



— Page references correspond to locations of Extra Examples icons in the textbook.

p.257, icon at Example 2

#1. Find the prime factorization of:

- (a) 487.
- (b) 6600.

Solution:

(a) If we try to divide 487 by all primes from 2 to $\lfloor \sqrt{487} \rfloor = 22$ (that is, 2, 3, 5, 7, 11, 13, 17, 19), we find that none of these divides 487 without a remainder. Therefore 487 is prime.

(b) Begin by writing 6600 as any product of smaller positive factors, such as $6600 = 66 \cdot 100$. We continue this process until only primes are obtained:

$$\begin{aligned} 6600 &= 66 \cdot 100 \\ &= (6 \cdot 11)(10 \cdot 10) \\ &= (2 \cdot 3 \cdot 11) \cdot (2 \cdot 5 \cdot 2 \cdot 5) \\ &= 2^3 \cdot 3 \cdot 5^2 \cdot 11. \end{aligned}$$

If we initially factor 6600 in a different way, such as $6 \cdot 1100$, we would still arrive at the same product of prime factors.

p.262, icon at Example 6

#1. Suppose the odd primes 3, 5, 7, 11, 13, 17, ... in order of increasing size are p_1, p_2, p_3, \dots . Prove or disprove:

$$p_1 p_2 p_3 \dots p_k + 2 \text{ is prime, for all } k \geq 1.$$

Solution:

We begin by trying a few cases. Hopefully we will either get an idea of how to prove the result, or we will find a counterexample.

$$\begin{aligned} 3 + 2 &= 5 \\ 3 \cdot 5 + 2 &= 17 \\ 3 \cdot 5 \cdot 7 + 2 &= 107 \\ 3 \cdot 5 \cdot 7 \cdot 11 + 2 &= 1,157 \\ 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 + 2 &= 15,017 \\ 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 + 2 &= 255,257. \end{aligned}$$

We stop at this step because the number 255,257 is not prime; it can be factored as $47 \cdot 5,431$. Therefore we have a counterexample to the statement that $p_1 p_2 p_3 \dots p_k + 2$ is always prime.

p.262, icon at Example 6

#2. Suppose the odd primes $3, 5, 7, 11, 13, 17, \dots$ in order of increasing size are p_1, p_2, p_3, \dots . Prove or disprove:

$$p_i p_{i+1} + 2 \text{ is prime, for all } i \geq 1.$$

Solution:

We begin by trying a few cases. Hopefully we will either get an idea of how to prove the result, or we will find a counterexample.

$$\begin{aligned} 3 \cdot 5 + 2 &= 17 \\ 5 \cdot 7 + 2 &= 37 \\ 7 \cdot 11 + 2 &= 79 \\ 11 \cdot 13 + 2 &= 145. \end{aligned}$$

We stop here because 145 is not prime. The number 145 is a counterexample. Therefore the original statement is false.

p.262, icon at Example 6

#3. (Problem A1 from the 1989 William Lowell Putnam Mathematics Competition.) Consider the sequence of integers (in base 10): $101, 10101, 1010101, 101010101, 10101010101, \dots$. Prove that 101 is the only number in this sequence that is prime. (*Hint:* Use place value to write each number in terms of the sum of its digits; for example, $abcde = a10^4 + b10^3 + c10^2 + d10 + e$. Then examine how the sum might be factored.)

Solution:

It is easily checked that 101 is prime. Given any number of the form $10101\dots 01$ greater than 101, write the number in terms of its digits. Then there is an integer $n \geq 2$ such that

$$\begin{aligned} 10101\dots 01 &= 10^{2n} + 10^{2n-2} + \dots + 10^4 + 10^2 + 1 && \text{(this is a geometric series)} \\ &= \frac{10^{2n+2} - 1}{99} && \text{(the geometric series has this sum)} \\ &= \frac{(10^{n+1})^2 - 1}{99} && \text{(by the law of exponents } a^{bc} = (a^b)^c) \\ &= \frac{(10^{n+1}-1)(10^{n+1} + 1)}{99} && (10^{n+1}-1 \text{ is an integer of the form } 999\dots 9, \text{ which is} \\ & && \text{divisible by 9)} \\ &= \frac{a_n(10^{n+1} + 1)}{11}, \end{aligned}$$

where a_n is the integer that is a string of $n + 1$ 1's. The reader can verify that if n is odd, then $11|a_n$, and if n is even, then $11|(10^{n+1} + 1)$. In either case, $10101\dots 01$ is a product of two integers, each greater than 1. Therefore $10101\dots 01$ is not prime if $n > 1$.
