Solutions to Exercises

Chapter 1 The Apportionment Problem

1. The apportionment for methods LF, GD, MF, and EP is (1, 1, 2, 4); for SD the apportionment is (1, 2, 2, 3). **2.** This problem illustrates that ties can occur, thus violating the house size (this is not likely to occur if state populations are not essentially small integers). GD produces the apportionment (1, 1, 3, 4); EP and SD provide the apportionment (1, 2, 2, 4); but the allocation process under LF or MF encounters a tie rendering the apportionment $(1, 2, 3, 4)$ if an extra seat is assigned because of the tie.

3.

State	LF	GD	MF	EP	SD
NH	4	4	4	4	4
MA	11	12	11	11	11
RI	2	2	2	2	2
CT	7	7	7	7	7
NY	10	10	10	10	10
NJ	5	5	5	5	6
PA	13	13	13	13	13
DE	2	1	2	2	2
MD	8	8	8	8	8
VA	19	19	19	19	18
NC	10	11	10	10	10
SC	6	6	6	6	6
GA	3	3	3	3	3
VT	3	2	3	3	3
KY	2	2	2	2	2

4. Only EP and SD, because the denominators for the Huntington sequential method are zero, hence the ratios are infinite, until a seat has been assigned. **5.** **(a)** Since GD does not inherently assign even one seat per state, the natural method is to start the Huntington sequential method with $a_i = 3$ for each state. This is equivalent to employing $\sum max(3, \lfloor q_i/\lambda \rfloor) = H$ with the λ method. **(b)** The proof of equivalence is the same inductive proof given for the regular case in the text. **6.** The apportionment is $(3, 3, 4, 5)$, which can be contrasted to $(1, 3, 5, 6)$ if the three seat constraint does not exist. **7.** The proof is the same as the proof of Theorem 1 with the "+1" following the a's omitted. **8.** By Exercise 7, it suffices to consider the sequential algorithm; use induction on the number of seats assigned. Because this method assigns one seat per district before assigning subsequent seats (which is necessary in order that no p_i/a_i be infinite), assume that each state has one seat. The next seat is assigned to the state for which $p_i/1$ (the district size) is greatest; hence that state's district size must be reduced to minimize the maximum district size. After k of the seats have been assigned, the next state to receive a seat will be the one with the largest district size. This assignment is also necessary to reduce the maximum district size. If the maximum district size was as small as possible before the $(k+1)$st seat was assigned, any reallocation of seats will increase the maximum district size to at least the value with k seats assigned. **9.** **(a)** Let states $p_1 = 1000$, $p_2 = 10$, $p_3 = 10$, and $H = 3$. **(b)** No, because the choice $\lambda = 1$ in the λ method provides each state with its upper quota, but too large a House. Increasing λ to decrease the total House size cannot increase the size of any state's delegation. **10.** **(a)** For example, $\mathbf{p} = (24, 73, 83)$ with H changing from 18 to 19. **(b)** No, because with only two states, a fractional part greater than 0.5 is necessary and sufficient for a state to receive its upper quota. Increasing the House size will increase its quota, which will still have a fractional part greater than 0.5 unless it has increased its integer part. **11.** $104!/(90!14!) \approx 8 \times 10^{16}$. **12.** $434!/(385!49!)$. **13.** Based on the 1990 census figures, 21 states receive the same number of representatives under all of the apportionment methods discussed in this chapter. These states (with their numbers of representatives) are: AL(7), AK(1), AR(4), CO(6), CT(6), GA(11), HI(2), IN(10), IA(5), MD(8), MN(8), MO(9), NV(2), NH(2), OR(5), SC(6), UT(3), VT(1), VA(11), WV(3), WY(1). The numbers of seats for states whose assignment depends on the apportionment method are given for greatest divisors, equal proportions (which is the method currently employed), and smallest divisors: AZ(6,6,7), CA(54,52,50), DE(1,1,2), FL(23,23,22), ID(1,2,2), IL(21,20,19), KS(4,4,5), KY(6,6,7), LA(7,7,8), ME(2,2,3), MA(11,10,10), MI(17,16,16), MS(4,5,5), MT(1,1,2), NE(2,3,3), NJ(14,13,13), NM(2,3,3), NY(33,31,30), NC(12,12,11), ND(1,1,2), OH(19,19,18), OK(5,6,6), PA(21,21,20), RI(1,2,2), SD(1,1,2), TN(8,9,9), TX(31,30,29), WA(8,9,9), WI(8,9,9). The major fractions method provides the same apportionment as equal proportions, except that MA would get 11 seats and OK would only get five. Largest fractions provides the same apportionment as equal proportions except for MA(11), NJ(14), MS(4), and OK(5). The classification of states with ten or more representatives as large states and those with fewer than ten as small states bears out the characterization of the various apportionment methods as favoring large or small states. **14.(a)** We show $\log(\lim_{t\to 0}(0.5(a_i^t + (a_i+1)^t))^{1/t}) = \log\sqrt{a_i(a_i+1)}$, which is equivalent.

$$\lim_{t\to 0}\frac{1}{t}\log(0.5(a_i^t + (a_i+1)^t)) =$$

$$\lim_{t\to 0}\frac{1}{t}\log(0.5((1+t\log a_i) + (1+t\log(a_i+1)))) =$$

$$\lim_{t\to 0}\frac{1}{t}(0.5(t\log a_i + t\log(a_i+1))) =$$

$$\lim_{t\to 0} 0.5(\log a_i + \log(a_i+1)) = \log\sqrt{a_i(a_i+1)}.$$

(For each of the first two equalities, l'Hôpital's rule can be used to show that the ratio of the left side to the right side is 1.) **(b)** The limit is manifestly less than or equal to a_i+1 because $a_i < a_i+1$, and replacing the former with the latter would provide equality. Because $(a_i+1)^t < a_i^t + (a_i+1)^t$, the limit is greater than $\lim_{t\to\infty}(0.5(a_i+1)^t)^{1/t} = \lim_{t\to\infty} 0.5^{1/t}((a_i+1)^t)^{1/t} = a_i+1$, since $0.5^{1/t}$ has 1 as limit. **(c)** Letting $b_i = 1/a_i$ and $b_i' = 1/(a_i+1)$, this limit is the reciprocal of the limit in part (b) for the bs. Since $b_i > b_i'$, the limit is the reciprocal of b_i (i.e., a_i).

Chapter 2 Finite Markov Chains

1. a) Looking at $\mathbf{T}^2$ (in Example 13), we find $p_{13}^{(2)} = 0.16$. **b)** By Theorem 1, this is $q_2 p_{23} p_{31} p_{11} = 0.4 \times 0.2 \times 0.5 \times 0.3 = 0.012$. **c)** The possible outcomes comprising this event are: $s_1s_1s_2$, $s_1s_3s_2$, $s_3s_3s_2$, $s_3s_1s_2$. We compute the probability of each of these, as in b), and add them, to get 0.112. **2.** $\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1-p & 0 & p & 0 & 0 \\ 0 & 1-p & 0 & p & 0 \\ 0 & 0 & 1-p & 0 & p \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$. **3. a)** The state space here is $S = \{s_1 = 0, s_2 = 1, s_3 = 2\}$. The random process is a Markov chain since each day the marbles to be exchanged are chosen at random, meaning that previous choices are not taken into account. Thus, the probability of observing a particular number X_{k+1} of red marbles on day $k+1$ depends only on how many red marbles were in the jar on day k. **b)** $\mathbf{T} = \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0.09 & 0.82 & 0.09 \\ 0 & 0.2 & 0.8 \end{bmatrix}$. For example,

$p_{22} = 0.82$ is the probability that if each jar has one red marble in it, there will be one red marble in each after the exchange has been made. This event can occur if one *white* marble is chosen from each of the jars (each of which contains 9 white marbles); by the multiplication rule, there are a total of 100 possible pairs of choices and 81 ways to choose a white marble from each jar. This event would also occur if the one red marble in each jar is chosen; there is clearly only one way to do that. Thus, there are a total of 82 ways out of 100 possibilities of starting with a red in each jar, and ending with a red in each jar after the exchange is made (and 82/100=0.82). A similar use of the multiplication rule will yield the other probabilities found in the matrix. **c)** We begin by computing $\mathbf{T}^3 = \begin{bmatrix} 0.56 & 0.40 & 0.04 \\ 0.18 & 0.64 & 0.18 \\ 0.04 & 0.40 & 0.56 \end{bmatrix}$. We then see that $p_{32}^{(3)}$, the probability that after 3 days there is one red marble in the jar, given that we start out with 2 red marbles, is 0.4. **4.** p_{ij} is the probability that, starting in state s_i, the Markov chain moves to state s_j. Now the random process *must* move to one of the states s_j, $1 \le j \le N$, by definition of the state space. That is, it moves to exactly one of these states with probability 1, which (for the ith row) is $p_{i1} + p_{i2} + \cdots + p_{iN} = 1$, which is what is to be shown. **5.** The result is obtained by mathematical induction. The result is correct for $k = 1$ by the definition of p_{ij}. Assume it is true for $k = m$, i.e., for any i, l, $p(X_m = s_l | X_0 = s_i) = p_{il}^{(m)}$. We must then deduce the result for $k = m + 1$. We have

$$p_{ij}^{(m+1)} = P(X_{m+1} = s_j | X_0 = s_i) \qquad \text{by definition of } p_{ij}^{(m+1)}$$

$$= \frac{p(X_{m+1} = s_j, X_0 = s_i}{p(x_0 = s_i)} \qquad \text{by definition of conditional probability}$$

$$= \sum_{l=1}^{N} \frac{p(X_{m+1} = s_j, X_m = s_l, X_0 = s_i)}{p(X_0 = s_i)} \qquad \text{addition rule of probabilities}$$

$$= \sum_{l=1}^{N} \frac{p(X_m = s_l, X_0 = s_i)}{p(X_0 = s_i)} p(X_{m+1} = s_j | X_m = s_l, X_0 = s_i) \qquad \text{cond. prob.}$$

$$= \sum_{l=1}^{N} \frac{p(X_m = s_l, X_0 = s_i)}{p(X_0 = s_i)} p(X_{m+1} = s_j | X_m = s_l) \qquad \text{by (i)}$$

$$= \sum_{l=1}^{N} P(X_m = s_l | X_0 = s_i) p_{lj} \qquad \text{by (ii)}$$

$$= \sum_{l=1}^{N} p_{il}^{(m)} p_{lj} \qquad \text{by the induction assumption}$$

The last expression is the (i, j)th entry of $\mathbf{T}^{m+1}$. **6. a)** $\mathbf{A}$ is not, since the first row is always (1 0 0) (see Exercise 8). $\mathbf{B}$ is, since $\mathbf{B}^3$ has all positive entries.

b) By the definition of a regular Markov chain, for some k, $\mathbf{T}^k$ has all positive entries. We will show that if $m > 0$ then $\mathbf{T}^{m+k}$ has all positive entries, which amounts to what was to be shown. Now $\mathbf{T}^{m+k} = \mathbf{T}^m\mathbf{T}^k$ which has as its (i, j)th entry $\sum_{n=1}^{N} p_{in}^{(m)} p_{nj}^{(k)} \geq p_{in}^{(m)} p_{nj}^{(k)}$, for any choice of n, since all of the terms of the sum are nonnegative. By assumption, $p_{nj}^{(k)} > 0$; furthermore, at least one of the $p_{in}^{(m)}$ must be greater than 0 since (see Exercise 4) the rows have entries adding up to 1. Thus for some n, $p_{in}^{(m)} p_{nj}^{(k)} > 0$, which yields the desired conclusion. **7. a)** The state space $S = \{s_1, s_2\}$, where s_1 is the state that a women is a voter and s_2 that she is a nonvoter. Let X_k be the voting status of a woman in the kth generation of her family. This will be a Markov chain with matrix of transition probabilities $\mathbf{T} = \begin{bmatrix} 0.5 & 0.5 \\ 0.1 & 0.9 \end{bmatrix}$. **b)** $(p\ q) \begin{bmatrix} 0.5 & 0.5 \\ 0.1 & 0.9 \end{bmatrix} = (p\ q)$ is the system of equations $0.5p + 0.1q = p$, $0.5p + 0.9q = q$. We also have $p + q = 1$. Solving the system of three equations for the unknowns p and q yields $p = 0.167$ and $q = 0.833$. This may be interpreted to mean that in the long run (if these voting patterns remain unchanged) about 17% of women will be voters and the remaining 83% will be nonvoters. **8. a)** 0 and 4 are absorbing states in Example 1. **b)** Example 3 has none, since with positive probability, the process can leave any of the given states. This is of course true for *any* regular Markov chain. **c)** No, it is not. If the Markov chain has an absorbing state, say s_i, then the ith row of the matrix of transition probabilities will have a 1 as the (i, i)th entry and all other entries in that row will be 0. When we perform the matrix multiplication, this row never changes. The underlying phenomenon is this: in a regular Markov chain, it must eventually be possible to reach any state from a particular state, since all the entries of some power of the matrix of transition probabilities are all positive. This clearly cannot happen if there is an absorbing state, since by the definition of such a state, no state can be reached from it no matter how long we wait. **9. a)** Using the definition of conditional probability, the right hand side becomes $P(A)\frac{P(E\cap A)}{P(A)} + P(B)\frac{P(E\cap A)}{P(B)} = P(E \cap A) + P(E \cap B)$. Now, $E = (E \cap A) \cup (E \cap B)$ (since $A \cup B$ contains all possible outcomes) and $(E \cap A) \cap (E \cap B) = E \cap (A \cap B) = E \cap \emptyset = \emptyset$, so $P(E) = P(E \cap A) + P(E \cap B)$ ("addition rule" for probabilities), and we are done. **b)** Using the hint, we compute $P(E) = P(A)P(E|A) + P(B)P(E|B)$. Now, $P(A) = P(B) = 1/2$. Also, $P(E|A) = r_{k+1}$, since the only way to win (starting with k dollars), given that we win the first round, is to win, given that we start with $k + 1$ dollars. (This is a consequence of property (i) in Definition 1; in this context, we might say that the Markov chain "forgets" that we have just won and thinks that we are starting with $k+1$ dollars.) Similarly, $P(E|B) = r_{k-1}$, thus, $r_k = P(E) = \frac{1}{2}r_{k-1} + \frac{1}{2}r_{k+1}$, $0 < k < 4$, with $r_0 = 0$, $r_4 = 1$. **c)** The identical argument yields $r_k = \frac{1}{2}r_{k-1} + \frac{1}{2}r_{k+1}$, $0 < k < N$, $r_0 = 0$, $r_N = 1$. **d)** This is a straightforward verification by substitution. **10. a)** Label the vertices of a pentagon $1, 2, 3, 4, 5$; thus, the state space is $S = \{1, 2, 3, 4, 5\}$. We then have $\mathbf{T} = \begin{bmatrix} 0 & 1/2 & 0 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 0 & 1/2 & 0 \end{bmatrix}$. **b)** Let $\mathbf{Q} = [0.2 \;\; 0.2 \;\; 0.2 \;\; 0.2 \;\; 0.2]$. Then $\mathbf{QT} = [0.2 \;\; 0.2 \;\; 0.2 \;\; 0.2 \;\; 0.2] = \mathbf{Q}$. In the long run, the probability of finding the drunkard at any particular corner of the building is 0.2=1/5; that is, there is an equal chance of finding him at any particular corner. **c)** Let the state space here be $S = \{1, 2, 3, 4\}$. The matrix of transition probabilities is $\mathbf{T} = \begin{bmatrix} 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \end{bmatrix}$ and the equilibrium distribution is $(0.25 \;\; 0.25 \;\; 0.25 \;\; 0.25)$. **11. a)** $S = \{\text{win}, \text{lose}, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$ where $3, 4, \ldots, 11$ are the possible sums from which we can win (rolling a 2 or 12 immediately puts us in the "lose" state, since we must have rolled doubles). Note that once we are in one of the states $3, 4, \ldots, 11$, we stay there until we either win or lose. **b)** The initial distribution (where the order is that given in the above description of S) is $[0 \; 6/36 \; 2/36 \; 2/36 \; 4/36 \; 4/36 \; 6/36 \; 4/36 \; 4/36 \; 2/36 \; 2/36]$. For example, there are 6 ways to roll doubles initially out of 36 possible outcomes so (since we lose when we roll doubles) the probability of losing right away is 6/36. Of the 3 possible ways to roll a 4, $((1, 3); (2, 2); (3, 1))$, only two, $(1, 3)$ and $(3, 1)$, correspond to the state "4"; $(2, 2)$ corresponds to the state "lose". **c)**

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{2}{36} & \frac{6}{36} & \frac{28}{36} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{2}{36} & \frac{6}{36} & 0 & \frac{28}{36} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{2}{36} & \frac{6}{36} & 0 & 0 & \frac{26}{36} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{4}{36} & \frac{6}{36} & 0 & 0 & 0 & \frac{26}{36} & 0 & 0 & 0 & 0 & 0 \\ \frac{6}{36} & \frac{6}{36} & 0 & 0 & 0 & 0 & \frac{24}{36} & 0 & 0 & 0 & 0 \\ \frac{4}{36} & \frac{6}{36} & 0 & 0 & 0 & 0 & 0 & \frac{26}{36} & 0 & 0 & 0 \\ \frac{4}{36} & \frac{6}{36} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{26}{36} & 0 & 0 \\ \frac{2}{36} & \frac{6}{36} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{28}{36} & 0 \\ \frac{2}{36} & \frac{6}{36} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{28}{36} \end{bmatrix}.$$

The idea is that once we roll a certain sum initially, we stay in that state until we either win or lose. **12. a)** Reflexivity and symmetry are easy consequences of the definition of the relation. To show that the relation is transitive, we must prove that $s_i \leftrightarrow s_k$ and $s_k \leftrightarrow s_j$ implies $s_i \leftrightarrow s_j$. Since s_k is accessible from s_i, there is a number n such that $p_{ik}^{(n)} > 0$ and since s_j is accessible from s_k, $p_{kj}^{(m)} > 0$ for some number m. Now $\mathbf{T}^{n+m} = \mathbf{T}^n\mathbf{T}^m$, so the (i,j)th entry of $\mathbf{T}^{n+m}$ (which is, by Theorem 2, $p_{ij}^{(n+m)}$) can be computed as $p_{ij}^{(n+m)} = \sum_{l=1}^{N} p_{il}^{(n)} p_{lj}^{(m)} \geq p_{ik}^{(n)} p_{kj}^{(m)} > 0$. For $l = k$, we have $p_{ik}^{(n)} p_{kj}^{(m)} > 0$, so that the sum above (none of whose terms are negative) must in fact be larger than 0, which means that s_j is accessible from s_i. The identical argument demonstrates that s_i is accessible from s_j, so that these two states communicate. This establishes transitivity and hence that the relation $\leftrightarrow$ is an equivalence relation. **b)** If s_k is absorbing, $p_{kk} = 1$ and $p_{kj} = 0$ for $j \neq k$, i.e., it is impossible for the process to leave state s_k and enter another state s_j. This means that for $j \neq k$, s_j is *not* accessible from s_k, so they cannot be communicating states. **c)** No. A transient state like s_i is one that the process eventually leaves forever, whereas it returns over and over again to recurrent states (like s_j). **d)** In Example 1, the equivalence classes are $\{0\}$, $\{4\}$, $\{1,2,3\}$. In Example 2, there is only one equivalence class $\{s_1, s_2, s_3\}$. e) There is only one equivalence class. The fact that *all* entries of $\mathbf{T}^k$ are positive for some k means that for this k and for any i and j, we have $p_{ij}^{(k)} > 0$, which means that every pair of states communicate. **13.** Let $p, q \geq 0$ with $p + q = 1$. Then $\mathbf{Q} = (p\ 0\ 0\ 0\ q)$ is an initial probability distribution and it is easy to check that for such $\mathbf{Q}$ $\mathbf{Q} = \mathbf{QT}$. The idea is that there is a rather uninteresting equilibrium attained if we start out with probability p of having no money to start with and probability q of already at the beginning having the amount of money we want to end up with. **14. a)** Choose a number e to be the error tolerance. We must show that for any choice of an initial probability distribution $\mathbf{Q}$, we have $|\sum_{i=1}^{N} q_i p_{ij}^{(k)} - r_j| < e, \quad 1 \leq j \leq N$, since $\sum_{i=1}^{N} q_i p_{ij}^{(k)}$ is the jth entry of $\mathbf{Q}_k$, which is supposed to be close to $\mathbf{Q}_e$. Now the entries of the jth column of $\mathbf{T}^{(k)}$ are by assumption all within e units of r_j. Also, $\sum_{i=1}^{N} q_i = 1$. We thus have $\left|\sum_{i=1}^{N} q_i p_{ij}^{(k)} - r_j\right| = \left|\sum_{i=1}^{N} q_i p_{ij}^{(k)} - \sum_{i=1}^{N} q_i r_j\right| = \left|\sum_{i=1}^{N}[q_i p_{ij}^{(k)} - q_i r_j]\right| \leq \sum_{i=1}^{N}\left|q_i(p_{ij}^{(k)} - r_j)\right| = \sum_{i=1}^{N} q_i\left|(p_{ij}^{(k)} - r_j)\right| \leq \sum_{i=1}^{N} q_i e = e$. **b)** The $N \times N$ matrix $\mathbf{T}^2$ has N^2 entries. Each of these requires N additions and N multiplications, namely $p_{ij}^2 = \sum_{k=1}^{N} p_{ik} p_{kj}$. Thus, there are a total of N^3 additions and N^3 multiplications required to multiply two $N \times N$ matrices together. If this operation were to be repeated $k-1$ times, which is the number of matrix multiplications required to compute $\mathbf{T}^k$, there would be a total of $(k-1)N^3$ additions and $(k-1)N^3$ multiplications required. Finally, to compute $\mathbf{Q}_k$, we must perform N additions and N multiplications for each of the N entries of the $1 \times N$ matrix $\mathbf{Q}_k = \mathbf{QT}^k$ for a grand total of $(k-1)N^3 + N^2 \approx kN^3$ multiplications and additions to compute $\mathbf{Q}_k$. **c)** $\mathbf{Q}_k = \mathbf{QT}^k = \mathbf{QT}^{k-1}\mathbf{T} = \mathbf{Q}_{k-1}\mathbf{T}$, which is what is to be shown. To compute $\mathbf{Q}_k$ using this idea, we must perform k operations of multiplying a $1 \times N$ matrix by an $N \times N$ matrix. Each one of these matrix multiplications requires N^2 multiplications and additions, for a total of kN^2 multiplications and additions for the computation of $\mathbf{Q}_k$, a very significant improvement over the method of part b).

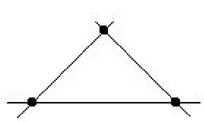
Chapter 3 Rational Election Procedures

1. One example is the ranking of students by letter grades. If two students have the same grade, they are related to each other (XRY and YRX), hence it is not a total order. **2.** The standard example of the power set of $\{a, b, c\}$ with a partial order defined by "is a subset of". It is not a weak order because $\{a\}$ and $\{b, c\}$ are not comparable. **3.** Specializing XRY or YRX to the case $X = Y$ provides XRX. **4.** Indifference is defined as $XIY \Leftrightarrow XRY \wedge YRX$. It is therefore symmetric. Since R is a weak order (which is transitive and reflexive), it is the intersection of two relations which are transitive and reflexive, hence is also transitive and reflexive. **5.** It is connected since if there is a tie, the candidates are related to each other; else the winner is related to the loser. It is transitive since it is reflexive and any reflexive relation on two elements is transitive. **6.** Majority rule with just two candidates satisfies PR since a stronger preference for candidate A entails giving him a vote previously assigned to B, which raises A's relative position, so he fares at least as well as before. IA is vacuously true since there are no candidates to withdraw from the contest if one wishes to compare the relative rankings of two. **7.** If a relation is a total order, it is connected. If it is an equivalence relation it is symmetric. Combining $XRY \vee YRX$ with $XRY \Rightarrow YRX$ implies that XRY for all X and Y, i.e., everything is related to everything. This violates antisymmetry if the relation is on a set with more than one element. **8.** Assume $X\widehat{P}_1Y$ and $Y\widehat{R}X$. $Y\widehat{P}_2X$ contradicts the hypothesis, hence $X\widehat{R}_2Y$. But $X\widehat{P}_1Y$, $X\widehat{P}_2Y$ and $Y\widehat{R}X$ contradicts Lemma 1, hence $X\widehat{I}_2Y$. By IA and PR, $X\widetilde{P}_1Y$ and $Y\widetilde{P}_2X$ provide $Y\widetilde{R}X$, which violates the hypothesis of Lemma 2. Therefore the assumption $X\widehat{P}_1Y$ and $Y\widehat{R}X$ cannot be true and Lemma 2 is proven. **9.** If XP_1Y, YP_2X, and XPY; by symmetry with respect to candidates YP_1X and XP_2Y implies YPX and by symmetry with respect to voters XP_2Y and YP_1X implies XPY; which two implications provide a contradiction. Assuming initially YPX leads to the same contradiction, hence indifference must hold. **10.** A has 4, B has 3, C has 2; hence none of the candidates has a majority of first place votes. A has a plurality. **11.** If C is eliminated, B receives 5 first place votes while A still receives only 4. Hence B wins. **12.** If only the favorite candidates are acceptable, approval voting is plurality voting for the favorite candidate, and A wins. If every voter approved his first two choices, A would still only get 4 votes, B would get six votes, and C would get 8 votes; hence C would win. **13.** B would defeat A. C would then defeat B. **14.** Yes, C would defeat either A or B in a two-way race. **15.** The scoring is $2 - 1 - 0$. A gets 8, B gets 9, and C gets 10; hence C wins. **16.** B receives 8, H receives 8, Y receives 2, and D receives 3; hence there is a tie between B and H. **17.** C must beat D by transitivity; either outcome of the B-D competition will be consistent. **18.** Three. No matter how many teams are in the tournament, if A defeats B, B defeats C, and C defeats A, transitivity will be violated. **19.** If the number of voters is divisible by three, assign one-third to each preference schedule; for four voters, giving two voters to one preference schedule will result in contradicting transitivity with indifference instead of preference; for more than four voters assign the one or two voters in excess of a multiple of three to different (if the excess is two) candidates and the argument presented remains valid. **20.** For the individual preferences BP_1C, CP_1A and CP_2A, AP_2B Lemma 1 provides CPA. BPC because it is true for P_1 (from above in the lemma). Hence BPA by transitivity. Lemma 2 (with IA and PR) provides this for all preferences compatible with BP_1A. Hence $BP_1A \Rightarrow BPA$. For CPA consider the preferences $C \succ B \succ A$ and $A \succ C \succ B$. **21.** If YRX when more voters prefer X, then, by PR, YRX when more voters prefer Y, but XRY by symmetry. This is a contradiction if there is no indifference, hence XRY (which is consistent with indifference). **22.** With three alternatives, there are a total of three two way contests; either each candidate wins one (no Copeland winner), or one candidate wins both contests in which he is involved (a Condorcet winner). If there are four candidates there are six two-way contests with each candidate involved in three of them. If no candidate wins three contests (hence there is no Condorcet winner), at least two candidates must win two contests (pigeonhole principle). **23.** 315. (There are 3 ways to order teams B, C, and D; 3 ways to order teams F, G, and H; and 35 ways to interleaf the orderings of BCD with $EFGH$. **24.** A *vs.* B and C *vs.* D or A *vs.* C and B *vs.* D. A *vs.* D initially if A is to lose.

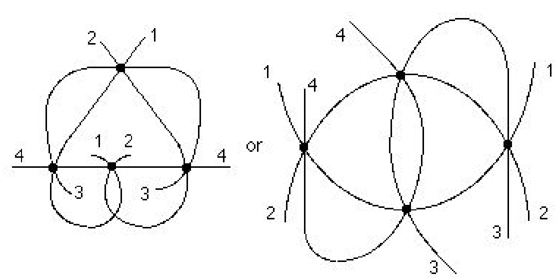
Chapter 4 Gödel's Undecidability Theorem

1. $14 = 3 + 11$, $16 = 3 + 13$, $18 = 5 + 13$, $20 = 3 + 17$, $22 = 3 + 19$, $24 = 5 + 19$, $26 = 3 + 23$, $28 = 5 + 23$, $30 = 7 + 23$, $32 = 3 + 29$, $34 = 3 + 31$, $36 = 5 + 31$, $38 = 7 + 31$, $40 = 3 + 37$, $42 = 5 + 37$, $44 = 3 + 41$, $46 = 3 + 43$, $48 = 5 + 43$, $50 = 3 + 47$. **2.** Starting with $n = 22$, repeated application of f gives the following sequence of integers: 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1. Hence $i = 15$. Starting with $n = 23$, repeated application of f gives: 23, 70, 35, 106, 53, 160, 80, 40, 20, 10, 5, 16, 8, 4, 2, 1. Hence $i = 15$.

3.

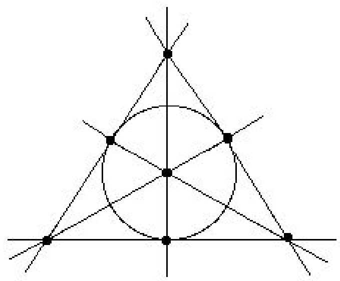


4.

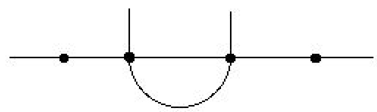


5. Same as solution to Exercise 3.

6.



7. Yes, as illustrated below.



8. Suppose points a, b, c lie on line l_1, and d lies on l_2. Since l_2 contains three points and there are only four points in the system, at least two of the points a, b, c must also lie on l_2. This contradicts axiom (ii). **9.** Suppose points a, b, c lie on line l_1. Since a lies on two lines, there is a line l_2 containing a. By axiom (ii), neither b nor c can lie on l_2, so l_2 must contain points d and e. Points b and d must also lie on another line, say l_3. The third point on l_3 cannot be a, c, or e; let f be this third point on l_3. Then a and f must lie on a line other than l_1 and l_2, which forces a to be on three lines, contradicting axiom (iii). **10.** If the statement is true, then it is false. If the statement is false, then it is true. **11. a)** If a liar makes the statement, then "I am lying" is false, and hence the speaker is telling the truth (which contradicts the fact that the speaker never tells the truth). If a truth-teller makes the statement, then "I am lying" is true, and the speaker is lying (which contradicts the fact that the speaker always tells the truth). **b)** This is a paradox. If the person is a liar, then the statement is false. So I will know that he is telling the truth and so therefore he is a truth teller. But this truth teller says that I cannot know that he is truthful, which is an untrue statement since I do know. So, he is not a truth teller either. **12. a)** $2^{21}3^{19}5^{3}7^{3}11^{1}$ **b)** $2^{5}3^{11}5^{11}7^{21}11^{19}13^{1}17^{13}19^{7}23^{11}29^{23}31^{19}37^{1}41^{13}43^{13}$ **c)** $2^{9}3^{21}5^{17}7^{23}11^{11}13^{5}17^{11}19^{3}23^{21}29^{19}31^{23}37^{13}41^{13}$

d) $2^{9}3^{21}5^{11}7^{11}11^{21}13^{19}17^{3}19^{1}23^{13}29^{7}31^{5}37^{11}41^{21}43^{19}47^{3}53^{1}59^{13}61^{13}$

13. a) $2^{k_1}3^{k_2}5^{k_3}$, where $k_1 = 2^{21}3^{19}5^{3}7^{1}$, $k_2 = 2^{23}3^{19}5^{3}7^{21}$, and

$k_3 = 2^{23}3^{19}5^{3}7^{3}11^{1}$. **b)** $2^{k_1}3^{k_2}5^{k_3}$, where $k_1 = 2^{17}3^{23}5^{11}7^{21}11^{19}13^{3}17^{23}19^{13}$, $k_2 = 2^{9}3^{23}5^{11}7^{5}11^{11}13^{1}17^{19}19^{3}23^{23}29^{13}31^{13}$, and $k_3 = 2^{5}3^{11}5^{21}7^{19}11^{1}13^{13}$.

14. a) $\neg(x = y)$. **b)** $\forall x \exists y (x = fy)$. **c)** $(x = 0) \vee \neg(y = 0)$.

15. a) $\left\{ \begin{array}{c} x = ff0 \\ \neg(y = x) \\ \neg(y = ff0) \end{array} \right\}$. **b)** $\left\{ \begin{array}{c} \forall y \exists x \neg(x = y) \\ y = f0 \\ \exists x \neg(x = f0) \end{array} \right\}$. **16.** Gödel's Undecidability Theorem predicts that there is some new statement G' in the larger system which is true but which cannot be proven within that system.

Chapter 5 Coding Theory

1. a) no **b)** yes **c)** yes **d)** yes. **2. a)** 5 **b)** 3 **c)** 4 **d)** 6. **3. a)** 0.9509900499 **b)** 0.0096059601 **c)** 0.0000970299 **d)** 0.0000009801 **e)** 0.9990198504 **4. a)** detect 6, correct 3 **b)** detect 1, correct 0 **c)** detect 4, correct 2. **5.** 0.999999944209664279880021 **6.** Let C be a binary code with $d(C) = 4$. We can correct one error and detect two errors as follows. Let $\mathbf{y}$ be a bit string we receive. If $\mathbf{y}$ is a codeword $\mathbf{x}$ or has distance 1 from a codeword $\mathbf{x}$ we decode it as $\mathbf{x}$ since, as can be shown using the triangle inequality, every other codeword would have distance at least three from $\mathbf{y}$. If $\mathbf{y}$ has distance at least two from every codeword, we detect at least two errors, and we ask for retransmission. **7.** 11 **8.** The minimum distance of this code in n. We find that $2^n/(\sum_{j=0}^{(n-1)/2} C(n,j)) = 2^n/2^{n-1} = 2$, since $\sum_{j=0}^{(n-1)/2} C(n,j) = (\sum_{j=0}^{n} C(n,j))/2 = 2^n/2 = 2^{n-1}$. Since there are two codewords in this code and this is the maximum number of codewords possible by the sphere packing bound, this code is perfect. **9.** There is a 1 in a specified position in $\mathbf{x}+\mathbf{y}$ if there is a 1 in this position in exactly one of $\mathbf{x}$ and $\mathbf{y}$. Note that $w(\mathbf{x}) + w(\mathbf{y})$ is the sum of the number of positions where $\mathbf{x}$ is 1 and the number of positions where $\mathbf{y}$ is 1. It follows that the number of positions where exactly one of $\mathbf{x}$ and $\mathbf{y}$ equals 1 is this sum minus the number of positions where both $\mathbf{x}$ and $\mathbf{y}$ are 1. The result now follows. **10.** $\mathbf{H} = (1\ 1\ 1\ 1\ 1)$. **11.** $\mathbf{H} = \begin{pmatrix} 1&0&0&1&0&0&0&0&0\\ 0&1&0&0&1&0&0&0&0\\ 0&0&1&0&0&1&0&0&0\\ 1&0&0&0&0&0&1&0&0\\ 0&1&0&0&0&0&0&1&0\\ 0&0&1&0&0&0&0&0&1 \end{pmatrix}$. **12.** $\mathbf{H} = \begin{pmatrix} 1&1&0&1&1&0&0\\ 1&0&1&1&0&1&0\\ 1&1&1&0&0&0&1 \end{pmatrix}$. **13.** $\mathbf{G} = \begin{pmatrix} 1&0&1&1&0\\ 0&1&0&1&1 \end{pmatrix}$. **14.** 0000000, 0001111, 0010110, 0011001, 0110011, 0100101, 0101010, 0111100, 1000011, 1001100, 1010101, 1011010, 1100110, 1101001, 1110000, 1111111. **15. a)** Suppose that when the codeword $\mathbf{x}$ is sent, $\mathbf{y}$ is received, where $\mathbf{y}$ contains l erasures, where $l \le d-1$. Since the minimum distance between codewords is d there can be at most one codeword that agrees with $\mathbf{y}$ is the $n-l$ positions that were not erased. This codeword is $\mathbf{x}$. Consequently, we correct $\mathbf{y}$ to be $\mathbf{x}$. **b)** Suppose that when the codeword $\mathbf{x}$ is sent, $\mathbf{y}$ is received and $\mathbf{y}$ contains at m errors and l erasures, such that $m \le t$ and $l \le r$. Suppose that S is the set of bit strings that agree with $\mathbf{y}$ in the $n-l$ positions that were not erased. Then there is a bit string $\mathbf{s}_1 \in S$ with $d(\mathbf{x}, \mathbf{s}_1) \le t$. We will show that $\mathbf{x}$ is the only such codeword. Suppose that there was another codeword $\mathbf{z} \in S$. Then $d(\mathbf{z}, \mathbf{s}_2) \le t$ for some string $\mathbf{s}_2 \in S$. By the triangle inequality, we then would have $d(\mathbf{x}, \mathbf{z}) \le d(\mathbf{x}, \mathbf{s}_1) + d(\mathbf{s}_1, \mathbf{s}_2) + d(\mathbf{s}_2, \mathbf{z})$. But since the first and third terms in the sum of the right hand side of this inequality equal m and the middle term is no larger than l, it follows that $d(\mathbf{x},\mathbf{z}) \le m+l+m = 2m+l \le 2t+r$, which is a contradiction since both $\mathbf{x}$ and $\mathbf{z}$ are codewords and the minimum distance between codewords is $d = 2t+r+1$.

Chapter 6 Stirling Numbers

1. Using $S(7,k) = S(6,k-1) + k\,S(6,k)$ and the values of $S(6,k)$ in Table 1, we get 1, 63, 301, 350, 140, 21, 1 as the values of $S(7,k)$. For the $s(7,k)$, use $s(7,k) = s(6,k-1) - 6\,s(6,k)$ and Table 2 to get the values 720, −1764, 1624, −735, 115, −21, 1. **2.** $(x)_4 = x(x-1)(x-2)(x-3) = x^4 - 6x^3 + 11x^2 - 6x$. **3.** Write $x^4 = (x)_4 + a_3(x)_3 + a_2(x)_2 + a_1(x)_1 = (x^4 - 6x^3 + 11x^2 - 6x) + (x^3 - 3x^2 + 2x)a_3 + (x^2 - x)a_2 + xa_1$ for unknown coefficients a_i. Equating the coefficients of x^j above, we get the equations $-6 + a_3 = 0$, $11 - 3a_3 + a_2 = 0$, $-6 + 2a_3 - a_2 + a_1 = 0$, and find the solution $a_3 = 6 = S(4,3)$, $a_2 = 7 = S(4,2)$, $a_3 = 1 = S(4,1)$. **4.** The number of onto functions from an 8-element set to a 5-element set is, by (3), $\frac{1}{5!}\sum_{i=0}^{5}(-1)^i C(5,i)(5-i)^8 = 1050$. **5.** Writing each function f as an ordered triple $f(1)f(2)f(3)$, the functions f with $|f(N)| = 1$ are the $S(3,1)(3)_1 = 1\cdot 3 = 3$ functions *aaa*, *bbb*, *ccc*. Those with $|f(N)| = 2$ are the $S(3,2)(3)_2 = 3\cdot 6 = 18$ functions *aab*, *aba*, *baa*, *aac*, *aca*, *caa*, *bba*, *bab*, *abb*, *bbc*, *bcb*, *cbb*, *cca*, *cac*, *acc*, *ccb*, *cbc*, and *bcc*. The $S(3,3)(3)_3 = 1\cdot 6 = 6$ functions f with $|f(N)| = 3$ are *abc*, *acb*, *bac*, *bca*, *cab*, and *cba*. **6.** Since the balls are distinguishable and the boxes are identical, we can interpret the balls in a nonempty box as a class of a partition of the set of balls. Then the numbers are given by: **a)** $S(7,4) = 350$; **b)** $S(7,1) + S(7,2) + S(7,3) + S(7,4) = 1 + 63 + 301 + 350 = 715$. **7.** **a)** $2^6/3^6 = 64/729 = 0.088$. **b)** $S(6,3)\,3!/3^6 = 90\cdot 6/729 = 0.741$. **c)** $S(6,2)(3)_2 = 31\cdot 6/729 = 0.255$. **8.** Using Theorem 1 with $k = 3$, recursively compute $S(n,3)$ for $n = 3,4,\ldots,12$. Then the number of partitions of the set of 12 people into three subsets is $S(12,3) = 86,526$. **9.** After $n-1$ rolls, all but one of the k numbers have appeared, and it, say x, appears on roll n. The first $n-1$ rolls then define a function from the set $\{1,2,\ldots,n-1\}$ onto the set of $k-1$ numbers other than x. Since x can be chosen in k ways, by Theorem 2 and the product rule the number of possible sequences is $S(n-1,k-1)k!$. **10.** The sequence corresponds to a function from the set of n positions to the set of 10 digits. The function is arbitrary in a) and onto in b), so the numbers are: **a)** 10^n; **b)** $S(n,10)\cdot 10!$. **11.** Here the pockets are the cells of the distribution. **a)** The distribution corresponds to a function from the set of 15 numbered balls to the set of six pockets, so the number is 6^{15}. **b)** The distributions in the four identical corner pockets correspond to partitions of the set of 15 distinguishable balls into at most four classes, while the distributions in the two identical center pockets correspond to partitions into at most two classes.

By the sum rule, the number of distributions is $\sum_{k=1}^{4} S(15,k) + \sum_{k=1}^{2} S(15,k)$. **12.** The $S(n+1,k)$ partitions into k classes of an $(n+1)$-set $S = \{0,1,\ldots,n\}$ can be classified according to the number of elements other than 0 in the class containing 0. The number of these partitions in which the class containing 0 has size $j+1$ is obtained by choosing a j-element subset J of the n-set $\{1,2,\ldots,n\}$ to put in a class with 0 and then choosing a $(k-1)$-class partitions of the set of $n-j$ remaining elements. Apply the product rule and sum over j. **13. a)** A partition of an n-set N into $n-1$ classes must have one class of size two, and the remaining classes of size one. Since it is determined uniquely by the class of size two, the number of these partitions equals $C(n,2)$, the number of 2-subsets of N. **b)** Since $s(n,n-1)$ is the coefficient of x^{n-1} in the expansion of $(x)_n = x(x-1)\ldots(x-n+1)$, it is equal to -1 times the sum of the positive integers $1,2,\ldots,n-1$, which is $-n(n-1)/2 = -C(n,2)$. **14. a)** The number of ordered partitions (A,B) of an n-element set into two nonempty subsets is 2! times the number $S(n,2)$ of such unordered partitions, since the two classes can be ordered in 2! ways. But there are $2^n - 2$ nonempty proper subsets A of the n-set, and then B is uniquely determined as the complement of A. Thus $S(n,2) = (2^n-2)/2! = 2^{n-1}-1$. **b)** The coefficient $s(n,1)$ of x in the expansion of $(x)_n = x(x-1)\cdots(x-n+1)$ is the product of the terms $-1,-2,\ldots,-(n-1)$ in the last $n-1$ factors, which is $(-1)^{n-1}(n-1)!$ **15.** This follows immediately from the fact that the $S(n,k)$, $s(n,k)$ are the entries for exchanging between the two bases $\{x\}$, $\{(x)_n\}$ of the vector space. From the definitions $x^m = \sum_{k=0}^{m} S(m,k)(x)_k = \sum_{k=0}^{m} S(m,k)\sum_{n=0}^{k} s(k,n)x^n = \sum_{n=0}^{m}(\sum_{k=n}^{m} S(m,k)s(k,n))x^n$, where we have used the fact that $0 \le n \le k \le m$ and $S(m,k) = 0$ for $k > m$, $s(k,n) = 0$ for $k > n$. Equating the coefficients of x^n on both sides gives the result. **16.** Replace x by $-x$ in (8), and note that $(-x)_n = (-1)^n x(x+1)\cdots(x+n-1) = (-1)^n (x)^n$, where $(x)^n = x(x+1)\cdots(x+n-1)$ is the **rising factorial**. This gives $(x)^n = \sum_{k=0}^{n}(-1)^{n-k}s(n,k)x^k = \sum_{k=0}^{n} t(n,k)x^k$ and the coefficients of x^k in the expansion of $(x)^n$ are clearly positive. **17.** From Exercise 16 we have $t(n,k) = (-1)^{n-k}s(n,k)$, which is equivalent to $s(n,k) = (-1)^{n-k}t(n,k)$. Substituting for $s(n,k)$ in (12) gives $t(n,k) = t(n-1,k-1) + (n-1)t(n-1,k)$. **18.** Let $p(n,k)$ be the number of permutations of an n-element set with k cycles in their cycle decomposition. For a fixed k, assume that $p(n-1,k) = t(n-1,k)$. Then a k-cycle permutation σ of the n-set $\{1,2,\ldots,n\}$ can be obtained from either: (i) a $(k-1)$-cycle permutation of the $(n-1)$-set $\{1,2,\ldots,n-1\}$ by adding (n) as a cycle of length one (fixed point), or (ii) a k-cycle permutation of $\{1,2,\ldots,n-1\}$ by choosing one of the $n-1$ elements a and inserting n as the new image of a, with the image of n being the previous image of a. The two cases are exclusive. There are $p(n-1,k-1)$ of type (i) and $(n-1)p(n-1,k)$ of type (ii). Thus $p(n,k)$ satisfies the recurrence relation satisfied by $t(n,k)$ (Exercise 17). Since $p(n,k) = t(n,k)$ for the initial values with $n = k, k+1$, the two sequences are equal. **19.** Since the permutations are given as random, we assume each of the $n!$ permutations has equal probability $1/n!$. Let X be the number of cycles. **a)** By Exercise 18 the number of permutations with k cycles is $t(n,k)$, so the probability is $p(X=k)=t(n,k)/n!$. **b)** The total number of permutations is $8! = 40{,}320$. Using results from Exercises 1, 16, and 17, $t(8,2)=t(7,1)+7t(7,2)=720+7\cdot 1764 = 13{,}068$, $t(8,3)=t(7,2)+7t(7,3) = 1764+7\cdot 1624 = 13{,}132$, so the probabilities are $p(X=2)=13{,}068/40{,}320 = 0.3241$, $p(X=3)=13{,}132/40{,}320 = 0.3257$. It is slightly more likely that the number of cycles is 3 than it is 2. **20.** Let $C(n,k)$ be the set over which the sum is taken, so $a(n,k)=\sum\limits_{(c_1,c_2,\ldots,c_k)\in C(n,k)} 1^{c_1}2^{c_2}\cdots k^{c_k}$. Partition $C(n,k)$ into two classes $A(n,k)=\{(c_1,c_2,\ldots,c_k)|c_k=0\}, B(n,k)=\{(c_1,c_2,\ldots,c_k)|c_k\geq 1\}$. Then $a(n,k)=\sum\limits_{A(n,k)} 1^{c_1}2^{c_2}\cdots k^{c_k} + \sum\limits_{B(n,k)} 1^{c_1}2^{c_2}\cdots k^{c_k}$. But the first sum is unchanged if we omit the k^{c_k} factor in each term, since $k^{c_k}=1$ when $c_k=0$. But then it is equal to $\sum\limits_{C(n,k-1)} 1^{c_1}2^{c_2}\cdots (k-1)^{c_{k-1}} = a(n-1,k-1)$ since $c_1+c_2+\cdots+c_{k-1} = (n-1)-(k-1)$. In the second sum where $c_k\geq 1$, factor out k to reduce k^{c_k} to k^{c_k-1} inside the summation. Then $c_1+c_2+\cdots+(c_k-1) = n-k-1 = (n-1)-k$, and all the new c_i's are non-negative, so $\sum\limits_{B(n,k)} 1^{c_1}2^{c_2}\cdots k^{c_k} = k\sum\limits_{C(n-1,k)} 1^{c_1}2^{c_2}\cdots k^{c_k} = k\,a(n-1,k)$. Thus $a(n,k)=a(n-1,k-1)+k\,a(n-1,k)$ which is equivalent to (1). For the initial conditions, the sum is vacuous if $k>n$, so $a(n,k)=0$ when $k>n$. When $k=1$, the only term in the sum is $1^{n-1}=1$, so $a(n,1)=1$. Since $a(n,k)$ and $S(n,k)$ satisfy the same recurrence relation and have the same initial values, it follows that $S(n,k)=a(n,k)=\sum\limits_{C(n,k)} 1^{c_1}2^{c_2}\cdots k^{c_k}$. **21. a)** Dividing both sides by $S(n,k)$ of the recurrence relation in (1) gives $\frac{S(n+1,k)}{S(n,k)} = \frac{S(n,k-1)}{S(n,k)}+k$, so $\lim\limits_{n\to\infty}\frac{S(n+1,k)}{S(n,k)} = \lim\limits_{n\to\infty}\frac{S(n,k-1)}{S(n,k)}+k = k$. **b)** Here we divide the recurrence relation in (12) by $s(n,k)$ to get $\frac{s(n+1,k)}{s(n,k)}=\frac{s(n,k-1)}{s(n,k)}-n$, and on taking limits we get $\lim\limits_{n\to\infty}\frac{s(n+1,k)}{s(n,k)} = \lim\limits_{n\to\infty}\frac{s(n,k-1)}{s(n,k)}-n=-n$. **22.** By the Binomial Theorem, $\frac{1}{k!}(e^x-1)^k = \frac{1}{k!}\sum\limits_{j=0}^{k}(-1)^jC(k,j)e^{(k-j)x}$. Now use the fact that $e^{ax}=\sum\limits_{n=0}^{\infty}a^n\frac{x^n}{n!}$ to get $\frac{1}{k!}(e^x-1)^k = \frac{1}{k!}\sum\limits_{j=0}^{k}(-1)^jC(k,j)\sum\limits_{n=0}^{\infty}(k-j)^n\frac{x^n}{n!} =$

$\sum\limits_{n=0}^{\infty}(\sum\limits_{j=0}^{k}\frac{(-1)^j}{k!}C(k,j)(k-j)^n)\frac{x^n}{n!} = \sum\limits_{n=0}^{\infty}S(n,k)\frac{x^n}{n!}$ by (4). **23.** The polynomial $p_k(x)=\prod\limits_{i=1}^{k}(1-ix)$ can be written as $p_k(x)=\prod\limits_{i=0}^{k}(1-ix)$ since the added term for $i=0$ doesn't change the product. Then divide each of the $k+1$ factors by x and multiply $p_k(x)$ by x^{k+1}, to get $p_k(x)=x^{k+1}\prod\limits_{i=0}^{k}\left(\frac{1}{x}-i\right) =$

$$x^{k+1}(x^{-1})_{k+1} = x^{k+1}\sum_{j=0}^{k+1} s(k+1,j)x^{-j} = \sum_{j=0}^{k+1} s(k+1,j)x^{k+1-j} =$$

$$\sum_{j=0}^{k+1} s(k+1,k+1-j)x^{j}.$$

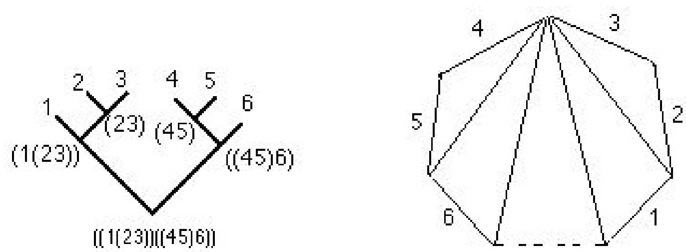
Chapter 7 Catalan Numbers

1. a) Substituting left parentheses for 1 and right parentheses for -1 gives the well-formed sequence (( )( )( ))(( )).

b)

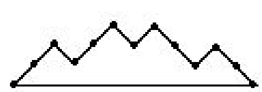
c) Starting with 123456[ ], and interpreting each + as a push and each − as a pop, the sequence successively produces: 12345[6],1234[56], 1234[6]5, 123[46]5, 123[6]45, 12[36]45,12[6]345, 12[ ]6345, 1[2]6345, [12]6345, [2]16345, [ ]216345. The stack permutation is therefore 216345. **2. a)** The string **q** of left parentheses and the first five variables is ((1(23((45. Adding the last variable on the right and closing parentheses from the left gives the well-parenthesized product **p** = ((1(23))((45)6)).

b)



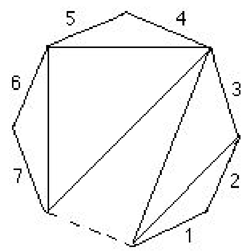
3. a) ((1((2(34))(56)))7). **b)** (+ + − + + − + − − + −−).

c)



d) 12345[6], 1234[56], 1234[6]5, 123[46]5, 12[346]5, 12[46]35, 1[246]35, 1[46]235, 1[6]4235, [16]4235, [6]14235, [ ]614235. **4. a)** (+++−−−++−−+−). **b)** 12345[6], 1234[56], 123[456], 123[56]4, 123[6]54, 123[ ]654, 12[3]654, 1[23]654, 1[3]2654, 1[ ]32654, [1]132654, [ ]132654.

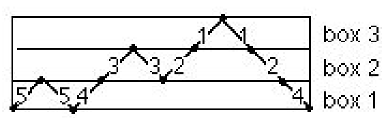
c)



5.

[ ]42135		
[4]2135	4 Popped	−
[24]135	2 Popped	−
[124]35	1 Popped	−
1[24]35	1 Pushed	+
12[4]35	2 Pushed	+
12[34]5	3 Popped	−
123[4]5	3 Pushed	+
1234[ ]5	4 Pushed	+
1234[5]	5 Popped	−
12345[ ]	5 Pushed	+

The admissible sequence is therefore $(+ - + + - + + - - -)$. The path is the following.



6. $c_n = \frac{1}{n+1}\frac{(2n)!}{(n!)^2} = \frac{2n(2n-1)}{(n+1)n^2}\frac{(2n-2)!}{((n-1)!)^2} = \frac{(4n-2)\cdot(2n-2)!}{(n+1)n\cdot((n-1)!)^2} = \frac{4n-2}{n+1}c_{n-1}$.

7. Substituting Stirling's approximation for the factorials in c_n gives $c_n = \frac{1}{n+1}\frac{(2n)!}{n!n!} \approx \frac{1}{n+1}\frac{\sqrt{4\pi n}\ (2n)^{2n}}{2\pi n\ n^{2n}} = \frac{2^{2n}}{(n+1)\sqrt{\pi n}} \approx \frac{4^n}{\sqrt{\pi}\ n^{3/2}} = O(n^{-3/2}\ 4^n)$. **8.** If $v_iv_j \in D$, then the polygon P' with vertices $v_i, v_{i+1}, \ldots, v_j$ is triangulated by the set D' of diagonals of D that are also diagonals of P'. With v_iv_j serving as the excluded side of P', the product of sides $s_{i+1}s_{i+2}\cdots s_j$ is well-parenthesized into $\mathbf{p}'$ by recursively parenthesizing the two polygon sides that are sides of outer triangles, and then reducing the polygon. The order of reducing the polygons does not affect the well-parenthesized product finally obtained, so $\mathbf{p}'$ will be a subsequence of consecutive terms of $\mathbf{p}$. Conversely, if $s_{i+1}s_{i+2}\cdots s_j$ is well-parenthesized in p as $\mathbf{p}'$, then $\mathbf{p}'$ will appear on the diagonal joining the end vertices of the path formed by the sides in $\mathbf{p}'$. But that diagonal is v_iv_j. **9.** Suppose that D contains the diagonal v_kv_{n+1}. Then after putting $i = k, j = n+1$ in Exercise 8, the product $s_{k+1}s_{k+2}\cdots s_{n+1}$ is well-parenthesized as a sequence $\mathbf{p}'$. Corresponding to $\mathbf{p}'$ is a nonnegative path from the origin to $(2(n-k), 0)$. Translating this path to the right $2k$ units identifies it with the segment of the path corresponding to T from $(2k, 0)$ to

$(2n, 0)$. Conversely, suppose the nonnegative path corresponding to T meets the x-axis at the point $(2k, 0)$. On removing s_{n+1} and the n right parentheses from the corresponding well-parenthesized sequence $\mathbf{p}$ of the product $s_1 s_2 \cdots s_{n+1}$, we obtain a sequence $\mathbf{q}$ with n left parentheses interlaced with the product $s_1 s_2 \cdots s_n$. Since the path passes through the point $(2k, 0)$, the first $2k$ terms of $\mathbf{q}$ has k left parentheses interlaced with $s_1 s_2 \cdots s_k$. Hence the subsequence $\mathbf{q}'$ consisting of the last $2(n-k)$ terms of $\mathbf{q}$ interlaces $n-k$ left parentheses with $s_{k+1} s_{k+2} \cdots s_n$. Since the path is nonnegative, the segment corresponding to $\mathbf{q}'$ is nonnegative, and so corresponds to a triangulation of the $(n-k+2)$-gon with sides $s_{k+1} s_{k+2} \cdots s_{n+1}$ and the diagonal (of P) joining the end vertices of the path. But that diagonal is $v_k v_{n+1}$. **10.** If D contains the diagonal $v_0 v_n$, then $v_n v_{n+1} v_0$ is an outer triangle, so no diagonals on v_{n+1} can be in D. By Exercise 9 the corresponding path cannot meet the x-axis at any point $(2k, 0)$ for $1 \le k \le n-1$. Since the path starts on the x-axis, it can only return to the x-axis after an even number $2k$ of steps. Thus the path is positive. Conversely, if the path is positive, then D contains no diagonals $v_k v_{n+1}$ for $1 \le k \le n-1$. Since every side of P is in exactly one triangle of T, s_n, s_0 must be in the same triangle. But the outer triangle $v_n v_{n+1} v_0$ is the only triangle that can contain both. Thus T contains the diagonal $v_0 v_n$.

Chapter 8 Ramsey Numbers

1. There are 4 colorings of K_3 to consider: all 3 edges red, 2 red and 1 green, 1 red and 2 green, all 3 edges green. The first 3 colorings yield a red K_2, while the last yields a green K_3. The coloring of K_2 where both edges are green fails to have a red K_2 or a green K_3. **2.** Draw K_6 as a hexagon with all its diagonals. Color the edges of the outer hexagon red and all diagonals green. Clearly, there is no red K_3. To see that there is no green K_4, number the vertices clockwise with $1, \ldots, 6$. If vertex 1 were a vertex of a green K_4, then vertices 3,4,5 must also be in K_4 (since K_4 has 4 vertices). But vertices 3 and 4, for example, are not joined by a green edge, and hence no K_4 subgraph using vertex 1 is possible. Similar reasoning shows that none of the vertices $2, \ldots, 6$ can be in a green K_4. **3.** The coloring of Figure 1(a) shows that 5 does not have the $(3, 3)$-Ramsey property. Suppose $m < 5$, and choose any m vertices from this figure and consider the subgraph K_m of K_5 constructed on these m vertices. This gives a coloring of the edges of K_m that has no red K_3 or green K_3, since the original coloring of the edges of K_5 has no red K_3 or green K_3. **4.** Every coloring of the edges of K_k either has a red K_2 (i.e., a red edge) or else every edge is colored green (which gives a green K_k). Therefore $R(2, k) \le k$. If we color every edge of K_{k-1} green, then K_{k-1} does not have either a red K_2 or a green K_k. Therefore $R(2, k) > k-1$, and hence $R(2, k) = k$. **5.** Let $R(i, j) = n$. We show that $R(j, i) \le R(i, j)$ and $R(i, j) \le R(j, i)$. To show that $R(j,i) \leq R(i,j)$, we show that every coloring of the edges of K_n also contains either a red K_j or a green K_i. Choose any coloring, C, of K_n and then reverse the colors to get a second coloring, D. Since $R(i,j) = n$, the coloring D contains either a red K_i or a green K_j. Therefore, coloring C contains either a red K_j or a green K_i. This shows that $R(j,i) \leq n = R(i,j)$. A similar argument shows that $R(i,j) \leq R(j,i)$. **6.** Suppose m has the (i,j)-Ramsey property and $n > m$. Consider any coloring of the edges of K_n with red and green. Choose any m vertices of K_n and consider the subgraph K_m determined by these m vertices. Then K_m, with the coloring that is inherited from the coloring of K_n, must have a subgraph that is a red K_i or a green K_j. Hence K_n also has a red K_i or a green K_j. Therefore n has the (i,j)-Ramsey property. **7.** Suppose n did have the (i,j)-Ramsey property. By Exercise 6, every larger integer also has the (i,j)-Ramsey property. In particular, m must have the (i,j)-Ramsey property, which is a contradiction of the assumption. **8.** We will show that if m has the (i_1,j)-Ramsey property, then m also has the (i_2,j)-Ramsey property. If m has the (i_1,j)-Ramsey property, then every coloring of the edges of K_m with red and green contains a red K_{i_1} or a green K_j. If K_m has a red K_{i_1}, then it also has a red K_{i_2} (since $i_1 \geq i_2$). Therefore K_m has either a red K_{i_2} or a green K_j. Hence K_m has the (i_2,j)-Ramsey property and therefore $m \geq R(i_2,j)$. This shows that every integer m with the (i_1,j)-Ramsey property also satisfies $m \geq R(i_2,j)$. Therefore, the smallest integer m with this property, namely $R(i_1,j)$, must also satisfy $R(i_1,j) \geq R(i_2,j)$. **9.** We assume that $|B| \geq R(i,j-1)$ and must show that m has the (i,j)-Ramsey property. Since $|B| \geq R(i,j-1)$, the graph $K_{|B|}$ contains either a red K_i or a green K_{j-1}. If we have a green K_{j-1}, we add the green edges joining v to the vertices of K_{j-1} to obtain a green K_j. Thus, $K_{|B|}$, and hence K_m, has either a red K_i or a green K_j. Therefore m has the (i,j)-Ramsey property. **10.** The proof follows the proof of Lemma 1. We know that either $|A| \geq R(i,j-1)$ or $|B| \geq R(i-1,j)$. But m is even, so $\deg(v)$ is odd. Therefore, either $|A|$ or $|B|$ is odd. Let $m = R(i,j-1)+R(i-1,j)-1$, which is an odd number. Suppose the edges of K_m are colored red or green. Choose any vertex v and define the sets A and B as in the proof of the Lemma. It follows that either $|A| \geq R(i,j-1)$ or $|B| \geq R(i-1,j)-1$. But $\deg(v)$ is even since v is joined to the $m-1$ other vertices of K_m. Since $|A \cup B| = |A| + |B| = m-1$ (which is even), either $|A|$ or $|B|$ are both even or both odd. If $|A|$ and $|B|$ are both even, then $|B| > R(i-1,j)-1$, since $R(i-1,j)-1$ is odd. If $|A|$ and $|B|$ are both odd, then $|A| > R(i,j-1)$, since $R(i,j-1)$ is even. Therefore $m-1 = |A| + |B| > R(i-1,j) + R(i,j-1) - 1$. **11.** Suppose that the vertices of the graph in Figure 2(b) are numbered $1,\ldots,8$ clockwise. By the symmetry of the figure, it is enough to show that vertex 1 cannot be a vertex of a K_4 subgraph. Suppose 1 were a vertex of a K_4 subgraph. Since vertex 1 is adjacent to only 3,4,6,7, we would need to have three of these four vertices all adjacent to each other (as well as to 1). But this is impossible, since no three of the vertices 3,4,6,7 are all adjacent to each other. **12.** Suppose that the graph had a red K_3 and vertex 1 were a vertex of a red triangle. Vertex 1 is adjacent to vertices 2,6,9,13, but none of these four vertices are adjacent. Therefore, 1 is not a vertex of a red triangle. Similar reasoning applies to each of the other twelve vertices. Thus, K_{13} has no red K_3. We next show that K_{13} has no green K_5. Suppose it did, and vertex 1 were a vertex of the K_5. Then the other four vertices must come from the list 3,4,5,7,8,10,11,12. If vertex 3 were also in K_5, then the other two vertices must come from 5,7,10,12 (since there are the only vertices adjacent to 1 and 3). But we cannot choose both 5 and 10 or 7 and 12 since they are not adjacent. Thus, we can only pick 4 vertices: 1,3, and two from 5,7,10,12. Hence there is no K_5 if we use both 1 and 3. Similar reasoning shows that we cannot build a K_5 by using 1 and 4, 1 and 5, 1 and 7, etc. Thus vertex 1 cannot be a vertex in a K_5 subgraph. By the symmetry of the graph, the same reasoning shows that none of the vertices $2, 3, \ldots, 13$ can be part of K_5. Therefore, the graph contains no green K_5. **13.** $R(4,4) \leq R(4,3) + R(3,4) = 9 + 9 = 18$. **14.** Consider G as a subgraph of K_9. Color the edges of G red and color the edges of $\overline{G}$ (also a subgraph of K_9) green. Since $R(3,4) = 9$, K_9 has either a red K_4 or a green K_3. But a red K_4 must be a subgraph of G and a green K_3 must be a subgraph of $\overline{G}$. **15.** Basis step: $R(2,2) = 2 \leq C(2,1)$. Induction step: We must prove that if $i + j = n + 1$, then $R(i,j) \leq C(i + j - 2, i - 1)$. Using (1), $R(i,j) \leq R(i,j-1) + R(i-1,j) \leq C(i+j-3,i-1) + C(i+j-3,i-2) = C(i+j-2,i-1)$. **16.** We first show that 2 has the $(2,\ldots,2;2)$-Ramsey property. K_2 has only 1 edge, so 1 of the colors $1,\ldots,n$ must have been used to color this edge. If this edge has color j, then K_2 has a subgraph K_2 of color j. Thus $R(2,\ldots,2;2) \leq 2$. But $R(2,\ldots,2;2) > 1$ since K_1 has 0 edges and thus cannot have a subgraph K_2 of any color. **17.** We first show that 7 has the $(7,3,3,3,3;3)$-Ramsey property. Let $|S| = 7$ and suppose that its 3-element subsets are partitioned into 5 collections $C_1, C_2, \ldots, C_5$. If all the 3-element subsets are in C_1, we can take S itself as the subset that satisfies the definition of the $(7,3,3,3,3;3)$-Ramsey property. If some C_j (for some $j > 1$) has at least one 3-element subset in it, then this 3-element subset of S satisfies the definition. Therefore 7 has the $(7,3,3,3,3;3)$-Ramsey property. We now show that 6 fails to have the $(7,3,3,3,3;3)$-Ramsey property. Suppose $|S| = 6$ and all 3-element subsets are placed in C_1. The value $j = 1$ does not work in the definition since S has no subset of size $i_1 = 7$. The values $j = 2,3,4,5$ do not work since these C_j contain no sets. Therefore $R(7,3,3,3,3;3) = 7$. **18.** Suppose we have a partition with no monochromatic solution. We must have either $1 \in R$ or $1 \in G$. Assume $1 \in R$. Then $2 \in G$ (otherwise we have a red solution $1 + 1 = 2$). Since $2 \in G$, we must have $4 \in R$ (otherwise we have a green solution $2 + 2 = 4$). Then we must have $3 \in G$ to avoid the red solution $1 + 3 = 4$. If $5 \in R$, we have the red solution $1 + 4 = 5$, and if $5 \in G$, we have the green solution $2 + 3 = 5$. This contradicts the assumption that we have no monochromatic solution. A similar argument holds if $1 \in G$. **19.** Use one color for each of the following sets: $\{1,4,10,13\}, \{2,3,7,11,12\}, \{5,6,8,9\}$. **20.** Suppose the 5 points are a, b, c, d, e. Form the smallest convex polygon P that contains these 5 points either inside P or on P itself. If at least 4 of the 5 points lie on P, then we obtain a convex polygon by joining these 4 points. If P contains only 3 of the 5 points, say a, b, and c, the points d and e lie inside P. Draw the straight line through d and e. Two of the points a, b, c lie on one side of the line. Then these two points, together with d and e determine a convex 4-gon **21. (a)** If the game ended in a draw, we would have K_6 colored with red and green with no monochromatic K_3, contradicting the fact that $R(3,3) = 6$. **(b)** The coloring of the graph of Figure 1 shows how the game could end in a draw. **(c)** There must be a monochromatic K_3 since $R(3,3) = 6$. Therefore someone must lose. **22. (a)** Example 4 shows that $R(3,3,3;2) = 17$. Therefore there is a coloring of K_4 with 3 colors that contains no monochromatic K_3. Thus, the game can end in a draw. **(b)** Since $R(3,3,3;2) = 17$, there must be a monochromatic triangle, and therefore a winner. **23.** We could conclude that $R(5,5) > 54$. (If this does happen, please inform your instructor.) **24.** To show that $R(K_{1,1}, K_{1,3}) \leq 4$, take K_4 and color its edges red and green. If any edge is red, K_4 has a red $K_{1,1}$. If no edges are red, then K_4 has a green $K_{1,3}$. But $R(K_{1,1}, K_{1,3}) > 3$ since we can take K_3 and color its edges green, showing that K_3 has no red $K_{1,1}$ or green $K_{1,3}$.

Chapter 9 Arrangements with Forbidden Positions

1. For B_1, there are 3 ways to place 1 rook and 1 way to place 2 rooks ($(1,S)$ and $(4,F)$), so $R(x, B_1) = 1 + 3x + x^2$. For B_2, one rook can be placed in 6 ways, so $r_1 = 6$. Two rooks can be placed on $(2,K)$ and any of the squares $(3,W)$, $(5,D)$, $(5,W)$, on $(D,2)$ and any of the squares $(3,K)$, $(3,W)$, $(5,W)$, on $(3,K)$ and either $(5,D)$ or $(5,W)$, or $(3,W)$ and $(5,D)$; thus $r_2 = 9$. Three rooks can be placed on $(2,K)$, $(3,W)$, $(5,D)$ or on $(2,D)$, $(3,K)$, $(5,W)$; thus $r_3 = 2$. Therefore, $R(x, B_2) = 1 + 6x + 9x^2 + 2x^3$. **2.** Since there is only 1 way to place one rook on B_1, $R(x, B_1) = 1 + x$. For board B_2, there are 4 ways to place 1 rook on a forbidden square, so $r_1 = 4$. There are 4 ways to place 2 rooks—on (blue, tan) and (brown, blue), or on (blue, tan) and (plaid, yellow), or on (brown, blue) and (plaid, yellow), or on (brown, blue) and (plaid, tan)—so $r_2 = 4$. There is only 1 way to place 3 rooks—on (blue, tan), (brown, blue), (plaid, yellow)—so $r_3 = 1$. Therefore, $R(x, B_2) = 1 + 4x + 4x^2 + x^3$. **3.** Suppose B' is a subboard of B and (5,6) is in B'. This forces row 5, and hence columns 8 and 9 to be in B'. This forces row 3 to be in B'. This in turn forces column 7 to be in B', which forces column 10 to be in B'. Thus, all five columns must be in B', and hence $B' = B$. **4.** We can take B' to be the board with rows 1 and 3, and columns 1 and 2. Then $R(x, B') = 1 + 3x + x^2$. For B''(rows 2 and 4, and columns 3 and 4), we have $R(x, B'') = 1 + 2x$. Hence $R(x, B) = R(x, B') \cdot R(x, B'') = (1 + 3x + x^2)(1 + 2x) = 1 + 5x + 7x^2 + 2x^3$. The number of arrangements is $4! - (5 \cdot 3! - 7 \cdot 2! + 2 \cdot 1!) = 6$. **5.** This board cannot be split into disjoint subboards. Assume that the rows (and columns) are numbered $1, 2, 3, 4$. Choose the square (4,1). The rook polynomial for the board with row 4 and column 1 removed is $R(x, B') = 1 + 4x + 3x^2$. Also, $R(x, B'') = 1+6x+11x^2+7x^3+x^4$. Therefore, $R(x, B) = x(1+4x+3x^2)+(1+6x+11x^2+7x^3+x^4) = 1+7x+15x^2+10x^3+x^4$, and the number of arrangements is $4! - (7 \cdot 3! - 15 \cdot 2! + 10 \cdot 1! - 1 \cdot 0!) = 3$. **6.** The board cannot be split. Using square (1,1), the rook polynomial for the board with row 1 and column 1 removed is $R(x, B') = 1+5x+7x^2+2x^3$. Also, $R(x, B'') = 1+7x+16x^2+13x^3+3x^4$. Therefore $R(x, B) = xR(x, B') + R(x, B'') = 1 + 8x + 21x^2 + 20x^3 + 5x^4$. Thus, the number of arrangements is $5{\cdot}4{\cdot}3{\cdot}2-[8{\cdot}4{\cdot}3{\cdot}2-21{\cdot}3{\cdot}2+20{\cdot}2-5{\cdot}1] = 19$. **7.** Split the board, using rows 1 and 2 and columns 1 and 2 for board B'. Therefore, $R(x, B) = R(x, B') \cdot R(x, B'') = (1 + 3x + x^2)(1 + 3x + 3x^2 + x^3) = 1+6x+13x^2+13x^3+6x^4+x^5$. Therefore, the number of allowable arrangements is $5!-6 \cdot 4!+13 \cdot 3!-13 \cdot 2!+6 \cdot 1!-1 \cdot 0! = 33$. **8.** Deleting row 5 and column 9 from B'' yields a board with rook polynomial $1+4x+3x^2$. When only square (5,9) is deleted, we obtain a board B'''. This board can be broken up, using row 5 and columns 6 and 8 for the first board. Its rook polynomial is $1 + 2x$. The rook polynomial for the other part is $1 + 5x + 6x^2 + x^3$. Therefore, the rook polynomial for B''' is $(1 + 2x)(1 + 5x + 6x^2 + x^3)$. Theorem 3 then gives the rook polynomial for B''. **9.** The rook polynomial is $1 + 4x + 4x^2 + x^3$. Therefore, the number of permutations is $4! - 4 \cdot 3! + 4 \cdot 2! - 1 \cdot 1! = 7$. **10.** Using Theorem 3 with the square (3,DeMorgan), we obtain $x(1 + 5x + 7x^2 + 3x^3) + (1 + 2x)(1 + 6x + 9x^2 + 2x^3)$. (The second product is obtained by deleting the square (3,DeMorgan) and breaking the remaining board into 2 disjoint subboards, using row 2 and the De Morgan and Hamilton columns.) Therefore, the rook polynomial is $1+9x+26x^2+27x^3+7x^4$ and the number of arrangements is $6{\cdot}5{\cdot}4{\cdot}3-[9{\cdot}5{\cdot}4{\cdot}3-26{\cdot}4{\cdot}3+27{\cdot}3-7{\cdot}1] = 58$. **11.** The rook polynomial is $1+4x$. Therefore, the number of arrangements is $4! - [4 \cdot 3!] = 0$. **12.** The rook polynomial is 1. Therefore, the number of arrangements is $4! - [0 \cdot 3! - 0 \cdot 2! + 0 \cdot 1! - 0 \cdot 0!] = 4!$. **13.** This is Exercise 9 in another form. The answer is 7. **14.** Break the board into 2 disjoint subboards, using rows Nakano and Sommer, and columns numerical analysis and discrete math for the first subboard. This yields $(1+3x+x^2)(1+4x+4x^2) = 1+7x+17x^2+16x^3+4x^4$. Therefore, the number of arrangements is $5! - 7 \cdot 4! + 17 \cdot 3! - 16 \cdot 2! + 4 \cdot 1! = 26$. **15.** Choose any seating arrangement for the 4 women (1, 2, 3, 4) at the 4 points of the compass (N, E, S, W). Suppose that the 4 women are seated in the order 1, 2, 3, 4, clockwise from north. Also suppose that the men are named 1, 2, 3, 4 and the empty chairs for the men are labeled A, B, C, D (clockwise), where chair

A is between woman 1 and woman 2., etc. Using Theorem 3 with the square (1,A), the rook polynomial is $x(1+5x+6x^2+x^3)+(1+7x+15x^2+10x^3+x^4) = 1+8x+20x^2+16x^3+2x^4$. Therefore, Theorem 2 shows that the number of arrangements is $4!-8\cdot 3!+20\cdot 2!-16\cdot 1!+2\cdot 0! = 2$. Since there are 4! ways in which the 4 women could have been seated, the answer is $4!\cdot 2 = 48$.

Chapter 10 Block Designs and Latin Squares

1. Interchange rows 2 and 3, then interchange columns 2 and 3, and finally reverse the names 2 and 3 to obtain $\begin{pmatrix}1&2&3&4\\2&3&4&1\\3&4&1&2\\4&1&2&3\end{pmatrix}$. **2.** Move columns 4 and 5 to columns 2 and 3, then rearrange rows to obtain $\begin{pmatrix}1&2&3&4&5\\2&3&4&5&1\\3&4&5&1&2\\4&5&1&2&3\\5&1&2&3&4\end{pmatrix}$.

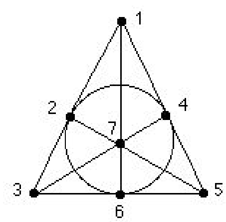
3. a) Reflexivity is clear. **b)** Symmetry is shown by reversing the steps that show L_1 equivalent to L_2. **c)** Execute the steps that show L_1 equivalent to L_2, followed by the steps that show L_2 equivalent to L_3; this shows L_1 equivalent to L_3. **4.** $\begin{pmatrix}1&3&4&2\\4&2&1&3\\2&4&3&1\\3&1&2&4\end{pmatrix}$. **5. a)** $\begin{pmatrix}1&2&3&4\\2&1&4&3\\3&4&1&2\\4&3&2&1\end{pmatrix}$, $\begin{pmatrix}1&3&4&2\\4&2&1&3\\2&4&3&1\\3&1&2&4\end{pmatrix}$, $\begin{pmatrix}1&4&2&3\\3&2&4&1\\4&1&3&2\\2&3&1&4\end{pmatrix}$. **b)** $4^2+4+1 = 17$ points. **c)** $4+1 = 5$ points on a line. **d)** points $\{a_{ij} | 1 \le i,j \le 4\}, R, C, P_1, P_2, P_3$; lines: $a_{11}, a_{12}, a_{13}, a_{14}, R$; $a_{21}, a_{22}, a_{23}, a_{24}, R$; $a_{31}, a_{32}, a_{33}, a_{34}, R$; $a_{41}, a_{42}, a_{43}, a_{44}, R$; $a_{11}, a_{21}, a_{31}, a_{41}, C$; $a_{12}, a_{22}, a_{32}, a_{42}, C$; $a_{13}, a_{23}, a_{33}, a_{43}, C$; $a_{14}, a_{24}, a_{34}, a_{44}, C$;
from L_1: $a_{11}, a_{22}, a_{33}, a_{44}, P_1$; $a_{12}, a_{21}, a_{34}, a_{43}, P_1$; $a_{13}, a_{24}, a_{31}, a_{42}, P_1$; $a_{14}, a_{23}, a_{32}, a_{41}, P_1$;
from L_2: $a_{11}, a_{23}, a_{34}, a_{42}, P_2$; $a_{14}, a_{22}, a_{31}, a_{43}, P_2$; $a_{12}, a_{24}, a_{33}, a_{41}, P_2$; $a_{13}, a_{21}, a_{32}, a_{44}, P_2$;
from L_3: $a_{11}, a_{24}, a_{31}, a_{44}, P_3$; $a_{13}, a_{22}, a_{34}, a_{41}, P_3$; $a_{14}, a_{21}, a_{33}, a_{42}, P_3$; $a_{12}, a_{23}, a_{31}, a_{44}, P_3$; extra R, C, P_1, P_2, P_3. **6.** Since $b = 69$, $v = 24$, and $k = 8$, $bk = vr$ gives $69(8) = 24r$, so $r = 23$. Then $\lambda(23) = 23(7)$, so $\lambda = 7$. Thus, this is a $(69, 24, 23, 8, 7)$-design. **7.** Deleting column 4 yields the 3-design $\{\{1,2,3\},\{2,1,4\},\{3,4,1\},\{4,3,2\}\}$. Clearly $b = 4$, $v = 4$, $k = 3$. Hence $r = 3$ $(bk = vr)$, so $3\lambda = 3(2)$. Thus $\lambda = 2$. This is a $(4,4,3,3,2)$-design. **8.** Let

a_{11} be 1, a_{12} be 2,..., a_{33} be 9, R be 10, C be 11, P_1 be 12, P_2 be 13. Then the varieties are $1, 2, \ldots, 13$. The blocks are $\{1,2,3,10\}$, $\{4,5,6,10\}$, $\{7,8,9,10\}$, $\{1,4,7,11\}$, $\{2,5,8,11\}$, $\{3,6,9,11\}$, $\{1,6,8,12\}$, $\{2,4,9,12\}$, $\{3,5,7,12\}$, $\{1,5,9,13\}$, $\{2,6,7,13\}$, $\{3,4,8,13\}$, $\{10,11,12,13\}$. This is a $(13,13,4,4,1)$-design. **9.** Since $\lambda(v-1) = r(k-1)$, $\lambda = 1$, $k = 3$, and $v = 6n+3$, we have $6n+2 = 2r$, so $r = 3n+1$. Also $bk = vr$, so $b = vr/k = (6n+3)(3n+1)/3 = (2n+1)(3n+1)$. **10. a)** Taking group 0: $\{14,1,2\}$, $\{3,5,9\}$, $\{11,4,0\}$, $\{7,6,12\}$, $\{13,8,10\}$, and adding $2i$ to each number not equal to 14, then taking the result mod 14 to obtain group i gives:

group 1: $\{14,3,4\}$, $\{5,7,11\}$, $\{13,6,2\}$, $\{9,8,0\}$, $\{1,10,12\}$;
group 2: $\{14,5,6\}$, $\{7,9,13\}$, $\{1,8,4\}$, $\{11,10,2\}$, $\{3,12,0\}$;
group 3: $\{14,7,8\}$, $\{9,11,1\}$, $\{3,10,6\}$, $\{13,12,4\}$, $\{5,0,2\}$;
group 4: $\{14,9,10\}$, $\{11,13,3\}$, $\{5,12,8\}$, $\{1,0,6\}$, $\{7,2,4\}$;
group 5: $\{14,11,12\}$, $\{13,1,5\}$, $\{7,0,10\}$, $\{3,2,8\}$, $\{9,4,6\}$;
group 6: $\{14,13,0\}$, $\{1,3,7\}$, $\{9,2,12\}$, $\{5,4,10\}$, $\{11,6,8\}$. **b)** Thursday is group 4, when 5 walks with 12 and 8. **c)** 14 and 0 walk with 13 in group 6 (Saturday). **11.** This amounts to finding a Kirkman triple system of order 9 with 12 blocks partitioned into four groups (weeks). Here is one solution (devised from letting $A, \ldots, I$ correspond to $0, \ldots, 8$; finding an original group; and adding $2i$ mod 8 to obtain other groups).

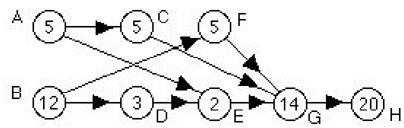
	Mon	Wed	Fri
Week 1	I, A, B	C, E, H	D, F, G
Week 2	I, C, D	E, G, B	F, H, A
Week 3	I, E, F	G, A, D	H, B, C
Week 4	I, G, H	A, C, F	B, D, E.

12. a) Since $k = 3$, $\lambda = 1$, $v = 7$, from $\lambda(v-1) = r(k-1)$ we obtain $6 = 2r$, so $r = 3$. Hence $b = 7$ ($bk = vr$), so this is a $(7,7,3,3,1)$-design. **b)** The blocks are $\{1,6,7\}$, $\{2,5,7\}$, $\{3,5,6\}$, $\{1,2,3\}$, $\{1,4,5\}$, $\{2,4,6\}$, $\{3,4,7\}$. **c)** The blocks are the lines in this projective plane of order 2.

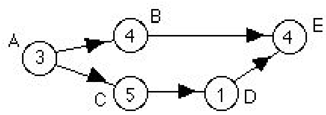


Chapter 11 Scheduling Problems and Bin Packing

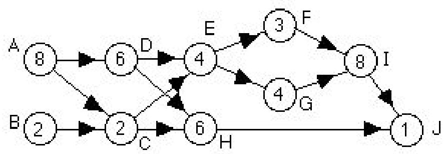
1.



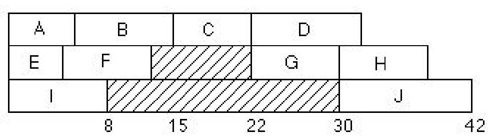
2.



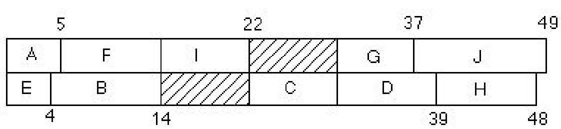
3. Order requirement digraph:



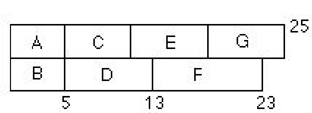
Critical path $A \to D \to E \to E \to G \to G \to I \to J$; length 31 weeks. Schedule (giving task and starting time for task in weeks after start of project): A, 0; B, 0; C, 8; D, 8; E, 14; F, 18; G, 18; H, 14; I, 22; J, 30. **4. a)** Critical path: $A \to B \to C \to G \to J$; length 42. Schedule (giving task and starting time for task after start of project): A, 0; B, 5; C, 15; D 22; E, 0; F, 4; G, 22; H, 30; I, 0; J, 30.



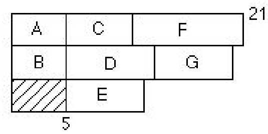
b)



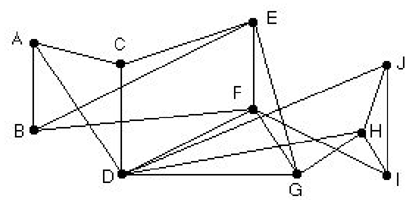
5. a)



b) This schedule is optimal for two processors. **c)** If there are unlimited processors, then the optimal schedule is the following:



6. The incomparability graph is:



A maximal set of disjoint edges is AB, CD, EF, GH, IJ. The optimal schedule for this set is:

A	C	E	G	I
B	D	F	H	J

7. First delete FC as redundant. Next D and H receive permanent labels 1 and 2. Next C receives temporary label (1), G gets temporary label (2) and K gets temporary label (2). Next C gets permanent label 3, G gets 4, and K gets 5. Now B gets temporary label (3), F gets (4), and J gets (4,5). By dictionary order, B is labeled 6, F is labeled 7, and J is labeled 8. Now A gets temporary label (6), E gets (6,7), and I gets (8). These vertices then get permanent labels 9, 10, and 11, respectively. The schedule is then:

I	A	B	K	C	D
E	J	F	G	H	

8. a) Using FF, the bins contain: bin 1 – 85, 55; bin 2 – 95, 25; bin 3 – 135; bin 4 – 65, 35, 40; bin 5 – 95, 35. **b)** Using FFD: bin 1 – 135; bin 2 – 95, 40; bin 3 – 95, 35; bin 4 – 85, 55; bin 5 – 65, 35, 25. **c)** Both of these are optimal packings since the total weight to be packed is 665 pounds and four cartons can hold a maximum of 560 pounds. **9. a)** 15 9-foot boards are needed for FF. **b)** 15 9-foot boards are needed for FFD. **c)** 13 10-foot boards are needed for FF and FFD. **10.** Next fit (NF) works like FF except that once a new bin is opened, the previous bin(s) is closed. This method might be preferable in the case of loading trucks from a single loading dock since only one bin at a time is used. Other list orders could be FFI (first-fit increasing) or (largest, smallest, next largest, next smallest,...) with FF.

Chapter 12 Burnside-Polya Counting Methods

1.

x	o	o
o	o	o
o	o	x

2. 90°: $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 4 & 1 & 8 & 5 & 2 & 9 & 6 & 3 \end{pmatrix}$;

180°: $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}$;

270°: $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 6 & 9 & 2 & 5 & 8 & 1 & 4 & 7 \end{pmatrix}$;

360°: $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{pmatrix}$.

3. Horizontal: $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 8 & 9 & 4 & 5 & 6 & 1 & 2 & 3 \end{pmatrix}$;

Vertical: $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 2 & 1 & 6 & 5 & 4 & 9 & 8 & 7 \end{pmatrix}$;

Main diagonal: $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 4 & 7 & 2 & 5 & 8 & 3 & 6 & 9 \end{pmatrix}$;

Minor diagonal: $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 6 & 3 & 8 & 5 & 2 & 7 & 4 & 1 \end{pmatrix}$.

4.

$$\begin{array}{l} \qquad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} \\ 180^\circ \begin{pmatrix} 3 & 1 & 4 & 2 \\ 2 & 4 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \\ \hline 270^\circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix} \end{array}$$

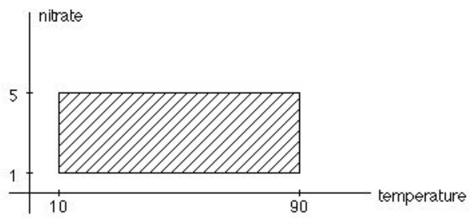
5.

$$\operatorname{inv}\overset{90^\circ}{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}} = \begin{pmatrix} 3 & 1 & 4 & 2 \\ 1 & 2 & 3 & 4 \end{pmatrix} = \overset{270^\circ}{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}}.$$

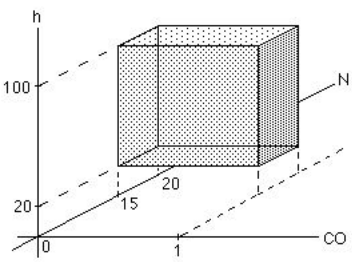
6. $(1/6)(t_1^3 + 3t_1t_2 + 2t_3)$. **7.** $(1/10)(t_1^5 + 5t_1t_2^2 + 4t_5)$. **8.** $(1/12)(t_1^6 + 3t_1^2t_2^2 + 4t_2^3 + 2t_3^2 + 2t_6)$. **9.** The appropriate cycle index is $(1/8)(t_1^8 + 4t_1^2t_2^3 + t_2^4 + 2t_4^2)$. Substitute 2 for every t_i to obtain 51. **10.** You can draw the 13 configurations, or you may calculate the coefficient of x^4 in the polynomial $(1/8)((1+x)^8 + 4(1+x)^2(1+x^2)^3 + (1+x^2)^4 + 2(1+x^4)^2)$.

Chapter 13 Food Webs

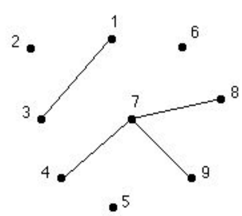
1. a)



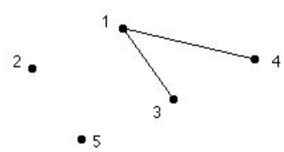
b)



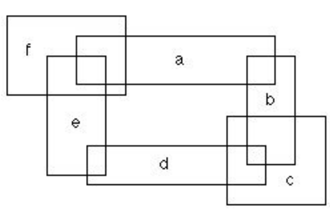
2. a)



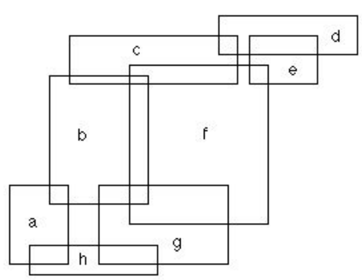
b)



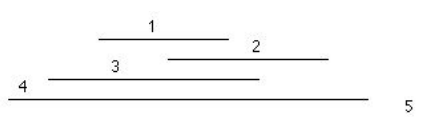
3. a)



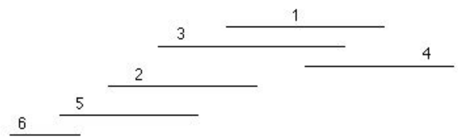
b)



4. a)

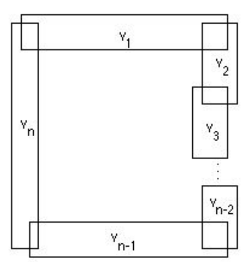


b)

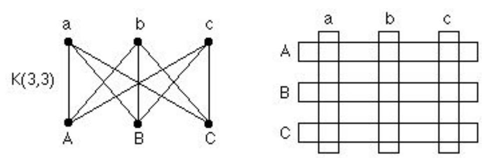


5. If G is an interval graph then it cannot have C_4 as a generated subgraph since otherwise the intervals corresponding to the vertices of that subgraph would constitute an interval representation of C_4, which is impossible. Using the orientation suggested by the hint establishes a transitive orientation of $\overline{G}$ since the ordering of the intervals on the line defined by the hint is transitive.

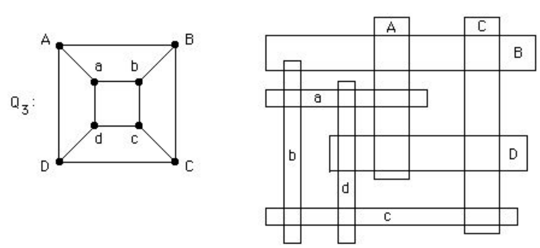
6. It is easy to find an interval representation of C_3; any set of three intervals, each intersecting the other two, will do. Thus C_3 has boxicity 1. If $n > 3$, let $v_1, \ldots, v_n$ be the vertices of G. The following diagram shows that the boxicity of G is no more than 2. C_n cannot have boxicity 1 by the same sort of argument used in the text to show that C_4 is not an interval graph.



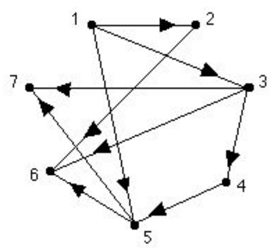
7.



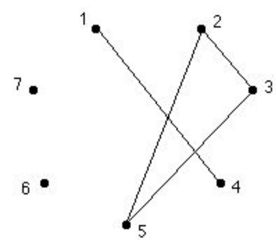
8. Q_3 is not an interval graph because the edges and corners of one face of the cube form a generated subgraph isomorphic to C_4. The following diagram shows a representation of Q_3 as the intersection graph of a family of boxes in two-space.



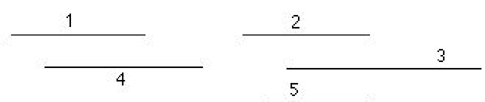
9. a)


1
2
3
4
5
6
7

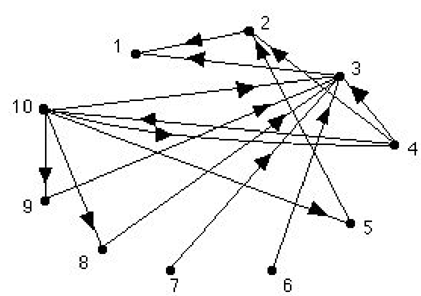
b)


1
2
3
4
5
6
7

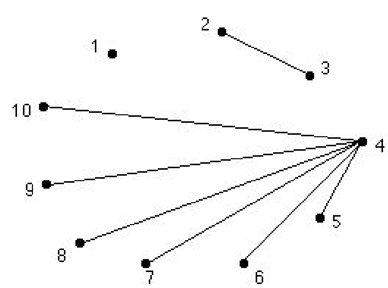
c)


1
2
3
4
5

10. a)


1
2
3
4
5
6
7
8
9
10

b)


1
2
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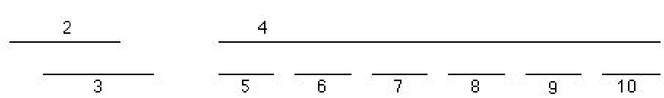
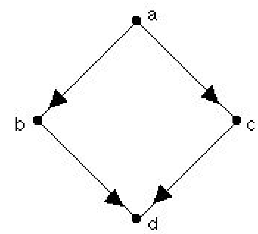
6
7
8
9
10

c)


2
4
3
5
6
7
8
9
10

11.



12. h_W via shortest path

a)

vertex	h_W
a	14
b	9
c	3
d	0
e	4
f	1
g	1
h	0

b)

vertex	h_W
a	12
b	1
c	0
d	6
e	0
f	2
g	0
h	0

h_W via longest path

a)

vertex	h_W
a	16
b	9
c	3
d	0
e	4
f	1
g	1
h	0

b)

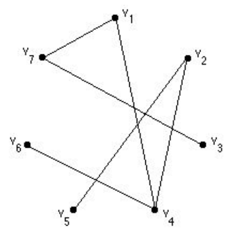
vertex	h_W
a	15
b	1
c	0
d	6
e	0
f	2
g	0
h	0

13. The second version of h_W given in the text satisfies this condition and is defined for any acyclic food web. **14.** Suppose there is a directed path from u to v. Since every direct or indirect prey of v is also an indirect prey of u, every term in the sum for $t_W(v)$ is also a term in the defining sum for $t_W(u)$. Since all the values of h are non-negative, it follows that $t_W(u) \geq t_W(v)$. Axiom 1 is satisfied trivially. Axiom 2 is not satisfied. Suppose u is a vertex and we add a new species v which is a direct prey of u, but which itself has no prey. Then $h_W(v) = 0$, so the value of the sum for $t_W(u)$ is not increased. Axiom 3 is satisfied since increasing the level of v relative to u increases $h_W(v)$ and hence increases the sum defining $t_W(u)$. **15.** This assumption *is* restrictive. An example of two species which are mutual prey would be man and grizzly bear.

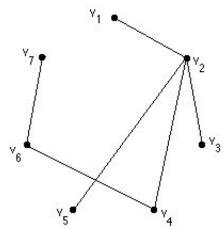
Chapter 14 Applications of Subgraph Enumeration

1. **(a)** 1957 **(b)** $8^6 = 262{,}144$ **(c)** 2520 **(d)** 105. **2.** **(a)** $(2, 2, 1, 5, 5)$ **(b)** $(1, 4, 1, 2, 1)$.

3. a)

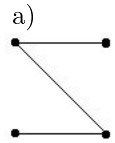


b)



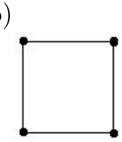
4. By Theorem 4, there are 15 perfect matches of K_6. Example 15 shows how to generate 5 perfect matches of K_6 using the matching $\{\{1,2\},\{3,4\}\}$ of K_4. The five perfect matches of K_6 generated from $\{\{1,3\},\{2,4\}\}$ are $\{\{5,3\},\{2,4\},\{1,6\}\}$, $\{\{1,5\},\{2,4\},\{3,6\}\}$, $\{\{1,3\},\{5,4\},\{2,6\}\}$, $\{\{1,3\},\{2,5\},\{4,6\}\}$, and $\{\{1,3\},\{2,4\},\{5,6\}\}$. The five perfect matches of K_6 generated from $\{\{1,4\},\{2,3\}\}$ are $\{\{5,4\},\{2,3\},\{1,6\}\}$, $\{\{1,5\},\{2,3\},\{4,6\}\}$, $\{\{1,4\},\{5,3\},\{2,6\}\}$, $\{\{1,4\},\{2,5\},\{3,6\}\}$, and $\{\{1,4\},\{2,3\},\{5,6\}\}$. **5.** Consider the longest path in the tree. Its two endpoints have degree 1. **6.** Suppose G is a graph with n vertices, $n-1$ edges, and no circuits. The proof is complete if we show that G is connected. Suppose G is not connected. Then G is a forest with k components, where $k > 1$. Let $C_1, C_2, \ldots, C_k$ be the k components and let $n_1, n_2, \ldots, n_k$ be the number of vertices in each component. Then $\sum_{i=1}^{k} n_i = n$. Since each C_i is a tree, the number of edges in each C_i is $n_i - 1$. This implies that the number of edges in G is $\sum_{i=1}^{k}(n_i - 1) = (\sum_{i=1}^{k} n_i) - k = n - k < n - 1$, a contradiction. **7.** 5
8. The path A, E, D, C has total length 39 and total cost \$45.

9. a)



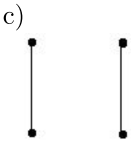
range 75

b)



weight 1010

c)



weight 360

10. Visit the villages in the order $ACEBD$. **11.** 157/1957. **12.** $1/n^{n-3}$. **13.** $2/((n-1)(n-2))$. **14.** $1/((n-1)(n-2))$. **15.** The number of perfect matchings in $K_{n,n}$ is $n!$, which may be generated by observing that there is a one-to-one correspondence between the perfect matches in $K_{n,n}$ and the permutations on $\{1, 2, \ldots, n\}$. **16.** Any perfect matching of $K_{n,n}$ may be described as a permutation of $\{1, 2, \ldots, n\}$. Hence, the perfect matches of $K_{n,n}$ can be generated by generating the permutations of $\{1, 2, \ldots, n\}$. A method for doing this is given in Section 4.7 of *Discrete Mathematics and Its Applications*.
17. Let $m = |M| = n/2$. Then the number of spanning trees of K_n containing M is $2^{m-1}m^{m-2}$. To see this, consider the m edges of M as vertices of K_m which must be spanned by a tree. By Cayley's Theorem there are m^{m-2} such trees. Since every pair of edges of M may be connected by two different edges of K_n, every spanning tree of K_m gives rise to 2^{m-1} spanning trees of K_n containing

M. **18. (a)** Generate all of the subsets of $\{1, 2, \ldots, n\}$ by generating the set of all r-combinations using the algorithm described in Section 5.6 of *Discrete Mathematics and Its Applications*. For each subset $\{i_1, i_2, \ldots, i_r\}$ compute $z = w_{i_1} + w_{i_2} + \cdots + w_{i_r}$, then select a subset on which the minimum of z is attained. **(b)** $C(n, r)$ candidates must be checked to solve the problem. **19. (a)** Use the linear search algorithm to find the smallest element of W, say x_1. Remove x_1 from W and repeat the algorithm to find the smallest element of the remaining set, say x_2. Continue repeating the linear search algorithm n times, removing the smallest element after each repetition. This gives $\{x_1, x_2, \ldots, x_n\} = W$ such that $x_1 \le x_2 \le \cdots \le x_n$. A subset of W of size r with the smallest possible sum is $\{x_1, x_2, \ldots, x_r\}$. **(b)** This algorithm requires at most $O(n^2)$ comparisons. **20.** $n = 10$. **21.** $n2^{n-1}$. **22.** Let e be the edge $\{n-1, n\}$. Show there is a one-to-one correspondence between the spanning trees of $K_n - e$ and the $(n-2)$-tuples $(a_1, a_2, \ldots, a_{n-2})$ such that each a_i is an integer, $1 \le a_i \le n$, for $i = 1, 2, \ldots, n-3$, and $1 \le a_{n-2} \le n-2$. The result follows from the fact that there are $(n-2)n^{n-3}$ such $(n-2)$-tuples. **23.** Show that the spanning trees of K_n such that vertex v_i has degree d_i are in one-to-one correspondence with the $(n-2)$-tuples $(a_1, a_2, \ldots, a_{n-2})$ such that the integer i appears $d_i - 1$ times. The result follows from counting all such $(n-2)$-tuples. **24.** Use the method suggested by the solution to Exercise 23. **25.** We use induction on $k = m + n$. The result can be checked directly for $m = 1$, $n = 3$, or $m = n = 2$. So, assume the result holds for all complete bipartite graphs with $m + n < k$, and let $K_{m,n}$ be one with $m + n = k$. Since every tree must have at least two vertices of degree 1, at least two of the d_i or f_j must be 1, say $d_1 = 1$. The number of spanning trees of $K_{m,n}$ such that vertex u_i has degree d_i, for all i, and vertex v_j has degree f_j, for all j, is equal to the number of spanning trees of $K_{m,n}$ such that vertex u_i has degree d_i and vertex v_j has degree f_j which contain edge $(1, 1)$, plus the number of such spanning trees which contain edge $(1, 2)$, plus the number of such spanning trees which contain $(1, 3)$, etc. The number of spanning trees such that vertex u_i has degree d_i, for all i, and vertex v_j has degree f_j, for all j, which contain edge $(1, k)$ is equal to 0 if $f_k = 1$, otherwise is equal to the number of spanning trees of $K_{m-1,n}$ such that vertex u_i has degree d_i (for $i = 2, 3, \ldots, m$), vertex v_k has degree f_{k-1}, and vertex v_j has degree f_j (for $j = 1, \ldots, k-1, k+1, \ldots, n$). By induction, the number of such trees is

$$\frac{(m-2)!(n-1)!}{(d_2-1)!\cdots(d_m-1)!(f_1-1)!\cdots(f_{k-1}-1)!(f_k-2)!(f_{k+1}-1)!\cdots(f_n-1)!}.$$

The total number of spanning trees such that vertex u_i has degree d_i and vertex v_j has degree f_j is obtained by summing the above terms over all $k = 1, 2, \ldots, n$ such that $f_k \ge 2$. This sum is simplified by multiplying each term by $(f_k - 1)/(f_k - 1)$. Summing gives

$$\frac{(m-2)!(n-1)![(f_1-1) + (f_2-1) + \cdots + (f_n-1)]}{(d_2-1)!\cdots(d_m-1)!(f_1-1)!\cdots(f_n-1)!}.$$

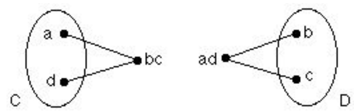
Since $(f_1-1)+(f_2-1)+\cdots+(f_n-1) = f_1+f_2+\cdots+f_n-n = m+n-1-n = m-1$. The solution is complete.

Chapter 15 Traveling Salesman Problem

1. a) f, e, a, b, c, d is forced, so the only Hamilton circuits are $f, e, a, b, c, d, g, h, f$ of length 22 and $f, e, a, b, c, d, h, g, f$ of length 21. The latter is the shorter, so the solution is $f, e, a, b, c, d, h, g, f$. **b)** a, b, c and d, e, f are forced, so the only Hamilton circuits are a, b, c, g, f, e, d, a of length 19 and a, b, c, f, e, d, g, a of length 21. The former is the shorter, so the solution is a, b, c, f, e, d, g, a. **c)** $h, d, c, b, a, e, f, g, h$ is the only Hamilton circuit. Its length is 30. **d)** $f, d, c, b, a, e, g, j, h, i, f$ is the only Hamilton circuit. Its length is 38. **e)** $d, c, b, a, e, h, i, g, f, d$ with length 36 and $d, c, b, a, e, f, g, h, i, d$ with length 23 are the only Hamilton circuits. The latter is shorter, so the solution is $d, c, b, a, e, f, g, h, i, d$. **f)** $a, b, c, d, i, f, e, g, h, a$ with length 27 and $a, b, c, d, i, g, f, e, h, a$ with length 37 are the only Hamilton circuits. The former is shorter, so the solution is $a, b, c, d, i, f, e, g, h, a$. **g)** $a, b, c, d, g, h, e, i, f, a$ with length 15 and $d, c, b, a, f, g, i, h, e, d$ with length 27 are the only Hamilton circuits. The former is shorter, so the solution is $a, b, c, d, g, h, e, i, f, a$. **h)** $a, b, c, d, i, g, f, e, h, a$ with length 23 and $d, c, b, a, e, h, f, i, g, d$ with length 34 are the only Hamilton circuits. The former is shorter, so the solution is $a, b, c, d, i, g, f, e, h, a$. **2.**

a) $\mathbf{M} = \begin{array}{c} \\ C \\ E \\ A \\ D \\ B \end{array} \begin{array}{c} \begin{array}{ccccc} c & e & a & b & d \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$. Thus C and E are in an industry performing tasks b and d, A shares task b with them and does task a alone, and B and D are in an industry doing tasks c and e. **b)** $\mathbf{M} = \begin{array}{c} \\ B \\ A \\ C \\ D \end{array} \begin{array}{c} \begin{array}{ccccc} d & e & a & b & c \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$.

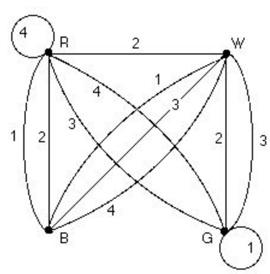
Thus A, B, and C are in a single industry performing tasks a, d, and e (although A does not do d and C does not do e), C and D are in an industry performing tasks b and c, and A shares task b with C and D. **3. a)** The tree with edges $\{a, b\}$, $\{a, d\}$, $\{b, c\}$, and $\{d, e\}$ yields circuit c, b, a, d, e, c with length 7. **b)** The tree with edges $\{a, b\}$, $\{a, f\}$, $\{c, d\}$, $\{d, e\}$, and $\{d, f\}$ yields circuit b, a, f, d, e, c, b with length 19. **4.** $l(a, c, b, d, e, a) = 15$, $l(a, c, d, b, e, a) = 14$, $l(a, c, d, e, b, a) = 13$, $l(b, c, d, e, a, b) = 14$. The shortest of these is a, c, d, e, b, a with length 13. The strategy used here was just to find all Hamilton circuits in the graph. The weakness is that the number of Hamilton circuits rises extremely fast, so that there is not time enough to do it on larger graphs. **5.** There are two directions around a Hamilton circuit. Starting at vertex a, the next vertex in one direction must be different from the next vertex in the other direction because there are at least three vertices in the graph. Hence the two permutations of the vertices are different. **6.** To maximize S_2, it suffices to minimize $-S_2 = \sum_{j=1}^{n-1}\sum_{i=1}^{m}(-a_{ij}a_{i,j+1})$. Given matrix $\mathbf{A}$, for each column j we introduce a vertex j. For any two columns k and ℓ, we join them by an undirected edge with weight $c_{k\ell} = \sum_{i=1}^{m}(-a_{ik}a_{i,\ell})$. In the resulting undirected graph G'_4, each Hamilton path h describes a permutation of the columns of $\mathbf{A}$. Further, if $\mathbf{A}' = [a'_{ij}]$ is formed from A by carrying out this permutation of the columns for a minimum weight Hamilton path, then $\sum_{j=1}^{n-1}\sum_{i=1}^{m}(-a'_{ij}a'_{i,j+1})$ is precisely the sum of the weights along h. Since h is a minimum weight Hamilton path in G'_4, this means that $\sum_{j=1}^{n-1}\sum_{i=1}^{m}(a'_{ij}a'_{i,j+1})$ is largest among all possible orderings of the columns of $\mathbf{A}$. Thus the maximum value of this half of $f(\mathbf{A})$ is found by finding a minimum weight Hamilton path in G'_4. To convert this method to the TSP, add one more vertex 0 to G'_4 and join 0 to each other vertex by an edge of weight 0, thus forming graph G_4. A solution of the TSP in G_4 corresponds to a permutation of the columns of $\mathbf{A}$ that maximizes S_2. **7.** Let us represent the Hamilton circuits by sequences of vertex labels. We consider two circuits to be 'different' if the corresponding sequences are 'different.' Let s_1 and s_2 be two of these sequences of vertex labels. In terms of these sequences, the meaning of the word 'different' that was used in the text was that the sequences are 'different' unless they are identical or satisfy one or the other of the following two conditions: (a) s_1 begins with a label a, and if by starting at the occurrence of a in s_2, proceeding from left to right, jumping to the beginning of s_2 when its end is reached (without repeating the end label), and stopping when a is reached again, we can produce s_1, or (b) s_1 begins with a label a, and if by starting at the occurrence of a in s_2, proceeding from right to left, jumping to the end of s_2 when its beginning is reached (without repeating the end label), and stopping when a is reached again, we can produce s_1. An alternative meaning is that two are 'different' unless they are identical or satisfy (a), but not (b). In this case, the number of 'different' sequences, and thus 'different' Hamilton circuits, is 6!. A third meaning is that two sequences are 'different' unless they are identical. In this case, the number of 'different' sequences, and thus 'different' Hamilton circuits, is 7!. The text's choice of the meaning of "different" is best because it accurately represents the geometric structure of the graph. **8.** Procedure Short Circuit fails because edges that it assumes are present are not in fact present. The procedure generates circuits in complete graphs, and the graph given is not complete. **9.** In the figure, any Hamilton circuit must pass through both edges $\{a, b\}$ and $\{c, d\}$, and it must contain a Hamilton path from d to a in C and another Hamilton path from b to c in D. the TSP here can be reduced to two smaller TSPs, one in which C has been reduced to a single vertex ad and one in which D has been reduced to a single vertex bc, as shown in the figure below. Once Hamilton circuits solving the TSP have been found in each of the reduced graphs, the vertex ad can be removed from the one through D, and the resulting path can replace bc in the one through C, thus producing a Hamilton circuit through the graph given.



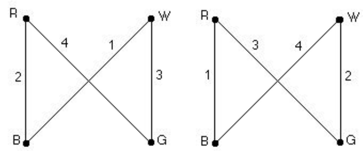
10. The banks are represented by vertices, and each safe route between two banks, or between the home base of the car and a bank, is represented by an edge joining the corresponding vertices, with the time it takes to travel the route as the weight on the edge. **11.** The cities are represented by vertices, and each possible route for the railroad between two cities is represented by an edge. The weight on an edge is the cost of the rail line on that route. To make this a TSP problem, another vertex is added, with an edge of zero weight to each of the vertices representing New York and San Francisco. **12.** There are several TSP problems, one for each truck. For a single truck, the destinations of the packages are represented by the vertices, and two vertices are joined by an edge if there is a route between the corresponding destinations. The weight on the edge is the time it takes a truck to travel the corresponding route. **13.** Each front line unit is assigned a vertex. Edges join two vertices whenever there is any route that connects the corresponding units. A scale for safety is assigned with 0 representing a perfectly safe route and higher numbers indicating greater risk on the route. These numbers are the weights assigned to the edges of the graph.

Chapter 16 The Tantalizing Four Cubes

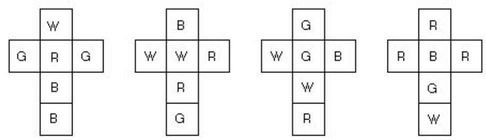
1. The underlying graph is



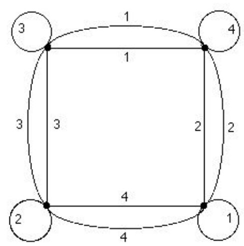
Two disjoint acceptable subgraphs are



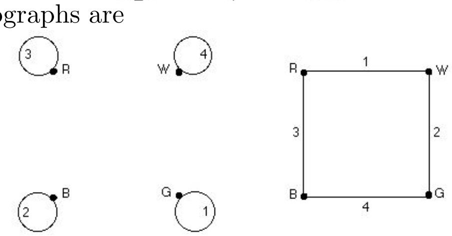
Hence a solution is



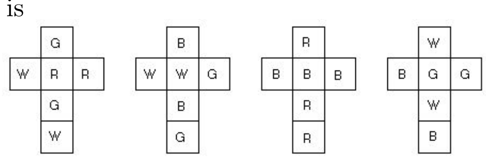
2. The underlying graph is



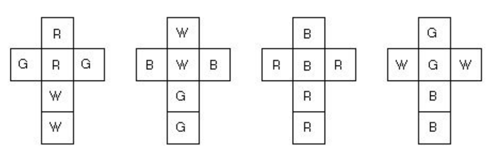
The acceptable subgraphs are



Thus one solution is



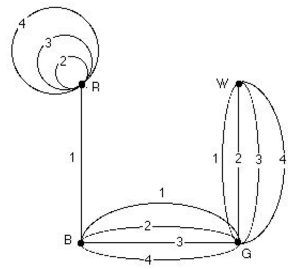
3. There are 2 disjoint versions of the second subgraph. Thus, another solution is



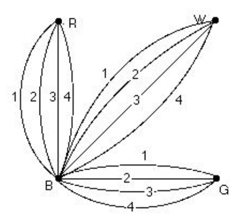
4. Two acceptable subgraphs are
Thus a solution is
The last left-right pair is R-W, but since the graph was given, not the cubes,

it is unknown which is left and which is right.

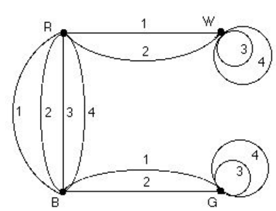
5. For example, there are no acceptable subgraphs in the following graph



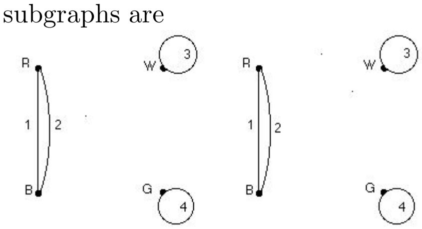
6. For example, there are no acceptable subgraphs in the following graph



7. In the following graph,

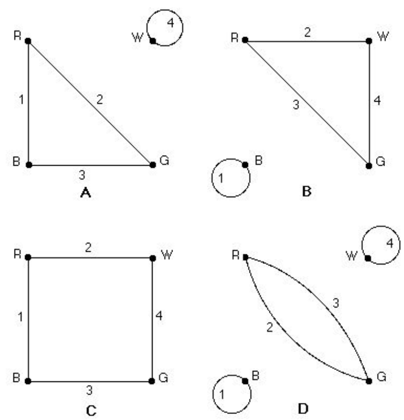


the only acceptable subgraphs are

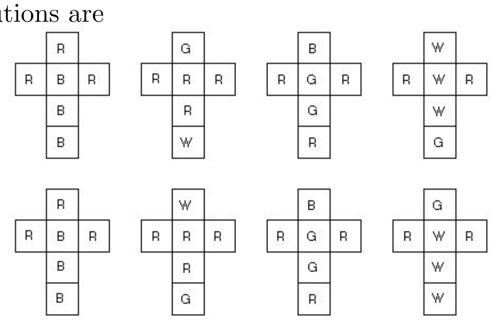


and they are not disjoint.

8. Again, the easiest way to find a puzzle with exactly two solutions is to construct the underlying graph. Here are two pairs of subgraphs



Of these four subgraphs, only the pairs AB and CD are disjoint. So they give two distinct solutions. If the additional four edges contribute no new acceptable subgraphs, these will be the only two solutions. So let all the other faces be red. The two solutions are



Note: To obtain the second solution from the first, leave cubes 1 and 3 the same, rotate the second cube 180° to the right and then 90° up, and rotate the fourth cube 90° down. **9. a)** If there are three cubes and three colors, the statement of the theorem remains the same. However, there are only three vertices and three labels. A typical acceptable subgraph is a triangle with three different labels, although a 2-cycle and a loop or three loops are possible. **b)** If there are three cubes and four colors, it is impossible for all colors to appear on the same side of a stack. **c)** If there are three cubes and five colors, the theorem remains the same. Now the graph has five vertices and five labels. A typical acceptable subgraph is a pentagon with five different labels, although numerous other configurations are possible.

Chapter 17 The Assignment Problem

1. 2143 is an optimal solution with $z = 20$. **2.** 2341 is an optimal solution with $z = 13$. **3.** 216354 is an optimal solution with $z = 18$. **4.** 321465 is an optimal solution with $z = -6$. **5.** 124356 is an optimal solution with $z = 65$. **6.** Assign Ken to the Butterfly, Rob to the Freestyle, Mark to the Backstroke, and David to the Breaststroke, the minimum time is 124.8. **7.** The following couples should marry: Jane and Joe, Mary and Hal, Carol and John, Jessica and Bill, Dawn and Al, and Lisa and Bud, giving a measure of anticipated "happiness" of 8. **8.** The route New York, Chicago, Denver, Los Angeles is optimal with total time of 62 hours. **9. a)** Edges $\{1,1\}$, $\{2,2\}$, $\{3,3\}$, $\{4,4\}$, and $\{5,5\}$ are a perfect matching. **b)** No perfect matching exists. If $W = \{3,4,5\}$, then $R(W) = \{4,5\}$ and $|R(W)| < |W|$. **c)** Edges $\{1,1\}$, $\{2,2\}$, $\{3,3\}$, $\{4,5\}$, and $\{5,4\}$ are a perfect matching. **10.** The perfect matching $\{1,4\}$, $\{2,1\}$, $\{3,2\}$, $\{4,3\}$ has the smallest possible weight, which is 12. **11.** a) Let $\mathbf{C} = [c_{ij}]$ where $c_{ij} = 0$ for all i and j. **b)** Let $\mathbf{C} = [c_{ij}]$ where $c_{ii} = 0$ for $i = 1,2,3,4,5$, and $c_{ij} = 1$ for $i \neq j$. **12.** Let $u_1 = 0$, $u_2 = 4$, $u_3 = 8$, $u_4 = 12$, and $v_1 = v_2 = v_3 = v_4 = 0$. Then apply Theorem 1 to obtain $\widehat{\mathbf{C}} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$. Then $\mathbf{C}$ and $\widehat{\mathbf{C}}$ have the same set of optimal solutions. But for any permutation σ of $\{1,2,3,4\}$, $\sum_{i=1}^{n} \widehat{c}_{i\sigma(i)} = 1+2+3+4 = 10$. So, any permutation is an optimal solution to the problem specified by $\widehat{\mathbf{C}}$, and hence is an optimal solution to the problem specified by $\mathbf{C}$. **13.** If $c_{ij} \geq 0$ for all i and j, then subtracting the smallest entry from each row can not result in a negative entry. If some row, say $(c_{11}c_{12}\cdots c_{1n})$ has negative entries and, say, c_{11} is the smallest entry, then $\widehat{c}_{1j} = c_{1j} - c_{11} = c_{1j} + |c_{11}| \geq 0$, for $j = 1,2,\ldots,n$ since either $c_{1j} \geq 0$ or if $c_{1j} < 0$, then $|c_{1j}| \geq |c_{11}|$. Thus, after subtracting the smallest entry in each row, the resulting matrix has all nonnegative entries. Now subtracting the smallest entry from each column cannot result in a negative entry. **14.** The problem may be solved by multiplying all entries in c by -1 since finding a permutation σ which maximizes $z = \sum_{i=1}^{n} c_{i\sigma(i)}$ is equivalent to finding a permutation σ which minimizes $-z = -\sum_{i=1}^{n} c_{i\sigma(i)}$. **15.** 352461 is an optimal solution with $z = 35$. **16.** Suppose σ^* is a permutation which solves the assignment problem specified by the matrix in Example 11. Suppose also that P is a path which has total delivery time less than $\sum_{i=1}^{n} c_{i\sigma^*(i)}$. Number the cities using $1,2,3,4,5,6,7$ representing New York, Boston, Chicago, Dallas, Denver, San Francisco, and Los Angeles respectively. Define the permutation σ as follows: if P visits city j immediately after P visits city i, then $\sigma(i) = j-1$; if P does not visit city i, then $\sigma(i) = i-1$. Then the total delivery time of P is $\sum_{i=1}^{n} c_{i\sigma(i)}$ since $c_{ii-1} = 0$, for $i = 2,3,4,5,6$. But this implies that $\sum_{i=1}^{n} c_{i\sigma(i)} < \sum_{i=1}^{n} c_{i\sigma^*(i)}$, a contradiction. **17.** The number of iterations is at most the sum of all of the entries of the reduced matrix. Let k be the largest entry in the reduced matrix, then the sum of all entries in the reduced matrix is at most kn^2. Thus the number of iterations is $O(n^2)$. **18.** Given a bipartite graph G = (V,E) define the matrix $\mathbf{A} = [a_{ij}]$ as follows. Let $a_{ij} = \begin{cases} 0 & \text{if edge } \{i,j\} \in E \\ 1 & \text{otherwise} \end{cases}$. Then matches correspond to independent sets of zeros in $\mathbf{A}$ and vertex covers correspond to line covers of $\mathbf{A}$. The result now follows from Theorem 3. **19.** Let $G = (V, E)$ be a bipartite graph, with $V = V_1 \cup V_2$. If G has a perfect matching M, then $|V_1| = |M| = |V_2|$. For any W contained in V_1, let $W = \{w_1, w_2, \ldots, w_k\}$, and for any $w_i \in W$, let r_i be the vertex in V_2 such that $\{w_i, r_i\} \in M$. Then $r_1, r_2, \ldots, r_k$ are all distinct, and $\{r_1, r_2, \ldots, r_k\}$ is contained in $R(W)$, so $|R(W)| \geq |W|$. Conversely, assume $|V_1| = |V_2|$ and $|R(W)| \geq |W|$, for all W contained in V_1. Since every edge in E joins a vertex in V_1 to one in V_2, V_1 is a vertex cover. The proof is complete if we show that V_1 is a cover of minimal size. For then, Exercise 18 implies there is a matching of size $|V_1|$, which must be a perfect matching. Suppose Q is a vertex cover of the edges of minimum size, and $Q \neq V_1$. Let $U_1 = Q \cap V_1$ and let $U_2 = V_2 - U_1$. By assumption, $|R(U_2)| \geq |U_2|$. However, $R(U_2)$ is contained in $Q \cap V_2$ because edges not covered by vertices in V_1 must be covered by vertices in V_2. Thus, $|U_2| \leq |R(U_2)| \leq |Q \cap V_2|$. This implies that $|V_1| = |U_1| + |U_2| \leq |U_1| + |Q \cap V_2| = |Q \cap V_1| + |Q \cap V_2| = |Q|$. Since $|V_1| \leq |Q|$ and Q is a cover of minimum size, V_1 must be a cover of minimum size. **20.** A stable set of four marriages is: David and Susan, Kevin and Colleen, Richard and Nancy, and Paul and Dawn. **21.** We prove the existence of a stable set of n marriages by giving an iterative procedure which finds a stable set of marriages.

Initially, let each man propose to his favorite woman. Each woman who receives more than one proposal replies "no" to all but the man she likes the most from among those who have proposed to her. However, she does not give him a definite "yes" yet, but rather a conditional "yes" to allow for the possibility that a man whom she likes better may propose to her in the future.

Next, all those men who have not received a conditional "yes" now propose to the women they like second best. Each woman receiving proposals must now choose the man she likes best from among the men consisting of the new proposals and the man who has her conditional "yes", if any.

Continue to iterate this procedure. Each man who has not yet received a conditional "yes" proposes to his next choice. The woman again says "no" to all but the proposal she prefers thus far.

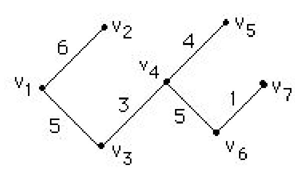
The procedure must terminate after a finite number of iterations since every woman will eventually receive a proposal. To see this, notice that as long as there is a woman who has not been proposed to, there will be rejections and new proposals. Since no man can propose to the same woman more than once, every woman will eventually receive a proposal. Once the last woman receives a proposal, each woman now marries the man who currently has her conditional "yes", and the procedure terminates.

The set of marriages produced is stable, for suppose David and Susan are not married to each other, but David prefers Susan to his own wife. Then David must have proposed to Susan at some iteration and Susan rejected his proposal in favor of some man she preferred to David. Thus Susan must prefer her husband to David. So there can be no instability.

Chapter 18 Shortest Path Problems

1. The length is 90. The path is v_2, v_1, v_9, v_8. **2. a)** v_7 comes from v_6, v_6 from v_4, v_4 from v_3, v_3 from v_1. Thus the path is v_1, v_3, v_4, v_6, v_7. **b)** v_5 comes from v_4, v_4 from v_3, v_3 from v_1. Thus the path is v_1, v_3, v_4, v_5.

c)



There could be many other edges. For instance, any edge from v_5 to v_7 of weight greater than 2 would not affect the vectors. **3. a)** There are $O(n^4)$ operations if $\mathbf{A}^{n-1}$ must be computed. **b)** If, on the average, no shortest paths were longer than $n/2$, the algorithm would be only half as long as before, but still $O(n^4)$. If no shortest path had more than 4 edges, $\mathbf{A}^5$ would be the last computation, so $4O(n^3) = O(n^3)$ operations would do.

4. $$\mathbf{A} = \begin{pmatrix} 0 & 2 & 3 & * & * & * \\ 2 & 0 & * & 5 & 2 & * \\ 3 & * & 0 & * & 5 & * \\ * & 5 & * & 0 & 1 & 2 \\ * & 2 & 5 & 1 & 0 & 4 \\ * & * & * & 2 & 4 & 0 \end{pmatrix}, \quad \mathbf{A}^5 = \mathbf{A}^4 = \begin{pmatrix} 0 & 2 & 3 & 5 & 4 & 7 \\ 2 & 0 & 5 & 3 & 2 & 5 \\ 3 & 5 & 0 & 6 & 5 & 8 \\ 5 & 3 & 6 & 0 & 1 & 2 \\ 4 & 2 & 5 & 1 & 0 & 3 \\ 7 & 5 & 8 & 2 & 3 & 0 \end{pmatrix}.$$

5. $$\mathbf{A} = \begin{pmatrix} 0 & 4 & 3 & * & * & * & * & * \\ 4 & 0 & 2 & 5 & * & * & * & * \\ 3 & 2 & 0 & 2 & 6 & * & * & * \\ * & 5 & 2 & 0 & 1 & 5 & * & * \\ * & * & 6 & 1 & 0 & * & 7 & * \\ * & * & * & 5 & * & 0 & 2 & 7 \\ * & * & * & * & 7 & 2 & 0 & 4 \\ * & * & * & * & * & 7 & 4 & 0 \end{pmatrix}, \quad \mathbf{A}^6 = \mathbf{A}^5 =$$

$$\begin{pmatrix} 0 & 4 & 3 & 5 & 6 & 10 & 12 & 16 \\ 4 & 0 & 2 & 4 & 5 & 9 & 11 & 15 \\ 3 & 2 & 0 & 2 & 3 & 7 & 9 & 13 \\ 5 & 4 & 2 & 0 & 1 & 5 & 7 & 11 \\ 6 & 5 & 3 & 1 & 0 & 6 & 7 & 11 \\ 10 & 9 & 7 & 5 & 6 & 0 & 2 & 6 \\ 12 & 11 & 9 & 7 & 7 & 2 & 0 & 4 \\ 16 & 15 & 13 & 11 & 11 & 6 & 4 & 0 \end{pmatrix}.$$

6. $\mathbf{A}^3 = \begin{pmatrix} 0 & 2 & 3 & 5 & 4 & 8 \\ 2 & 0 & 5 & 3 & 2 & 5 \\ 3 & 5 & 0 & 6 & 5 & 8 \\ 5 & 3 & 6 & 0 & 1 & 2 \\ 4 & 2 & 5 & 1 & 0 & 3 \\ 8 & 5 & 8 & 2 & 3 & 0 \end{pmatrix}$

$a^3_{3k} = (3,5,0,6,5,8)$; $a_{k4} = (*,5,*,0,1,2)$; $5+1 = 6$, so there is an edge of weight 1 at the end from v_5 to v_4. $a_{k5} = (*,2,5,1,0,4)$. Only $5+0$ and $0+5$ arise, so there is an edge of weight 5 from v_3 to v_4. The path is v_3, v_5, v_4. **7.** $a^3_{1k} = (0,2,3,5,4,8)$; $a_{k6} = (*,*,*,2,4,6)$; $5+2 = 7$, so there is an edge of weight 2 from v_4 to v_6. $a_{k4} = (*,5,*,0,1,2)$; $4+1 = 5$, so there is an edge of weight 1 from v_5 to v_4. $a_{k5} = (*,2,5,1,0,4)$; $2+2 = 4$, so there is an edge of weight 2 from v_2 to v_5. $a_{k2} = (2,0,*,5,2,*)$. Only $2+0$ and $0+2$ arise, so there is an edge of weight 2 from v_1 to v_2. The path is v_1, v_2, v_5, v_4, v_6.

8. $$\mathbf{A}^6 = \mathbf{A}^5 = \begin{pmatrix} 0 & 4 & 3 & 5 & 6 & 10 & 12 & 16 \\ 4 & 0 & 2 & 4 & 5 & 9 & 11 & 15 \\ 3 & 2 & 0 & 2 & 3 & 7 & 9 & 13 \\ 5 & 4 & 2 & 0 & 1 & 5 & 7 & 11 \\ 6 & 5 & 3 & 1 & 0 & 6 & 7 & 11 \\ 10 & 9 & 7 & 5 & 6 & 0 & 2 & 6 \\ 12 & 11 & 9 & 7 & 7 & 2 & 0 & 4 \\ 16 & 15 & 13 & 11 & 11 & 6 & 4 & 0 \end{pmatrix},$$

$a^4_{1k} = (0,4,3,5,6,10,12,16)$; $a_{k8} = (*,*,*,*,*,7,4,0)$ $16 = 12+4$, a vertex of weight 4 from v_7 to v_8; $a_{k7} = (*,*,*,*,7,2,0,4)$ $12 = 10+2$, a vertex of weight 2 from v_6 to v_7; $a_{k6} = (*,*,*,5,*,0,2,7)10 = 5+5$, a vertex of weight 5 from v_4 to v_6; $a_{k4} = (*,5,2,0,1,5,*,*)5 = 3+2$, a vertex of weight 2 from v_3 to v_4; $a_{k3} = (3,2,0,2,6,*,*,*)3 = 0+3$, a vertex of weight 3 from v_1 to v_3. The path is $v_1, v_3, v_4, v_6, v_7, v_8$. **9. a)** Only half the calculations would be necessary at each stage. So, half the time could be saved.

b)

for $i := 1$ **to** n
  for $j := 1$ **to** i $\{j \leq i$ always$\}$
    $A(1, i, j) = w(v_i, v_j)$
$i := 1$
repeat
  $flag := true$
  $t := t + 1$
  for $i := 1$ **to** n
    for $j := 1$ **to** i
    $A(t, i, j) := A(t - 1, i, j)$
    for $k := 1$ **to** $j - 1$
    $A(t, i, j) := \min\{A(t, i, j), A(t - 1, i, k) + A(1, j, k)\}$
    for $k := j$ **to** $i - 1$
    $A(t, i, j) := \min\{A(t, i, j), A(t - 1, i, k) + A(1, k, j)\}$
    for $k := i$ **to** n
    $A(t, i, j) := \min\{A(t, i, j), A(t - 1, k, i) + A(1, k, j)\}$
if $A(t, i, j) \neq A(t - 1, i, j)$ **then** $flag := false$
until $t = n - 1$ **or** $flag = true$.

10. If the underlying graph is a directed graph, then edges are one-way, and the distance from i to j may not be the same as the distance from j to i. So, nonsymmetric are necessary.

Chapter 19 Network Survivability

1. a)

edge	circuit or cut edge	edge	circuit or cut edge
$\{a, b\}$	a, b, c, a	$\{a, c\}$	a, b, c, a
$\{b, c\}$	a, b, c, a	x	c, d, c
x'	c, d, c	$\{d, e\}$	cut edge
$\{e, f\}$	e, f, h, e	$\{e, h\}$	e, f, h, e
$\{f, g\}$	f, g, h, f	$\{f, h\}$	f, g, h, f
y	g, h, g	y'	g, h, g

b)

edge	circuit or cut edge	edge	circuit or cut edge
$\{a, b\}$	cut edge	x	b, j, b
x'	b, j, b	$\{b, c\}$	b, c, d, b
$\{b, d\}$	b, c, d, b	$\{c, d\}$	b, c, d, b
$\{b, e\}$	cut edge	$\{e, g\}$	e, f, g, e
y	e, f, e	y'	e, f, e
$\{f, g\}$	e, f, g, e	$\{g, h\}$	g, h, i, g
$\{g, i\}$	g, h, i, g	$\{h, i\}$	g, h, i, g

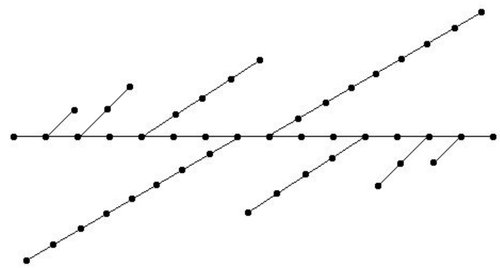
2. **a)** $\gamma(G) = \eta(G) = 9/6 = 3/2$. **b)** $\gamma(G) = \eta(G) = 12/5$. **3.** **a)** $\gamma(G) = 7/4$, $\eta(G) = 3/2$, and G_0 is the triangle K_3. (One contraction is needed to find G_0.) **b)** $\gamma(G) = 5/3$, $\eta(G) = 5/4$, and G_0 is the circuit C_5. (Two contractions are needed to find G_0.) **4.** Let G' be formed by erasing a and all edges of G incident with a. Suppose there are vertices v and v' of G which are distinct from a, are in the same component of G, and such that every path joining v with v' includes a. Then there can be no paths joining v and v' in G'. Thus v and v' are in different components of G'. But any component of G that does not contain a is a component of G'. Since v and v' are in the same component of G, they must be in the same component as a, and that component must have split into at least two components. Thus G' has more components than G, and a is a cut vertex of G. Now suppose a is a cut vertex of G. Then G' has a component H not in G. By the construction of G', the vertex a must be incident in G with an edge whose other end v is in H. But a cannot be adjacent only to vertices in H, for otherwise G and G' would have the same number of components. Hence there is a component H' of G' different from H that has a vertex v' adjacent to a in G. Now we have a path (v, a, v') in G joining v with v', whereas there are no such paths in G', since the two vertices are in different components of G'. Thus v and v' are in the same component of G, and a must be on every path joining v with v'. Lemma 1 follows. **5.** For $k = 2$, suppose $p_1/q_1 \le p_2/q_2$. Then $p_1q_2 \le p_2q_1$, so $p_1q_1 + p_1q_2 \le p_1q_1 + p_2q_1$. Dividing by $q_1(q_1 + q_2)$, we get $p_1/q_1 \le (p_1 + p_2)/(q_1 + q_2)$. But from $p_1q_2 \le p_2q_1$ we also get $p_1q_2 + p_2q_2 \le p_2q_1 + p_2q_2$. Dividing through this by $q_2(q_1 + q_2)$, we get $(p_1 + p_2)/(q_1 + q_2) \le p_2/q_2$. These two results show Lemma 2 for $k = 2$. Suppose the lemma is true for $k = n$, so that $\min_{1\le i\le n} p_i/q_i \le (p_1 + p_2 + \cdots + p_n)/(q_1 + q_2 + \cdots + q_n) \le \max_{1\le i\le n} p_i/q_i$. We will use the fraction in the middle of this last inequality as a single fraction to complete the proof. By the induction hypothesis and the first part of this proof, $\min_{1\le i\le n+1} p_i/q_i = \min\bigl(\min_{1\le i\le n}(p_i/q_i), p_{n+1}/q_{n+1}\bigr) \le \min\bigl(\frac{p_1+p_2+\cdots+p_n}{q_1+q_2+\cdots+q_n}, \frac{p_{n+1}}{q_{n+1}}\bigr) \le \frac{p_1+p_2+\cdots+p_{n+1}}{q_1+q_2+\ldots+q_{n+1}} \le \max\bigl(\frac{p_1+p_2+\cdots+p_n}{q_1+q_2+\cdots+q_n}, \frac{p_{n+1}}{q_{n+1}}\bigr) \le \max\bigl(\max_{1\le i\le n}\bigl(\frac{p_i}{q_i}\bigr), \frac{p_{n+1}}{q_{n+1}}\bigr) = \max_{1\le i\le n+1}\bigl(p_i/q_i\bigr)$. **6.** The graph shown is not induced, and adding the missing edge (edge $\{a, h\}$) gives a larger value of g. Other examples can be obtained from any subgraph shown in Figure 7 that includes the two edges joining a and b by leaving one of them out of the subgraph. **7.** By definition, each component of a forest is a tree and so has one fewer edges than vertices. Thus if forest F has k components, then $|E(F)| = |V(F)| - k = |V(F)| - \omega(F)$. But every subgraph of a forest is a forest. Thus, if H is any subgraph of forest F, and if H has an edge (so that $|V(H)| - \omega(H) > 0$), then $g(H) = |E(H)|/(|V(H)| - \omega(H)) = (|V(H)| - \omega(H))/(|V(H)| - \omega(H)) = 1$. Thus $\gamma(F) = \max_{H\subseteq G} g(H) = \max_{H\subseteq G} 1 = 1$. Now $\gamma(F) = 1 = g(F)$, so F is its η–reduced graph G_0. Thus $\eta(F) = \gamma(G_0) = \gamma(F) = 1$, and the claim is proved. **8.** Consider an arbitrary set F of edges of G. Let us erase the edges of F one at a time from G. We start with $\omega(G)$ components. Each time we erase an edge of F, the number of components either is unchanged or is increased by one because the edge has only two ends. Hence $\omega(G-F) \leq \omega(G) - |F|$. This gives us $|F| \geq \omega(G) - \omega(G-F)$. Thus we have $F/(\omega(G) - \omega(G-F)) \geq 1$, for any set F for which $\omega(G) - \omega(G-F) > 0$. But $\eta(G)$ is the minimum of all such ratios, so $\eta(G) \geq 1$ also. **9.** Let G be a plane triangulation with e edges, v vertices, and f faces. Since each edge is on exactly two faces (one face counted twice if the both sides of the edge are on the same face), and since each face has exactly three edges on its boundary, we have $2e = 3f$, or $f = (2/3)e$. By Euler's formula, $v - e + f = 2$. Hence $v - e + (2/3)e = 2$. simplifying and solving for e, we get $e = 3v - 6$, which is the claim. **10.** Since any subgraph of a plane triangulation is a planar graph, we have $|E(H)| \leq 3|V(H)| - 6$ for every connected subgraph of G. Let H be a subgraph of G, and let the components of H be $H_1, H_2, \cdots, H_k$. Then $|V(H)| = \sum_{i=1}^{k} |V(H_i)|$ and $|E(H)| = \sum_{i=1}^{k} |E(H_i)|$. Hence, since $\omega(H) = k$ and $k \geq 1$, $|E(H)| = \sum_{i=1}^{k} |E(H_i)| \leq \sum_{i=1}^{k}(3|V(H_i)| - 6) = 3\sum_{i=1}^{k}(|V(H_i)| - 2) = 3(|V(H)| - 2k) = 3(|V(H)| - k) - 3k \leq 3(|V(H)| - \omega(H)) - 3 = f(|V(H)| - \omega(H))$.

Chapter 20 The Chinese Postman Problem

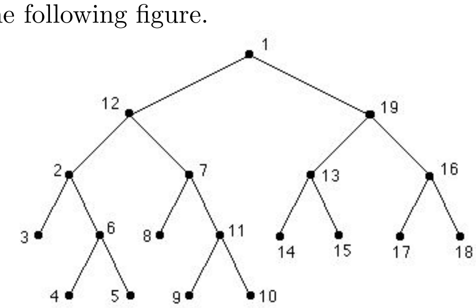
1. 227 minutes. The minimum weight deadheading edges are $\{D, F\}$ and $\{F, G\}$, resulting in a ten-minute deadheading time. Any Euler circuit in the graph of Figure 3 gives a minimal route. **2.** 73 minutes. Path G, D, E, B is the path of minimum weight joining G and B. **3.** 68 minutes. The shortest path between B and E is the path B, E, H, G, yielding a deadheading time of 15. **4.** 2. Since we are counting the *number* of edges retraced, we can treat the graph as a weighted graph where every edge has weight 1. There are two perfect matchings: $\{\{A, B\}, \{C, D\}\}$ and $\{\{A, D\}, \{B, C\}\}$. **5. a)** 3 **b)** 0 if n is odd, $n/2$ if n is even **c)** 2 **d)** 0 if m and n are even, n if m is odd and n is even, m if m is even and n is odd, the larger of m and n if m and n are both odd. **6.** $A, D, C, F, E, D, C, B, A$. **7.** 5. There are 15 matchings to consider. The matchings $\{\{2, 5\}, \{3, 8\}, \{10, 11\}\}$ and $\{\{2, 3\}, \{5, 10\}, \{8, 11\}\}$ have minimum total weight. **8.** 0.74 seconds. This is Exercise 7 again. The actual printing time is 0.58 seconds and the deadheading time is 0.16 seconds. There are six odd vertices. If we label them a, b, c, d, e, f clockwise from the top of the figure, the perfect matchings of minimum weight are: $\{\{a, b\}, \{c, d\}, \{e, f\}\}$ and $\{\{b, c\}, \{d, e\}, \{f, a\}\}$. Each of these matchings requires three vertical and two horizontal deadheading edges. **9.** 217 minutes. Each block is traversed twice (once for each side of the street). Therefore the graph to consider is one where each street in the original map is replaced by a pair of multiple edges. The degree of each vertex is even, so the graph has an Euler circuit. Following any Euler circuit will yield the minimal time. **10.** 16. Of the 28 odd vertices, 24 are adjacent in pairs. The remaining four can be at best joined by pairs of length two.

Chapter 21 Graph Layouts

1. Let v be a vertex of degree k, and let $N(v)$ be the set of its neighbors in G. Then in any numbering f of G at least half the points of $N(v)$ must be mapped either to the left or to the right of v. This shows that $\text{dil}(f) \geq 2$. Since this is true for any numbering, it follows that $B(G) \geq \lceil k/2 \rceil$. **2.** This follows from Exercise 1 since $K_4 - e$ has a point of degree 3. **3.** By part (iv) of Theorem 1 we have $B(G) \geq 5/2$, so $B(G) \geq 3$ since $B(G)$ is an integer. The opposite inequality $B(G) \leq 3$ follows by constructing a dilation 3 numbering of G. **4. a)** If we remove any k consecutive vertices of P_n^k other than the first k or the last k, then the resulting graph is disconnected. This shows that $\kappa(P_n^k) \leq k$. You can see that $\kappa(P_n^k) > k-1$ by noting that the removal of any set S of $k-1$ consecutive vertices from P_n^k still leaves a connected graph. This is true because the "long" edges $\{x, y\}$ of P_n^k joining x and y at distance k in P_n manage to tie together the two pieces of P_n that are left when S is removed. When the set S consists of nonconsecutive vertices, then even shorter edges of P_n^k tie the various pieces together. **b)** An independent set S of P_n^k of size $\lceil n/k \rceil$ consists of vertices $1, k+1, 2k+2$, etc., showing that $\beta(P_n^k) \geq \lceil n/k \rceil$. On the other hand, if T is any set of vertices containing two vertices x and y with $\text{dist}_{P_n}(x, y) \leq k$, then S is not independent in P_n^k. Thus, any independent set has size at most $|S|$, showing that $\beta(P_n^k) \leq \lceil n/k \rceil$. **c)** The degrees of the vertices of P_n^k $(n \geq 2k+1)$, going from left to right, are $k, k+1, k+2, \ldots, 2k-1, 2k, \ldots, 2k, 2k-1, \ldots, k+2, k+1, k$. Now, to find the number of edges, add these degrees and divide by 2. **5.** The smallest is 5, and the largest is 7. You get the smallest by using a numbering in which the highest number in a level is adjacent to the highest numbers (allowed by the edges) in the next level. You get the largest by using a numbering in which the lowest number in a level is adjacent to the highest numbers (allowed by the edges) in the next level. **6.** The reader may well have wondered whether the poor performance of the level algorithm on the trees $L(n)$ is really due to the level algorithm itself, or rather to the bad choice of a root in $L(n)$. Indeed, if we took the rightmost vertex of the path P in $L(n)$ as our root then the level algorithm would produce a numbering of $L(n)$ with correct dilation 2. But, in fact, the bad performance is inherent in the level algorithm, as can be seen in the following modified example. We form a tree $H(n)$ by gluing two copies of $L(n)$ along the path P in reverse order. That is, we glue the leftmost vertex of P in the first copy to the rightmost vertex of P in the second copy, and in general the kth vertex from the left in the first copy to the $2^n - k + 1$st vertex from the right in the second copy. The tree $H(4)$ is illustrated in the following graph.



It can then be shown that $B(H(n)) = 3$ but that the level algorithm applied to $H(n)$ gives dilation $\Omega(\log|H(n)|)$ regardless of which vertex of $H(n)$ is chosen as root. We leave the proof that $B(H(n)) = 3$ to the reader (just find a dilation 3 numbering). The proof that $\text{dil}(f) \geq \Omega(\log|H(n)|)$ for any level numbering regardless of the root is based on the same idea as the corresponding proof for $L(n)$, except that the gluing of the two copies of $L(n)$ now makes every vertex on the path P behave in essentially the same way. That is, you can show that now matter what vertex v along P is chosen as a root, there are integers i for which the size of S_i is at least logarithmic in i. **7.** The required embedding is illustrated in the following figure.



8. Just experiment and see how small you can make the area. You might notice that by allowing bigger area you can get smaller dilation. **9. a) – d)** To see that $S(P_n) = n - 1$, map P_n to itself by the identity map. This gives the upper bound for $S(P_n)$. The upper bounds for $S(C_n)$ and for $S(K_{1,n})$ follow from the same maps described in the text that we used to calculate $B(C_n)$ and $B(K_{1,n})$. For $S(K_n)$, observe that since every pair of vertices in K_n is joined by an edge, all maps $f : K_n \to P_n$ have the same $\text{sum}(f)$. To calculate $\text{sum}(f)$, we let $x \in V(K_n)$ and calculate $S(x) = \sum_{y \neq x} \text{dist}(f(x), f(y))$. The desired $\text{sum}(f)$ is $\sum_{x \in V(K_n)} S(x)$. To calculate $S(x)$, suppose that $f(x) = i$. Then $S(x) = \sum_{f(y)<i} \text{dist}(f(x), f(y)) + \sum_{f(y)>i} \text{dist}(f(x), f(y)) = \sum_{t=1}^{i-1} t + \sum_{t=1}^{n-i} t = \frac{1}{2}((i-1)i + (n-i)(n-i+1))$. Now just sum this over all i to get the desired result. We leave the proofs of the lower bounds to the reader. **10.** As in Exercise 1, in any numbering f of G at least half the points of $N(v)$ must be mapped either to the left or to the right of v. This forces an overlap of size at least half of k at the interval $(f(v), f(v)+1)$ or at the interval $(f(v)-1, f(v))$. Thus value$(f) \geq \lceil k/2 \rceil$, and since f was arbitrary this shows that $c(G) \geq \lceil k/2 \rceil$. **11.** For $c(P_n) = 1$, map P_n by the identity map. If we denote the vertices of C_n by $1, 2, \ldots, n$ as we traverse C_n cyclically, then to get $c(C_n) \leq 2$ map vertex i of C_n to vertex i of P_n. The map which shows that $c(K_{1,n}) \leq \lfloor n/2 \rfloor$ puts the vertex in $K_{1,n}$ of degree n in the middle and the remaining vertices (of degree 1) anywhere arbitrarily. As in the calculation of $S(K_n)$, all maps f of K_n have the same value(f). The fact that $c(K_n) = \lfloor n^2/4 \rfloor$ follows since cut$(\lfloor n/2 \rfloor) = \lfloor n^2/4 \rfloor$ because every vertex x with $f(x) \leq \lfloor n/2 \rfloor$ is joined to every vertex x with $f(x) > \lfloor n/2 \rfloor$ causing an overlap of size $\lfloor n^2/4 \rfloor$ over the interval $(\lfloor n/2 \rfloor, \lfloor n/2 \rfloor + 1)$. We leave the lower bounds to the reader; Exercise 10 can be used here. **12. a)** The matrix $\begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$ has a band of 4 enclosing all the 1s, and is obtained by a permutation of the rows and columns of A. **b)** The graph $G(A)$ is $K_{4,4}$, and it has bandwidth 4. **13.** Let x be the vertex of degree k in G, and let $f : G \to P_n$ be any one-to-one map. Let $S(x) = \sum_{y \in N(x)} \text{dist}(f(x), f(y))$, where again $N(x)$ is the set of neighbors of x in G. Clearly $S(x)$ is minimized when half the vertices of $N(x)$ are mapped immediately to the left of $f(x)$ and the other half immediately to the right. Thus we get sum$(f) \geq S(x) \geq 2(\sum_{t=1}^{k/2} t) = \frac{k}{2}(k+2)$. **14.** The biggest dilation occurs when the vertex with smallest number in level $k-1$ is joined to the vertex with biggest number in level k. The smallest number is $1 + 2^{k-2} - 1 = 2^{k-2}$ (i.e. one more than the number of vertices in T_{k-2}), and the biggest number is the number of vertices in T_k which is $2^k - 1$. The dilation that we get in that case is the difference between these two numbers, which is $2^k - 2^{k-2} - 1$. The smallest dilation occurs when the vertex with biggest number in level $k-1$ is joined to the vertex with biggest number in level k, and smaller numbered vertices in level $k-1$ are joined to smaller numbered vertices in level k. The biggest in level $k-1$ is $2^{k-1} - 1$, and the biggest in level k is $2^k - 1$. Hence the dilation in this case is $2^k - 2^{k-1}$.

Chapter 22 Graph Multicolorings

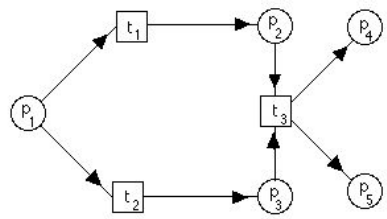
1. a) 8 **b)** 4 **c)** 5 **d)** 6 **e)** 7 **f)** 4. **2.** Station 1: channels 1,2,3; station 2: channels 4,5,6; station 3: channels 7,8,9; Station 4: channels 4,5,6; station 5: channels 7,8,9. **3.** Graph Algorithms: Monday AM, Monday PM; Operating Systems: Tuesday AM, Tuesday PM; Automata Theory: Monday AM, Wednesday AM; Number Theory: Tuesday PM, Wednesday AM; Computer Security: Monday PM, Tuesday AM; Compiler Theory: Tuesday PM, Wednesday PM; Combinatorics: Monday AM, Monday PM. **4.** Let the edges in G be partitioned into disjoint nonempty subsets V_1 and V_2 such that every edge connects a vertex in V_1 and an edge in V_2. Assign colors $1, 2, ..., n$ to each vertex in V_1 and colors $n+1, n+2, ..., 2n$ to each vertex in V_2. This produces an n-tuple coloring using $2n$ colors. Therefore $\chi_n(G) \leq 2n$. But any edge in G will have $2n$ colors used for its endpoints, so $\chi_n(G) \geq 2n$. Hence $\chi_n(G) = 2n$. **5. a)** 3 **b)** 4 **c)** 3 **d)** 5. **6.** Graphs in parts a, b, and d are weakly γ-perfect. **7.** It is easy to see that the clique number of a bipartite graph with at least one edge is 2. Since a bipartite graph has chromatic number equal to 2, such a graph is weakly γ-perfect. If no edge is present, both the clique number and chromatic number of the graph equal 1, so that graph is weakly γ-perfect. **8. a)** 6 **b)** 8 **c)** 8 **d)** 10. **9.** Suppose the vertices of K_4 are a, b, c, and d. By assigning color 1 to a, color 3 to b, color 8 to c, and color 10 to d, we get a T-coloring with span 9. To see that no T-coloring has smaller span, note that without loss of generality we can assume that a is assigned color 1. If the span is smaller than 9, since $T = \{0, 1, 3, 4\}$, the only available colors for b, c, and d are 3,6,7,8, and 9. But no three of these numbers have the property that all pairs of differences of the numbers are not in T. **10. a)** T-chromatic number: 3, T-span: 4. **b)** T-chromatic number: 3, T-span: 4. **c)** T-chromatic number: 3, T-span: 6. **11. a)** T-chromatic number: 4, T-span: 6. **b)** T-chromatic number: 4, T-span 6. **c)** T-chromatic number: 4, T-span: 9 (see Exercise 9). **12. a)** 5 **b)** 7 **c)** 7 **d)** 10. **13. a)** v_1: colors 1 and 2; v_2: colors 3 and 4; v_3: colors 1 and 2; v_4: colors 3 and 4; v_5: colors 3 and 4; v_6: colors 5 and 6; v_7: colors 7 and 8; v_8 colors 1 and 2; v_9: colors 3 and 4; v_{10}: colors 1 and 2. **b)** It uses 6 colors for a 2-tuple coloring of C_5. **14.** First, order the vertices as v_1, v_2,...,v_n and represent colors by positive integers. Assign as many colors as specified to v_1. Once having assigned as many colors as specified to each of v_1,v_2,...,v_k, assign the smallest numbered colors of the quantity specified to v_{k+1} such that no color assigned is the same as a color assigned to a vertex adjacent to v_{k+1} that already was assigned a set of colors. **15. a)** v_1: color 1; v_2: color 3; v_3: color 1; v_4: color 3; v_5: color 3; v_6: color 9; v_7: color 11; v_8: color 1; v_9: color 3; v_{10}: color 1. **b)** It produces a span of 8 for C_5, but the T-span of C_5 is 4. **16.** A list coloring is needed to model examination scheduling when particular examinations can only be given during restricted periods, to model maintenance scheduling when certain vehicles can only be maintained during particular times, or to model task scheduling when certain tasks can only be performed during specified times. **17.** An I-coloring is needed to model assignments of frequency bands for mobile radios, to model space assignment in a maintenance facility laid out along a linear repair dock, or to model task scheduling when tasks take an interval of time to complete. **18.** A J-coloring is needed to model assignments of several frequency bands for each station or to model task scheduling when tasks are completed during more than one interval of time.

Chapter 23 Network Flows

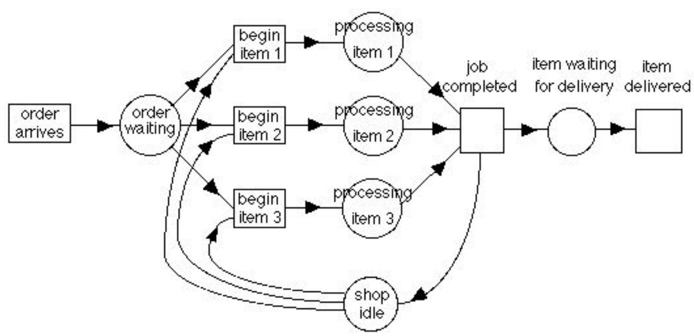
1. The maximum flow is 13 with cut $\{(d,t),(d,e),(b,e),(c,e),(c,f)\}$. **2.** The maximum flow is 7 with cut $\{(a,d),(b,d),(f,e),(c,g)\}$. **3.** The maximum flow is 13 with cut $\{(d,t),(d,e),(b,e),(c,e),(c,f)\}$. **4.** The maximum flow is 7 with cut $\{(a,d),(b,d),(f,e),(c,g)\}$. **5.** The maximum flow is 4 with cut $\{\{a,c\},\{a,d\},\{b,d\}\}$. **6.** The maximum flow is 4 with cut $\{\{a,d\},\{c,d\},\{d,e\},\{e,t\}\}$. **7.** The maximum flow is 7 with a minimum cut consisting of edges (d,g), (e,g), (e,h), (f,h), and (f,l). **8. a)** The maximum flow is 5 with minimum cut containing the edges (s,S_1), (s,S_2), (s,S_4), (C_2,t), and (C_3,t). Interpreting a maximum flow, we find that an optimum assignment has each of Students 1, 2, and 4 grading a section of Course 1, and Student 3 grading one section of each of Courses 2 and 3. There are not enough graders competent to grade all of the sections of Course 1, and there are not enough sections in courses other than Course 1 to satisfy the desires of all of the graders. (Notice that the edges of the minimum cut are directed from $A=\{s,S_3,C_2,C_3\}$ to $V(G)-A$. Other edges between these two sets of vertices are directed toward A.) **b)** The maximum flow is 7 with minimum cut containing the edges (s,S_1), (s,S_2), (s,S_3), and (s,S_4). Examining a maximum flow shows that an optimum assignment has Student 1 grading one section of Course 1, Student 2 grading one section of each of Courses 1 and 2, Student 3 grading the other section of Course 2, and Student 4 grading two sections of Course 3 and the one section of Course 4. Both the students' desires and the needs of the courses are met with this assignment. **9.** Let G be an undirected bipartite graph, and suppose $V(G)$ is the disjoint union of nonempty sets V_1 and V_2 such that every edge joins a vertex in V_1 with a vertex in V_2. Form a directed capacitated s,t-graph H as follows: Direct every edge of G from its end in V_1 to its end in V_2 and place a capacity of ∞ on each such edge. Add a source vertex s and connect it to every vertex in V_1 with an edge directed away from s of capacity 1. Add a sink vertex t and connect each vertex in V_2 to t with an edge directed toward t of capacity 1. Now find a maximum flow f in H and the corresponding minimum cut $(A,V(H)-A)$. Since each unit of flow must travel from s to a vertex x in V_1 through an edge of capacity 1, from x to a vertex y in V_2 through an edge originally in G, and, because of the directions of the edges originally in G, must then go on directly to t through another edge of capacity 1, the edges originally in G which are used for flow must form a matching in G whose number of edges is the same as the number of units of flow in H. Further, if M is a matching in G, then using the corresponding edges of H as the middle edges of a flow, there must be a flow in H whose number of units is equal to the size of M. Since f is a maximum flow, the edges originally in G used by the flow form a maximum matching of G. But the cut $(\{s\}, V(H) - \{s\})$ has capacity equal to the number of vertices in V_1, so the minimum cut $(A, V(H) - A)$ is finite. Hence it consists solely of edges of H incident with the vertices s and t, and so there is a one-to-one correspondence between the edges in $(A, V(H) - A)$ and vertices of G. Further, for each path s, m, n, t carrying a unit of flow, one of (s, m) or (n, t) must be in $(A, V(H) - A)$. Let the set of vertices of G met by edges in $(A, V(H) - A)$ be C, so that the size of C is the same as the size of $(A, V(H) - A)$. Suppose there is an edge e of G which does not meet a vertex in C, and let the edge of H corresponding to e be (m, n). Suppose $f(s, m) = 1$. Since m and n are not in C, neither (s, m) nor (n, t) is in $(A, V(H) - A)$, so the unit of flow from s to m must continue on to a vertex n' in V_2 with $n' \neq n$, and $(n', t) \in (A, V(H) - A)$. Hence n' received a label in the final labeling of the algorithm. Since $f(m, n') > 0$, m has a label, and since $c(m, n) = \infty$, n has a label. But t does not have a label, so $(n, t) \in (A, V(H) - A)$, contrary to our supposition. Thus $f(s, m) = 0$. Now in the final labeling of the algorithm, m and n receive labels, so $(n, t) \in (A, V(H) - A)$, again contrary to our assumption that neither m nor n is in C. Since we have a contradiction in either case, it follows that e cannot exist. Thus C is a cover of G. But by Theorem 4, the number of elements in $(A, V(H) - A)$ is the same as the number of units of flow in f, and those numbers are the same as the numbers of elements in C and M, respectively. The theorem follows. **10.** Suppose G is 2-connected. Let $v, w \in V(G)$ with $v \neq w$. As suggested in the hint, replace each vertex x other than v and w with two new vertices x_1 and x_2 and a directed edge (x_1, x_2) with capacity 1. We will call this edge a "vertex-generated edge" generated by x. For every edge (x, y) not meeting v or w, replace it with the directed edges (x_2, y_1) and (y_2, x_1), both with capacity infinity. If edge $\{v, w\}$ is in $E(G)$, replace it with edge (v, w) having capacity 1. For each edge $\{v, x\}$ with $x \neq w$, replace it with an edge (v, x_1) having capacity infinity. Similarly, for each edge $\{w, x\}$ with $x \neq v$, replace it with an edge (x_2, w) having capacity infinity. The resulting directed graph H is a capacitated v, w-graph. Let f be a maximum flow in H from v to w. Since every unit of flow from v to w must pass through either edge (v, w) of capacity 1 or some vertex-generated edge (x_1, x_2) of capacity 1, f must have a finite value. Further, no two units of flow can have paths sharing a vertex other than v or w since (v, w) if it exists and all vertex-generated edges have capacities 1. By Theorem 4, the value of a maximum flow equals the capacity of a minimum cut in H between v and w. Since minimum cuts must be made of vertex-generated edges and (v, w), and since the removal of $\{v, w\}$ alone or of one vertex alone cannot disconnect G, there must be at least two edges in a minimum cut of H. Hence there must be at least two units of flow in f and thus at least two paths from v to w in H which share no vertices other than v and w. But these paths are easily translated back into paths in G with the same property, proving the theorem in one direction. Suppose that, for any distinct vertices v and w of G, there are two simple paths joining v with w which have only the vertices v and w in common. Suppose, for the sake of contradiction, that G has a cut vertex x. Form $G-x$ by erasing x and all edges incident with x in G. By the definition of a cut vertex, there must be vertices v and w in different components of $G-x$ which are in the same component of G. Let p_1 and p_2 be paths in G which join v and w and which share only those two vertices. Then x can be on at most one of p_1 or p_2; suppose x is not on the path p_2. Then p_2 is a path in $G-x$ joining v with w, so v and w are not in different components of $G-x$, contrary to the choice of v and w. It follows that G has no cut vertices. But G has at least three vertices, so it must have at least two edges. Thus G is 2-connected. **11.** This is solved exactly as the problem of assigning graders to sections of classes is solved. The workers play the part of graders and the machines are the sections of the classes. Thus, we have a vertex for each worker and one for each type of machine, and we join a vertex for a worker to one for a type machine by an edge if the worker is competent to run the machine. We add a source s and a sink t. We join s to the vertex for each worker by an edge of capacity 1 (since each worker works on just one machine). We join the vertex for each type of machine to t by an edge of capacity equal to the number of machines of that type. Then a maximum flow in the graph will give the necessary assignment. **12.** Let the CIC office be a vertex, introduce two vertices and a directed edge from one (the tail) to the other (the head) for each switch in the telephone network, and add one more sink vertex t. Give the capacity of a switch to the edge corresponding to that switch. Let the source vertex s be the CIC office. As suggested, join the head vertex for each switch to t by an edge directed toward t whose capacity is the number of local users tied to that switch. If there is a direct telephone link from one switch a to another b, introduce an edge from the head of the edge corresponding to a to the tail of the edge corresponding to b; since any call passing out of a switch reaches the next one, give a capacity of infinity to such edges. Join s to the tails of edges corresponding to switches to which the CIC office is directly connected by edges directed toward the switches, with capacity equal to the number of calls from that switch the company can accept. Then, given the reported trouble with calls coming to CIC, a minimum cut in the resulting graph will include edges corresponding to switches that are bottlenecks for the system. The max-flow min-cut algorithm will find such a minimum cut.

Chapter 24 Petri Nets

1.

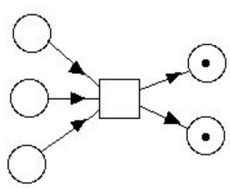


2.

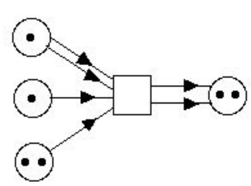


3. a is not enabled. The upper left place needs at least two tokens. b and c are enabled.

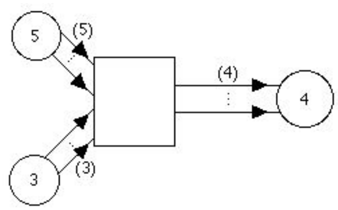
4. a)



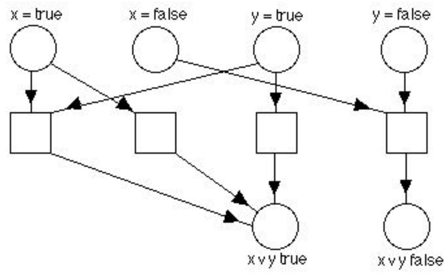
b)



c)

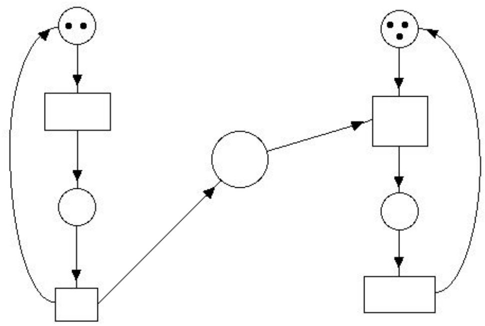


5.

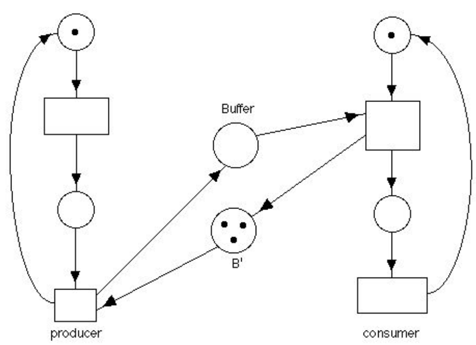


6. $\neg x \wedge \neg y$ is equivalent to $\neg(x \vee y)$, so the required net can be constructed from those for negation and disjunction, similarly to the way the net for $\neg(x \wedge y)$ was constructed in the text. **7.** $x \rightarrow y$ is equivalent to $\neg x \vee y$. Thus, the required net can be constructed using those for negation and disjunction. **8.** The access control place should have only two tokens. There should be five "reader" tokens and three "writer" tokens. Instead of three multiple edges at the access control place, there should be two.

9.

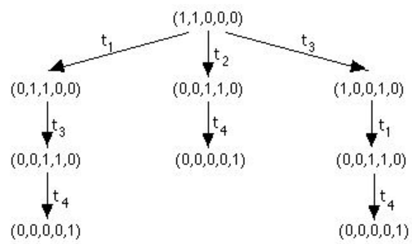


10. Add another place in the buffer which starts out with three tokens. Each time an object is put in the buffer, one token is removed from the new place. Each time an object is consumed, one token is added to the new place. See the following diagram, where B' is the new place.



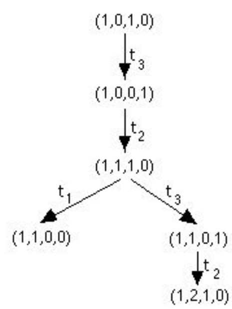
11. **a)** The net is not safe. Firing t_3 produces two tokens at p_1. **b)** The net is bounded. No place ever has more than two tokens. **c)** The net is not conservative. Firing t_1 puts one token in p_2 and removes one from p_1. Then, firing t_4 puts one token in p_1 and removes one from each of p_2 and p_3, reducing the number of tokens in the net. **12.** Firing t_2 first and then t_5 puts one token in p_3 and p_2. From here repeat firings of t_5 increase the number of tokens in p_2 each time by one. Thus, the net is not safe, not bounded, and not conservative.

13.



a) The net is safe, by inspection. **b)** The net is not conservative since t_4 decreases the number of tokens. **c)** $(0, 0, 1, 0, 1)$ is not reachable from $(1, 1, 0, 0, 0)$, by inspection.

14.



The net is not bounded. The number of tokens in place p_2 can be increased without bound.

15.