



p.84, icon below start of "Understanding How Theorems Are Stated" subsection

#1. Sometimes quantifiers in statements are understood, but do not actually appear in the words of the statement. Explain what quantifiers are understood in the statement “The product of two negative numbers is positive.”

Solution:

What is really meant is “For all pairs of negative numbers, the product is positive.” In symbols,

$$\forall x \forall y [(x < 0) \wedge (y < 0) \rightarrow (xy > 0)].$$

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#2. Consider the theorem “If x ends in the digit 3, then x^3 ends in the digit 7.” What quantifier is understood, but not written?

Solution:

The universal quantifier “ $\forall x$ ” (where the universe for x consists of all integers) is understood. That is, we have

$$\forall x ((x \text{ ends in the digit } 3) \rightarrow (x^3 \text{ ends in the digit } 7)).$$

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#3. Consider the theorem “No squares of integers end in the digit 8.” What quantifier is understood, but not written?

Solution:

The universal quantifier “ $\forall x$ ” (where the universe for x consists of all integers) is understood, if we read the statement as “for every x we choose, x^2 does not end in the digit 8”. That is, we have

$$\forall x (x^2 \text{ does not end in the digit } 8).$$

However, we can read the given statement equivalently as “there does not exist an integer x such that x^2 ends in the digit 8”. If we do this, the existential quantifier is used:

$$\neg \exists x (x^2 \text{ ends in the digit } 8).$$

Both statements are equivalent.

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#4. Consider the theorem "The average of two numbers can be 0." What quantifier is understood, but not written?

Solution:

Two existential quantifiers " $\exists x$ " and " $\exists y$ " (where the universe for x and y consists of all real numbers) are understood. That is, we have

$$\exists x \exists y \left(\frac{x + y}{2} = 0 \right).$$

p.85, icon at Example 1

#1. Using the definitions of even integer and odd integer, give a direct proof that this statement is true for all integers n :

If n is odd, then $5n + 3$ is even.

Solution:

We give a proof using two columns, one for statements and one for reasons:

Suppose n is odd.	given
Therefore $n = 2k + 1$ for some integer k	definition of odd integer
Therefore $5n + 3 = 5(2k + 1) + 3$	substitution of $2k + 1$ for n
Therefore $5n + 3 = 10k + 8$	algebra
Therefore $5n + 3 = 2(5k + 4)$	algebra
Therefore $5n + 3$ is even	$5n + 3$ is an integer multiple of 2

p.86, icon at Example 3

#1. Using the definitions of even integer and odd integer, give a proof by contraposition that this statement is true for all integers n :

If $3n - 5$ is even, then n is odd.

Solution:

We begin by assuming that n is not odd and try to conclude that $3n - 5$ is not even. You should supply reasons for each step.

Suppose n is not odd.
Therefore n is even and hence $n = 2k$.
Therefore $3n - 5 = 3(2k) - 5 = 6k - 5 = 2(3k - 3) + 1$.
Therefore $3n - 5$ is odd, and hence is not even.

p.86, icon at Example 3

#2. Suppose we need to prove that this statement is true for all integers n , using the definitions of even integer and odd integer:

If $7n - 5$ is odd, then n is even.

Solution:

Suppose we try a direct proof:

Suppose $7n - 5$ is odd.

Therefore $7n - 5 = 2k + 1$ for some integer k .

Therefore $7n = 2k + 6 = 2(k + 3)$.

Therefore $n = \frac{2(k+3)}{7}$.

At this point it is not clear how to proceed in order to conclude that n is even. (We could fashion such a proof, but it would require some additional knowledge about primes and divisibility.) It would be wise to look at a proof by contraposition.

Suppose n is not even. Therefore n is odd.

Therefore $n = 2k + 1$ for some integer k .

Therefore $7n - 5 = 7(2k + 1) - 5 = 14k + 2 = 2(7k + 1)$.

Therefore $7n - 5$ is even.

As a general rule, it is usually better to try to proceed from simple to complicated. For example, in the proof of “If $7n - 5$ is odd, then n is even” a proof by contraposition (beginning with “Suppose n is not even”) is easier than a direct proof (beginning with “Suppose $7n - 5$ is odd”).

p.88, icon at Example 7

#1. Suppose a , b , and c are odd integers. Prove that the roots of $ax^2 + bx + c = 0$ are not rational.

Solution:

In general, it is not very easy to give a direct proof that numbers are not rational. Here we would need to show that neither root can be written as a quotient of integers. It might be easier to try a proof by contradiction because we can make the assumption that the roots are rational and hence can be written as fractions. That is, there is a very specific form in which the roots can be written.

Suppose the roots of $ax^2 + bx + c = 0$ are rational — say the roots are e/f and g/h . Then we can write $(x - \frac{e}{f})(x - \frac{g}{h}) = 0$. Equivalently, this is $(\frac{fx - e}{f}) \cdot (\frac{hx - g}{h}) = 0$, or $(fx - e)(hx - g) = 0$.

Multiplying together the two terms yields:

$$(fh)x^2 - (fg + eh)x + (eg) = 0.$$

Because this must have the form $ax^2 + bx + c = 0$, the corresponding coefficients must be equal:

$$fh = a, \quad -(fg + eh) = b, \quad eg = c.$$

Therefore f and h must be odd because their product is the odd integer a . Likewise, e and g must be odd because their product is the odd integer c . But this forces $fg + eh$ to be even because it is the sum of the odd integers fg and eh . Therefore $b = -(fg + eh)$ must be even, a contradiction of the fact that b is odd.

p.89, icon at Example 9

#1. Give a proof by contradiction of: “If n is an even integer, then $3n + 7$ is odd.”

Solution:

To give a proof by contradiction, we assume that the hypothesis “ n is an even integer” is true, but the conclusion “ $3n + 7$ is odd” is false, and show that this results in a contradiction (a proposition that is never true).

Suppose n is even but $3n + 7$ is not odd.

Therefore $n = 2k$ and $3n + 7 = 2l$ for some integers k and l .

Therefore $3(2k) + 7 = 2l$ by substituting $2k$ for n .

Therefore $6k + 7 = 2l$.

Therefore $2l - 6k = 7$.

Therefore $2(l - 3k) = 7$.

But in this equation the left side is even and the right side is odd, a contradiction.

Therefore, if n is an even integer, we cannot have $3n + 7$ even.

That is: if n is an even integer, then $3n + 7$ is odd.

p.90, icon at Example 12

#1. Prove that this statement is true for all integers n : n is odd if and only if $5n + 3$ is even.

Solution:

We must prove that two statements are true: n is odd if $5n + 3$ is even, and n is odd only if $5n + 3$ is even. That is,

(a) If $5n + 3$ is even, then n is odd, and

(b) If n is odd, then $5n + 3$ is even.

It is easy to give a proof by contraposition of (a):

Suppose n is not odd, and therefore is even.

Therefore $n = 2k$ for some integer k .

Therefore $5n + 3 = 5(2k) + 3 = 10k + 3 = 2(5k + 1) + 1$.

Therefore $5n + 3$ is odd.

It is also easy to give a direct proof of (b):

Suppose n is odd.

Therefore $n = 2k + 1$ for some integer k .

Therefore $5n + 3 = 5(2k + 1) + 3 = 10k + 8 = 2(5k + 4)$.

Therefore $5n + 3$ is even.

p.91, icon at Example 14

#1. Show that the statement “Every integer is less than its cube” is false by finding a counterexample.

Solution:

Note that $-2 > (-2)^3 = -8$, so that -2 is not less than its cube, -8 . Because we have found a counterexample, we know that the statement “Every integer is less than its cube” is false. (Note that every negative integer is a counterexample to this statement.)
