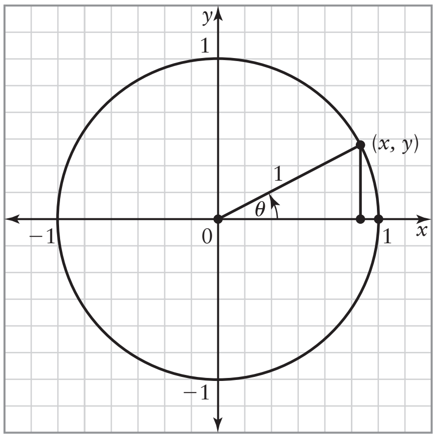
Chapter 1 Trigonometric Ratios

1.1 Sine, Cosine, and Tangent of Special Angles

KEY CONCEPTS

- Exact trigonometric ratios for 30°, 45°, and 60° angles can be determined using special triangles.

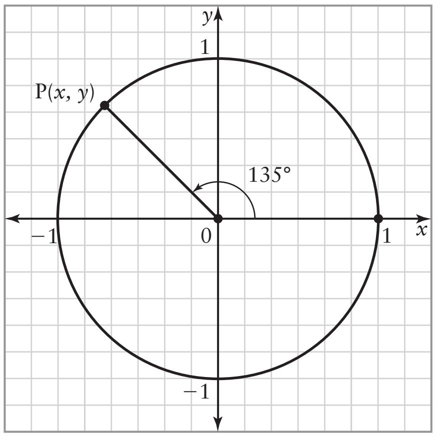


- Any point (x, y) on a unit circle can be joined to the origin to form a radius 1 unit long.
- A rotation angle θ, in standard position, is formed by proceeding counterclockwise from the initial arm on the positive x-axis to the terminal arm through (x, y).
- For any rotation angle, the reference angle is the acute angle between the terminal arm and the x-axis.
- Given a point (x, y) on a unit circle, $\cos\theta = x$, $\sin\theta = y$, and $\tan\theta = \frac{y}{x}$.

Example

Determine the exact values of the primary trigonometric ratios for 135°.

Solution



The measure of the reference angle is 180° −135°, or 45°.

Use the special triangles to determine the sine and cosine ratios for the reference angle.

$\sin 45° = \frac{1}{\sqrt{2}}$ $\cos 45° = \frac{1}{\sqrt{2}}$ $\tan 45° = 1$

Since the terminal arm of a 135° angle in standard position is in quadrant II, the x-coordinate of P is negative.

$\sin 135° = \frac{1}{\sqrt{2}}$ $\cos 135° = -\frac{1}{\sqrt{2}}$ $\tan 135° = -1$

A

Unless otherwise specified, give all answers as exact values.

1. Determine the sine, cosine, and tangent ratios for each angle.

a) 30° **b)** 45° **c)** 60°

2. **a)** Copy and complete the table.

	sin θ	
θ	**Exact**	**Calculator**
0°		
30°		
45°		
60°		
90°		

b) Compare the exact values of the trigonometric ratios to the values from the calculator.

3. Repeat question 2 for cos θ and tan θ.

4. **a)** What reference angle should be used to find the primary trigonometric ratios for 150°?

b) Determine the primary trigonometric ratios for 150°.

5. **a)** Draw a 225° angle in standard position on a unit circle.

b) What reference angle should be used to find the primary trigonometric ratios for 225°?

c) Find the primary trigonometric ratios for 225°.

6. **a)** Draw a 240° angle in standard position on a unit circle.

b) State two other angles that have the same reference angle.

c) Find the primary trigonometric ratios of 240°.

d) State the primary trigonometric ratios for the angles in part b).

7. **Use Technology** Use geometry software to construct a circle with radius 1 unit. Label the origin O. Construct and label point B on the circle in quadrant I. Determine the coordinates of point B.

a) Construct segment OB. Determine the length of OB. Find the measure of ∠AOB.

b) Drag B until ∠AOB = 30°. Record the coordinates of point B, the length of OB, and the measure of ∠AOB.

c) Reflect B and OB in the *x*-axis. Record the coordinates of B′ and the coordinate distance OB′.

d) Construct segment BB′. Determine the coordinate distance BB′. What type of triangle is △OBB′?

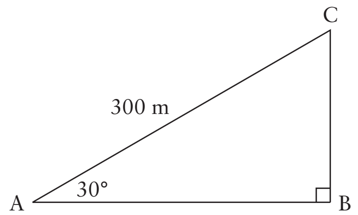
e) Construct the midpoint, D, of BB′. Then, construct segment OD. Determine the coordinate distances OD and DB.

f) How do the distances OD and DB compare to the coordinates of B? How do the distances OD and DB′ compare to the coordinates of B′?

g) Drag point B around the unit circle. What do you notice?

B

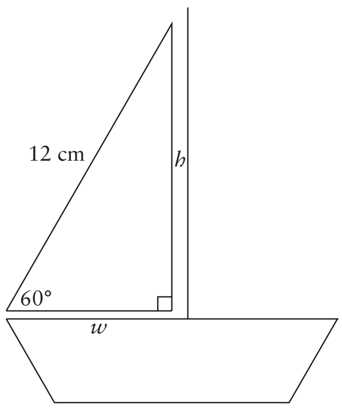
8. Devan started at point A and walked 300 m across a park to a store at point C. Sonal started at point A and walked east to point B and then walked north to the store at point C.



a) Determine the distance AB.

b) Determine the distance BC.

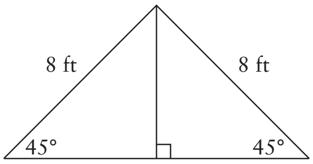
9. Doug designed a sail for a model sailboat. The sail is in the shape of a right triangle.



a) Determine the height, h, of the sail.

b) Determine the width, w, of the sail.

10. Alicia is on an overnight camping trip. At the front of her tent, the distance from the top of the tent to the ground on either side is 8 ft.



a) Determine the height of Alicia's tent.

b) Determine the width of the floor of her tent.

☆ 11. A 3-m long brace is placed against a wall so the bottom of the brace makes an angle of 60° with the ground.

a) Draw a diagram to represent this situation.

b) How far up the wall is the top of the brace?

12. A 4-m long ramp is placed against a wall. The ramp makes an angle of 30° with the ground. How far from the wall is the bottom of the ramp?

13. Boat A is 5 km north of a marina. Boat B is 5 km east of the marina.

a) Determine the distance between the two boats.

b) Describe an alternative method that can be used to solve this problem.

☆ 14. A patio in the shape of a regular hexagon has side lengths 4 m. Determine the area of the patio.

15. **Use Technology** Use geometry software to construct a circle with radius 1 unit. Construct point A on the circle in quadrant I. Construct a segment joining A to the origin, O, to form angle θ in standard position. Determine the coordinates of A.

a) Calculate the sine, cosine, and tangent ratios of $\angle\theta$, using x and y.

b) Drag point A around the circle. What do you notice?

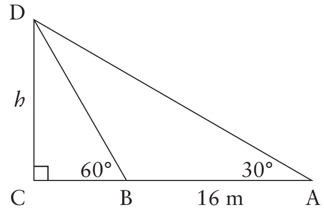
☆ 16. Determine the exact value of $\cos 30° \times \sin 240° + \sin 330°$.

C

17. Determine all the possible measures of θ, where $0° \leq \theta \leq 360°$.

a) $\cos\theta = \frac{\sqrt{3}}{2}$ b) $\sin\theta = \frac{1}{2}$

18. Determine h.

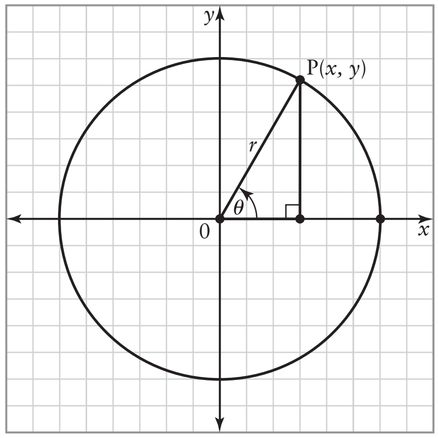


☆ 19. Given $\sin\theta = \frac{y}{r}$ and $\cos\theta = \frac{x}{r}$, show that $\frac{\sin\theta}{\cos\theta} = \tan\theta$.

20. Given $\sin\theta = \frac{y}{r}$, $\cos\theta = \frac{x}{r}$, and $x^2 + y^2 = r^2$, show that $\sin^2\theta + \cos^2\theta = 1$.

KEY CONCEPTS

- For any rotation angle θ in standard position that has a point P(x, y) on its terminal arm, the primary trigonometric ratios for the angle are $\sin\theta = \frac{y}{r}$, $\cos\theta = \frac{x}{r}$, and $\tan\theta = \frac{y}{x}$, where $r = \sqrt{x^2 + y^2}$.
- Pairs of related angles can be found using the coordinates of the endpoints of their terminal arms and a reference angle in quadrant I.

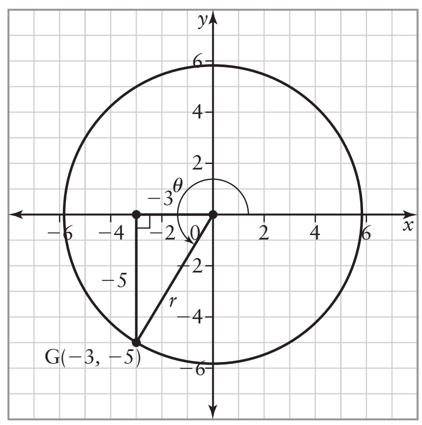


Example

The coordinates of a point on the terminal arm of an angle θ in standard position are G(−3, −5). Determine the exact primary trigonometric ratios for θ.

Solution

Plot G(−3, −5) on a coordinate grid.



Determine r.

$r = \sqrt{x^2 + y^2}$

$r = \sqrt{(-3)^2 + (-5)^2}$

$r = \sqrt{34}$

$$\sin\theta = \frac{y}{r} = \frac{-5}{\sqrt{34}} = -\frac{5}{\sqrt{34}}$$

$$\cos\theta = \frac{x}{r} = \frac{-3}{\sqrt{34}} = -\frac{3}{\sqrt{34}}$$

$$\tan\theta = \frac{y}{x} = \frac{-5}{-3} = \frac{5}{3}$$

A

Unless specified otherwise, all angles are between 0° and 360°.

1. Use a calculator. Evaluate each trigonometric ratio. Round decimal answers to four decimal places.

a) sin 58° **b)** tan 321°

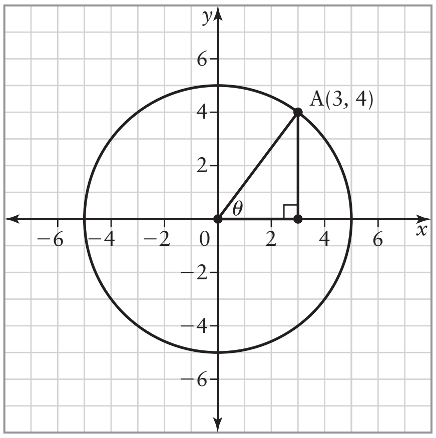
c) sin 144° **d)** cos 44°

e) cos 312° **f)** tan 68°

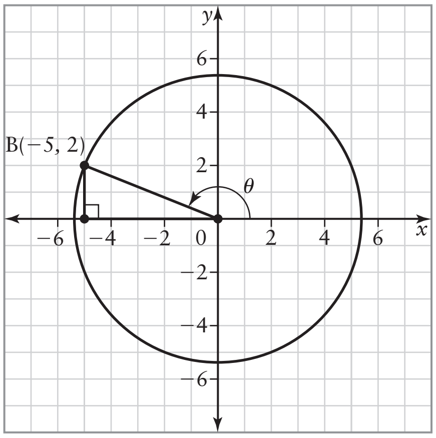
g) tan 152° **h)** sin 260°

2. The coordinates of a point on the terminal arm of an angle θ are shown. Determine the exact primary trigonometric ratios for θ.

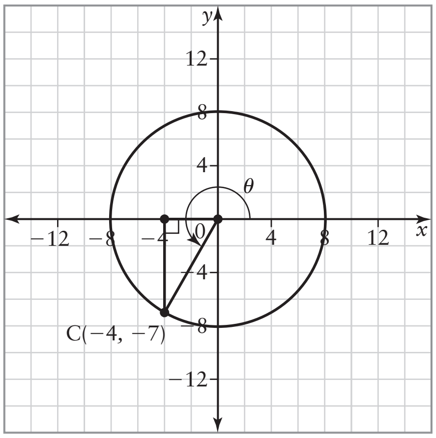
a) A(3, 4)



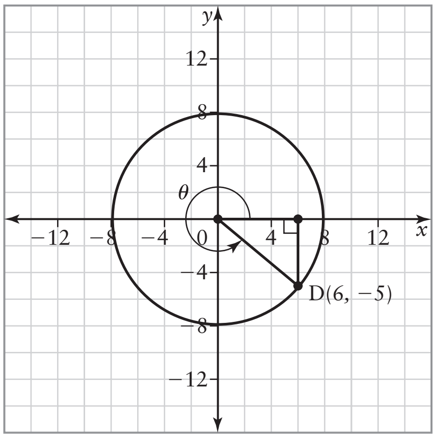
b) B(−5, 2)



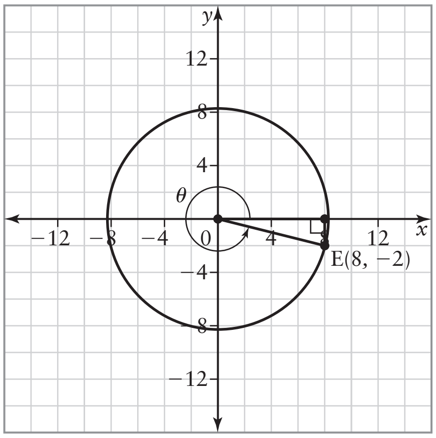
c) C(−4, −7)



d) D(6, −5)



e) E(8, −2)



3. The coordinates of a point on the terminal arm of an angle θ are given. Determine the exact primary trigonometric ratios for θ.

a) E(-5, 8)

b) F(7, -2)

c) G(-3, -5)

d) H(2, 3)

e) I(6, -1)

f) J(4, -6)

g) K(-7, 9)

h) L(5, -3)

4. One of the primary trigonometric ratios for an angle is given, as well as the quadrant in which the terminal arm lies. Determine the other two primary trigonometric ratios. Give exact answers.

a) $\sin A = \frac{4}{5}$, quadrant I

☆**b)** $\cos B = -\frac{5}{8}$, quadrant III

c) $\tan C = -\frac{6}{5}$, quadrant II

d) $\sin D = \frac{4}{9}$, quadrant I

e) $\cos E = \frac{4}{11}$, quadrant III

f) $\tan F = -\frac{6}{5}$, quadrant IV

g) $\tan G = -\frac{10}{3}$, quadrant IV

h) $\cos H = \frac{5}{9}$, quadrant IV

5. Determine the exact primary trigonometric ratios for each angle. You may wish to use a unit circle to help you.

a) $\angle A = 150°$

b) $\angle B = 90°$

c) $\angle C = 135°$

d) $\angle D = 300°$

e) $\angle E = 330°$

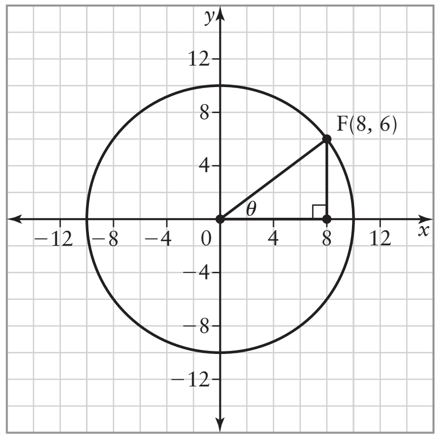
f) $\angle F = 240°$

g) $\angle G = 225°$

h) $\angle H = 180°$

B

☆**6.** The coordinates of a point F on the terminal arm of an angle θ are shown.



a) Determine the primary trigonometric ratios for angle θ.

b) Determine the coordinates of the endpoints of the terminal arm for each angle.

i) $180° - \theta$

ii) $180° + \theta$

iii) $360° - \theta$

c) Determine the primary trigonometric ratios for each of the angles in part b).

d) How do the trigonometric ratios for angle θ and the angles in part b) compare?

☆**7.** Consider $\angle A$ in quadrant I, such that $\sin A = \frac{12}{13}$.

a) Draw a diagram.

b) Determine exact values for cos A and tan A.

c) How would your answers to parts a) and b) change if the quadrant in which $\angle A$ was located was not specified?

8. Suppose $\angle B$ lies in quadrant II and $\cos B = -\frac{7}{25}$. Draw a diagram, and then determine exact values for sin B and tan B.

9. Given $\tan C = \frac{7}{4}$ with $\angle C$ in quadrant III, determine exact values for sin C and cos C.

10. Consider $\angle A$ in standard position on a coordinate grid such that $\sin A = \frac{9}{41}$ and $\tan A = -\frac{9}{40}$.

a) Can the ratios for sin A and tan A be used to determine the quadrant in which $\angle A$ is located? Explain.

b) Determine the exact value for cos A.

c) How would your answers for parts a) and b) change if the ratios were $\sin A = -\frac{9}{41}$ and $\tan A = \frac{9}{40}$?

d) How would your answers for parts a) and b) change if the ratios were $\sin A = -\frac{9}{41}$ and $\tan A = -\frac{9}{40}$?

11. Suppose $\angle D$ lies in quadrant IV and $\sin D = -\frac{5}{\sqrt{34}}$. Determine the exact values for cos D and tan D.

12. Determine one other angle that has the same trigonometric ratio as each given angle.

a) sin 45°

b) cos 300°

c) tan 225°

d) sin 230°

e) cos 115°

f) tan 310°

13. Are the angles in each pair related? Explain.

a) 135° and 315°

b) 40° and 320°

c) 45° and 215°

d) 150° and 210°

e) 210° and 330°

f) 125° and 245°

14. Use Technology

a) Use geometry software to construct a circle with radius 3 units. Plot a point A on the circle in quadrant I.

b) Measure the distance between point A and the origin, O. Construct segment OA and label it *r*. Determine the coordinates of point A.

c) Calculate the ratios $\frac{x}{r}$, $\frac{y}{r}$, and $\frac{y}{x}$.

d) Measure $\angle A$.

e) Calculate sin A, cos A, and tan A. Compare your answers to the ratios from part c).

C

15. An angle θ is in standard position on a coordinate grid. The terminal arm of θ is in quadrant III on the line defined by the given equation. Determine the exact values of the three primary trigonometric ratios for angle θ.

a) $y = x$

b) $y = 2x$

c) $y = 4x$

16. Refer to your answer to question 15. How does $\tan \theta$ relate to the equation of the line through the terminal arm?

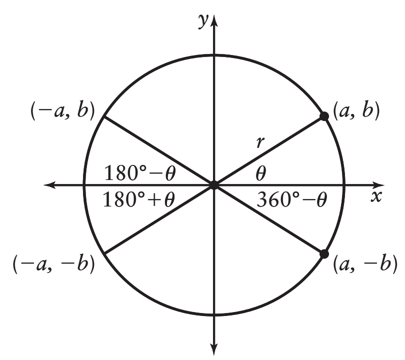
17. A 10-m guy wire is attached to a post at ground level and to the top of a nearby vertical antenna. The angle formed between the guy wire and the ground is 60°.

a) Determine the height of the antenna to the nearest tenth of a metre.

b) Determine the horizontal distance from the post to the antenna to the nearest tenth of a metre.

KEY CONCEPTS

- Exactly two angles between 0° and 360° have the same sine ratio.
  For example, $\sin\theta = \sin(180° - \theta) = \frac{b}{r}$
- Exactly two angles between 0° and 360° have the same cosine ratio.
  For example, $\cos\theta = \cos(360° - \theta) = \frac{a}{r}$
- Exactly two angles between 0° and 360° have the same tangent ratio.
  For example, $\tan\theta = \tan(180° + \theta) = \frac{b}{a}$



Example

Given $\cos\theta = \frac{4}{5}$, determine θ, where $0 \le \theta \le 360°$.
Then, determine $\sin\theta$ and $\tan\theta$.

Solution

Determine the measure of angle θ in quadrant I for which $\cos\theta = \frac{4}{5}$.

$$\cos\theta = \frac{4}{5}$$
$$\angle\theta = \cos^{-1}\left(\frac{4}{5}\right)$$
$$= 36.8698...°$$
$$\doteq 36.9°$$

The cosine ratio is positive in quadrants I and IV, so there is another angle for which $\cos\theta = \frac{4}{5}$ in quadrant IV.

$$\angle\theta = 360° - 36.9°$$
$$= 323.1°$$

Given $\cos\theta = \frac{4}{5}$, the angle θ is approximately 37° or 323°.

If $\cos\theta = \frac{4}{5}$, $x = 4$ and $r = 5$. Determine y.

$$r^2 = x^2 + y^2$$
$$5^2 = (4)^2 + y^2$$
$$25 = 16 + y^2$$
$$9 = y^2$$
$$\pm 3 = y$$

Write the sine and tangent ratios for $\angle\theta$.

$\sin\theta = \pm\frac{3}{5}$ $\quad$ $\tan\theta = \pm\frac{3}{4}$

A

Unless specified otherwise, all angles are between 0° and 360°.

1. Use a calculator to calculate each pair of ratios. Round decimal answers to four decimal places.
   - **a)** sin 58°, sin 122°
   - **b)** cos 117°, cos 243°
   - **c)** tan 238°, tan 58°
   - **d)** sin 310°, sin 230°
   - **e)** cos 82°, cos 278°
   - **f)** tan 266°, tan 86°
   - **g)** sin 65°, sin 115°
   - **h)** tan 109°, tan 289°

2. What do you notice about each pair of ratios in question 1? Explain.

3. Use a calculator to evaluate each ratio to four decimal places. Determine a second angle with the same ratio.
   - **a)** sin 89°
   - **b)** cos 335°
   - **c)** sin 132°
   - **d)** tan 140°
   - **e)** cos 155°
   - **f)** tan 305°
   - **g)** cos 307°
   - **h)** sin 13°

4. The coordinates of a point on the terminal arm of an angle θ are given. Determine the primary trigonometric ratios for θ. Round decimal answers to four decimal places.
   - **a)** A(5, 3) **b)** B(−4, 7)
   - **c)** C(−6, −2) **d)** D(2, −1)
   - **e)** E(10, 3) **f)** F(−5, −7)
   - **g)** G(−8, 6) **h)** H(−1, −2)

5. Use a calculator to determine the primary trigonometric ratios for each angle. Round decimal answers to four decimal places.
   - **a)** 80°
   - **b)** 110°
   - **c)** 200°
   - **d)** 324°
   - **e)** 47°
   - **f)** 192°
   - **g)** 217°
   - **h)** 345°
   - **i)** 13°
   - **j)** 270°

6. Find the values of θ, where $0° \leq \theta \leq 360°$.
   - **a)** $\sin \theta = \frac{\sqrt{3}}{2}$
   - **b)** $\cos \theta = \frac{1}{\sqrt{2}}$
   - **c)** $\tan \theta = \sqrt{3}$
   - **d)** $\sin \theta = 1$
   - **e)** $\cos \theta = \frac{\sqrt{3}}{2}$
   - **f)** $\tan \theta = 1$

B

☆7. Determine two angles between 0° and 360° that have a sine ratio of $\frac{\sqrt{3}}{2}$. Do not use a calculator.

8. Use a diagram to determine two angles between 0° and 360° that have a cosine ratio of $-\frac{1}{2}$. Do not use a calculator.

9. The tangent ratio of each of two angles between 0° and 360° is $-\frac{1}{\sqrt{3}}$. Without using a calculator, determine the angles.

10. Two angles between 0° and 360° have a tangent ratio of −1. Without using a calculator, determine the angles.

11. The point T(3, 4) is on the terminal arm of ∠B in standard position.

a) Draw and label a diagram.

b) Explain how you would determine the primary trigonometric ratios for ∠B.

c) Determine the three primary trigonometric ratios for ∠B.

d) Explain how you would determine the measure of ∠B.

e) Determine the measure of ∠B to the nearest degree.

f) How would the answer for parts a), c), and e) change if point T was reflected in the x-axis?

g) How would the answer for parts a), c), and e) change if point T was reflected in the y-axis?

12. Consider an angle, ∠C, that lies in quadrant III, such that tan C = 0.4663.

a) Draw a diagram to represent this situation.

b) Determine the measure of ∠C to the nearest degree. Explain how you determined the measure of ∠C.

13. Use a calculator to find the values of θ to the nearest degree, where $0° \leq \theta \leq 360°$.

a) $\sin \theta = 0.7312$ **b)** $\cos \theta = 0.4538$

c) $\tan \theta = -1.7321$ **d)** $\sin \theta = 0.9534$

e) $\cos \theta = 0.8862$ **f)** $\tan \theta = 1$

g) $\sin \theta = -0.7317$ **h)** $\cos \theta = -0.3640$

i) $\tan \theta = 2.4751$ **j)** $\sin \theta = -0.9511$

k) $\cos \theta = 0.1829$ **l)** $\tan \theta = 0.0543$

14. Determine another angle that has the same trigonometric ratio as each given angle. Draw a sketch with both angles labelled.

a) sin 75° **b)** cos 190°

c) tan 355° **d)** sin 252°

15. Draw a diagram, and then determine values for the other primary trigonometric ratios, to four decimal places.

a) sin A = 0.9138; ∠A lies in quadrant I

b) cos B = −0.2145; ∠B lies in quadrant II

c) tan C = −8.144; ∠C lies in quadrant IV

16. Determine the approximate measures of all angles from 0° to 360° in each case.

a) The sine ratio is 0.3195.

b) The tangent ratio is 1.4385.

c) The cosine ratio is −0.7431.

☆ **17. a)** If $\cos \theta = \frac{1}{3}$, find two possible values for $\sin \theta$.

b) For each value of $\sin \theta$ from part a), find the value(s) of θ.

☆ **18.** The point S(−5, −6) is on the terminal arm of ∠A.

a) Determine the primary trigonometric ratios for ∠A.

b) Determine the measure of ∠A.

c) Determine the primary trigonometric ratios for ∠B such that sin B = sin A.

d) Determine the measure of ∠B.

C

☆ **19. a)** Solve $2x^2 - x - 1 = 0$.

b) Explain how the equation in part a) is related to $2 \sin^2 \theta - \sin \theta - 1 = 0$.

c) Solve $2 \sin^2 \theta - \sin \theta - 1 = 0$.

20. Determine all the possible measures of θ, where $0° \leq \theta \leq 360°$.

a) $\cos^2 \theta - 1 = 0$ **b)** $\tan^2 \theta = 3$

21. Given $\tan A = \frac{a + b}{a - b}$ and ∠A in quadrant I, determine expressions for sin A and cos A. State any restrictions on the values of a and b.

KEY CONCEPTS

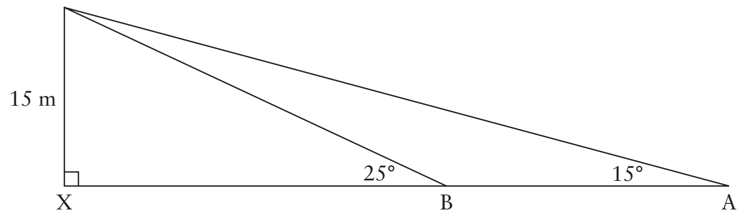
- The primary trigonometric ratios can be used to solve problems that involve right triangles.
- Problems involving right triangles in two dimensions and in three dimensions can be solved using the Pythagorean theorem and/or the primary trigonometric ratios.

Example

From an army tank, the angle of elevation to the top of a 15-m tower is 15°. After advancing toward the tower, the angle of elevation is 25°. How far did the tank advance?

Solution

Sketch and label a diagram.



Write an expression for the initial distance from the tank to the tower, AX.

$$\tan 15° = \frac{15}{AX}$$
$$AX = \frac{15}{\tan 15°}$$

Write an expression for the new distance from the tank to the tower, BX.

$$\tan 25° = \frac{15}{BX}$$
$$BX = \frac{15}{\tan 25°}$$

Determine the distance the tank advanced, AX − BX.

$$\begin{aligned} \text{Distance} &= AX - BX \\ &= \frac{15}{\tan 15°} - \frac{15}{\tan 25°} \\ &= 23.8131\ldots \\ &\doteq 23.8 \end{aligned}$$

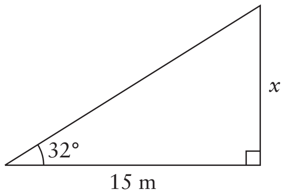
The tank advanced approximately 23.8 m.

A

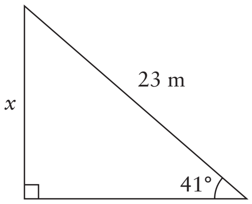
Unless specified otherwise, all angles are between 0° and 360°. Round all lengths to the nearest tenth of a unit and all angle measures to the nearest degree.

1. Determine the length of x.

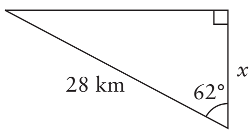
a)



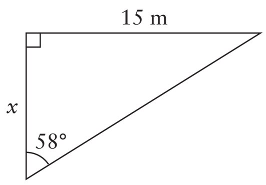
b)



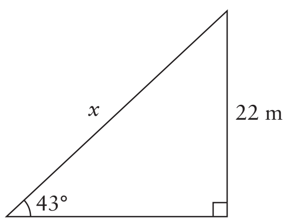
c)



d)



e)



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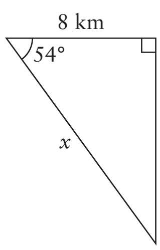
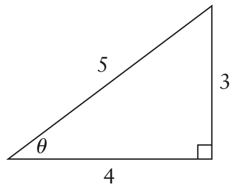


2. Determine sin θ, cos θ, and tan θ. Then, determine the measure of angle θ.

a)



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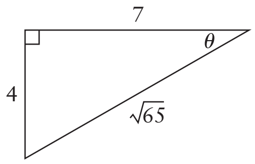

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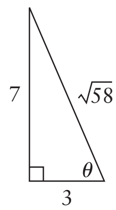

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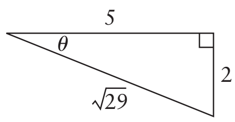
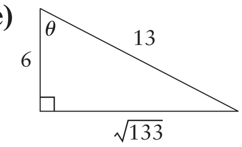
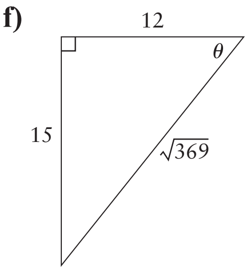


e)



f)



3. Sketch each triangle and solve if possible.

a) In $\triangle$ABC, $\angle$A = 90°, $\angle$B = 36°, and b = 15.3 m

b) In $\triangle$DEF, $\angle$D = 90°, $\angle$E = 48°, and e = 9.6 cm

c) In $\triangle$JKL, $\angle$J = 90°, $\angle$K = 62°, and j = 7.2 km

d) In $\triangle$PQR, $\angle$P = 53°, $\angle$Q = 37°, and p = 13.5 m

4. From a point 5 m from the base of a tree, the angle of elevation to the top of the tree is 38°. Draw a diagram, and then determine the height of the tree.

5. The shadow of a building that is 28 m in height measures 20 m in length. Draw a diagram, and then determine the angle of elevation of the sun.

6. If you know a triangle contains a right angle, what other information is required to solve the triangle?

B

7. A rectangle is 16 cm long and 12 cm wide.

   a) Determine the length of the diagonals of the rectangle.

   b) Determine the measure of the angle formed by the length of the triangle and a diagonal.

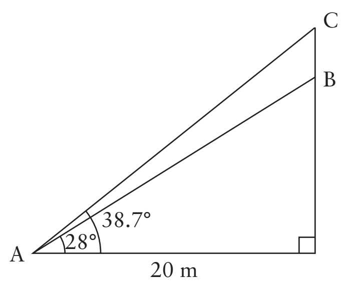
8. Two guy wires from the top of a radio tower are fixed to the ground 20 ft apart, at points A and B. The tower is halfway between points A and B. From both points, the angle of elevation to the top of the tower is 72°.

   a) Draw a diagram.

   b) Determine the height of the tower.

   c) Determine the total length of the two guy wires.

9. Kevin is standing at a point A, 20 m from the base of a building. From point A, the angle of elevation to point B is 28°, and the angle of elevation to point C is 38.7°. Determine the vertical distance between points B and C.



☆10. Zac is standing 10 m from the centre of a fountain. Water shoots straight up from the fountain's centre. Shortly after, the angle of inclination of the highest point of the water is 46°. Fifteen seconds later, the angle of inclination is 60°.

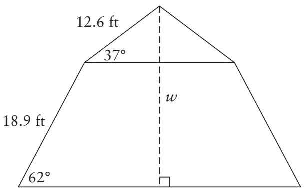
   a) Draw a diagram.

   b) Determine the height of the highest point of the water at each angle of inclination.

   c) Determine the average speed, in metres per second, at which the water shoots up.

11. Explain how you could find the height of a radio antenna on top of a tall observation tower using a measuring tape, a clinometer, and trigonometry.

☆12. A patio has the shape shown. Determine the total width, *w*, of the patio.

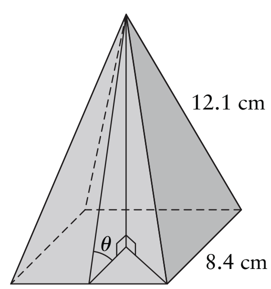


13. Boat A is 5 km northeast of a lighthouse, L, and boat B is 7 km southeast of the lighthouse.

   a) Determine the distance between the boats.

   b) Determine the measure of ∠ALB.

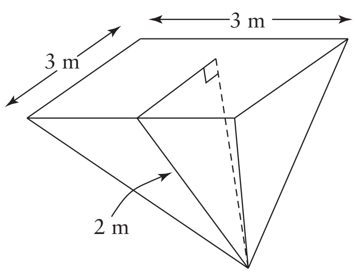
14. The base of a square-based pyramid has length 8.4 cm. The slant height of the pyramid is 12.1 cm.



   a) Determine the height of the pyramid.

   b) What is the angle between one triangular face of the pyramid and the base?

15. Peter designed a garden pond, which is in the shape of an inverted square-based pyramid. The side lengths of the base are 3 m, and the slant height is 2 m.

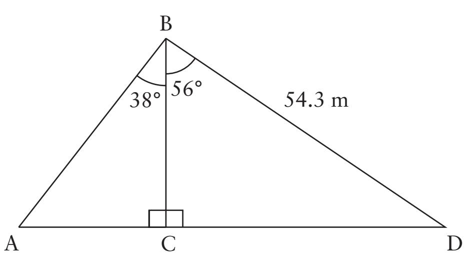


a) Determine the greatest depth of the garden pond.

b) Determine the maximum volume of water the pond can hold.

16. The roof of a house is constructed in the shape of a square-based pyramid, and the lengths of the sides of the base are 26 ft. The angle of elevation formed by each side of the roof with the top of the house is 32°. Determine the height of the roof.

17. Determine the distance AC.



18. Ella is standing on a bridge. From her location, the angle of elevation to the top of a nearby building is 28°, and the angle of depression to the base of the building is 42°. The least distance between Ella and the building is 40 m. Determine the height of the building.

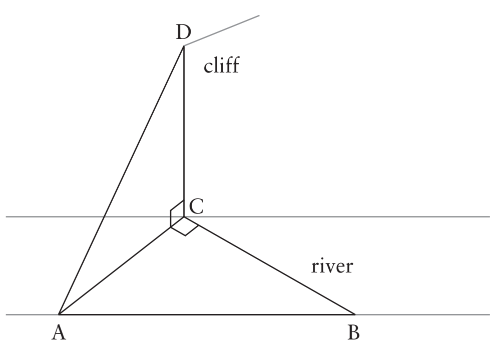
19. Adriana is standing at point A, directly across the river from point B, at the base of a cliff. From point A, the angle of elevation to point C at the top of the cliff is 60°. Dominic is standing at point D, on the same side of the river and 30 ft farther down from point A. AB = DB and ∠ABD = 90°.

a) Is there enough information to determine, BC? Explain.

b) If your answer to part a) is yes, determine BC. If your answer to part a) is no, explain what information is needed.

C

20. Louis is standing at point A on the shoreline of a river. From where he is standing, the angle of elevation to point D at the top of a cliff across the river is 25°. Louis walks to point B, which is 30 m east of point A, and determines that ∠ACB is 90° and ∠CAB is 38°. Determine the height of the cliff.



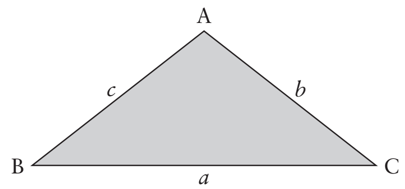
21. From point A on the level ground, the angle of elevation of the top of a building is 25°. From point B, 10 ft closer to the base of the building, the angle of elevation of the top of the building is 35°. Determine the height of the building.

KEY CONCEPTS

- For any $\triangle ABC$, the sine law states that

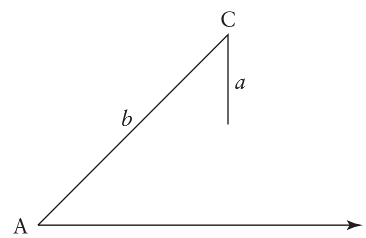
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

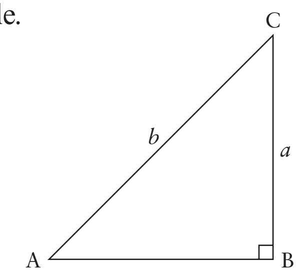


- A side length can be determined if the corresponding opposite angle and one other side-angle pair are known.
- An angle measure can be determined if the corresponding opposite side and one other side-angle pair are known.
- If the lengths of two sides and the measure of one angle are known, the ambiguous case is possible. Given $\triangle ABC$ with known side lengths a and b and known $\angle A$, if $a < b$, there are three possibilities:

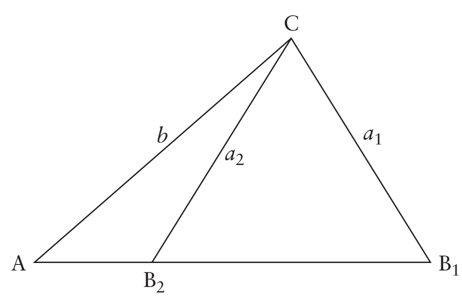
If $a < b \sin A$, then no triangle is possible.



If $a = b \sin A$, then only one right triangle is possible.

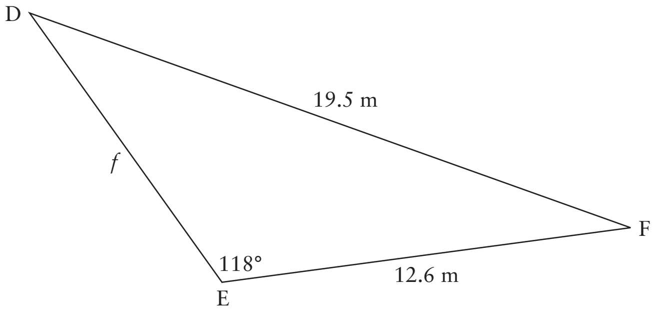


If $a > b \sin A$, then two triangles are possible.
This is the ambiguous case.



Example

Determine the length of f to one decimal place.



Solution

Two side lengths and one angle measure are given. Check for the ambiguous case.

Side e is opposite the known angle.

Since $e > d$, only one triangle is possible.

Determine ∠D.

$$\frac{\sin D}{d} = \frac{\sin E}{e}$$

$$\frac{\sin D}{12.6} = \frac{\sin 118°}{19.5}$$

$$\sin D = \frac{12.6 \sin 118°}{19.5}$$

$$\begin{aligned}\angle D &= \sin^{-1}\left(\frac{12.6 \sin 118°}{19.5}\right)\\ &= 34.7864\ldots°\\ &\doteq 34.8°\end{aligned}$$

Determine ∠F.

$$\begin{aligned}\angle F &= 180° - (118° + 34.8°)\\ &= 27.2°\end{aligned}$$

Determine f.

$$\frac{f}{\sin F} = \frac{e}{\sin E}$$

$$\frac{f}{\sin 27.2°} = \frac{19.5}{\sin 118°}$$

$$\begin{aligned}f &= \frac{19.5 \sin 27.2°}{\sin 118°}\\ &= 10.0950\ldots\\ &\doteq 10.1\end{aligned}$$

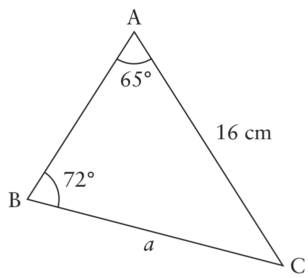
The length of side f is approximately 10.1 m.

A

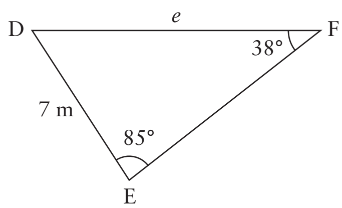
Unless specified otherwise, all angles are between 0° and 360°. Round all lengths to the nearest tenth of a unit and all angle measures to the nearest degree.

1. Determine the length of each indicated side.

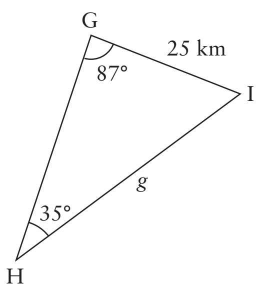
a)



b)



c)



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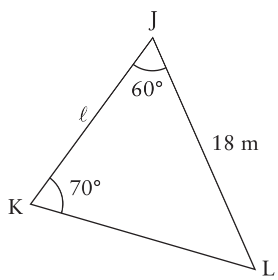
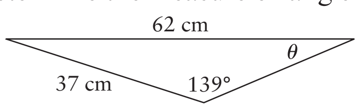
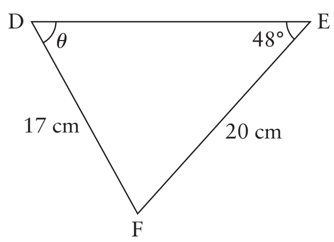


2. Determine the measure of angle θ.

a)



b)



3. Determine the length of the indicated side.

a) In △ABC, ∠A = 52°, ∠B = 43°, and b = 9 m. Determine a.

b) In △DEF, ∠D = 65°, ∠E = 49°, and e = 15.7 cm. Determine f.

4. Solve each triangle.

a) In △ABC, ∠A = 72°, ∠B = 47°, and a = 15 km.

b) In △DEF, ∠D = 65°, ∠F = 51°, and d = 8 m.

c) In △JKL, ∠K = 52°, k = 25 cm, and ℓ = 28 cm.

d) In △PQR, ∠R = 35°, p = 30 m, and r = 37 m.

5. In △HJK, h = 7.2 m, j = 8.4 m, and ∠H = 68°.

a) Calculate j sin H.

b) How many solutions are possible for △HJK?

6. In △ABC, a = 4.5 cm, b = 5.2 cm, and ∠A = 37°.

a) Calculate b sin A.

b) How many different △ABC are possible?

c) Solve △ABC.

7. In △PQR, p = 5.3 km, q = 10.6 km, and ∠P = 30°.

a) Determine the number of possible solutions for △PQR.

b) Solve △PQR.

B

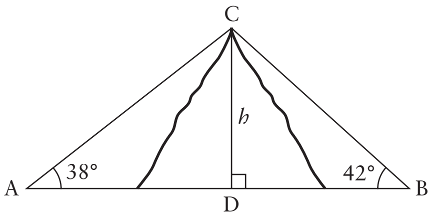
☆**8.** Given △MNP with m = 6.7 cm, n = 12.4 cm, and ∠M = 26°, determine p.

9. Sketch each triangle. Then, calculate the length of the third side.

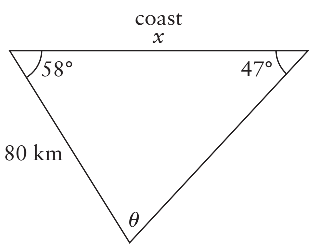
a) In △ABC, c = 10 cm, b = 12 cm, and ∠B = 44°.

b) In △RST, r = 6.2 m, s = 8.1 m, and ∠R = 39°.

☆ **10.** Sheila is standing at point A west of a mountain in the Kootenay region of British Columbia. From point A, the angle of elevation of the top of the mountain is 38°. From point B, which is 10 560 ft to the east of point A, the angle of elevation of the top of the same mountain is 42°. Determine the height of the mountain.



☆ **11.** A triangular delta has been formed at the mouth of a large river by sediment deposits. The distance from the coast to the starting point of the delta is 80 km, and the angles formed by the coastline and two sides of the triangular delta are 58° and 47°. Determine the length of the delta along the coastline.



12. A triangular deck is to be added to the back wall of a house. The length of the side of the deck that will be attached to the house is 20 ft, and the other two sides form angles of 32° and 65° with the side that is attached to the wall of the house.

a) Draw a diagram to represent this situation.

b) Determine the length of the shortest side of the deck.

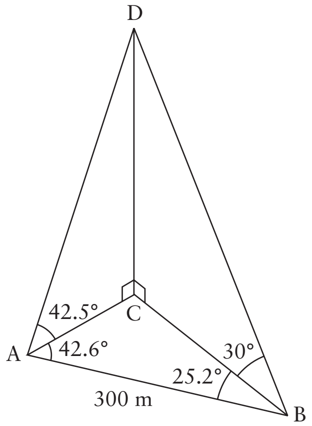
13. Michael is an architect who has designed a cottage with a roof that has two sides with different slopes. The shorter side of the roof is 12 ft in length and makes an angle of 68° with the top of the building. The longer side of the roof makes an angle of 28° with the top of the building.

a) Draw a diagram to represent this situation.

b) Determine the length of the other side of the roof.

C

14. A rocket, launched vertically from point C, is tracked by two tracking stations at A and B. Data from the launch were recorded according to the diagram below.



a) Determine the height, *h*, of the rocket as calculated from tracking station A.

b) Determine the height, *h*, of the rocket as calculated from tracking station B.

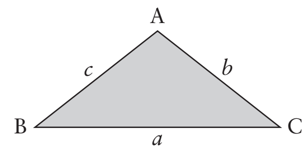
c) Are the approximate results for the calculated heights of the rocket the same or different for parts a) and b)? Explain.

15. Two boats, A and B, are in a harbour close to a marina at point C. Boat A is 38 ft from the marina. The angle between AC and BC is 51°, and the angle between AC and AB is 65°. From boat A, the angle of depression to the anchor directly below boat B is 50°. Determine the distance from boat A to the anchor directly below boat B.

1.6 Solving Problems Using the Cosine Law

KEY CONCEPTS

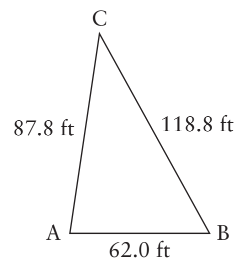
- The cosine law states that for any △ABC,
  $a^2 = b^2 + c^2 - 2bc \cos A$
  $b^2 = a^2 + c^2 - 2ac \cos B$
  $c^2 = a^2 + b^2 - 2ab \cos C$



- A side length can be determined if the lengths of the other two sides and the measure of the contained angle are known.
- An angle measure can be determined if all three side lengths are known.

Example

The lengths of the sides of a triangular property on Sydenham Lake are 62.0 ft, 87.8 ft, and 118.8 ft. Determine the angles formed between the sides of the triangular property to the nearest tenth of a degree.



Solution

For △ABC, the side lengths are $a = 118.8$, $b = 87.8$, and $c = 62.0$.
Use the cosine law to determine ∠A.

$$\begin{aligned}\cos A &= \frac{b^2 + c^2 - a^2}{2bc} \\ &= \frac{87.8^2 + 62.0^2 - 118.8^2}{2(87.8)(62.0)} \\ &= -0.2351\ldots \\ \angle A &= 103.6030\ldots^\circ\end{aligned}$$

Check for the ambiguous case. Given ∠A and $a > b$, only one triangle is possible.
Use the sine law to determine ∠B.

$$\begin{aligned}\frac{\sin B}{87.8} &= \frac{\sin 103.6030\ldots^\circ}{118.8} \\ \sin B &= \frac{87.8 \sin 103.6030\ldots^\circ}{118.8} \\ &= 0.7183\ldots \\ \angle B &= 45.9164\ldots^\circ\end{aligned}$$

Determine ∠C.

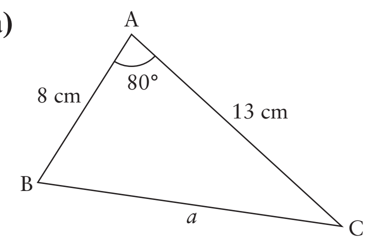
$$\begin{aligned}\angle C &= 180^\circ - (103.6030\ldots^\circ + 45.9164\ldots^\circ) \\ &= 30.4805\ldots^\circ \\ &\doteq 30.5^\circ\end{aligned}$$

A

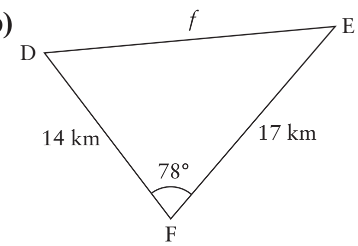
Unless specified otherwise, all angles are between 0° and 360°. Round all lengths to the nearest tenth of a unit and all angle measures to the nearest degree.

1. Determine the length of the indicated side.

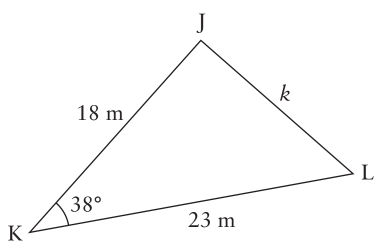
a)



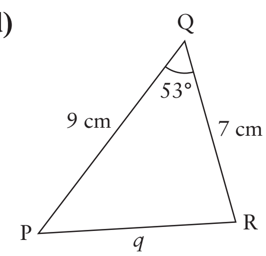
b)



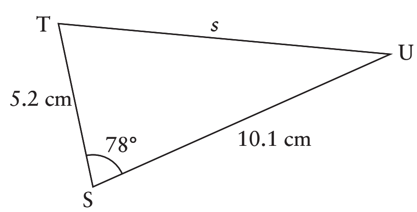
c)



d)

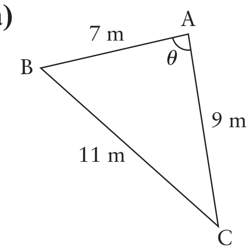


e)

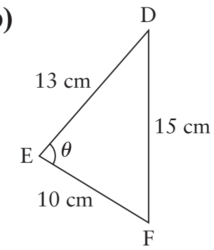


2. Determine the measure of angle θ.

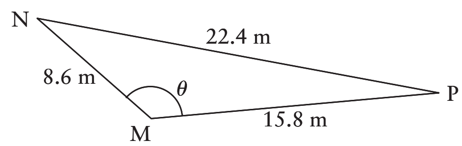
a)



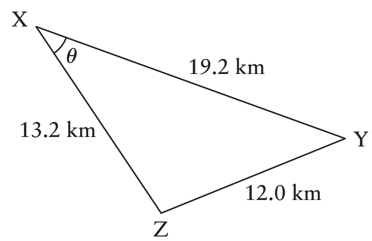
b)



c)



d)



3. Determine the length of the indicated side.

a) In $\triangle ABC$, $\angle A = 32°$, $b = 22$ cm, and $c = 15$ cm. Determine a.

b) In $\triangle DEF$, $\angle F = 71°$, $d = 9$ m, and $e = 5$ m. Determine f.

4. Sketch, and then solve if possible.

a) In $\triangle ABC$, $\angle A = 48°$, $b = 18$ cm, and $c = 12$ cm.

b) In $\triangle JKL$, $j = 23$ cm, $k = 25$ cm, and $l = 27$ cm.

c) In $\triangle PQR$, $p = 15$ m, $q = 17$ m, and $r = 20$ m.

d) In $\triangle DEF$, $\angle D = 62°$, $e = 12$ m, and $f = 10$ m.

B

5. An air traffic controller is tracking two airplanes, flying at the same altitude, on the radar screen. Airplane A, en route to Saskatoon, is 23.8 km away in a direction 60° west of north. Airplane B, en route to Calgary, is 25.2 km away in a direction 70° west of north.
   a) Draw a diagram to illustrate this situation.
   b) How far apart are the two airplanes?

☆6. Building plans show a triangular cottage property with side lengths 42.7 m, 49.4 m, and 58.2 m.
   a) Sketch a diagram of this property.
   b) Determine the angles formed between the sides of the triangular property.

7. A funnel is in the shape of a right circular cone. The angle at the vertex of the cone is 36°. Find the diameter of the funnel at a point on the face 6 in. from the vertex.

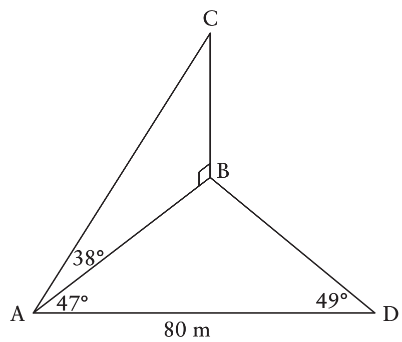
☆8. Two ships, A and B, both started out at point C. Ship A is now 40 km from point C in a direction 30° west of north. Ship B is now 60 km from point C in a direction 40° west of south. Determine the distance between the two ships.

☆9. Two checkpoints, A and B, in an orienteering course are 5 km apart. Another checkpoint, C, is 3 km from checkpoint A. The angle between AB and AC is 25°. Determine the distance between checkpoints B and C.

10. The pendulum on a clock is 30 cm long. The pendulum moves a horizontal distance of 5 cm from one end of each swing to the other. Find the angle through which the pendulum swings.

11. Bill started at point A and walked 22 m to a point B, then walked 17 m to a point C, and then walked back to point A. The path of his walk formed a triangle with $\angle BAC = 42°$. Leslie calculated that the distance Bill walked from point C to point A was 24.9 m. Kelly calculated that the distance that Bill walked from point C to point A was 7.8 m.
   a) Do you think that Leslie's, Kelly's, or both girls' calculations are correct?
   b) Draw a diagram to represent Leslie's calculations for the distance Bill walked from point C to point A.
   c) Draw a diagram to represent Kelly's calculations for the distance Bill walked from point C to point A.
   d) Explain the differences between Leslie's calculations and Kelly's calculations.

12. Faiza would like to calculate the height of a cliff. From point A where she is standing, the angle of elevation to point C at the top of the cliff is 38°. If Faiza walks 80 m east to point D, $\triangle ABD$ is formed, where B is the base of the cliff, such that $\angle DAB = 47°$ and $\angle ADB = 49°$.



   a) Determine the height, BC, of the cliff.
   b) Determine the distance from point A, where Faiza is standing, to point C at the top of the cliff.

13. A drill bit is in the shape of a cone. The angle at the vertex of the cone is 15°. The length of the slanted side of the drill bit is 13 mm. Determine the diameter at the top of the drill bit.

14. In order to measure the height of a cliff, AB, a surveyor uses a baseline, CD, and records the following data: $\angle BCD = 64.3°$, $\angle BDC = 55.2°$, CD = 240 m, and $\angle ACB = 28°$. Draw a diagram to illustrate this situation, and then determine the height of the cliff.

C

15. Two roads intersect at an angle of 38°. Two cars, A and B, leave the intersection at the same time and both travel in an easterly direction. After 2 h, car A has travelled 140 km and car B has travelled 160 km. At this time, a hot-air balloon is directly above the line between the two cars. The balloonist notices that the angle of depression to the faster car, which is 1 km away from the hot-air balloon, is 25°. Determine the distance from the hot-air balloon to the slower car.

Chapter 1: Checklist

By the end of this chapter, I will be able to:

- determine the values of the trigonometric ratios for angles between 0° and 360°
- solve problems using the primary trigonometric ratios, the sine law, and the cosine law
- determine the exact values of the sine, cosine, and tangent ratios of the special angles 0°, 30°, 45°, 60°, 90°, and their multiples
- determine the values of the sine, cosine, and tangent ratios of angles from 0° to 360°, through investigation using a variety of tools and strategies
- determine the measures of two angles from 0° to 360° for which the value of a given trigonometric ratio is the same
- solve multi-step problems in two and three dimensions, including those that arise from real-world applications by determining the measures of the sides and angles of right triangles, using the primary trigonometric ratios
- solve problems involving oblique triangles, including those that arise from real-world applications, using the sine law (including the ambiguous case) and the cosine law