

Rosen, Discrete Mathematics and Its Applications, 7th edition, Global Edition  
Extra Examples  
Section 1.5—Nested Quantifiers



— Page references correspond to locations of Extra Examples icons in the textbook.

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**p.53, icon at Example 1**

**#1.** Write the following statements in English, using the predicate  $S(x, y)$ : “ $x$  shops in  $y$ ”, where  $x$  represents people and  $y$  represents stores:

(a)  $\forall y S(\text{Margaret}, y)$ .

(b)  $\exists x \forall y S(x, y)$ .

**Solution:**

(a) The predicate states that if  $y$  is a store, then Margaret shops there. That is, “Margaret shops in every store.”

(b) The predicate states that there is a person  $x$  with the property that  $x$  shops in every store  $y$ . That is, “There is a person who shops in every store.” [Note that part (a) is obtained from part (b) by taking a particular value, *Margaret*, for the variable  $x$ . If we do this, we do not need to quantify  $x$ .]

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**p.53, icon at Example 1**

**#2.** Write in symbols using predicates and quantifiers: “Every Junior in this class scored above 90 on the first exam.”

**Solution:**

The solution depends on what we take for the universe for the variable. If we take all Juniors in this class as the universe, we can write the proposition as

$$\forall x S(x)$$

where  $S(x)$  is the predicate “ $x$  scored above 90 on the first exam.”

However, if we take all students in this class as the universe, then we can write the proposition as

$$\forall x (J(x) \rightarrow S(x))$$

where  $J(x)$  is the predicate “ $x$  is a Junior.”

We can extend the universe still further. Suppose we take all students as the universe. Then we need to introduce a third predicate  $C(x)$  to mean “ $x$  is in this class.” In this case, the proposition becomes

$$\forall x ((C(x) \wedge J(x)) \rightarrow S(x)).$$

If we also wish to distinguish among possible scores on the first exam, we can use “nested quantifiers”, discussed later in this section of the book. We can replace  $S(x)$  by  $S(x, y)$  where  $S(x, y)$  means “ $x$  received a score of  $y$  on the first exam” and the universe for  $y$  is the set of all possible exam scores. In this case the proposition becomes

$$\forall x \exists y (C(x) \wedge J(x) \rightarrow (y > 90) \wedge S(x, y)).$$

Note that we used a predicate,  $y > 90$ , without giving it a name.

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**p.53, icon at Example 1**

**#3.** Write the following statement in English, using the predicates

$$\begin{aligned} S(x, y): & \text{“}x \text{ shops in } y\text{”} \\ T(x): & \text{“}x \text{ is a student”} \end{aligned}$$

where  $x$  represents people and  $y$  represents stores:

$$\exists y \forall x (T(x) \rightarrow \neg S(x, y)).$$

**Solution:**

The statement  $\exists y \forall x (T(x) \rightarrow \neg S(x, y))$  says that “there is a store  $y$  with a certain property, namely, if  $x$  is any student whatever, then  $x$  does not shop in  $y$ .” We have “There is a store in which no student shops.”

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**p.53, icon at Example 1**

**#4.** Write the following statement in English, using the predicates

$$\begin{aligned} S(x, y): & \text{“}x \text{ shops in } y\text{”} \\ T(x): & \text{“}x \text{ is a student”} \end{aligned}$$

where  $x$  represents people and  $y$  represents stores:

$$\forall y \exists x (T(x) \wedge S(x, y)).$$

**Solution:**

The statement  $\forall y \exists x (T(x) \wedge S(x, y))$  asserts that for every store  $y$  that can be chosen, there is a person  $x$  who is a student and who shops in  $y$ . Therefore: “Every store has at least one student who shops in it.”

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**p.53, icon at Example 1**

**#5.** Write the following statement in English, using the predicate  $S(x, y)$  for “ $x$  shops in  $y$ ”, where  $x$  represents people and  $y$  represents stores:

$$\exists x_1 \exists y \forall x_2 [S(x_1, y) \wedge (x_1 \neq x_2 \rightarrow \neg S(x_2, y))].$$

**Solution:**

The statement  $S(x_1, y) \wedge (x_1 \neq x_2 \rightarrow \neg S(x_2, y))$  tells us two things: person  $x_1$  shops in store  $y$ , and if  $x_2$  is any other person then  $x_2$  does not shop in  $y$ . Therefore, we have “There is a store in which exactly one person shops.”

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**p.53, icon at Example 1**

**#6.** Write the following statement in English, using the predicates

$$\begin{aligned}C(x): & \text{“}x \text{ is a Computer Science major”} \\M(y): & \text{“}y \text{ is a math course”} \\T(x, y): & \text{“}x \text{ is taking } y\text{”}\end{aligned}$$

where  $x$  represents students and  $y$  represents courses:

$$\forall x \exists y (C(x) \rightarrow M(y) \wedge T(x, y)).$$

**Solution:**

The statement  $\forall x \exists y (C(x) \rightarrow M(y) \wedge T(x, y))$  asserts that for every student  $x$  there is a course  $y$  such that if  $x$  is a major in Computer Science then  $x$  is taking  $y$  and  $y$  is a math course. Therefore, “Every Computer Science major is taking at least one math course.”

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**p.53, icon at Example 1**

**#7.** Write the following statement in English, using the predicates

$$\begin{aligned}C(x): & \text{“}x \text{ is a Computer Science major”} \\T(x, y): & \text{“}x \text{ is taking } y\text{”}\end{aligned}$$

where  $x$  represents students and  $y$  represents courses:

$$\forall y \exists x (\neg C(x) \wedge T(x, y)).$$

**Solution:**

The statement  $\forall y \exists x (\neg C(x) \wedge T(x, y))$  says that for every course  $y$  there is a student  $x$  such that  $x$  is not a Computer Science major and  $x$  is taking  $y$ . That is, “Every course has a student in it who is not a Computer Science major.”

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**p.53, icon at Example 1**

**#8.** Write the following statement in English, using the predicates

$$\begin{aligned}F(x): & \text{“}x \text{ is a Freshman”} \\M(y): & \text{“}y \text{ is a math course”} \\T(x, y): & \text{“}x \text{ is taking } y\text{”}\end{aligned}$$

where  $x$  represents students and  $y$  represents courses:

$$\neg \exists x [F(x) \wedge \forall y (M(y) \rightarrow T(x, y))].$$

**Solution:**

First examine part of the statement,  $\forall y (M(y) \rightarrow T(x, y))$ . This says that “if  $y$  is a math course, then  $x$  is taking  $y$ ”, or, equivalently, “ $x$  is taking every math course”. The given statement says that there is no

student with this property:  $F(x) \wedge \forall y (M(y) \rightarrow T(x, y))$ ; that is, there is no student who is both a freshman and who is taking every math course. Therefore, we have “No Freshman is taking every math course.”

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**p.53, icon at Example 1**

**#9.** Write the following statement using quantifiers and the predicate  $S(x, y)$  for “ $x$  shops in  $y$ ”, where the universe for  $x$  consists of people and the universe for  $y$  consists of stores:

“Will shops in Al’s Record Shoppe.”

**Solution:**

Using “Will” for  $x$  and “Al’s Record Shoppe” for  $y$ , we have

$S(\text{Will}, \text{Al’s Record Shoppe})$ .

No quantifiers are needed.

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**p.53, icon at Example 1**

**#10.** Write the following statement using quantifiers and the predicates

$S(x, y)$ : “ $x$  shops in  $y$ ”  
 $T(x)$ : “ $x$  is a student”

where the universe for  $x$  consists of people and the universe for  $y$  consists of stores:

“There is no store that has no students who shop there.”

**Solution:**

We can begin by stating that “It is false that there exists a store  $y$  with the property that no students shop in  $y$ .” Saying that “no students shop in  $y$ ” is saying  $\forall x (T(x) \rightarrow \neg S(x, y))$ . Completely written in symbols, we have

$\neg \exists y \forall x (T(x) \rightarrow \neg S(x, y))$ .

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**p.53, icon at Example 1**

**#11.** Write the following statement using quantifiers and the predicates

$S(x, y)$ : “ $x$  shops in  $y$ ”  
 $T(x)$ : “ $x$  is a student”

where the universe for  $x$  consists of people and the universe for  $y$  consists of stores:

“The only shoppers in some stores are students.”

**Solution:**

The given statement asserts that “There is at least one store,  $y$ , such that only students shop there.” Saying

that “only students shop in  $y$ ” means that  $\forall x (S(x, y) \rightarrow T(x))$ . Putting these together gives

$$\exists y \forall x (S(x, y) \rightarrow T(x)).$$

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**p.53, icon at Example 1**

**#12.** Suppose that the universe for  $x$  and  $y$  is  $\{1, 2, 3\}$ . Also, assume that  $P(x, y)$  is a predicate that is true in the following cases, and false otherwise:  $P(1, 3), P(2, 1), P(2, 2), P(3, 1), P(3, 2), P(3, 3)$ . Determine whether each of the following is true or false:

- (a)  $\forall y \exists x (x \neq y \wedge P(x, y))$ .
- (b)  $\forall x \exists y (x \neq y \wedge \neg P(x, y))$ .
- (c)  $\forall y \exists x (x \neq y \wedge \neg P(x, y))$ .

**Solution:**

(a) True. We need to consider three cases:  $y = 1, y = 2, y = 3$ .

If  $y = 1$ , we can take  $x = 2$ , obtaining the true statement  $2 \neq 1 \wedge P(2, 1)$ .

If  $y = 2$ , we can take  $x = 3$ , obtaining the true statement  $3 \neq 2 \wedge P(3, 2)$ .

If  $y = 3$ , we can take  $x = 1$ , obtaining the true statement  $1 \neq 3 \wedge P(1, 3)$ .

Therefore, the statement  $\exists x (x \neq y \wedge P(x, y))$  is true for all possible choices of  $y$ . Hence,  $\forall y \exists x (x \neq y \wedge P(x, y))$  is true.

(b) False. Take  $x = 3$ . The statements  $P(3, 1), P(3, 2),$  and  $P(3, 3)$  are true; that is, the statements  $\neg P(3, 1), \neg P(3, 2),$  and  $\neg P(3, 3)$  are false. Therefore, there is no value  $y$  such that  $3 \neq y \wedge \neg P(3, y)$  is true.

(c) False. Take  $y = 1$ . We need to consider  $x = 1, x = 2,$  and  $x = 3$ . The conjunctions  $1 \neq 1 \wedge \neg P(1, 1), 2 \neq 1 \wedge \neg P(2, 1),$  and  $3 \neq 1 \wedge \neg P(3, 1)$  are all false.

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**p.53, icon at Example 1**

**#13.** Suppose that the universe for  $x$  and  $y$  is  $\{1, 2, 3, 4\}$ . Assume that  $P(x, y)$  is a predicate that is true in the following cases and false otherwise:  $P(1, 4), P(2, 1), P(2, 2), P(3, 4), P(4, 1), P(4, 4)$ . Determine whether each of the following is true or false:

- (a)  $\exists y \forall x P(x, y)$ .
- (b)  $\forall x P(x, x)$ .
- (c)  $\forall x \exists y (x \neq y \wedge P(x, y))$ .

**Solution:**

(a) False. If we take  $y = 1$ , not all four statements  $P(x, 1)$  are true. (Take  $x = 1$  for example.) If we take  $y = 2$ , not all four statements  $P(x, 2)$  are true. (Take  $x = 1$  for example.) If we take  $y = 3$ , not all four statements  $P(x, 3)$  are true. (Take  $x = 1$  for example.) If we take  $y = 4$ , not all four statements  $P(x, 4)$  are true. (Take  $y = 2$ .)

(b) False.  $P(1, 1)$  is false.

(c) True. For every  $x$  we can find a value  $y \neq x$  such that  $P(x, y)$  is true:  $P(1, 4), P(2, 1), P(3, 4),$  and  $P(4, 1)$ .

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**p.53, icon at Example 1**

**#14.** Consider this sentence, which is Amendment 3 to the U.S. Constitution: “No soldier shall, in time of peace, be quartered in any house, without the consent of the owner, nor in time of war, but in a manner to be prescribed by law.”

(a) The sentence has the form of a conjunction of two conditional sentences. Write the given sentence in this form.

(b) Using the six predicates,  $S(x)$ : “ $x$  is a soldier,”  $P(t)$  “ $t$  is a peaceful time,”  $Q(x, y, h)$ : “ $x$  is required to allow  $y$  to be quartered in  $h$ ,”  $O(x, h)$ : “ $x$  owns  $h$ ,”  $C(x, y, h)$ : “ $x$  consents to quarter  $y$  in  $h$ ,”  $A(x, h)$ : “the law allows  $x$  to be quartered in  $h$ ,” where the universe for  $x$  and  $y$  consists of all people, the universe for  $t$  consists of all points in time, and the universe for  $h$  consists of all houses, rewrite the sentence using quantifiers and predicates.

**Solution:**

(a) The sentence has the form “If it is a time of peace, then . . . , and, if it is a time of war, then . . . .” Written in full, the sentence is “If it is a time of peace, then no soldier shall be quartered in any house without the consent of the owner, and, if it is a time of war, then no soldier shall be quartered in any house except in a manner to be prescribed by law.”

(b) The statement is a conjunction; it has the form “(if  $P(t)$ , then . . . )  $\wedge$  (if  $\neg P(t)$ , then . . .).”

Let us examine the case when it is a time of peace. The statement says that “if the owner of a house does not give consent, then no soldier shall be quartered in that house.” That is, if person  $x$  owns house  $h$  and does not consent to quarter soldier  $y$  in  $h$ , then  $x$  is not required to quarter  $y$  in  $h$ . In symbols, we have

$$((O(x, h) \wedge S(y) \wedge \neg C(x, y, h)) \rightarrow \neg Q(x, y, h)).$$

Similarly, in the case when it is not a time of peace, we have

$$((O(x, h) \wedge S(y) \wedge A(y, h)) \rightarrow Q(x, y, h)).$$

Completely written in symbols we have

$$\forall t \forall x \forall y \forall h \{ [P(t) \rightarrow ((O(x, h) \wedge S(y) \wedge \neg C(x, y, h)) \rightarrow \neg Q(x, y, h))] \wedge [\neg P(t) \rightarrow ((O(x, h) \wedge S(y) \wedge A(y, h)) \rightarrow Q(x, y, h))] \}.$$

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**p.53, icon at Example 1**

**#15.** Consider these lines of code from a C++ program:

```
if (!(x!=0 && y/x < 1) || x==0)
    cout << "True";
else
    cout << "False"
```

(a) Express the code in this statement as a compound statement using the logical connectives  $\neg, \vee, \wedge, \rightarrow$ , and these predicates

$$\begin{aligned} E(x): & x = 0 \\ L(x, y): & y/x < 1 \\ A(z): & \text{“}z \text{ is assigned to cout”} \end{aligned}$$

where  $x$  and  $y$  are integers and  $z$  is a Boolean variable (with values True and False).

(b) Use the laws of propositional logic to simplify the statement by expressing it in a simpler form.

(c) Translate the answer in part (b) back into C++.

**Solution:**

(a) First we insert the predicates into the code, obtaining

```
if (!(!E(x) && L(x, y)) || E(x))
    A(True)
else
    A(False).
```

Next change to the usual logical connective symbols, keeping in mind that C++ code of the form “if  $p$  then  $q$  else  $r$ ” is really a statement of the form  $(p \rightarrow q) \wedge (\neg p \rightarrow r)$ :

```
[¬(¬E(x) ∧ L(x, y)) ∨ E(x)] →
    A(True)
```

```
∧
¬[¬(¬E(x) ∧ L(x, y)) ∨ E(x)] →
    A(False), or
```

```
[¬(¬E(x) ∧ L(x, y)) ∨ E(x)] → A(True) ∧ ¬[¬(¬E(x) ∧ L(x, y)) ∨ E(x)] → A(False).
```

Because this statement applies to all numbers  $x$  and  $y$ , we have

$$\forall x \forall y \left( [\neg(\neg E(x) \wedge L(x, y)) \vee E(x)] \rightarrow A(\text{True}) \right) \wedge \left( \neg[\neg(\neg E(x) \wedge L(x, y)) \vee E(x)] \rightarrow A(\text{False}) \right).$$

(b) Using one of De Morgan’s laws on the negation of the conjunction, the statement becomes

$$\forall x \forall y \left( [(E(x) \vee \neg L(x, y)) \vee E(x)] \rightarrow A(\text{True}) \right) \wedge \left( \neg[(E(x) \vee \neg L(x, y)) \vee E(x)] \rightarrow A(\text{False}) \right),$$

which can be simplified to give

$$\forall x \forall y \left( (E(x) \vee \neg L(x, y)) \rightarrow A(\text{True}) \right) \wedge \left( \neg(E(x) \vee \neg L(x, y)) \rightarrow A(\text{False}) \right).$$

(c) Translating the statement in (b) into C++ yields

```
if (x==0 || y/x >= 1)
    cout << "True"
else
    cout << "False".
```

**p.54, icon at Example 3**

**#1.** What are the truth values of each of these? Assume that in each case the universe consists of all real numbers.

- (a)  $\exists x \exists y (xy = 2)$
- (b)  $\exists x \forall y (xy = 2)$
- (c)  $\forall x \exists y (xy = 2)$
- (d)  $\forall x \forall y (xy = 2)$

**Solution:**

(a) This statement asserts that there are numbers  $x$  and  $y$  such that  $xy = 2$ . This is true because we can take  $x = 2$  and  $y = 1$ , for example.

(b) This statement asserts that there is a number  $x$  such that when we multiply this particular  $x$  by every possible number  $y$  we obtain  $xy = 2$ . There is no such number  $x$ . (If there were such a number  $x$ , then  $xy = 2$  for all  $y$ . If we take  $y = 0$ , the product  $xy$  cannot equal 2.) Therefore the statement is false.

(c) This statement asserts that for every number  $x$  we choose, we can find a number  $y$  such that the  $xy = 2$ . This is almost always the case, except if we choose  $x = 0$ . If we take  $x = 0$ , there is no number  $y$  such that  $xy = 2$ . Therefore the statement is false. (Note that the statement would be true if the universe for  $x$  consisted of all *nonzero* real numbers.)

(d) This statement claims that no matter what numbers  $x$  and  $y$  we choose, we obtain  $xy = 2$ . Clearly, this is false, because we could choose  $x = y = 1$ .

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**p.54, icon at Example 3**

**#2.** Write the following statements in English, using the predicate  $S(x, y)$ : “ $x$  shops in  $y$ ”, where  $x$  represents people and  $y$  represents stores:

(a)  $\exists y \forall x S(x, y)$ .

(b)  $\forall x \exists y S(x, y)$ .

**Solution:**

(a) The sentence states that there is a store  $y$  such that every person  $x$  shops there. Thus, “There is a store in which everyone shops.”

(b) The sentence states that for every person  $x$  there is a store  $y$  in which  $x$  shops. Therefore, we have “Everyone shops somewhere.”

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**p.54, icon at Example 3**

**#3.** Suppose that the universe for  $x$  and  $y$  is  $\{1, 2, 3\}$ . Also, assume that  $P(x, y)$  is a predicate that is true in the following cases, and false otherwise:  $P(1, 3), P(2, 1), P(2, 2), P(3, 1), P(3, 2), P(3, 3)$ . Determine whether each of the following is true or false:

(a)  $\exists x \forall y (y < x \rightarrow P(x, y))$ .

(b)  $\forall y \exists x (y < x \vee P(x, y))$ .

(c)  $\exists x \exists y (P(x, y) \wedge P(y, x))$ .

(d)  $\forall y \exists x (P(x, y) \rightarrow \neg P(y, x))$ .

**Solution:**

(a) True. We can take  $x = 1$ . Because there is no  $y$  such that  $y < 1$ , the hypothesis of the implication  $y < x \rightarrow P(x, y)$  is false, making the implication true.

(b) True. We need to consider the cases  $y = 1$ ,  $y = 2$ , and  $y = 3$ .

If  $y = 1$ , then the statement  $\exists x (y < x \vee P(x, y))$  is true for  $x = 2$  (because  $1 < 2$ ).

If  $y = 2$ , then the statement  $\exists x (y < x \vee P(x, y))$  is true for  $x = 3$  (because  $2 < 3$ ).

If  $y = 3$ , then the statement  $\exists x (y < x \vee P(x, y))$  is true for  $x = 1$  (because  $P(1, 3)$  is true).

(c) True. Take  $x = y = 2$ , for example.



(d) True. We need to consider the cases  $y = 1$ ,  $y = 2$ , and  $y = 3$ . This means that we must examine the three statements

$\exists x (P(x, 1) \rightarrow \neg P(1, x))$  (true for  $x = 2$  because  $P(2, 1) \rightarrow \neg P(1, 2)$  is true)

$\exists x (P(x, 2) \rightarrow \neg P(2, x))$  (true for  $x = 1$  because  $P(1, 2) \rightarrow \neg P(2, 1)$  is true)

$\exists x (P(x, 3) \rightarrow \neg P(3, x))$  (true for  $x = 2$  because  $P(2, 3) \rightarrow \neg P(3, 2)$  is true).

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**p.54, icon at Example 3**

**#4.** Suppose  $P(x, y, z)$  is a predicate where the universe for  $x$ ,  $y$ , and  $z$  is  $\{1, 2\}$ . Also suppose that the predicate is true in the following cases  $P(1, 1, 1)$ ,  $P(1, 2, 1)$ ,  $P(1, 2, 2)$ ,  $P(2, 1, 1)$ ,  $P(2, 2, 2)$ , and false otherwise. Determine the truth value of each of the following quantified statements:

(a)  $\forall x \exists y \exists z P(x, y, z)$ .

(b)  $\forall x \forall y \exists z P(x, y, z)$ .

(c)  $\forall y \forall z \exists x P(x, y, z)$ .

(d)  $\forall x \exists y \forall z P(x, y, z)$ .

**Solution:**

(a) True. For every value of  $x$  ( $x = 1$  and  $x = 2$ ) there are  $y$  and  $z$  such that  $P(x, y, z)$  is true. In both cases we can choose both  $y = z = 2$ .

(b) True. For each choice of values for  $x$  and  $y$ , we can find  $z$  such that  $P(x, y, z)$  is true. We need to consider four cases.

(1)  $x = y = 1$ : we take  $z = 1$ ,

(2)  $x = 1$  and  $y = 2$ : we can take  $z$  to be 1 or 2,

(3)  $x = 2$ ,  $y = 1$ : we take  $z = 1$ ,

(4)  $x = y = 2$ : we take  $z = 2$ .

(c) False. If we take  $y = 1$  and  $z = 2$ , there is no value of  $x$  such that  $P(x, 1, 2)$  is true.

(d) False. Take  $x = 2$ . There is no value of  $y$  such that  $\forall z P(2, y, z)$  is true.

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**p.54, icon at Example 3**

**#5.** Suppose  $P(x, y, z)$  is a predicate where the universe for  $x$ ,  $y$ , and  $z$  is  $\{1, 2\}$ . Also suppose that the predicate is true in the following cases  $P(1, 1, 1)$ ,  $P(1, 2, 1)$ ,  $P(1, 2, 2)$ ,  $P(2, 1, 1)$ ,  $P(2, 2, 2)$ , and false otherwise. Determine the truth value of each of the following quantified statements:

(a)  $\exists x \forall y \forall z P(x, y, z)$ .

(b)  $\forall x \exists z \forall y P(x, y, z)$ .

(c)  $\forall y \exists x \exists z \neg P(x, y, z)$ .

(d)  $\exists x \forall z \neg \forall y P(x, y, z)$ .

**Solution:**

(a) False. If we take  $x = 1$ , we do not have  $P(1, y, z)$  true for all possible values of  $y$  and  $z$  —  $P(1, 1, 2)$  is false. If we take  $x = 2$ , we do not have  $P(2, y, z)$  true for all possible values of  $y$  and  $z$  —  $P(2, 1, 2)$  and  $P(2, 2, 1)$  are both false.

(b) False. Take  $x = 2$ . Then  $\exists z \forall y P(2, y, z)$  is false. To see this, suppose we try  $z = 1$ ; then  $P(2, y, 1)$  is false for  $y = 1$ . If we try  $z = 2$ ,  $P(2, y, 2)$  is false for  $y = 1$ .

(c) True. We must consider the cases where  $y = 1$  and  $y = 2$ . If we take  $y = 1$ . Then  $\exists x \exists z \neg P(x, 1, z)$  is true if  $x = z = 2$ , that is,  $\neg P(2, 1, 2)$  is true. If we take  $y = 2$ . Then  $\exists x \exists z \neg P(x, 2, z)$  is true if  $x = 2$  and

$z = 1$ , that is,  $\neg P(2, 2, 1)$  is true.

(d) True. The given statement is equivalent to  $\neg \forall x \exists z \forall y P(x, y, z)$ , which is the negation of the statement in part (b). Because the statement in part (b) is false, this statement must be true.

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**p.54, icon at Example 3**

**#6.** Suppose that the universe for  $x$  and  $y$  is  $\{1, 2, 3, 4\}$ . Assume that  $P(x, y)$  is a predicate that is true in the following cases and false otherwise:  $P(1, 4), P(2, 1), P(2, 2), P(3, 4), P(4, 1), P(4, 4)$ . Determine whether each of the following is true or false:

- (a)  $\forall x \exists y P(x, y)$ .
- (b)  $\forall y \exists x P(x, y)$ .
- (c)  $\exists x \forall y P(x, y)$ .

**Solution:**

(a) True. For every value of  $x$  taken from the universe, there is a value  $y$  such that  $P(x, y)$  is true:  $P(1, 4), P(2, 1), P(3, 4),$  and  $P(4, 1)$  are all true.

(b) False. If  $y = 3$ , there is no value of  $x$  such that  $P(x, 3)$  is true.

(c) False. If we take  $x = 1$ , not all four statements  $P(1, y)$  are true. (Take  $y = 1$  for example.) If we take  $x = 2$ , not all four statements  $P(2, y)$  are true. (Take  $y = 3$  for example.) If we take  $x = 3$ , not all four statements  $P(3, y)$  are true. (Take  $y = 1$  for example.) If we take  $x = 4$ , not all four statements  $P(4, y)$  is true. (Take  $y = 2$  for example.)

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**p.56, icon at Example 6**

**#1.** Write this fact about numbers using predicates and quantifiers: “Given a number, there is a number greater than it.”

**Solution:**

The statement says that “For every number  $x$  we choose, there is a number  $y$  such that  $y > x$ .” That is,

$$\forall x \exists y (y > x)$$

where the universe for  $x$  and  $y$  consists of all numbers.

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**p.56, icon at Example 6**

**#2.** Express the following statement using predicates and quantifiers: “The product of two positive numbers is positive.”

**Solution:**

Using the universe consisting of all real numbers for  $x$  and  $y$ , we are saying that “If  $x$  and  $y$  are greater than zero, then  $xy$  is greater than zero. That is,

$$\forall x \forall y [(x > 0 \wedge y > 0) \rightarrow (xy > 0)].$$

If we use all *positive* real numbers as the universe for  $x$  and  $y$ , we can write the statement more simply:

$$\forall x \forall y (xy > 0).$$

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**p.56, icon at Example 6**

**#3.** Write these statements in symbols using the predicates:

$$S(x): x \text{ is a perfect square}; \quad N(x): x \text{ is negative.}$$

Assume that the variable  $x$  is an integer.

- (a) No perfect squares are negative.
- (b) No negative numbers are perfect squares.

**Solution:**

(a) We are saying that it is not possible to have a perfect square that is negative. That is,  $\neg \exists x (S(x) \wedge N(x))$ . Equivalently, we could say that if  $x$  is a perfect square, then  $x$  is not negative. That is,

$$\forall x (S(x) \rightarrow \neg N(x)).$$

We could rewrite this as its contrapositive: If  $x$  is negative, then  $x$  is not a perfect square. That is,

$$\forall x (N(x) \rightarrow \neg S(x)).$$

(b) This statement is equivalent to (a). This statement says that it is not possible to have a negative number that is a perfect square. That is,

$$\neg \exists x (N(x) \wedge S(x)).$$

You should use the various laws of logic to show that  $\neg \exists x (N(x) \wedge S(x))$  is indeed equivalent to  $\forall x (S(x) \rightarrow \neg N(x))$

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**p.56, icon at Example 6**

**#4.** Write the following statement in symbols using the predicates

$$S(x): x \text{ is a perfect square} \quad P(x): x \text{ is positive}$$

where the universe for  $x$  is the set of all integers:

“Perfect squares are positive.”

**Solution:**

Note that “for all” is implied. When we say “Perfect squares are positive” we are really saying that “For all integers  $x$  we choose, if  $x$  is a perfect square, then  $x$  is positive.” In symbols we have

$$\forall x (S(x) \rightarrow P(x)).$$

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**p.56, icon at Example 6**

**#5.** Write the following statement in symbols using the predicate  $P(x)$  to mean “ $x$  is positive”, where the universe for  $x$  is the set of all integers.

“Exactly one number is positive.”

**Solution:**

We are making a two-part statement:

- (1) there is a number  $x$  that is positive, that is,  $\exists x P(x)$ ; and  
(2)  $x$  is the only number with this property; that is, if  $y$  is any number different from  $x$ , then  $y$  is not positive. This can be written as  $\forall y (y \neq x \rightarrow \neg P(y))$ .

Forming the conjunction of these two statements, we have

$$\exists x [P(x) \wedge \forall y (y \neq x \rightarrow \neg P(y))],$$

or

$$\exists x \forall y [P(x) \wedge (y \neq x \rightarrow \neg P(y))].$$

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**p.56, icon at Example 6**

**#6.** Write the following statements in symbols, using  $P(x)$  to mean “ $x$  is positive” and  $F(x)$  to mean “ $x$  ends in the digit 5”. Assume that the universe for  $x$  is the set of all integers.

- (a) Some positive integers end in the digit 5.  
(b) Some positive integers end in the digit 5, while others do not.

**Solution:**

(a) We are asserting that there is an integer  $x$  that has two properties: (1) it is positive, (2) it ends in the digit 5. That is,  $\exists x (P(x) \wedge F(x))$ .

(b) This statement begins with the statement for (a) and then asserts that there is a different positive integer that does not end in the digit 5. That is,

$$\exists x (P(x) \wedge F(x)) \wedge \exists y ((y \neq x) \wedge P(y) \wedge \neg F(y)).$$

Equivalently, we could write

$$\exists x \exists y [(x \neq y) \wedge P(x) \wedge P(y) \wedge F(x) \wedge \neg F(y)].$$

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**p.56, icon at Example 6**

**#7.** Write in symbols: There is no smallest positive number.

**Solution:**

Using all positive real numbers as the universe for  $x$  and  $y$ , we are saying that “For every number  $x$  we can choose, there is a number  $y$  that is smaller than  $x$ .” In symbols,

$$\forall x \exists y (y < x).$$

If we use all real numbers as the universe for  $x$  and  $y$ , we are saying that “For every positive real number  $x$  we can choose, there is a real number  $y$  that is positive and smaller than  $x$ .” In symbols,

$$\forall x (x > 0 \rightarrow \exists y (0 < y < x)).$$

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**p.56, icon at Example 6**

**#8.** Write in symbols: If  $a < b$ , then  $\frac{a+b}{2}$  lies between  $a$  and  $b$ .

**Solution:**

Note that it is understood that the predicate applies to all  $a$  and  $b$  chosen from some universe. Using all real numbers as the universe for  $a$  and  $b$ , we have

$$\forall a \forall b (a < b \rightarrow a < \frac{a+b}{2} < b).$$

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**p.56, icon at Example 6**

**#9.** Write in symbols: For all choices of  $a$  and  $b$ ,  $\frac{a+b}{2}$  lies between  $a$  and  $b$ .

**Solution:**

Note that we cannot write  $\forall a \forall b (a < \frac{a+b}{2} < b)$  because we do not know that  $a < b$ . (It may be the case that  $a = b$  or that  $a > b$ .) We can write

$$\forall a \forall b \left( \left( a \leq \frac{a+b}{2} \leq b \right) \vee \left( b \leq \frac{a+b}{2} \leq a \right) \right).$$

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**p.59, icon at Example 14**

**#1.** Write the negation of the statement  $\exists x \forall y (xy = 0)$  in symbols and in English. Determine the truth or falsity of the statement and its negation. Assume that the universe for  $x$  and  $y$  is the set of all real numbers.

**Solution:**

We take the negation and then move the negation sign inside:

$$\neg(\exists x \forall y (xy = 0)) \equiv \forall x (\neg \forall y (xy = 0)) \equiv \forall x \exists y \neg(xy = 0) \equiv \forall x \exists y (xy \neq 0).$$

The original statement says that “There is a number with the property that no matter what number we multiply it by, we obtain 0.” (The statement is true because the number 0 is such a number  $x$ .) The negation

states that “No matter what number is chosen, there is a number such that the product is nonzero.” (As expected, the negation is false because it is the negation of a true statement. To see that the negation is false, take  $x$  to be 0. Then no matter what value we take for  $y$ , the product  $xy = 0$ .)

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**p.59, icon at Example 14**

**#2.** Write the statement “There is a largest number” using predicates and quantifiers. Then give its negation in symbols.

**Solution:**

Taking the universe for  $x$  and  $y$  to consist of all real numbers, we are stating that there is a number  $x$  such that, no matter what number  $y$  is chosen, we have  $x \geq y$ . Therefore.

$$\exists x \forall y (x \geq y).$$

Its negation can be formed using these steps:

$$\neg(\exists x \forall y (x \geq y)) \equiv \forall x \exists y \neg(x \geq y) \equiv \forall x \exists y (x < y).$$

(This says that there is no largest number.)

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