

Rosen, Discrete Mathematics and Its Applications, 7th edition, Global Edition  
Extra Examples  
Section 9.5—Equivalence Relations



— Page references correspond to locations of Extra Examples icons in the textbook.

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**p.588, icon at Example 2**

**#1.** (a) Verify that the following is an equivalence relation on the set of real numbers:

$$aRb \leftrightarrow \lfloor a \rfloor = \lfloor b \rfloor, \text{ where } \lfloor x \rfloor \text{ is the floor of } x.$$

(b) Describe the equivalence classes arising from the equivalence relation in part (a).

**Solution:**

(a)  $R$  is reflexive:  $\lfloor a \rfloor = \lfloor a \rfloor$  is true for all real numbers.

$R$  is symmetric: suppose  $\lfloor a \rfloor = \lfloor b \rfloor$ ; then  $\lfloor b \rfloor = \lfloor a \rfloor$ .

$R$  is transitive: suppose  $\lfloor a \rfloor = \lfloor b \rfloor$  and  $\lfloor b \rfloor = \lfloor c \rfloor$ ; from transitivity of equality of real numbers, it follows that  $\lfloor a \rfloor = \lfloor c \rfloor$ .

(b) Two real numbers,  $a$  and  $b$ , are related if they have the same floor. This happens if and only if  $a$  and  $b$  lie in the same interval  $[n, n + 1)$  where  $n$  is an integer. That is, the equivalence classes are the intervals  $\dots, [-2, -1), [-1, 0), [0, 1), [1, 2), [2, 3), \dots$

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**p.588, icon at Example 2**

**#2.** Let  $A$  be the set of all points in the plane with the origin removed. That is,

$$A = \{(x, y) \mid x, y \in \mathbf{R}\} - \{(0, 0)\}.$$

Define a relation on  $A$  by the rule:

$$(a, b)R(c, d) \leftrightarrow (a, b) \text{ and } (c, d) \text{ lie on the same line through the origin.}$$

(a) Prove that  $R$  is an equivalence relation.

(b) Describe the equivalence classes arising from the equivalence relation  $R$  in part (a).

(c) If  $A$  is replaced by the entire plane, is  $R$  an equivalence relation?

**Solution:**

(a)  $R$  is reflexive:  $(a, b)$  and  $(a, b)$  lie on the same line through the origin, namely on the line  $y = bx/a$  (if  $a \neq 0$ ), or else on the line  $x = 0$  (if  $a = 0$ ).

$R$  is symmetric: if  $(a, b)$  and  $(c, d)$  lie on the same line through the origin, then  $(c, d)$  and  $(a, b)$  lie on the same line through the origin.

$R$  is transitive: suppose  $(a, b)$  and  $(c, d)$  lie on the same line  $L$  through the origin and  $(c, d)$  and  $(e, f)$  lie on the same line  $M$  through the origin. Then  $L$  and  $M$  both contain the two distinct points  $(0, 0)$  and  $(c, d)$ . Therefore  $L$  and  $M$  are the same line, and this line contains  $(a, b)$  and  $(e, f)$ . Therefore  $(a, b)$  and  $(e, f)$  lie on the same line through the origin.

*Note:* The proof that  $R$  is an equivalence relation can be carried out using analytic geometry: if  $(a, b)$  and  $(c, d)$  lie on the same nonvertical line through the origin, then the slope must equal  $b/a$  because the line passes through  $(0, 0)$  and  $(a, b)$  and the slope must also equal  $d/c$  because the line passes through  $(0, 0)$  and

$(c, d)$ ; thus,  $b/a = d/c$ , or  $ad = bc$ . If  $(a, b)$  and  $(c, d)$  lie on the same vertical line through the origin, then the points must have the form  $(0, b)$  and  $(0, d)$ , and again it must happen that  $ad = bc$ . Therefore,  $(a, b)R(c, d)$  means that  $ad = bc$ . This equation can be used to verify that  $R$  is reflexive, symmetric, and transitive.

(b) Each equivalence class is the set of points of  $A$  on a line of the form  $y = mx$  or the vertical line  $x = 0$ .

(c) If  $A$  is replaced by the entire plane,  $R$  is not an equivalence relation. It fails to satisfy the transitive property; for example,  $(1, 2)R(0, 0)$  and  $(0, 0)R(2, 2)$ , but  $(1, 2) \not R(2, 2)$  because the line passing through  $(1, 2)$  and  $(2, 2)$  does not pass through the origin.

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