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#1. How many positive integers less than or equal to 100 are divisible by 6 or 9?

Solution:

Let A be the set of integers from 1 to 100 divisible by 6 and let B be the set of integers from 1 to 100 divisible by 9. By the inclusion-exclusion principle, the number of positive integers from 1 to 100 divisible by 6 or 9 is

$$\begin{aligned} |A \cup B| &= |A| + |B| - |A \cap B| \\ &= \left\lfloor \frac{100}{6} \right\rfloor + \left\lfloor \frac{100}{9} \right\rfloor - \left\lfloor \frac{100}{18} \right\rfloor \\ &= 16 + 11 - 5 \\ &= 22. \end{aligned}$$

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#2. How many positive integers less than or equal to 100 are relatively prime to 15?

Solution:

An integer is relatively prime to 15 if and only if it is not divisible by 3 and not divisible by 5. Let A be the set of integers from 1 to 100 divisible by 3 and let B be the set of integers from 1 to 100 divisible by 5. Then the number of integers from 1 to 100 that are *not* relatively prime to 15 is

$$\begin{aligned} |A \cup B| &= |A| + |B| - |A \cap B| \\ &= 33 + 20 - 6 \\ &= 47. \end{aligned}$$

Hence, the number of integers from 1 to 100 that are relatively prime to 15 is

$$|\overline{A \cup B}| = 100 - 47 = 53.$$

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#3. Find the number of elements in $A_1 \cup A_2 \cup A_3 \cup A_4$ if each set has size 50, each intersection of two sets has size 30, each intersection of three sets has size 10, and the intersection of all four sets has size 2.

Solution:

Using the inclusion-exclusion principle,

$$\begin{aligned}
|A_1 \cup A_2 \cup A_3 \cup A_4| &= \sum_{i=1}^4 |A_i| - \sum_{i \neq j} |A_i \cap A_j| + \sum_{i < j < k} |A_i \cap A_j \cap A_k| - |A_1 \cap A_2 \cap A_3 \cap A_4| \\
&= 4 \cdot 50 - \binom{4}{2} \cdot 30 + \binom{4}{3} \cdot 10 - 2 \\
&= 200 - 180 + 40 - 2 \\
&= 58.
\end{aligned}$$

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#4.

- (a) Find the number of permutations of $1, 2, \dots, 8$ that begin with 52 or end with 387.
(b) Find the number of permutations of $1, 2, \dots, 8$ that begin with 52 or end with 327.

Solution:

(a) Let A be the set of permutations of $1, 2, \dots, 8$ that begin with 52 and let B be the set of permutations of $1, 2, \dots, 8$ that end with 387. Using the inclusion-exclusion principle,

$$|A \cup B| = 6! + 5! - 3! = 720 + 120 - 6 = 834.$$

(b) Let A be the set of permutations of $1, 2, \dots, 8$ that begin with 52 and let B be the set of permutations of $1, 2, \dots, 8$ that end with 327. In this case $A \cap B = \emptyset$ because the digit 2 cannot occur in both the string 52 and 327. Therefore,

$$|A \cup B| = 6! + 5! = 720 + 120 = 840.$$

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#5. Find the number of permutations of all 26 letters of the alphabet that contain at least one of the words FIGHT, BALKS, MOWER.

Solution:

Let F , B , and M be the sets of permutations of all 26 letters in the alphabet that contain FIGHT, BALKS, and MOWER, respectively. Then the set of all permutations of the 26 letters of the alphabet that contain at least one of these words is $|F \cup B \cup M|$. Using the inclusion-exclusion principle,

$$\begin{aligned}
|F \cup B \cup M| &= |F| + |B| + |M| - |F \cap B| - |F \cap M| - |B \cap M| + |F \cap B \cap M| \\
&= 22! + 22! + 22! - 18! - 18! - 18! + 14! \\
&= 3 \cdot 22! - 3 \cdot 18! + 14!.
\end{aligned}$$

(We obtain $|F| = 22!$ by treating the five letters F, I, G, H, T as being glued together on one card, leaving 21 other cards with single letters. Similarly we obtain the values for the other sets.)

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#6. Find the number of permutations of all 26 letters of the alphabet that contain at least one of the words CAR, CARE, SCARE, SCARED.

Solution:

Let A , B , C , and D be the sets of permutations of the 26 letters of the alphabet that contain the words CAR, CARE, SCARE, and SCARED, respectively. Then $D \subseteq C \subseteq B \subseteq A$. Hence,

$$|A \cup B \cup C \cup D| = |A| = 24!.$$

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#7. Suppose $|U| = n$ and A and B are subsets of U such that $|A| > n/2$, and $|B| > n/2$. Prove that $A \cap B \neq \emptyset$.

Solution:

(a) Suppose $A \cap B = \emptyset$. By the inclusion-exclusion principle

$$|A \cup B| = |A| + |B| - |A \cap B| = |A| + |B| > n/2 + n/2 = n.$$

Therefore $|A \cup B| > n$. But this is not possible because $A \cup B \subseteq U$ and every subset of U has size at most n . Therefore $A \cap B \neq \emptyset$.

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#8. Let n be an odd positive integer and let a_1, a_2, \dots, a_n represent an arbitrary arrangement of the integers $1, 2, 3, \dots, n$. Prove that the product $(a_1 - 1)(a_2 - 2) \dots (a_n - n)$ is an even integer.

Solution:

We can use the inclusion-exclusion principle in (a) to give a proof.

We need to show that the product $(a_1 - 1)(a_2 - 2) \dots (a_n - n)$ is even. As a strategy to consider here, note that this will happen if at least one of the factors is even. Thus, reasoning backwards, it is enough to show that at least one of the factors is even. This will happen if the product has a factor of the form “even-even” or “odd-odd”. The proof of this depends on the simple observation that if n is odd, the set $\{1, 2, 3, \dots, n\}$ has one more odd integer than even integer.

Each factor in the product has the form “ $a_i - i$ ”. We do not know whether a_i and i are both even, both odd, or have opposite parity. Let

$$A = \{a_i - i \mid a_i \text{ odd}\} \quad \text{and} \quad B = \{a_i - i \mid i \text{ odd}\}.$$

Because n is an odd integer, the number of odd integers in $\{1, 2, 3, \dots, n\}$ and in $\{a_1, a_2, \dots, a_n\}$ is one more than the number of even integers in $\{1, 2, 3, \dots, n\}$. Therefore $|A| > n/2$ and $|B| > n/2$. By part (a) we know that $A \cap B \neq \emptyset$. This means that at least one of the factors $a_i - i$ must be in $A \cap B$, and hence must have both a_i and i odd. Therefore $a_i - i$ is even, which forces the product $(a_1 - 1)(a_2 - 2) \dots (a_n - n)$ to be even.
