

Rosen, Discrete Mathematics and Its Applications, 7th edition, Global Edition
Extra Examples
Section 2.2—Set Operations



— Page references correspond to locations of Extra Examples icons in the textbook.

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#1. Prove that the following is true for all sets A , B , and C : if $A \cap C \subseteq B \cap C$ and $A \cap \overline{C} \subseteq B \cap \overline{C}$, then $A \subseteq B$.

Solution:

Let $x \in A$. We need to show that $x \in B$. We will construct a proof by cases, depending on whether $x \in C$ or $x \notin C$.

Case 1: $x \in C$. If $x \in C$, then by the original hypothesis ($x \in A$) we know that $x \in A \cap C$. But it is given that $A \cap C \subseteq B \cap C$. Therefore $x \in B \cap C$, and hence $x \in B$.

Case 2: $x \notin C$. Then $x \in \overline{C}$. Then by the original hypothesis ($x \in A$) we know that $x \in A \cap \overline{C}$. But it is given that $A \cap \overline{C} \subseteq B \cap \overline{C}$. Therefore $x \in B \cap \overline{C}$, and hence $x \in B$.

Therefore, in either case, if $x \in A$, then $x \in B$. Therefore $A \subseteq B$.

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#2. Prove that the following is true for all sets A , B , and C : if $A \cap C = B \cap C$ and $A \cup C = B \cup C$, then $A = B$.

Solution:

We will show $A \subseteq B$ and $B \subseteq A$.

Proof that $A \subseteq B$: Let $x \in A$. We need to show that $x \in B$. We will give a proof by cases, depending on whether or not $x \in C$.

Case 1: $x \in C$. In this case $x \in A \cap C$. Because $A \cap C = B \cap C$, we have $x \in B \cap C$, and hence $x \in B$.

Case 2: $x \notin C$. In this case $x \in A \cup C$ (because $x \in A$). Because $A \cup C = B \cup C$, we have $x \in B \cup C$. But $x \notin C$. Therefore we must have $x \in B$.

Cases 1 and 2 show that if $x \in A$, then $x \in B$, or $A \subseteq B$.

A similar proof can be given to show that $B \subseteq A$.

Because $A \subseteq B$ and $B \subseteq A$, $A = B$.

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#3. Use logical equivalence to show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Solution:

We begin with $A \cap (B \cup C)$ and show that this is the same as $(A \cap B) \cup (A \cap C)$.

$$\begin{aligned}
A \cap (B \cup C) &= \{x \mid x \in A \wedge x \in B \cup C\} && \text{definition of intersection} \\
&= \{x \mid x \in A \wedge (x \in B \vee x \in C)\} && \text{definition of union} \\
&= \{x \mid (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)\} && \text{distributive law} \\
&= \{x \mid (x \in A \cap B) \vee (x \in A \cap C)\} && \text{definition of intersection} \\
&= (A \cap B) \cup (A \cap C) && \text{definition of union}
\end{aligned}$$

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#4. Prove: if $A \subseteq \overline{B}$, then $B \subseteq \overline{A}$.

Solution:

We will use a direct proof. Suppose $A \subseteq \overline{B}$. We must show that $B \subseteq \overline{A}$. To show that $B \subseteq \overline{A}$, assume that $x \in B$ and show that $x \in \overline{A}$.

Suppose $x \in B$.

Therefore $x \notin \overline{B}$.

Therefore $x \notin \overline{A}$ (because $A \subseteq \overline{B}$).

Therefore $x \in \overline{A}$.

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#5. Prove: If $A \subseteq B$ and $C \subseteq D$, then $A \cap C \subseteq B \cap D$.

Solution:

We assume that $A \subseteq B$ and $C \subseteq D$, and we must show that $A \cap C \subseteq B \cap D$. In terms of predicates and quantifiers, the statement we need to prove has the form

$$\forall x (x \in A \cap C \rightarrow x \in B \cap D)$$

where the universe for x is the universal set U (any set containing A , B , C , and D). To show that this statement is true, suppose that $x \in A \cap C$. (Note that the only thing we know about x is that it is an arbitrary element of $A \cap C$.) Therefore, $x \in A$ and $x \in C$, by definition of intersection of sets. Therefore, $x \in B$ and $x \in D$ (because $A \subseteq B$ and $C \subseteq D$). This says that $x \in B \cap D$. Hence, if x is any element of $A \cap C$, then x is also an element of $B \cap D$.

Therefore, $A \cap C \subseteq B \cap D$.

We can write the proof more briefly as:

$$\begin{aligned}
&\text{Let } x \in A \cap C. \\
&\therefore x \in A \text{ and } x \in C. \\
&\therefore x \in B \text{ and } x \in D. \\
&\therefore x \in B \cap D.
\end{aligned}$$
