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#1. Let $\mathbf{A} = \begin{pmatrix} 2 & 7 \\ -1 & 5 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & -3 \\ 4 & 2 \end{pmatrix}$. Find the products \mathbf{AB} and \mathbf{BA} .

Solution:

The order in which the two matrices appear matters.

$$\mathbf{AB} = \begin{pmatrix} 2 & 7 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 2 \cdot 1 + 7 \cdot 4 & 2 \cdot (-3) + 7 \cdot 2 \\ (-1) \cdot 1 + 5 \cdot 4 & (-1) \cdot (-3) + 5 \cdot 2 \end{pmatrix} = \begin{pmatrix} 30 & 8 \\ 19 & 13 \end{pmatrix}.$$

$$\mathbf{BA} = \begin{pmatrix} 1 & -3 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 2 & 7 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} 1 \cdot 2 + (-3) \cdot (-1) & 1 \cdot 7 + (-3) \cdot 5 \\ 4 \cdot 2 + 2 \cdot (-1) & 4 \cdot 7 + 2 \cdot 5 \end{pmatrix} = \begin{pmatrix} 5 & -8 \\ 6 & 38 \end{pmatrix}.$$

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#2. Determine whether the following is true for all 2×2 matrices: $(\mathbf{A} + \mathbf{B})^2 = \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2$.

Solution:

Using the distributive law to multiply the left side yields $(\mathbf{A} + \mathbf{B})^2 = (\mathbf{A} + \mathbf{B})(\mathbf{A} + \mathbf{B}) = \mathbf{A}^2 + \mathbf{AB} + \mathbf{BA} + \mathbf{B}^2$. Therefore, the original equation is true if and only if $\mathbf{BA} = \mathbf{AB}$.

But $\mathbf{AB} \neq \mathbf{BA}$. For example, we can take $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then $\mathbf{AB} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ but $\mathbf{BA} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

Therefore, the original equation is not always true.

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#3. Suppose a matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with real numbers as entries commutes under multiplication with all real 2×2 matrices. That is, $\mathbf{AB} = \mathbf{BA}$ all 2×2 matrices \mathbf{B} with real numbers as entries. What form must \mathbf{A} have?

Solution:

Although we could begin by multiplying \mathbf{A} by a general matrix $\mathbf{B} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$ and examining the corresponding resulting sums of products when we form \mathbf{AB} and \mathbf{BA} and assume $\mathbf{AB} = \mathbf{BA}$, we should consider exploiting the fact that in this case $\mathbf{AB} = \mathbf{BA}$ holds for all matrices \mathbf{B} . In particular, $\mathbf{AB} = \mathbf{BA}$ must hold for very simple matrices \mathbf{B} . If we take the matrix \mathbf{B} to be very simple, the products \mathbf{AB} and \mathbf{BA} may be

simple and allow to easily draw conclusions about the entries of \mathbf{A} .

Suppose we take $\mathbf{B} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then

$$\mathbf{AB} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & 0 \\ d & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{BA} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix}.$$

But we are told that $\mathbf{AB} = \mathbf{BA}$. Therefore

$$\begin{pmatrix} b & 0 \\ d & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix}$$

Because corresponding entries must match, we must have $b = 0$ and $d = a$. This says that the matrix \mathbf{A} must have the form $\begin{pmatrix} a & 0 \\ c & a \end{pmatrix}$. We can obtain a condition of c by taking $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Again forming the products \mathbf{AB} and \mathbf{BA} we have

$$\mathbf{AB} = \begin{pmatrix} a & 0 \\ c & a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{BA} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ c & a \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

This forces $c = 0$. Therefore \mathbf{A} must have the form $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$.

We can check that $\mathbf{AB} = \mathbf{BA}$ for all 2×2 matrices \mathbf{B} :

$$\mathbf{AB} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} aw & ax \\ ay & az \end{pmatrix} \quad \text{and} \quad \mathbf{BA} = \begin{pmatrix} w & x \\ y & z \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} aw & ax \\ ay & az \end{pmatrix}.$$

Therefore if a matrix \mathbf{A} commutes with all 2×2 matrices, then \mathbf{A} must have the form $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$.
