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#1. We know that the set of rational numbers is countable. Are the irrational numbers (the real numbers that cannot be written as fractions a/b where a and b are integers and $b \neq 0$) also countable, or are they uncountable?

Solution:

We will give a proof by contradiction that the irrational numbers are uncountable.

Suppose the irrational numbers were countable; then they can be listed as b_1, b_2, b_3, \dots . But we know that the rational numbers are also countable, and hence can be listed as a_1, a_2, a_3, \dots . “Interlace” the two lists as $a_1, b_1, a_2, b_2, a_3, \dots$ to obtain a countable set. But this is equal to the set of real numbers, because every real number is either rational or irrational. This says that the set of real numbers is a countable set, which contradicts the fact that the real numbers form an uncountable set. Therefore the irrational numbers are uncountable.

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#2. Show that the set $\{x \mid 0 < x < 1\}$ is uncountable by showing that there is a one-to-one correspondence between this set and the set of all real numbers.

Solution:

We first show that there is a one-to-one correspondence between the interval $\{x \mid -\pi/2 < x < \pi/2\}$ and \mathbf{R} . We can use the function $f(x) = \arctan x$ (the inverse tangent function), which is a one-to-one function from $\{x \mid -\pi/2 < x < \pi/2\}$ onto \mathbf{R} .

We can then use the function $g: (0, 1) \rightarrow (-\pi/2, \pi/2)$ defined by $g(x) = \frac{\pi}{2}(2x - 1)$ (which is a one-to-one correspondence) and form the composition $f \circ g: (0, 1) \rightarrow \mathbf{R}$.

This gives the desired one-to-one correspondence from the interval $(0, 1)$ to \mathbf{R} . Because we know that \mathbf{R} is uncountable, so is $(0, 1)$.
