Shigley's Mechanical Engineering Design
Tutorial 3 –14: Pressure Vessel Design

PRESSURE VESSEL DESIGN MODELS FOR CYLINDERS:

1. Thick-walled Cylinders
2. Thin-walled Cylinders

THICK-WALL THEORY

- Thick-wall theory is developed from the Theory of Elasticity which yields the state of stress as a continuous function of radius over the pressure vessel wall. The state of stress is defined relative to a convenient cylindrical coordinate system:

    1. σ_t — Tangential Stress
    2. σ_r — Radial Stress
    3. σ_l — Longitudinal Stress

- Stresses in a cylindrical pressure vessel depend upon the ratio of the inner radius to the outer radius (r_o / r_i) rather than the size of the cylinder.
- Principal Stresses ($\sigma_1, \sigma_2, \sigma_3$)

    1. Determined without computation of Mohr's Circle;
    2. Equivalent to cylindrical stresses ($\sigma_t, \sigma_r, \sigma_l$)

- Applicable for any wall thickness-to-radius ratio.

Cylinder under Pressure

Consider a cylinder, with capped ends, subjected to an internal pressure, p_i, and an external pressure, p_o,

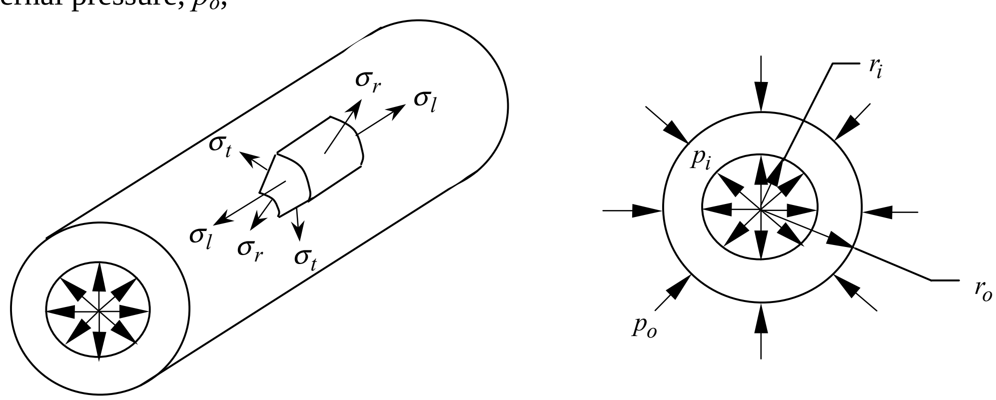


FIGURE T3-14-1

† *Text Eq.* refers to *Shigley's Mechanical Engineering Design,* 8th edition text by Richard G. Budynas and J. Keith; Nisbett; equations and figures with the prefix *T* refer to the present tutorial.

The cylinder geometry is defined by the inside radius, r_i, the outside radius, r_o, and the cylinder length, *l*. In general, the stresses in the cylindrical pressure vessel (σ_t, σ_r, σ_l) can be computed at any radial coordinate value, *r*, within the wall thickness bounded by r_i and r_o, and will be characterized by the ratio of radii, $\zeta = r_o / r_i$. These cylindrical stresses represent the principal stresses and can be computed directly using Eq. 3-49 and 3-51. Thus we do not need to use Mohr's circle to assess the principal stresses.

Tangential Stress:

$$\sigma_t = \frac{p_i r_i^2 - p_o r_o^2 - r_i^2 r_o^2 (p_o - p_i)/r^2}{r_o^2 - r_i^2} \quad \text{for } r_i \le r \le r_o \qquad \text{(Text Eq. 3-49)}$$

Radial Stress:

$$\sigma_r = \frac{p_i r_i^2 - p_o r_o^2 + r_i^2 r_o^2 (p_o - p_i)/r^2}{r_o^2 - r_i^2} \quad \text{for } r_i \le r \le r_o \qquad \text{(Text Eq. 3-49)}$$

Longitudinal Stress:

- Applicable to cases where the cylinder carries the longitudinal load, such as capped ends.
- Only valid far away from end caps where bending, nonlinearities and stress concentrations are not significant.

$$\sigma_l = \frac{p_i r_i^2 - p_o r_o^2}{r_o^2 - r_i^2} \quad \text{for } r_i \le r \le r_o \qquad \text{(Modified Text Eq. 3-51)}$$

Two Mechanical Design Cases

1. Internal Pressure Only ($p_o = 0$)
2. External Pressure Only ($p_i = 0$)

Design Case 1: Internal Pressure Only

- Only one case to consider — the critical section which exists at $r = r_i$.
- Substituting $p_o = 0$ into Eqs. (3-49) and incorporating $\zeta = r_o / r_i$,the largest value of each stress component is found at the inner surface:

$$\sigma_t (r = r_i) = \sigma_{t,\max} = p_i \frac{r_o^2 + r_i^2}{r_o^2 - r_i^2} = p_i \frac{\zeta^2 + 1}{\zeta^2 - 1} = p_i C_{ti} \qquad \text{(T-1)}$$

where $C_{ti} = \dfrac{\zeta^2 + 1}{\zeta^2 - 1} = \dfrac{r_o^2 + r_i^2}{r_o^2 - r_i^2}$ is a function of cylinder geometry only.

$$\sigma_r(r = r_i) = \sigma_{r,\max} = -p_i \qquad \textit{Natural Boundary Condition} \qquad \text{(T-2)}$$

- Longitudinal stress depends upon end conditions:

$$\sigma_l = \begin{cases} p_i C_{li} & \text{Capped Ends} \qquad \text{(T-3a)} \\ 0 & \text{Uncapped Ends} \qquad \text{(T-3b)} \end{cases}$$

where $C_{li} = \dfrac{1}{\zeta^2 - 1}$.

Design Case 2: External Pressure Only

- The critical section is identified by considering the state of stress at two points on the cylinder: $r = r_i$ and $r = r_o$. Substituting $p_i = 0$ into Text Eqs. (3-49) for each case:

$\boldsymbol{r = r_i}$ $\quad \sigma_r(r = r_i) = 0 \qquad$ *Natural Boundary Condition* $\qquad$ (T-4a)

$$\sigma_t\left(r = r_i\right) = \sigma_{t,\max} = -p_o \frac{2r_o^2}{r_o^2 - r_i^2} = -p_o \frac{2\zeta^2}{\zeta^2 - 1} = -p_o C_{to} \qquad \text{(T-4b)}$$

where, $C_{to} = \dfrac{2\zeta^2}{\zeta^2 - 1} = \dfrac{2r_o^2}{r_o^2 - r_i^2}$.

$\boldsymbol{r = r_o}$ $\quad \sigma_r(r = r_o) = \sigma_{r,\max} = -p_o \qquad$ *Natural Boundary Condition* $\qquad$ (T-5a)

$$\sigma_t(r = r_o) = -p_o \frac{r_o^2 + r_i^2}{r_o^2 - r_i^2} = -p_o \frac{\zeta^2 + 1}{\zeta^2 - 1} = -p_o C_{ti} \qquad \text{(T-5b)}$$

- Longitudinal stress for a closed cylinder now depends upon external pressure and radius while that of an open-ended cylinder remains zero:

$$\sigma_l = \begin{cases} -p_o C_{lo} & \text{Capped Ends} \qquad \text{(T-6a)} \\ 0 & \text{Uncapped Ends} \qquad \text{(T-6b)} \end{cases}$$

where $C_{lo} = \dfrac{\zeta^2}{\zeta^2 - 1}$.

Example T3.14.1: Thick-wall Cylinder Analysis

Problem Statement: Consider a cylinder subjected to an external pressure of 150 MPa and an internal pressure of zero. The cylinder has a 25 mm ID and a 50 mm OD, respectively. Assume the cylinder is capped.

Find:

1. the state of stress (σ_r, σ_t, σ_l) at the inner and outer cylinder surfaces;
2. the Mohr's Circle plot for the inside and outside cylinder surfaces;
3. the critical section based upon the estimate of $\tau_{\max}$.

Solution Methodology:

Since we have an external pressure case, we need to compute the state of stress (σ_r, σ_t, σ_l) at both the inside and outside radius in order to determine the critical section.

1. As the cylinder is closed and exposed to external pressure only, Eq. (T-6a) may be applied to calculate the longitudinal stress developed. This result represents the average stress across the wall of the pressure vessel and thus may be used for both the inner and outer radii analyses.
2. Assess the radial and tangential stresses using Eqs. (T-4) and (T-5) for the inner and outer radii, respectively.
3. Assess the principal stresses for the inner and outer radii based upon the magnitudes of (σ_r, σ_t, σ_l) at each radius.
4. Use the principal stresses to calculate the maximum shear stress at each radius.
5. Draw Mohr's Circle for both states of stress and determine which provides the critical section.

Solution:

1. Longitudinal Stress Calculation:

$$r_o = \frac{\text{OD}}{2} = \frac{50\,\text{mm}}{2} = 25\,\text{mm}; \quad r_i = \frac{\text{ID}}{2} = \frac{25\,\text{mm}}{2} = 12.5\,\text{mm}$$

Compute the radius ratio, ζ

$$\zeta = \frac{r_o}{r_i} = \frac{25\text{ mm}}{12.5\text{ mm}} = 2.0$$

Then,

$$C_{lo} = \frac{\zeta^2}{\zeta^2 - 1} = \frac{(2)^2}{(2)^2 - 1} = \mathbf{1.3333\ mm^2}$$

$$\sigma_l(r = r_i) = \sigma_l(r = r_o) = -p_o \frac{\zeta^2}{\zeta^2 - 1} = -p_o C_{lo} = (-150\text{MPa})(1.3333\text{ mm}^2)$$

$$\boldsymbol{\sigma_l} = \mathbf{-200\,MPa}$$

2. Radial & Tangential Stress Calculations:

Inner Radius ($r = r_i$)

$$C_{to} = \frac{2\zeta^2}{\zeta^2 - 1} = \frac{2(2)^2}{(2)^2 - 1} = \mathbf{2.6667}$$

$$\sigma_t(r = r_i) = \sigma_{t,\max} = -p_o \frac{2r_o^2}{r_o^2 - r_i^2} = -p_o C_{to} = (-150\,\text{MPa})(2.6667)$$

$$\sigma_t(r = r_i) = -400\text{ MPa} \quad \textit{Compressive}$$

$$\boldsymbol{\sigma_r(r = r_i) = 0} \quad \textit{Natural Boundary Condition for } p_i = 0$$

Outer Radius ($r = r_o$)

$$C_{ti} = \frac{\zeta^2 + 1}{\zeta^2 - 1} = \frac{(2)^2 + 1}{(2)^2 - 1} = \mathbf{1.6667}$$

$$\sigma_t(r = r_o) = \sigma_{t,\min} = -p_o \frac{r_o^2 + r_i^2}{r_o^2 - r_i^2} = -p_o C_{ti} = (-150\,\text{MPa})(1.6667)$$

$$\boldsymbol{\sigma_t(r = r_o) = -250\,\text{MPa}} \quad \textit{Compressive}$$

$$\boldsymbol{\sigma_r(r = r_i) = -p_o = -150\,\text{MPa}} \quad \textit{Natural Boundary Condition}$$

3. Define Principal Stresses:

Inner Radius ($r = r_i$)	*Outer Radius ($r = r_o$)*
$\sigma_1 = \sigma_r = 0$ MPa	$\sigma_1 = \sigma_r = -150$ MPa
$\sigma_2 = \sigma_l = -200$ MPa	$\sigma_2 = \sigma_l = -200$ MPa
$\sigma_3 = \sigma_t = -400$ MPa	$\sigma_3 = \sigma_t = -250$ MPa

4. Maximum Shear Stress Calculations:

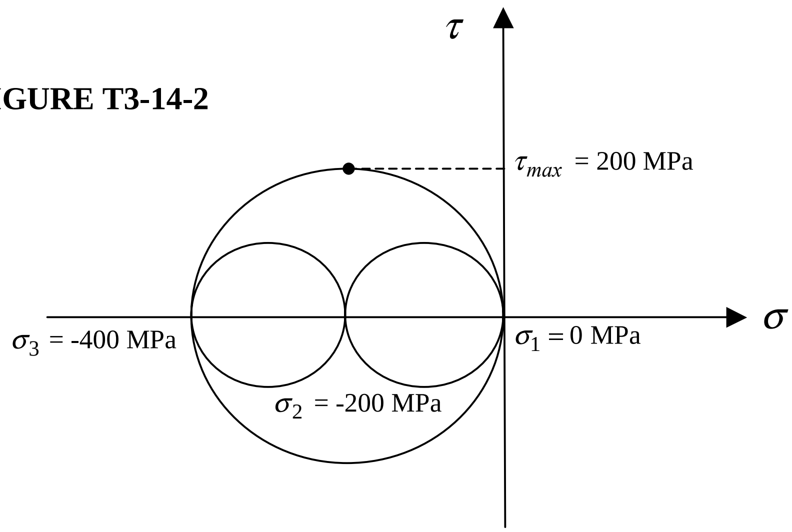
Inner Radius ($r = r_i$) $\quad \tau_{\max}(r = r_i) = \frac{\sigma_1 - \sigma_3}{2} = \frac{0 - (-400)}{2} = \mathbf{200\ MPa}$

Outer Radius $(r = r_o)$ $\tau_{\max}(r = r_o) = \dfrac{\sigma_1 - \sigma_3}{2} = \dfrac{(-150) - (-250)}{2} = \mathbf{50\ MPa}$

5. Mohr's Circles:

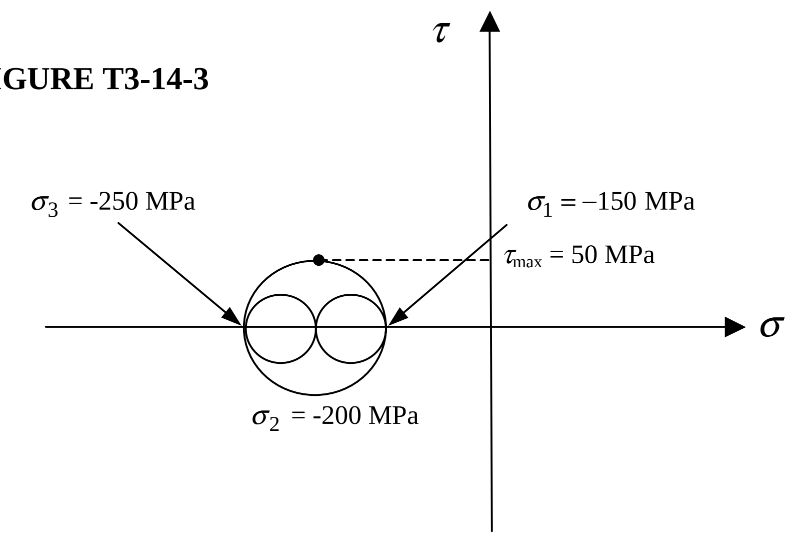
Inner Radius $(r = r_i)$

FIGURE T3-14-2



Outer Radius $(r = r_o)$

FIGURE T3-14-3



Critical Section

$\boldsymbol{\tau_{\max}(r = r_i) = 200\ \textbf{MPa}} \quad \Leftarrow \quad$ ***Critical Section is at Inside Radius!***

Thin-wall Theory

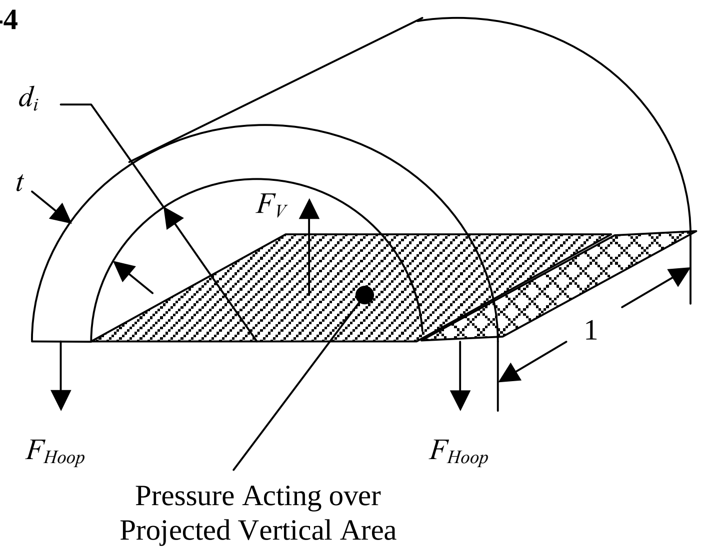
- Thin-wall theory is developed from a Strength of Materials solution which yields the state of stress as an average over the pressure vessel wall.
- Use restricted by wall thickness-to-radius ratio:
  - ➢ According to theory, Thin-wall Theory is justified for $\frac{t}{r} \le \frac{1}{20}$
  - ➢ In practice, typically use a less conservative rule, $\frac{t}{r} \le \frac{1}{10}$

- State of Stress Definition:

  1. Hoop Stress, σ_t , assumed to be uniform across wall thickness.
  2. Radial Stress is insignificant compared to tangential stress, thus, $\sigma_r \doteq 0$.
  3. Longitudinal Stress, σ_l
     - Exists for cylinders with capped ends;
     - Assumed to be uniformly distributed across wall thickness;
     - This approximation for the longitudinal stress is only valid far away from the end-caps.
  4. These cylindrical stresses $(\sigma_t, \sigma_r, \sigma_l)$ are principal stresses $(\sigma_t, \sigma_r, \sigma_l)$ which can be determined without computation of Mohr's circle plot.

- Analysis of Cylinder Section

FIGURE T3-14-4



The internal pressure exerts a vertical force, F_V, on the cylinder wall which is balanced by the tangential hoop stress, F_{Hoop}.

$$F_V = pA_{proj} = p\{(d_i)(1)\} = pd_i$$
$$F_{Hoop} = \sigma_t A_{stressed} = \sigma_t\{(t)(1)\} = \sigma_t t$$

$$\sum F_y = 0 = F_V - 2F_{Hoop} = pd_i - 2\sigma_t t$$

Solving for the tangential stress,

$$\sigma_t = \frac{pd_i}{2t} \quad \text{Hoop Stress} \qquad \text{(Text Eq.3-52)}$$

- Comparison of state of stress for cylinder under internal pressure verses external pressure:

Internal Pressure Only

$$\sigma_t = \frac{pd_i}{2t} \qquad \textit{Hoop Stress}$$
$$\sigma_r = 0 \qquad \textit{By Definition}$$
$$\sigma_l = \frac{pd_i}{4t} = \frac{\sigma_t}{2} \qquad \textit{Capped Case} \qquad \text{(Text Eq.3-54)}$$

External Pressure Only

$$\sigma_t = \frac{pd_o}{2t} \qquad \textit{Hoop Stress}$$
$$\sigma_r = 0 \qquad \textit{By Definition}$$
$$\sigma_l = \frac{pd_o}{4t} = \frac{\sigma_t}{2} \qquad \textit{Capped Case}$$

Example T3.14.2: Thin-wall Theory Applied to Cylinder Analysis

Problem Statement: Repeat Example T1.1 using the Thin-wall Theory (p_o = 150 MPa, p_i = 0, ID = 25 mm, OD = 50 mm).

Find: The percent difference of the maximum shear stress estimates found using the Thick-wall and Thin-wall Theories.

Solution Methodology:

1. Check *t/r* ratio to determine if Thin-wall Theory is applicable.
2. Use the Thin-wall Theory to compute the state of stress
3. Identify the principal stresses based upon the stress magnitudes.
4. Use the principal stresses to assess the maximum shear stress.
5. Calculate the percent difference between the maximum shear stresses derived using the Thick-wall and Thin-wall Theories.

Solution:

1. Check *t/r* Ratio: $\frac{t}{r} = \frac{12.5 \text{ mm}}{25 \text{ mm}} = \frac{1}{2} \rangle \frac{1}{20} \quad or \quad \frac{1}{10}$

   The application of Thin-wall Theory to estimate the stress state of this cylinder is thus <u>not justified</u>.

2. Compute stresses using the Thin-wall Theory to compare with Thick-wall theory estimates.

   a. Hoop Stress (average stress, uniform across wall)

   $$\sigma_t = \frac{-p_o d_o}{2t} = \frac{-(150 \text{ MPa})(50 \text{ mm})}{2(12.5 \text{ mm})} = \mathbf{-300\ MPa}$$

   b. Radial Stress $\sigma_\text{r} = \mathbf{0}$ by definition

   c. Longitudinal Stress (average stress, uniform across wall)

   $$\sigma_l = \frac{-p_o d_o}{4t} = \frac{\sigma_t}{2} = \mathbf{-150\ MPa}$$

3. Identify Principal Stresses in terms of "Average" Stresses:

   $$\sigma_1 = \sigma_r = 0 \text{ MPa}$$
   $$\sigma_2 = \sigma_l = -150 \text{ MPa}$$
   $$\sigma_3 = \sigma_t = -300 \text{ MPa}$$

4. Maximum Shear Stress Calculation:

   $$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{0 - (-300 \text{ MPa})}{2} = +150 \text{ MPa}$$

5. Percent Difference between Thin- and Thick-wall Estimates for the Critical Section:

$$\% \, Difference = \frac{\tau_{\text{max,Thin}} - \tau_{\text{max,Thick}}}{\tau_{\text{max,Thick}}} * 100\%$$

$$= \frac{(+150) - (+200)}{(+200)} * (100\%) = \mathbf{-25\%}$$

$\Rightarrow$ **Thin -wall estimate is 25% low!** $\Leftarrow$