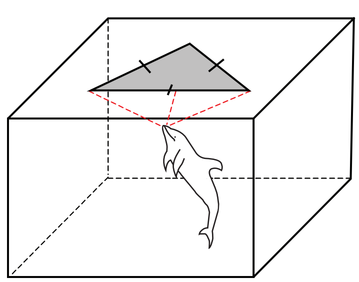
28. When you are doing push-ups, which situation requires a smaller muscular force? Justify your response.

a) Your hands are 0.25 m apart.

b) Your hands are 0.5 m apart.

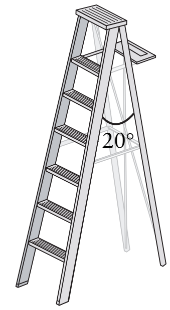
29. In the glass atrium at the entrance to the city aquarium, a designer wants to suspend a 2400-N sculpture of a dolphin. It will be secured by three chains, each of length 4 m. The chains are anchored to the ceiling at three points, spaced 3 m apart, that form an equilateral triangle.

a) Determine the magnitude of the tension in each chain.

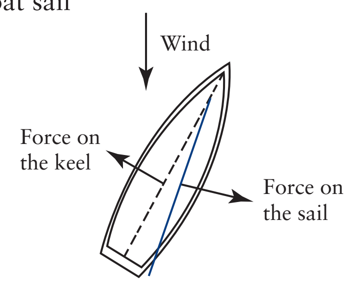


b) Why would the designer choose to use three anchor points rather than just one or two?

30. A ladder is supported by four legs that are braced with two crosspieces. The angle separating the legs at the top of the ladder is 20°. When a heavy object of mass 180 kg is placed on the top platform, what is the tension in each crosspiece?



31. How can a sailboat sail upwind? The wind exerts a force that is approximately perpendicular to the sail, as shown. The keel of the boat prevents it from sliding sideways through the water, such that the water exerts a force perpendicular to the long axis of the boat, as shown. The vector sum of these forces points in the direction that the boat will move. Note that this is a simplification of the actual situation. In reality, other forces are involved.



The wind is blowing from the north. The force on the sail is 200 N from a bearing of 100°. The force on the keel is 188 N from a bearing of 300°. Determine the vector sum of these forces, and show that the boat will move upwind.

32. **Math Contest** If $\vec{a}$ and $\vec{b}$ are perpendicular, $|\vec{a}| = 2|\vec{b}|$, and $|\vec{a} - \vec{b}| = 12$, then the angle between $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ is closest to

A 23°

B 33°

C 43°

D 53°

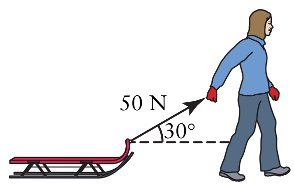
33. **Math Contest** The curves $x^2 + y^2 - 4x + 8y + 11 = 0$ and $x^2 - 4x - 4y - 24 = 0$ intersect at three distinct points, A, B, and C. Determine the area of $\triangle ABC$.

6.5 Resolution of Vectors Into Rectangular Components

We often think of just a single force acting on an object, but a lifting force and a horizontal force can act together to move the object. In Section 6.4, you added two rectangular components to determine the resultant vector. In this section, you will investigate how to determine the rectangular components of a given vector. This process is needed in order to understand the method of expressing vectors in Cartesian form in Chapter 7.

Investigate Horizontal and vertical components of a force

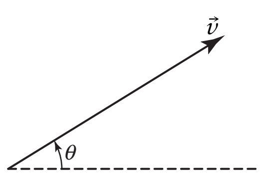
A girl pulls on the rope attached to her sled with a 50-N force at an angle of 30° to the horizontal. This force is actually the sum of horizontal and vertical forces, which are pulling the sled forward and upward, respectively.

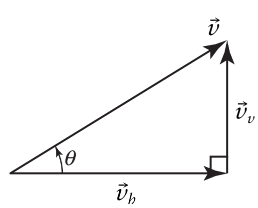


1. Draw a vector diagram breaking down the 50-N force into unknown horizontal and vertical components.
2. Explain why the 50-N force is the resultant.
3. Use trigonometry to find the length of each of the two unknown sides of the triangle.
4. **Reflect** Write a sentence to describe the horizontal and vertical forces.

A vector can be **resolved** into two perpendicular vectors whose sum is the given vector. This is often done when, for example, both vertical and horizontal forces are acting on an object. These are called the **rectangular components** of the force.

Consider a vector $\vec{v}$ at an angle of θ to the horizontal. It can be resolved into rectangular components $\vec{v}_h$ (horizontal component) and $\vec{v}_v$ (vertical component), where $\vec{v} = \vec{v}_h + \vec{v}_v$.





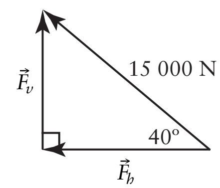
Example 1 Find the Rectangular Components of a Force

A tow truck is pulling a car from a ditch. The tension in the cable is 15 000 N at an angle of 40° to the horizontal.

a) Draw a diagram showing the resolution of the force into its rectangular components.

b) Determine the magnitudes of the horizontal and vertical components of the force.

Solution



a) The tension can be resolved into two rectangular components: vertical, $\vec{F}_v$, and horizontal, $\vec{F}_h$.

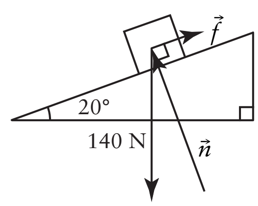
b)
$$\frac{|\vec{F}_h|}{\cos 40°} = 15\,000 \qquad \frac{|\vec{F}_v|}{\sin 40°} = 15\,000$$
$$|\vec{F}_h| = 15\,000\cos 40° \qquad |\vec{F}_v| = 15\,000\sin 40°$$
$$\doteq 11\,490 \qquad \doteq 9642$$

The magnitude of the horizontal component is about 11 500 N, and the magnitude of the vertical component is about 9600 N.

Example 2 Find Rectangular Components That Are Not Horizontal and Vertical

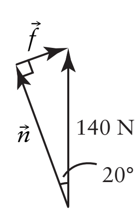
A box weighing 140 N is resting on a ramp that is inclined at an angle of 20°. Resolve the weight into rectangular components that keep the box at rest.

Solution



The box is being held at equilibrium by two rectangular components: a force, $\vec{n}$, perpendicular to the ramp, and a force of friction, $\vec{f}$, parallel to the surface of the ramp.

Redraw the diagram showing the sum of the vector components.

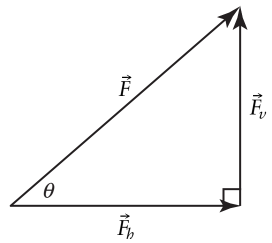


$$|\vec{n}| = 140\cos 20° \qquad |\vec{f}| = 140\sin 20°$$
$$\doteq 131.6 \qquad \doteq 47.9$$

The box is kept at rest by a force of 131.6 N perpendicular to the surface of the ramp and by friction of 47.9 N parallel to the surface of the ramp.

KEY CONCEPTS

- Any vector can be resolved into its rectangular (perpendicular) components.
- The given force, $\vec{F}$, is the resultant of the rectangular components $\vec{F}_h$ and $\vec{F}_v$.
- The horizontal component can be calculated using $|\vec{F}_h| = |\vec{F}|\cos\theta$.
- The vertical component can be calculated using $|\vec{F}_v| = |\vec{F}|\sin\theta$.



Communicate Your Understanding

C1 Draw a diagram resolving a 500-N force at 15° counterclockwise from the horizontal into its rectangular components.

C2 Explain how you would solve the following problem using rectangular components. Your friend swims diagonally across a 25 m by 10 m pool at 1 m/s. How fast would you have to walk around the edges of the pool to get to the same point at the same time as your friend?

C3 An airplane is flying at an airspeed of 150 km/h and encounters a crosswind. Will the plane's groundspeed be more than, equal to, or less than 150 km/h? Explain.

CONNECTIONS

A crosswind is a wind that blows at 90° to the heading.

A Practise

1. Determine the horizontal and vertical components of each force.

a) magnitude 560 N, $\theta = 21°$ counterclockwise from the horizontal

b) magnitude 21 N, $\theta = 56°$ counterclockwise from the horizontal

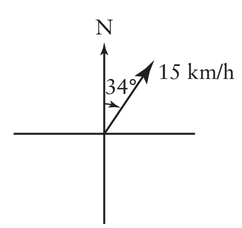
c) magnitude 1200 N, $\theta = 43°$ counterclockwise from the horizontal

d) magnitude 17 N, $\theta = 15°$ clockwise from the vertical

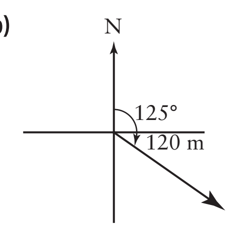
e) magnitude 400 N, $\theta = 12°$ clockwise from the vertical

2. Calculate the rectangular components of each velocity.

a)



b)



c) 880 km/h at 70° to the horizontal

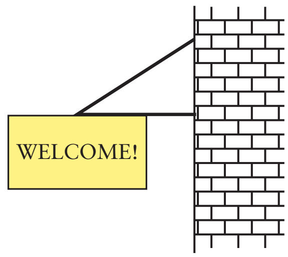
d) 135 m/s^2 at 40° to the vertical

B Connect and Apply

3. **a)** Resolve a 100-N force into two equal rectangular components.

b) Is there more than one answer to part a)? Explain.

4. A sign is supported as shown in the diagram.



The tension in the slanted rod supporting the sign is 110 N at an angle of 25° to the horizontal.

a) Draw a vector diagram showing the components of the tension vector.

b) What are the vertical and horizontal components of the tension?

5. A 35-N box is resting on a ramp that is inclined at an angle of 30° to the horizontal. Resolve the weight of the box into the rectangular components keeping it at rest.

6. It is important for aerospace engineers to know the vertical and horizontal components of a rocket's velocity. If a rocket is propelled at an initial velocity of 120 m/s at 80° from the horizontal, what are the vertical and horizontal components of the velocity?

7. A space probe is returning to Earth at an angle of 2.7° from the horizontal with a speed of 29 000 km/h. What are the horizontal and vertical components of the velocity? Round your answers to the nearest 100 km/h.

8. An airplane is climbing at an angle of 14° from the horizontal at an airspeed of 600 km/h. Determine the rate of climb and horizontal groundspeed.

9. A jet is 125 km from Sudbury airport at quadrant bearing N24.3°E, measured from the airport. What are the rectangular components of the jet's displacement?

10. The curator of an art gallery needs to hang a painting of mass 20 kg with a wire, as shown. The nail is at the midpoint of the wire. Find the tension in each part of the wire. Remember that tension is the force counteracting the force of gravity, so that you must multiply the mass of the painting by the acceleration due to gravity.

11. The handle of a lawnmower you are pushing makes an angle of 60° to the ground.

a) How could you increase the horizontal forward force you are applying to the lawnmower without increasing the total force?

b) What are some of the disadvantages of your method in part a)?

12. Anna-Maria is pulling a wagon loaded with paving stones with a total mass of 100 kg. She is applying a force on the handle at an angle of 25° with the ground. The horizontal force on the handle is 85 N.

a) Draw a diagram of the situation.

b) Find

i) the total force on the handle

ii) the vertical component of the force on the handle

Round your answers to the nearest tenth of a newton.

13. **Chapter Problem** A pilot is set to take off from an airport that has two runways, one at due north and one at 330°. A 30-km/h wind is blowing from a bearing of 335°.

a) What are the components of the wind vector for each runway?

b) An airspeed of 160 km/h is required for takeoff. What groundspeed is needed for each runway?

c) Pilots prefer to take off into the wind, where possible. Which runway should be used? Explain.

d) The aircraft manual mandates a maximum crosswind component of 20 km/h. Could the pilot safely select either runway for takeoff? Justify your answer.

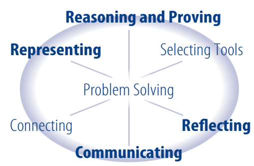
14. A force of 200 N is resolved into two components of 150 N and 80 N. Are these rectangular components? Justify your response. If they are not, determine the directions of the components.

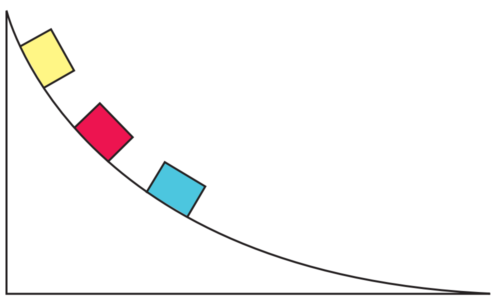
15. Resolve a 500-N force into two rectangular components such that the ratio of their magnitudes is 2:1. Calculate the angle between the greater component and the 500-N force.

16. Two cars, one travelling north and one travelling west, collide at an intersection. The resulting momentum of the two cars together after the collision is 32 000 kg·m/s N30°W. Find the momentum of each car before the collision.

C Extend and Challenge

17. A 50-N box is placed on a frictionless ramp as shown. Three positions of the box are shown in different colours.





a) Draw vector diagrams of the components of the weight vector as the box slides down the ramp.

b) In which position will the box have greatest acceleration? Use your vector diagrams to justify your conjecture.

18. In the absence of air resistance, the horizontal component of projectile motion remains constant, but the vertical component is constantly changing. Explain why this would occur.

19. A road is inclined at an angle of 5° to the horizontal. What force must be applied at a further 5° to the roadbed in order to keep a 15 000-N car from rolling down the hill?

20. A box with mass 10 kg is on a frictionless ramp that is inclined at 30°. Determine the acceleration of the box as it slides down the ramp.

21. For the horizontal components of projectile motion, the equations $\vec{x} = \vec{v}_{ix}t + \frac{1}{2}\vec{a}_x t^2$ and $\vec{v}_x = \vec{v}_{ix} + \vec{a}_x t$ are used. For the vertical components, the equations $\vec{y} = \vec{v}_{iy}t + \frac{1}{2}\vec{a}_y t^2$ and $\vec{v}_y = \vec{v}_{iy} + \vec{a}_y t$ are used.

a) Explain each part of the equations.

b) Relate these equations to the calculus applications of projectile motion.

22. A football is kicked with an initial velocity of 30 m/s at an angle of 30° to the horizontal. Determine the total time of the flight of the ball, the horizontal displacement when it lands, and the maximum height of the ball. Use vector methods from question 21 and calculus to solve this problem.

23. Math Contest Given $|\vec{c}| = 10$, $|\vec{d}| = 21$, and $|\vec{c} - \vec{d}| = 17$, determine $|\vec{c} + \vec{d}|$.

24. Math Contest Determine all acute angles x such that $\log_{\sqrt{3}}(\sin x) - \log_{\sqrt{3}}(\cos x) = 1$.

Chapter 6 REVIEW

6.1 Introduction to Vectors

1. For which of the following situations would a vector be a suitable mathematical model? Provide a reason for your decision.

a) A car is travelling at 70 km/h northeast.

b) A boy is walking at 5 km/h.

c) A rocket takes off at an initial speed of 800 km/h at 80° from the horizontal.

d) An airplane is sighted 20 km away.

e) A man's height is 180 cm.

2. Convert each true bearing to its equivalent quadrant bearing.

a) 130°

b) 080°

c) 250°

3. Use an appropriate scale to draw each vector. Label the magnitude, direction, and scale.

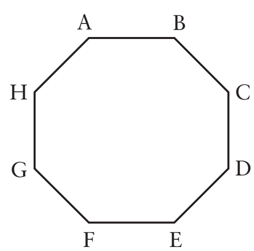
a) velocity of 140 km/h due west

b) acceleration of 20 m/s^2 at a bearing of 105°

c) force of 100 N upward

6.2 Addition and Subtraction of Vectors

4. The diagram shows a regular octagon. Write a single vector that is equivalent to each vector expression.



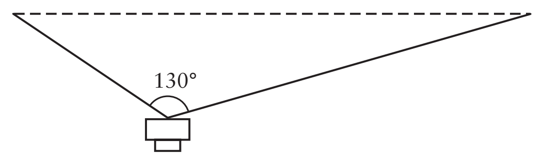
a) $\overrightarrow{HA} + \overrightarrow{AB}$

b) $\overrightarrow{GH} - \overrightarrow{FG}$

c) $\overrightarrow{FE} + \overrightarrow{BA}$

d) $\overrightarrow{GA} - \overrightarrow{EH} + \overrightarrow{DG}$

5. A camera is suspended by two wires over a football field to get shots of the action from above. At one point, the camera is closer to the left side of the field. The tension in the wire on the left is 1500 N, and the tension in the wire on the right is 800 N. The angle between the two wires is 130°.



a) Draw a vector diagram of the forces, showing the resultant.

b) Determine the approximate magnitude and direction of the resultant force.

6.3 Multiplying a Vector by a Scalar

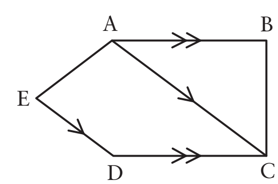
6. Express each sentence in terms of scalar multiplication of a vector.

a) An apple has a weight of 1 N, and a small car has a weight of 10 000 N.

b) A boat is travelling at 25 km/h northbound. It turns around and travels at 5 km/h southbound.

c) Acceleration due to gravity on Earth is 9.8 m/s^2, and on the Moon it is 1.63 m/s^2.

7. ABCDE is a pentagon such that $\overrightarrow{AB} = \overrightarrow{DC}$ and $\overrightarrow{AC} = 2\,\overrightarrow{ED}$. Write each vector in terms of $\overrightarrow{AB}$ and $\overrightarrow{AC}$.



a) $\overrightarrow{EC}$

b) $\overrightarrow{CE}$

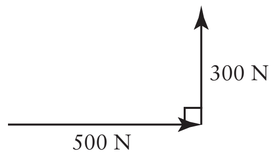
c) $\overrightarrow{CB}$

d) $\overrightarrow{AE}$

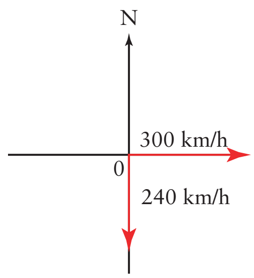
6.4 Applications of Vector Addition

8. Find the resultant of each pair of vectors.

a)



b)



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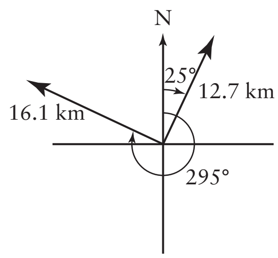
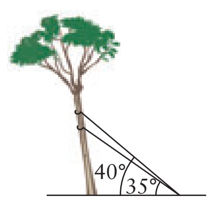


9. During a wind storm, two guy wires supporting a tree are under tension. One guy wire is inclined at an angle of 35° and is under 500 N of tension. The other guy wire is inclined at 40° to the horizontal and is under 400 N of tension. Determine the magnitude and direction of the resultant force on the guy wires.

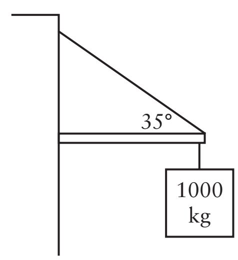


10. Three forces act on a body. A force of 100 N acts toward the north, a force of 120 N acts toward the east, and a force of 90 N acts at N20°E.
   a) Describe a method for finding the resultant of these three forces.
   b) Use your method to determine the resultant.

6.5 Resolution of Vectors Into Rectangular Components

11. In basketball, "hang time" is the time a player remains in the air when making a jump shot. What component(s) does hang time depend on? Explain.

12. A 1000-kg load is suspended from the end of a horizontal boom. The boom is supported by a cable that makes an angle of 35° with the boom.
   a) What is the weight of the load?
   b) What is the tension in the cable?
   c) What is the horizontal force on the boom?
   d) What is the vertical equilibrant component of the tension in the cable?



CHAPTER PROBLEM WRAP-UP

A small plane is heading north at 180 km/h. Its altitude is 2700 m.

a) Draw a labelled scale vector diagram of the effects of a 90-km/h wind from the west. Include the resultant in your diagram.

b) Determine the ground velocity of the airplane.

c) The airplane descends to 2000 m over a period of 2 min, still flying at the same groundspeed. What is the horizontal component of the change in displacement?

d) The airplane then enters turbulent air, which is falling at 30 km/h. The turbulence does not affect the groundspeed of the airplane. What is the airplane's resultant velocity?

e) The airplane then enters more turbulent air. The air mass is moving upward at 20 km/h and moving N30°W at 60 km/h, while the plane maintains its airspeed. Determine the vectors that represent the turbulent air and the resultant velocity of the airplane.

Chapter 6 PRACTICE TEST

For questions 1 to 4, choose the best answer.

1. Why is speed considered a scalar quantity and velocity a vector quantity?

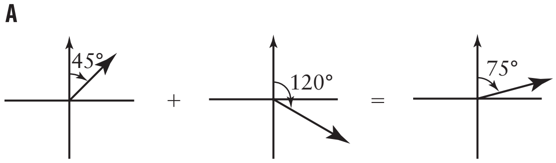
- **A** Speed has both magnitude and direction associated with it.
- **B** Velocity and speed are the same thing.
- **C** Velocity has both magnitude and direction associated with it.
- **D** Velocity has only magnitude associated with it.

2. You are driving on a curved highway on-ramp. Assuming you are driving at the speed limit of 70 km/h, which is the correct statement?

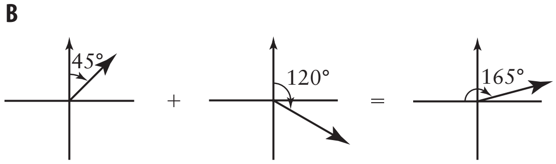
- **A** You are driving at a velocity of 70 km/h.
- **B** Your speed is constantly changing as you drive along the ramp.
- **C** Your velocity is constantly changing as you drive along the ramp.
- **D** None of the above.

3. Which is correct?

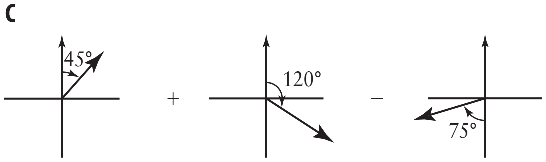
A



B



C



- **D** The resultant depends on the magnitude of the vectors.

4. A package is dropped from an airplane travelling horizontally at 160 km/h to accident victims on a desert island. The package will land

- **A** directly below where it was dropped
- **B** at a point depending on both the force of gravity and the velocity of the airplane
- **C** at a point depending only on the velocity of the airplane
- **D** at a point depending on the force of gravity, the wind velocity, and the velocity of the airplane

5. True or false? If the only force acting on a projectile is gravity, the horizontal component of its velocity is constant. Explain.

6. Convert each quadrant bearing to its equivalent true bearing.

a) N50°W **b)** N10°E **c)** S40°E

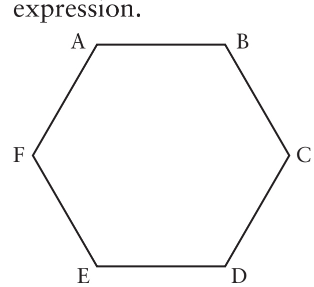
7. Use an appropriate scale to draw each vector. Label the magnitude, direction, and scale.

a) momentum of 50 kg·m/s south

b) velocity of 15 km/h at a quadrant bearing of N30°E

c) displacement of 120 m at a bearing of 075°

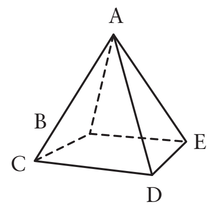
8. The diagram shows a regular hexagon. Write a single vector that is equivalent to each vector expression.



a) $\overrightarrow{AE} + \overrightarrow{EB}$ **b)** $\overrightarrow{AC} - \overrightarrow{BC}$

c) $\overrightarrow{CE} + \overrightarrow{DB} + \overrightarrow{AD}$ **d)** $\overrightarrow{DB} - \overrightarrow{EA} - \overrightarrow{DE}$

9. The diagram shows a square-based right pyramid. State a single vector equal to each expression.



a) $\overrightarrow{CB}$

b) $\overrightarrow{AB} + \overrightarrow{BD}$

c) $\overrightarrow{AB} - \overrightarrow{AD}$

d) $\overrightarrow{AE} - \overrightarrow{CD} + \overrightarrow{BD} - \overrightarrow{AD}$

10. In a soccer game, two opposing players kick the ball at the same time: one with a force of 200 N straight along the sidelines, and the other with a force of 225 N directly across the field. Calculate the magnitude and direction of the resultant force.

11. A ship's course is set at a heading of 143° at 18 knots. A 10-knot current flows at a bearing of 112°. What is the ground velocity of the ship?

12. A 150-N crate is resting on a ramp that is inclined at an angle of 10° to the horizontal.

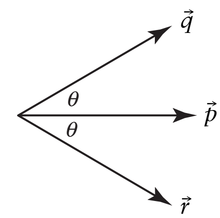
a) Resolve the weight of the crate into rectangular components that keep it at rest.

b) Describe these components so that a non-math student could understand them.

13. An airplane is flying at an airspeed of 400 km/h on a heading of 220°. A 46-km/h wind is blowing from a bearing of 060°. Determine the ground velocity of the airplane.

14. Devon is holding his father's wheelchair on a ramp inclined at an angle of 20° to the horizontal with a force of magnitude 2000 N parallel to the surface of the ramp. Determine the weight of Devon's father and his wheelchair and the component of the weight that is perpendicular to the surface of the ramp.

15. Given vectors $\vec{p}$, $\vec{q}$, and $\vec{r}$, such that $|\vec{p}| = |\vec{q}| = |\vec{r}| = 3$ units, and $\vec{p}$ bisects the angle formed by $\vec{q}$ and $\vec{r}$, express $\vec{p}$ as a linear combination of $\vec{q}$ and $\vec{r}$.



16. The Victoria Day fireworks at Daffodil Park are fired at an angle of 82° with the horizontal. The technician firing the shells expects them to explode about 100 m in the air, 4.8 seconds after they are fired.

a) Find the initial velocity of a shell fired from ground level.

b) Safety barriers will be placed around the launch area to protect spectators. If the barriers are placed 100 m from the point directly below the explosion of the shells, how far should the barriers be from the point where the fireworks are launched?

c) What assumptions are you making in part b)?

17. A hang-glider loses altitude at 0.5 m/s as it travels forward horizontally at 9.3 m/s. Determine the resultant velocity of the hang-glider, to one decimal place. Explain your result.

18. The force at which a tow truck pulls a car has a horizontal component of 20 000 N and a vertical component of 12 000 N.

a) What is the resultant force on the car?

b) Explain the importance of knowing these components.

19. A 100-N box is held by two cables fastened to the ceiling at angles of 80° and 70° to the horizontal.

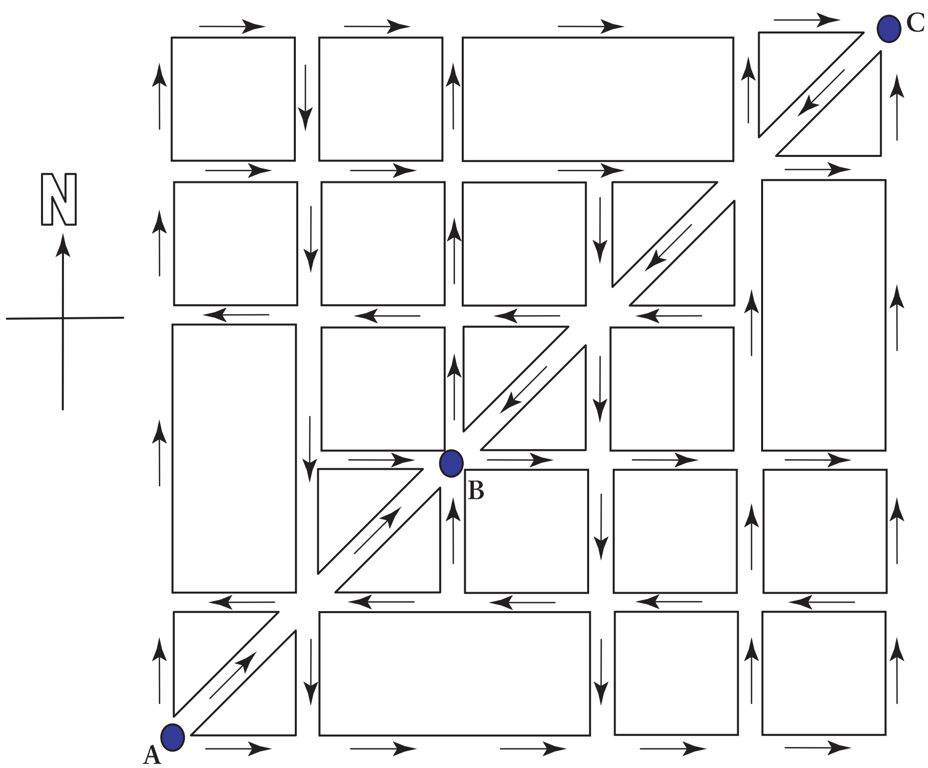
a) Draw a diagram of this situation.

b) Determine the tension in each cable.

c) If the cable hanging at 70° to the horizontal were lengthened, what would happen to the tensions? Justify your response.

TASK

Taxi Cab Vectors



A taxi has three passengers when it starts at A. It must drop off two people at B and the third at C. The arrows represent one-way streets.

a) Using vectors, find two different routes that go from A to C via B.

b) Show that the total displacement is equal in each case.

In the taxi, travelling northbound takes 12 min per block, travelling southbound takes 5 min per block, travelling westbound takes 6 min per block, travelling eastbound takes 8 min per block, and travelling northeast or southwest takes 10 min per block.

c) Which of your routes takes less time?

d) Is there a best route? Is it unique?

e) Identify which vector properties are used in your solution.

f) If the taxi charges for mileage are \$0.50/rectangular block and the time charges are \$0.10/minute, what is the cheapest route from A to C? How much should each passenger pay?