This is an unedited, uncorrected chapter.

The final chapter will be available in time for fall.

NOTE: Figures and tables appear at the end of the chapter.

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WEB CHAPTER 9

Adequately Braced Compact Beams

W9.1 Plastic Analysis of Beams

W9.1.1 Behavior of a Simply Supported Beam Under Central Concentrated Load

Figure W9.1.1: Behavior of a simply supported beam under a central concentrated load.

Consider a simply supported beam AB of length L and of a rectangular cross section, subjected to a concentrated load Q at midspan (Fig. W9.1.1). The structure is determinate and the bending moment diagram is triangular with the maximum bending moment occurring at the center C:

$$M_{\max} = \frac{QL}{4} \tag{W9.1.1}$$

As the load is increased, the most highly stressed section, namely the section at the center of the span, changes successively from: a completely elastic state (Fig. 9.2.2*a*) under a load of say, Q_1; to the fully elastic state (Fig. 9.2.2*b*) under the load Q_y; to a partially plastic state (Fig. 9.2.2*c*) under a load of say, Q_2; and finally to the fully plastic state (Fig. 9.2.2*d*) under the load Q_{pl}. Here,

$$0 < Q_1 < Q_y < Q_2 < Q_{pl} \tag{W9.1.2}$$

As the load approaches Q_{pl}, and the central section C approaches the fully plastic state, other nearby sections (such as F to the left) become partially plastic, while to the left of D (and to the right of E) the beam remains fully elastic. Although ductile yielding has begun between sections D and E, the load point will not deflect uncontrollably as long as section C can absorb some additional moment resulting from an increase in load. Only the central section C reaches the fully plastic state. Once that section becomes fully plastic, all fibers yield without further increase in stress, thereby permitting the two segments of the beam on either side of C to rotate freely relative to one other and a plastic hinge is said to form at C. A ***plastic hinge*** can be defined as a yielded section of a beam which acts as if it were hinged with a constant moment M_p.

Static equilibrium of the loaded beam cannot be maintained once the load reaches Q_{pl} because the moment capacity of the section is limited to M_p. The moment-curvature relationship of Fig. 9.2.5 indicates that once M_p has been reached, the angle change at the point of loading may take on infinite values. Due to this hinge-like action (hence the term, ***plastic hinge***), the beam will continue to deflect at constant load Q_{pl} and so fails by plastic collapse. The load at which this occurs is called the ***plastic limit load*** or the ***plastic collapse load*** and is denoted by Q_{pl}. The constancy of the load during collapse and therefore of the bending moments over the segments AC and BC implies constancy of curvatures during collapse over these segments. The increases of deflection during collapse are due therefore solely to the rotation at the central plastic hinge.

In Fig. W9.1.1*e* curve i represents the deflected form of the beam just before the collapse load Q_{pl} is attained, but before any rotation has occurred at the central plastic hinge. Curve ii is the position of the beam after the central hinge has undergone rotation through an arbitrary angle 2θ

and each half of the beam undergoes a rigid body motion of θ about the supports. The curved shape of each half of the beam is the same in case ii and in case i. Fig. W9.1.1*f* shows the changes of deflection which have occurred during collapse and represent the difference between the deflections in case ii and case i; each half of the beam is straight in this figure. The deformations occurring during collapse, shown in Fig. W9.1.1*f*, are due solely to the rotation at the central plastic hinge. This figure represents the collapse mechanism for the simple beam considered.

The shaded area in Fig. W9.1.1*b* denotes that part of the beam that has been strained into the plastic range. At section D, the outside fibers have just reached yield stress, but the stress distribution is still linearly elastic. Applying the flexure formula, the resisting moment at this section is:

$$M_{\mathrm{D}} = M_y = SF_y = \frac{bd^2}{6}F_y \tag{W9.1.3}$$

At section F, the section is elastic over a depth $2y_e$, but plastic beyond this depth, as shown by the stress distribution in sketch 9.2.2*c*. Using Eq. 9.2.13 the resisting moment at section F can be written as:

$$M_{\mathrm{F}} = \left[1 - \frac{1}{3}\left(\frac{y_e}{c}\right)^2\right]M_p \tag{W9.1.4}$$

At section C, the beam is fully plastic. The resisting moment at this section is given by:

$$M_{\mathrm{C}} = M_p = ZF_y = \frac{1}{4}bd^2F_y \tag{W9.1.5}$$

Plastic zones extend over the region where the bending moment exceeds the yield moment M_y. If the value of the shape factor is α, the length of that region (from similar triangles of Fig.W9.1.1*b*), is

$$\Delta L = \frac{(M_p - M_y)}{M_p} L = \frac{(\alpha - 1)}{\alpha} L = \frac{(1.5 - 1)}{1.5} L = 0.33 L \qquad \text{(W9.1.6)}$$

In view of the shape of the moment-curvature diagram OABD of Fig. 9.2.3, the curvature remains very small near ends D and E of the plastic region of Fig. W9.1.1*c*. On the other hand, in the neighborhood of point C, where the force is applied, the curvature is extremely high. The beam therefore deforms very nearly as if it consisted of two rigid segments AC and BC connected by a hinge at C (Fig. W9.1.1*f*).

W9.1.2 Behavior of a Fixed-Ended Beam Under Uniformly Distributed Load

Consider a fixed-ended beam AB of length, L, subjected to a uniformly distributed load of intensity, q (Fig. W9.1.2*a*). The beam has a uniform cross section with an idealized bilinear, moment-curvature relationship such as is defined by curve OED shown in Fig. 9.2.4. The load q is increased slowly from zero until collapse occurs at q_{pl}.

Figure W9.1.2: Behavior of a fixed-ended beam under uniformly distributed load.

In general, the degree of statical indeterminacy of a fixed-ended beam is 3, but because of the symmetry of loading and supports and the absence of axial forces, the indeterminacy I of the beam considered is one. Consider the unknown redundant to be the common value of the bending moments M_A and M_B at the ends of the beam.

a. Stage 1 Loading

As load q is increased steadily from zero, the beam behavior is, at first, wholly elastic. The bending distribution will be parabolic, with the maximum negative bending moments occurring at the supports A and B and the maximum positive bending moment occurring at the center C, as shown in Fig. W9.1.2*b*. The bending distribution can be determined using standard methods for solving indeterminate structures. Alternatively, from Case 15 in LRFDM Table 5-17: Shears, Moments and Deflections:

$$M_A = \frac{qL^2}{12} = M_B \, ; \quad M_C = \frac{qL^2}{24} \, ; \quad \delta_C = \frac{qL^4}{384\,EI} \tag{W9.1.13}$$

where M_A, M_B and M_C are the magnitudes of the bending moments at A, B and C, respectively. Also, δ_C is the maximum deflection at the center of the beam. Note that the bending moment at C is only half the magnitude of the clamping moment.

From the bending moment diagram, it is evident that as the load is increased steadily, the first two plastic hinges form simultaneously at supports A and B, corresponding to a load $q = q_1$ (Fig. W9.1.2 *c*), and

$$\frac{q_1 L^2}{12} = M_p \quad \rightarrow \quad q_1 = \frac{12\,M_p}{L^2} \tag{W9.1.14}$$

With $q = q_1$ applied, one can consider that part of the moment capacity at section C has been used up (Fig. W9.1.2*d*). The available moment capacity remaining at section C is:

$$M_{aC} = M_p - \frac{q_1 L^2}{24} = M_p - \frac{M_p}{2} = \frac{M_p}{2} \qquad \text{(W9.1.15)}$$

Also, the central deflection under load $q = q_1$ is:

$$\delta_1 = \frac{q_1 L^4}{384\, EI} = \frac{1}{384}\left(\frac{12\, M_p}{L^2}\right)\frac{L^4}{EI} = \frac{M_p L^2}{32\, EI} \qquad \text{(W9.1.16)}$$

The stiffness of the beam, in this stage, is:

$$s_1 = \frac{q_1}{\delta_1} = \frac{384\, EI}{L^4} \qquad \text{(W9.1.17)}$$

To summarize Stage 1: load q_1, just sufficient to cause the development of M_p at the beam ends, but not yet enough load to induce any rotation of either A or B, has been determined. From the moment diagram in Fig. W9.1.2*c* it is seen that the remainder of the beam, i.e., the portion spanning between A and B, is still in the elastic domain. Finally, the central deflection corresponding to q_1 was calculated, and the effective stiffness of the beam during this stage was obtained.

b. Stage 2 Loading

If the load is increased by a small amount Δq above q_1, the two plastic hinges at the ends of the beam undergo rotation, while the bending moment at each of these hinges remains constant at the value M_p. Thus, for any additional load beyond q_1, the changes of bending moment at the ends of the beam are therefore zero. The changes of bending moment throughout the beam which are caused by increasing the load from q_1 to $q_1 + \Delta q$ must therefore be the same as the bending moments which would be produced by the application of a load Δq to the same beam if it were simply supported at its ends (Fig. W9.1.2*e*). This follows from the fact that in the case of a simply supported beam the end moments would of necessity remain zero, while end rotations would be freely permitted.

The structure is now statically determinate, i.e., the degree of indeterminacy has been reduced from 1 to 0, due to the formation of hinges at A and B. From Case 1 in LRFDM Table 5-17: Shears, Moments and Deflections (Fig. W9.1.3*f*):

$$\Delta M_C = \frac{\Delta q\, L^2}{8}; \qquad \Delta\delta_C = \frac{5}{384}\frac{\Delta q\, L^4}{EI} \tag{W9.1.18}$$

where ΔM_C is the additional bending moment and $\Delta\delta_C$ the additional deflection that develops at C under a Stage 2 loading Δq.

As the load Δq is increased, the available moment capacity of $M_{aC} = \frac{1}{2} M_p$ at C at the end of Stage 1 is exhausted and a third plastic hinge forms at a load $\Delta q = \Delta q_2$ such that:

$$\frac{\Delta q_2 L^2}{8} = \frac{M_p}{2} \quad \rightarrow \quad \Delta q_2 = \frac{4 M_p}{L^2} \tag{W9.1.19}$$

The additional deflection $\Delta\delta_2$ of point C, at the end of Stage 2, is:

$$\Delta\delta_2 = \frac{5}{384}\frac{\Delta q_2 L^4}{EI} = \frac{5}{384}\left(\frac{4 M_p}{L^2}\right)\frac{L^4}{EI} = \frac{5}{96}\frac{M_p L^2}{EI} \tag{W9.1.20}$$

The stiffness of the beam in Stage 2 is:

$$s_2 = \frac{\Delta q_2}{\Delta \delta_2} = \frac{384}{5}\frac{EI}{L^4} = \frac{s_1}{5} \tag{W9.1.21}$$

To summarize, at the end of Stage 2, there are three plastic hinges; namely at A, B, and C (Figs. W9.1.2 *g* and *h*). The total load on the beam is:

$$q_2 = q_1 + \Delta q_2 = \frac{16\,M_p}{L^2} \tag{W9.1.22}$$

and the total deflection at midspan is:

$$\delta_2 = \delta_1 + \Delta\delta_2 = \left(\frac{1}{32} + \frac{5}{96}\right)\frac{M_p\,L^2}{EI} = \frac{M_p\,L^2}{12\,EI} \tag{W9.1.23}$$

c. Stage 3 Loading

For additional load Δq beyond Stage 2 (i.e., for values of $q = q_2 + \Delta q$), the three plastic hinges at A, B and C undergo rotation while the bending moment at each of these hinges remains constant at the value M_p. The changes of bending moment throughout the beam which are caused by increasing the load from q_2 to $q_2 + \Delta q$ must therefore be the same as the bending moment that would be produced by the application of a load Δq to the same beam if it were simply supported at its ends with an additional hinge at C. This structure however is a mechanism (indeterminacy, $I = -1$) and so cannot carry any additional transverse load (i.e., $\Delta q = 0$). The plastic limit load or collapse load of the beam therefore equals q_2 and is given by Eq. W9.1.22, or

$$q_{\text{pl}} = \frac{16\,M_p}{L^2} \tag{W9.1.24}$$

Since the beam behaves elastically except where the bending moment is of magnitude M_p, the two segments of the beam between plastic hinges (i.e., segments AC and BC) do not undergo any change of curvature while collapse is occurring, as the bending moment distribution remains unchanged. Thus, the mechanism shown in Fig. W9.1.2*i* represents the changes of deformation which occur due to additional rotations θ at the end hinges during plastic collapse. The beam now enters the range of unrestricted plastic flow or fully plastic range.

The behavior of the beam can be summarized by a diagram of the midspan deflection, δ versus the total load q (Fig. W9.1.3). This diagram consists of three straight lines Oa, ab and bc, corresponding to elastic behavior (Stage 1), elastic-plastic behavior (Stage 2), and fully plastic behavior (Stage 3), respectively. Figure W9.1.3 shows that the stiffness of the beam reduces with the formation of each additional plastic hinge. The stiffness equals zero when the plastic limit load is reached and the structure is reduced to a mechanism.

Figure W9.1.3: Load-deflection relation for a fixed ended beam under uniformly distributed load.

W9.1.3 Behavior of a Fixed-Ended Beam Under Off-Center Concentrated Load

The behavior of a fixed-ended beam subjected to an off-center concentrated load will be explained below, with the help of a numerical example.

EXAMPLE W9.1.1 Step-by-Step Method

A W24×68 of A992 steel beam AB is 40 ft long. It is fixed at both ends and subjected to a single concentrated load Q at a section C, 16 ft from the end A as shown in Fig. WX9.1.1*a*. Determine the collapse load of the beam using step-by-step method.

Figure WX9.1.1: Behavior of a fixed-ended beam under a concentrated load.

Solution

a. Data

Span, $L =$ 40.0 ft; $a =$ 16.0 ft; $b =$ 24.0 ft

From Table 1-1 of the LRFDM, for a W24×68 of A992 steel:

$$I_x = 1830 \text{ in.}^4; \quad S_x = 154 \text{ in.}^3; \quad Z_x = 177 \text{ in.}^3$$

$$F_y = 50 \text{ ksi}$$

$$M_y = \frac{154\ (50)}{12} = 642 \text{ ft-kips}$$

$$M_p = \frac{177\ (50)}{12} = 738 \text{ ft-kips}$$

b. Stage 1 loading

The member is elastic and indeterminate to the second degree ($I = 2$). From Case 17 in LRFDM Table 5-17: Shears, Moments and Deflections:

$$M_A = \frac{Q\,a b^2}{L^2}; \quad M_C = \frac{2Q\,a^2 b^2}{L^3}; \quad M_B = \frac{Q\,a^2 b}{L^2}$$

$$\delta_C = \frac{Q\,a^3 b^3}{3EIL^3}$$

Substituting for a, b and L,

$$M_A = 5.76\,Q; \quad M_C = 4.61\,Q; \quad M_B = 3.84\,Q$$

From the bending moment diagram (Fig. WX9.1.1c), it is evident that the first plastic hinge occurs at support A, corresponding to a load $Q = Q_1$ such that

$$5.76\,Q_1 = M_A = M_p = 738 \text{ ft-kips} \quad \rightarrow Q_1 = 128 \text{ kips}$$

With $Q = Q_1 =$ 128 kips applied (Fig. WX9.1.1d), part of the available moment capacity at points C and B has been used up. The available moment capacity remaining at these points is

$$M_{a\text{C}} = M_p - 4.61\,Q_1 = 738 - 4.61\,(128) = 148 \text{ ft-kips}$$

$$M_{a\text{B}} = \text{M}_\text{p} - 3.84\,Q_1 = 738 - 3.84\,(128) = 246 \text{ ft-kips}$$

The deflection of load point C under load Q_1 is

$$\delta_1 = \frac{128\,(16^3)\,(24^3)\,(12^3)}{3\,(29{,}000)\,(1830)\,(40^3)} = 1.23 \text{ in.}$$

The stiffness of the beam during Stage 1 is

$$s_1 = \frac{Q_1}{\delta_1} = \frac{128}{1.23} = 104 \text{ kips/in.}$$

c. Stage 2 loading

As the load is increased above $Q = Q_1 = 128$ kips, the moment at support A remains constant at the plastic moment 738 ft-kips. So, the added load ΔQ, in Stage 2, acts on a different elastic system as shown in Fig. WX9.1.1*e*. The indeterminacy of the structure is now equal to one ($I = 1$). From Case 14 in LRFDM Table 5-17: Shears, Moments and Deflections:

$$\Delta M_\text{C} = \frac{\Delta Q\,ab^2}{2\,L^3}(a + 2L); \qquad \Delta M_\text{B} = \frac{\Delta Q\,ab}{2\,L^2}(a + L)$$

$$\Delta\delta_\text{C} = \frac{\Delta Q\,a^2 b^3}{12\,EIL^3}(3L + a)$$

or (Fig. WX9.1.1*f*):

$$\Delta M_\text{C} = 6.91\,\Delta Q; \qquad \Delta\text{M}_\text{B} = 6.72\,\Delta Q$$

As the load ΔQ is increased, the available moment capacity of $M_{a\text{C}} = 148$ ft-kips at C at the end of Stage 1 is exhausted and a second plastic hinge forms at C at a load $\Delta Q = \Delta Q_2$, such that:

$$6.91\,\Delta Q_2 = 148 \quad \rightarrow \quad \Delta Q_2 = 21.4 \text{ kips}$$

Corresponding to this load, the moment capacity left at support B is:

$$M_{a\mathrm{B}} = 246 - 6.72\,(21.4) = 102 \text{ ft-kips}$$

The additional deflection at the load point at the end of Stage 2 is:

$$\Delta\delta_2 = \frac{21.4\,(16^2)\,(24^3)\,(3\times 40 + 16)\,(12^3)}{12\,(29{,}000)\,(1830)\,(40^3)} = 0.437 \text{ in.}$$

The stiffness of the beam during Stage 2 is

$$s_2 = \frac{\Delta Q_2}{\Delta\delta_2} = \frac{21.4}{0.437} = 49.0 \text{ kips/in.} \rightarrow s_2 = 0.471\, s_1$$

At the end of Stage 2 the load on the beam is $Q_2 = Q_1 + \Delta Q_2 = 128 + 21.4 = 149.4$ kips as shown in Fig. WX9.1.1g.

d. Stage 3 loading

As the load Q on the beam is increased above Q_2, the moment at points A and C remains constant at the plastic moment 738 ft-kips. So, the added load ΔQ, in Stage 3, acts on a yet different elastic system as shown in Fig. WX9.1.1*h* (Segment BC now acts as a cantilever beam). The structure is now determinate, i.e., the indeterminacy I equals zero. From Case 21 in LRFDM Table 5-17: Shears, Moments and Deflections :

$$\Delta M_B = b\,\Delta Q; \qquad \Delta\delta_C = \frac{\Delta Q\, b^3}{3EI}$$

As the load ΔQ is increased the remaining moment capacity at B of 102 ft-kips, available at the end of Stage 2, is eventually exhausted and a third plastic hinge forms at B at à load $\Delta Q = \Delta Q_3$ such that (Fig. WX9.1.1*i*):

$$24.0\,\Delta Q_3 = 102 \quad \rightarrow \quad \Delta Q_3 = 4.25 \text{ kips}$$

The additional deflection at the load point at the end of Stage 3 is:

$$\Delta\delta_3 = \frac{4.25\,(24^3)\,(12^3)}{3\,(29{,}000)\,(1830)} = 0.638 \text{ in.}$$

The stiffness of the beam during Stage 3 is

$$s_3 = \frac{\Delta Q_3}{\Delta\delta_3} = \frac{4.25}{0.638} = 6.66 \text{ kips/in.} \rightarrow \qquad s_3 = 0.064\, s_1$$

So, at the end of Stage 3, the beam is subjected to a total load of $Q_3 = Q_2 + \Delta Q_3 = 149.4 + 4.25 = 154$ kips, and three plastic hinges have developed; namely, at A, C and B (Fig. WX9.1.1*j*). The cumulative bending moment is as shown in Fig. WX9.1.1*k*.

e. Stage 4 loading

For additional load ΔQ beyond Stage 3, the structure acts as a simple beam with an additional hinge at load point C (Fig. WX9.1.1*l*); that is, the beam has been transformed into a mechanism and no additional load ΔQ_4 can be supported ($\Delta Q_4 = 0$). So, the plastic limit load of the beam is 154 kips.

The behavior of the beam can be summarized by a diagram of the deflection, δ,versus the load Q (Fig. WX9.1.1*m*). This diagram consists of four straight lines corresponding to elastic behavior (Stage 1), elastic-plastic behavior (Stages 2 and 3), and fully plastic behavior (Stage 4). Figure WX9.1.1*m* shows that the stiffness of the beam reduces with the formation of each additional plastic hinge. The stiffness equals zero when the plastic limit load is reached and the structure is reduced to a mechanism.

W9.1.4 Plastic Limit Load

General

Consider a statically indeterminate rigid-jointed, planar, steel structure with degree of indeterminacy I subjected to a set of proportional loads. A set of ***proportional loads*** is a set in which all loads are kept in constant proportion to one another. Quite simply, proportional loading occurs when all loads are multiplied by the same (load) factor. As the loads increase, plastic hinges appear in succession at sections where the absolute value of the bending moment has a local maximum, equal to the plastic moment. If the structure does not carry any distributed loads, the only possible locations of plastic hinges are at the end sections of the members and at sections where concentrated loads are applied. Once a plastic hinge forms at a section, the magnitude of the bending moment at this section remains constant at the known value $M_p = ZF_y$ and the degree of redundancy of the structure is reduced by one. The structure therefore becomes statically determinate when the I-th plastic hinge forms. The next plastic hinge transforms this statically determinate system into a mechanism with one degree of freedom which can deform under virtually constant load. Thus, the formation of the $(I+1)$th hinge represents the collapse of the structure. The load at which the $(I+1)$th hinge appears is known as the ***plastic limit load** or* ***collapse load***. Note that for symmetrically loaded symmetric structures, hinges that do not lie on an axis of symmetry will always form in symmetric pairs. There may therefore be more than $(I+1)$ plastic hinges at failure.

The plastic limit load is greater than the elastic limit load because complete plastificatin of a cross section requires more load than what is merely needed to initiate yielding at the extreme fibers. Moreover (assuming that a mechanism has not yet developed), redistribution of bending moment within the structure occurs as each plastic hinge develops and transforms the structure into one which possesses one less degree of indeterminancy. The redistribution process is dependent upon: the overall geometry and support conditions of the structure; the geometry and material properties of member cross sections; and the particular arrangement of the applied loads. Therefore the ratio of collapse load to yield load varies from structure to structure and for a given structure from load to load, and must be determined (in most instances at least) by performing a structural analysis.

Direct Calculation of the Plastic Limit Load

Plastic analysis of planar structures is based on many of the assumptions employed for elastic analysis. The structure is assumed to be composed of straight, rod-like members. Each member contains at least one axis of symmetry: this axis and all external loads and reactions are assumed to lie in the same plane. The displacements and rotations are small, and their effect upon the equations of equilibrium is negligible. The primary forces are flexural; the effects of axial and shear forces are not considered insofar as stresses on the cross section are concerned. The material is assumed to be homogenous and isotropic. Furthermore, the material must also be ductile so that members possess adequate rotation capacity in the yield range.

Unlike elastic analysis in which stress is always proportional to strain, the behavior of members in the inelastic domain is nonlinear and thus precludes the use of the principle of superposition. Thus, only proportional loadings are valid. ***Proportional loadings*** occur when the proportional ratios which exist among the various given loads are maintained throughout all subsequent loadings by use of a common (i.e., single) load factor, applied simultaneously to all loads.

The disposition of plastic hinges, together with any given elastic hinges, that is effective at collapse is known as the ***collapse mechanism***. It is clearly defined for any given structure and proportional loading. Real (or elastic) hinges are assumed to be ideal frictionless pins and are indicated in figures by the customary hollow circles. Solid circles are used for plastic hinges, where the hinge moment equals M_p. Plastic hinges are likely to occur at support points, under concentrated loads, or where the shear in a member is zero. Generally the locations of possible plastic hinges may be determined by inspection.

The basic beam mechanism occurs when three hinges (at least one of which must be a plastic hinge) transform a previously stable flexural member into two adjoining hinged beam segments. The end hinges are fixed against transverse displacement while the middle hinge is not. (All hinges may translate laterally, as would occur for an upper beam of a frame which experiences sway movement due to wind or other lateral loads). The three hinges permit rotation of either segment to freely occur without any corresponding increase in load. This rigid body displacement, or displacement without strain, is tantamount to collapse.

There are three conditions which must be satisfied by a collapse mechanism [ASCE, 1971]: the ***equilibrium condition*** where the applied loads and reactions must be in equilibrium; the ***mechanism condition*** which requires that sufficient plastic hinges form to convert the structure (or part of it) into a mechanism; and the ***plastic moment condition*** which dictates that the moment nowhere may exceed the plastic moment M_p.

There are two basic methods or approaches used to determine the collapse mechanism for a given structure, namely, the equilibrium method, and the virtual work method.

The ***static*** or ***equilibrium method*** consists of constructing an equilibrium moment diagram in which $M \leq M_p$ throughout the structure and which contains sufficient number and locations of plastic hinges to form a mechanism.

The ***mechanism method*** or ***the method of virtual displacements*** consists of equating the internal work absorbed by the plastic hinges to the external work done by the loads when a virtual displacement is given to the mechanism. Note that, when two members meeting at a plastic hinge rotate through an angle θ relative to each other, the work absorbed by the plastic hinge equals $M_p\theta$, where M_p is the plastic moment of the section.

In applying the mechanism method, a trial mechanism is first assumed. Equations expressing the external work done by the given loads and the internal work absorbed at the plastic hinges are then equated, resulting in a value for the collapse load that corresponds to the mechanism assumed. The bending moment diagram is then constructed using the value of the calculated collapse load. If the plastic moment condition is everywhere satisfied (i.e., $M \leq M_p$) for the resultant bending moment diagram, then the assumed mechanism is the true collapse mechanism, and the calculated collapse load is the true plastic limit load. When there are two or more possible mechanisms, the true ***collapse mechanism*** is the one that yields the greatest value of M_p (where M_p is expressed in terms of the given loading). Alternatively, the true ***collapse mechanism*** may be identified as the one that is associated with the smallest collapse load (where the collapse load is expressed in terms of M_p).

EXAMPLE W9.1.2 Mechanism Method

Calculate the plastic limit load, Q_{pl}, for the beam of Example W9.1.1 using the mechanism method.

Figure WX9.1.2

Solution

From Table 1-1 of the LRFDM, the plastic section modulus of a W24×68 shape is 177 in.[3] The plastic moment of this section, for F_y = 50 ksi, is:

$$M_p = Z_x F_y = \frac{177\,(50)}{12} = 738 \text{ ft-kips}$$

A fixed-ended beam is indeterminate to the third degree. As there are no externally applied horizontal loads, the indeterminacy reduces to two. Assume that plastic hinges form at support points A and B and also at load point C, converting the structure into a mechanism ($I = -1$). Give a virtual displacement to the beam mechanism so that the rigid segment BC rotates through an angle θ (Fig. WX9.1.2*a*). The virtual displacement, δ, at C equals 24θ, and the rotation at A of the rigid segment AC therefore equals $24\theta/16 = 1.5\theta$. The angle at C is the sum of the angles at A and B; thus the angle at C equals 2.5θ.

The virtual work done by the external load is

$$Q_{pl}\,\delta = Q_{pl}\,(24\theta) = 24\,Q_{pl}\theta$$

The virtual work absorbed by the plastic hinges

$$= M_p\,(1.5\theta) + M_p\,(2.5\theta) + M_p\,(\theta) = 5M_p\theta$$

Equating the external work to the internal work

$$24Q_{pl}\theta = 5M_p\theta = 5\,(738\theta)$$

$$\rightarrow Q_{pl} = \frac{5(738)}{24} = 154 \text{ kips} \qquad \text{(Ans.)}$$

The bending moment diagram at the plastic limit load is as shown in Fig. WX9.1.2*b*. As the bending moment at any point does not exceed the plastic moment of the beam, the mechanism selected is the true collapse mechanism.

Note:

1. The virtual rotation θ cancels out from the work equation, as is the case with all applications of the virtual work principle.

2. The value of 154 kips obtained for the plastic limit load tallies with the value obtained from the step-by-step method of Example W9.1.1.

EXAMPLE W9.1.3 **Mechanism Method**

Calculate the plastic moment for a fixed-ended beam of span, L, under uniformly distributed plastic limit load, q_{pl}. Use the mechanism method.

Figure WX9.1.3

Solution

A fixed-ended beam under transverse loads only, is indeterminate to the second degree (I = 2), and thus requires three plastic hinges to convert the structure into a mechanism (I = − 1). Two of these form at the supports A and B and a third at midspan (in view of the symmetry of the structure and loading), converting the structure into a mechanism. Give a virtual displacement to the beam mechanism (Fig. WX9.1.3*a*), so that the rigid segment AB rotates through an angle θ. The virtual displacement, δ, at C equals $(L/2)\theta$ and the rotation at B of the rigid segment BC equals θ. The angle at C is the sum of the angles at A and B, and equals 2θ.

Consider an elemental load, $q_{pl}\,dz$, acting at a distance z from A where the deflection equals δ_z. Virtual work done by the elemental load is

$$q_{pl}\,dz\,\delta_z \;=\; q_{pl}\,dz\,\frac{\delta}{L/2}z \;=\; \frac{2\delta}{L}\,q_{pl}\,(z\,dz)$$

Virtual work of the external load on segment AC

$$=\; \frac{2\delta}{L}\,q_{pl}\int_o^{L/2} z\,dz \;=\; \left(\frac{q_{pl}L}{2}\right)\frac{\delta}{2}$$

Virtual work of the external load on the beam AB therefore equals

$$2\left(\frac{q_{\mathrm{pl}}L}{2}\right)\frac{\delta}{2} = \left(\frac{q_{\mathrm{pl}}L}{2}\right)\left(\frac{L}{2}\theta\right) = \frac{q_{\mathrm{pl}}L^2}{4}\theta$$

Virtual work absorbed by the plastic hinges is

$$M_p\theta + M_p(2\theta) + M_p\theta = 4M_p\theta$$

Equating the internal work to the external work done by the loads, we obtain

$$4M_p\theta = \frac{q_{\mathrm{pl}}L^2}{4}\theta \quad \rightarrow \quad M_p = \frac{q_{\mathrm{pl}}L^2}{16} \qquad \text{(Ans.)}$$

The bending moment diagram at collapse is shown in Fig. WX9.1.3*b*. As the bending moment at any point does not exceed the plastic moment of the beam, the mechanism selected is the true collapse mechanism.

Note:

The M_p value obtained here corresponds with the value derived in Section W9.1.2, using the step-by-step method.

W9.2 Open Web Steel Joists and Joist Girders

W9.2.1 Introduction

Open web steel joists are standardized, prefabricated, welded steel trusses used as simply supported beams briefly introduced in Section 3.5.2 (Fig. 3.5.2). They are particularly well suited for single story structures with high ceilings such as gymnasiums, factories , and shopping centers, where fire proofing and acoustic needs are minimal. The 41st edition of the Standard

Specifications, Load Table & Weight Tables for Steel Joists and Joist Girders [SJI, 2002] developed by the Steel Joist Institute (SJI) give details for three types of open web steel joists (Standard K-series, longspan LH-series, and deep longspan DLH- series), and for open web joist girders (G-series). For any particular span length and loading, the structural engineer simply selects an appropriate joist from load tables that have been developed with due consideration for moments, shears, and deflections involved in simple spans. To prevent lateral buckling during construction, lateral bracing known as ***bridging*** is provided at regular intervals.

W9.2.2 Comparative Summary of Joists and Joist Girders

Figure W9.2.1: Longspan and deep longspan steel joists [SJI, 2002].

The following is a comparative summary of K-, LH- and DLH-series open web joists and G-series open web joist girders [SJI, 2002]:

1. K-series and LH-series joists are used for the direct support of roof or floor decks in buildings. DLH joists are used for the direct support of roof decks only. Joist girders are used for the support of equally spaced concentrated loads (reactions of roof or floor joists) acting at the panel points of the joist girders.
2. There are 64 separate designations in the load tables for K-series joists representing depths from 8 in. through 30 in. in 2inch increments and spans from 8 ft through 60 ft. Longspan series joists have been standardized in depths from 18 in. to 48 in. (with 4 inch increments between 20 in. to 48 in.), for clear spans from 25 to 96 feet. Deep longspan series joists have been standardized in depths from 52 in. to 72 in. in 4 inch increments for clear spans from 89 to 144 ft. Joist girders have been standardized for depths from 20

in. to 72 in. and spans from 20 to 60 ft.

3. Joists are designed as simple span trusses under gravity loads, when they are located between members of a primary bracing system. The depth-to-span ratio of a joist must not be less than 1/24. It is commonly 1/20.

4. For K-series joists, chords are essentially parallel. Longspan and deep longspan steel joists can be furnished with parallel chords or with single or double pitched top chords to provide sufficient slope for roof drainage (Fig. W9.2.1). Standard pitch, if desired, is ⅛ in. per foot. The nominal depth (and the joist designation) is determined by its depth at the center of the span.

5. For K-series joists, camber may be provided at the manufacturer's option; for other series it is required (standard).

6. K-series joists are furnished with underslung ends. LH and DLH joists can be furnished with either underslung or square ends. Square ends are primarily intended for bottom chord bearing. Joist girders are furnished with underslung ends and lower chord extensions.

7. K-series joists have a 2½ in. end bearing depth so that, regardless of the overall joist depths, the tops of all the joists lie in the same plane, facilitating the placement of the deck. The depth of the end bearing portion of the underslung joists is 5 in. for LH- series. For the DLH-series, the end bearing depths is 5 in. for chord sizes through 17, and 7½ in. for chord sizes 18 and 19. For joist girders the depth of the bearing portion is 6 inches.

8. The bottom chord is designed as an axially loaded tension member.

9. For design, the top chords of K-, LH-, and DLH-series joists are considered to be fully (continuously) braced against lateral buckling by the roof or floor deck construction. The top chord of a joist girder is considered as stayed laterally by the steel joists which are

located at panel points only.

10. The top chord of a joist series is designed for only axial compressive force when the panel length ℓ does not exceed 24 in. When the panel length exceeds 24 in., the top chord is designed as a continuous member subjected to combined axial load and bending moments. The top chord of a joist girder is designed as an axially loaded compression member laterally supported by roof or floor joists. The radius of gyration of the top chord about the vertical axis shall not be less than $\ell/575$.

11. Ends of K-series joists resting on steel supports shall be attached thereto with a minimum of two ⅛ in. fillet welds 1 in. long, or with a ½ in. bolt. The corresponding values for LH- or DLH-series joists and joist girders are two ¼ in. fillet welds 2 inches long, or two ¾ in. bolts. Where columns are not framed in at least two directions with structural steel members, joists and joist girders at column lines shall be field bolted to the column to provide lateral stability during construction.

12. The ends of K-series joists shall extend a distance of not less than 2½ inches over the steel supports. The corresponding bearing length for LH- and DLH- series is 4 inches. When the bearing is less than the above criteria, such as in the case of two opposite joists resting on a narrow steel beam, special joist end attachments must be provided with attachments to the support by bolting or welding. Due consideration of the end reactions and all other vertical and lateral forces shall be taken by the specifying engineer in the design of the steel support.

13. The deflection due to the design live load shall not exceed 1/360 of span for floors and roofs where a plaster ceiling is attached or suspended, and shall not exceed 1/240 of span for all other cases of roof support.

W9.2.3 Bridging

Steel joists are slender flexural members. Under gravity loads, the top chord of a joist is in compression and the bottom chord is in tension. In the finished stage, the metal decking or concrete slab acts as a continuous lateral bracing to the top flange. However, they require positive lateral bracing to the top flange, prior to placement of the decking. This bracing, called ***bridging***, stiffens the steel joists sufficiently to support the construction stage loads safely. The bridging spans transversely between the steel joist spans, and also helps keep the joists in their desired positions as shown in the plans. If joists are subjected to a net uplift (under load combination LC-6), the bottom chords of the joists will be in compression and bridging is required to provide lateral stability to the bottom chord. The specifications require joists subjected to uplift have a line of bridging near each of the first bottom chord panel points. Depending on the actual amount of uplift, additional bridging may be required.

There are two types of bridging, namely, horizontal and diagonal [see SJI, 2002]. Horizontal bridging consists of two continuous horizontal steel members, one attached to the top chord and the other attached to the bottom chord. The bridging is attached to the joists by welding or mechanical means. The ends of all bridging lines, terminating at a wall or a beam, should be anchored. The bolts or welds should be capable of resisting a minimum horizontal force of 0.7 kips. The maximum spacing and minimum number of rows of bridging are given in the specifications. In the case of bottom chord bearing joists, the ends of the joists must be restrained laterally by a row of diagonal bridging near the support to provide lateral stability. All bridging and bridging anchors shall be completely installed before construction loads are placed on the joists. Where diagonal bridging is required, the hoisting cables shall not be released until the diagonal bridging is completely installed and anchored. The ends of the joists shall be bolted or welded to the support as required.

W9.2.4 LRFD Load Tables for Steel Joists

In SJI publications [SJI, 2002; and SJI, 2000], designation of a K-series joist consists of a number (indicating the depth in inches) and the letter K followed by a number indicating the chord size. Example: 18K6. The longspan and deep longspan joists are designated by two digits (indicating the depth in inches) and the letters LH or DLH (indicating the series), followed by one or two digits (indicating the chord size designation). Examples: 24LH06, 56DLH16. Joist girders are designated by two digits (indicating the depth in inches) and the letter G (indicating joist girder), followed by one or two digits and an N (indicating the number of joist spaces), followed by one or two digits and a K (indicating the number of kips of each concentrated factored panel load). For example, a designation 60G10N12K indicates a 60 inch deep joist girder with 10 joist spaces, having panel point concentrated loads of 12 kips each.

Figure W9.2.2: LRFD load table and weight table for steel joists [SJI, 2000].

The existing design specification for steel joists and joist girders is based on the ASD format [SJI, 2002]. However, SJI currently offers and allows the use of either the ASD Load Tables [SJI, 2002] or the LRFD Load Tables [SJI, 2000] for the selection of joists and joist girders. Only use of LRFD Load Tables is described and used in this text book. In these load tables, there are two numbers that are listed for a given joist and span length. In the actual load tables found in the SJI Guide [SJI, 2000], the top number is in black and the bottom number is in red. The top number represents the total uniformly distributed factored load, q_{LRFD}, in pounds per linear foot, that this joist can safely support. It is obtained by multiplying the nominal strength of the joist, q_n (usually determined from full scale tests), by a resistance factor (= 0.9). The bottom number (in red, in Load Tables) is the service live load (in plf) that would produce an approximate deflection of $L/360$, the maximum live load deflection permitted for floors and roofs having attached or suspended plaster ceilings. Live loads which will produce a deflection of $L/240$, the maximum permissible live load deflection for cases of roofs with other than plastered ceilings, may be obtained by multiplying the tabulated figures in red by a factor of 1.5. Special deflection requirements may be necessary for perimeter steel joists loaded with building cladding and interior joists carrying folding partition walls. Figure W9.2.2 shows a typical page from the LRFD Load Table for the K-series joists.

The following procedure is used in the SJI LRFD Load Tables [SJI, 2000]. Let

q_{LRFD} = SJI LRFD Load Table load

= design strength of the joist, LRFD method

q_u = required strength of the joist, LRFD method

We have

$$qu = \max\,[1.4\,D;\ \ 1.2\,D + 1.6\,(L,\ \text{or}\ L_r,\ \text{or}\ S,\ \text{or}\ R\,)] \qquad \text{(W9.2.1)}$$

where $D, L, S,$ . . . etc. are the service dead load, live load, snow load, etc. as defined in Section 4.10.3. From the LRFD format, we have

$$q_{\text{LRFD}} \geq q_u \qquad \text{(W9.2.2)}$$

Enter the SJI LRFD Load Table at the given span with a value of q_u for the load, and select a suitable joist that satisfies Eq. W9.2.2. The SJI Specification recommends a maximum span-to-depth ratio for joists of all series equal to 24 (and a preferred value of 20). The designer should also examine bridging requirements for the selected joists. It is possible that by selecting a slightly heavier joist, a line of bridging can be eliminated, thus resulting in a substantial decrease in the total cost of the erected steel. If possible, for joists spanning less than 40 ft, selection should be made so that X-bracing is not required. The optimum joist girder depth in inches is approximately equal to the span of the girder in feet. The designer should generally follow this rule of thumb. However, for expensive wall systems a one foot savings in height of the structure may prove to be more economical as compared to the extra cost of shallower joist girders. Wide spacing of joists maximizing the roof deck or floor slab design, often results in fewer pieces to erect translating to a more economical solution.

EXAMPLE W9.2.1 Open Web Steel Joist and Joist Girder Selection

Select open web steel joists and joist girders for a roof system of a school building, as shown in Fig. WX9.2.1. The plan dimensions of the building are 252 ft×110 ft, with a 36ft×44 ft column grid. The building is located in Green Bay, Wisconsin. Use ASCE Standards, SJI and LRFD Specifications. Use A325-N type ¾-dia. bolts and E70 electrodes. Limit the depth of joists to 24 in. Gross wind uplift is 25 psf.

Figure WX9.2.1

Solution

a. Loads

If a standing seam roof is used, a 5 foot clear spacing between joists is to be used. This is due to the fact that UL 90 uplift requirements for most standing seam roof systems can only be met with a 5 ft. clear spacing for joists. Assume 5' 6" center-to-center spacing for joists.

The roof dead load is estimated, with the help of ASCES Tables C3-1 and C3-2, as follows:

Standing seam metal roof		2.0 psf
Wide rib steel deck 1½" - 22 gage		2.0 psf
3" rigid insulation boards (3 × 1.5)		4.5 psf
Suspended ceiling		8.0 psf
Mechanical, electrical, piping, say		3.5 psf
Total dead load	=	20.0 psf

Snow load (from ASCES Fig. 7.1 corresponding to Green Bay, Wisconsin) 40.0 psf

Roof live load (maintenance load from workers) 20.0 psf

b. Selection of steel joists

Spacing of joists = 5' 6"

Span of joists = 36' 0"

The dead load of joists is given as a minimum of 3 psf in the Code of Standard Practice. Whenever the actual weight of the joist exceeds this value the actual load must be used.

Joist loads:

Roof, deck, insulation and ceiling	20 psf × 5.5 = 110 plf
Joist, say	3 psf × 5.5 = 17 plf
Dead load, *D*	127 plf

Snow load, S $40 \text{ psf} \times 5.5 = 220 \text{ plf}$

Roof live load, L_r $20 \text{ psf} \times 5.5 = 110 \text{ plf}$

The maximum factored load on the joist is

$$q_u = 1.2D + 1.6(L, L_r, S, R) = 1.2\,(127) + 1.6\,(220) = 504 \text{ plf}$$

The SJI Specification recommends a maximum span-to-depth ratio for all series equal to 24 (and a preferred value of 20). For the 36 ft span the recommended minimum depth is $36\times12/24 = 18$ in. (and a preferred value of $36\times12/20 \approx 22$ in.). Entering Standard Load Table/ Open Web Steel Joists, for LRFD K-Series joists with a span of 36 ft and reading across until a load equal to or greater than the required load capacity of 504 plf is reached, the information given in the table below is obtained. The number of rows of bridging, n, is read from the Table on page 9 of the publication [SJI, 2000].

	q_{LRFD}	q_δ	weight	n	Remarks
20K10	547	*193*	12.2	2	NG
22K9	510	*201*	11.3	3	NG
24K8	**513**	**222**	**11.5** ←	**3**	**OK**
26K7	504	240	10.9	3	OK
28K7	544	280	11.8	3	OK

Joists 24K8, 26K7 and 28K7 are adequate for strength and serviceability, and require the same number of bridging. However, as the depth of the steel joists is to be limited to 24 inches, we will select a 24K8.

$q_{\text{LRFD}} = 513 \text{ plf} > q_{\text{req}} = 504 \text{ plf}$ O.K.

$q_\delta = 222 \text{ plf} > q_S = 220 \text{ plf}$ O.K.

Also, as the assumed weight of joists (= 16.5 plf) is greater than the actual weight (=

11.5 pℓf) of the joist selected, no redesign is necessary.

Net wind uplift from load combination LC-6 $= 1.6\,(25) - 0.9\,(20) = 22.0$ psf

$$= 22.0\,(5.5) = 121 \text{ plf}$$

The factored net uplift value of 121 plf must be specified on the contract documents.

So, select 24K8 open web steel joists. (Ans.)

c. Selection of joist girders

Span of joist girder = 44' 0" (centerline of column to centerline of column)

Panel length = joist spacing = 5' 6"

Number of actual joist spaces, $N = 8$

Contributory area for interior panel of joist girders $= 5.5\,(36) = 198$ ft^2

Panel load from:

Roof, deck, insulation, ceiling $= \dfrac{20\,(198)}{1000} = 3.96$ kips

Joist $= \dfrac{11.5\,(36)}{1000} = 0.414$ kips

Joist girder (say, 2 psf) $= \dfrac{2\,(198)}{1000} = 0.396$ kips

Total dead load, Q_D $= 4.77$ kips

Snow load, Q_S $= \dfrac{40\,(198)}{1000} = 7.92$ kips

Factored panel load, $Q_u = 1.2\,Q_D + 1.6\,Q_S = 1.2\,(4.77) + 1.6\,(7.92) = 18.4$ kips

The required SJI Joist/ Factored Load Table panel load, Q_u is taken as 19.5 kips. The optimum joist girder depth in inches is approximately equal to the span of the girder in feet. Deeper girders generally result in lighter girders but the increased cost resulting from the added height of structure (columns, walls, etc.) may nullify the cost savings in girder weight. Therefore, select a depth of 44 inches. The LRFD joist girder will then be designated as 44G8N19.5K

The LRFD Joist Girder Weight Table gives the weight for a 44G8N19.5K as 56 plf. As this value is less than the value 2 psf × 36 ft = 72 plf assumed in the load calculation, the girder selection is O.K. The live load deflection of girder is calculated next.

Live load on girder, $q_S = \dfrac{40.0\,(36.0)}{1000} = 1.44$ klf

Approximate joist girder moment of inertia,

$$I = 0.018\,N\,P\,L\,d = 0.018\,(8)\,(19.5)\,(44)\,(44) = 5{,}440 \text{ in.}^4$$

$$\text{Deflection, } \delta_L = 1.15\left(\frac{5qL^4}{384\,EI}\right)12^3 = \frac{1.15\,(5)\,(1.44)\,(44^4)\,(12^3)}{384\,(29{,}000)\,(5440)} = 0.885 \text{ in.}$$

$$\text{Allowable deflection, } \delta_a = \frac{L}{360} = \frac{44\,(12)}{360} = 1.47 \text{ in.} > \delta_L = 0.885 \text{ in.} \quad \text{O.K.}$$

So, select a 44G8N19.5K joist girder. (Ans.)

References

W9.1 ASCE [1971]: *Commentary on Plastic Design in Steel,* Joint Committee of Welding Research Council and the American Society of Civil Engineers, ASCE Manual No. 41, NY.

W9.2 Fisher, J. M., West, M. A. and Van de Pas, J. P. [1991]: *Designing with Steel Joists, Joist Girders, Steel Deck,* Vulcraft Company, Fort Payne, AL.

W9.3 SJI [2000]: *Guide for Specifying Joists with Load and Resistance Factor Design,* Steel Joist Institute, Myrtle Beach, SC.

W9.4 SJI [2002]: *Open Web, Longspan, and Deep Longspan Steel Joists, and Joist Girders,* Steel Joist Institute, Myrtle Beach, SC.

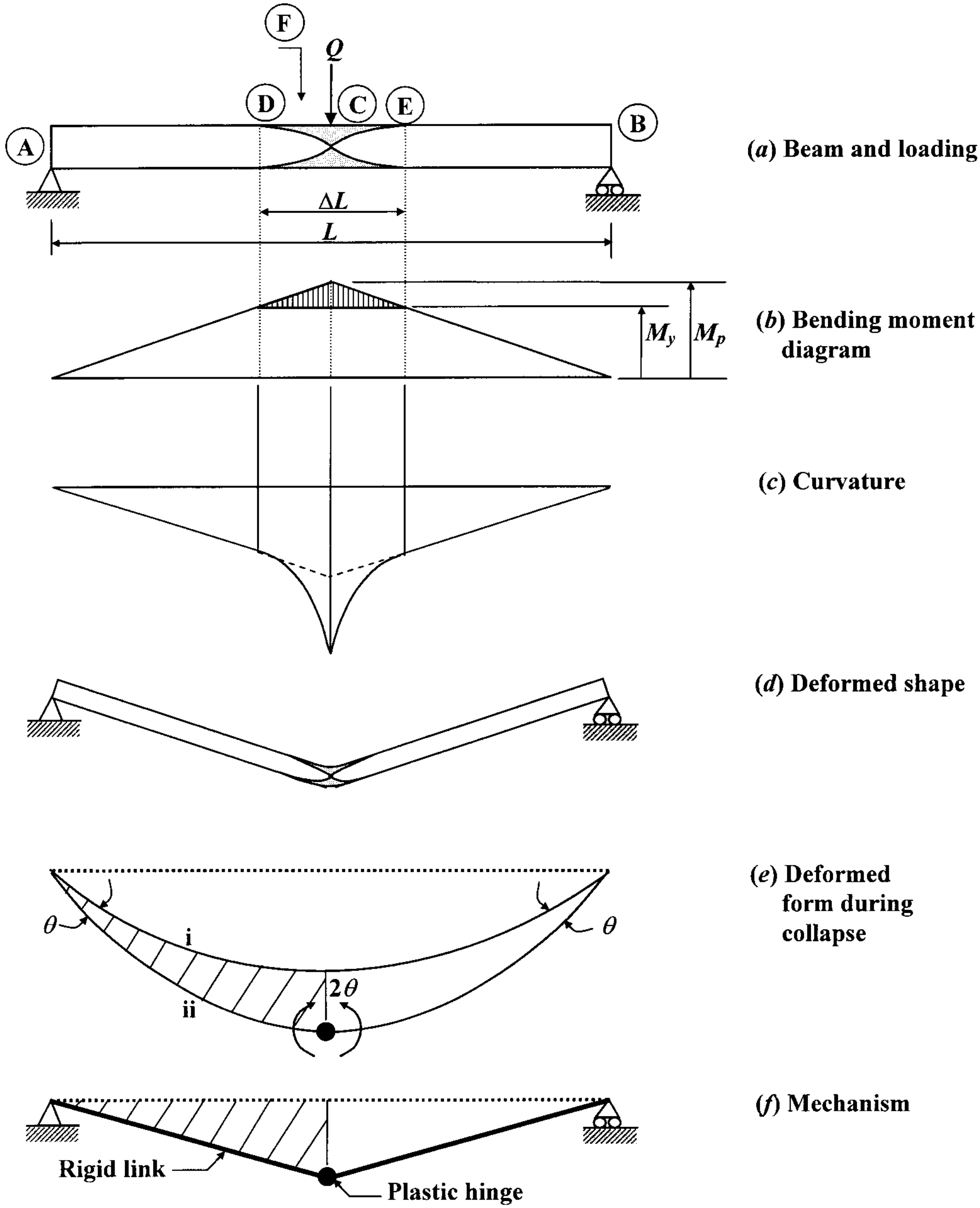


Figure W9.1.1: Behavior of a simply supported beam under a central concentrated load.

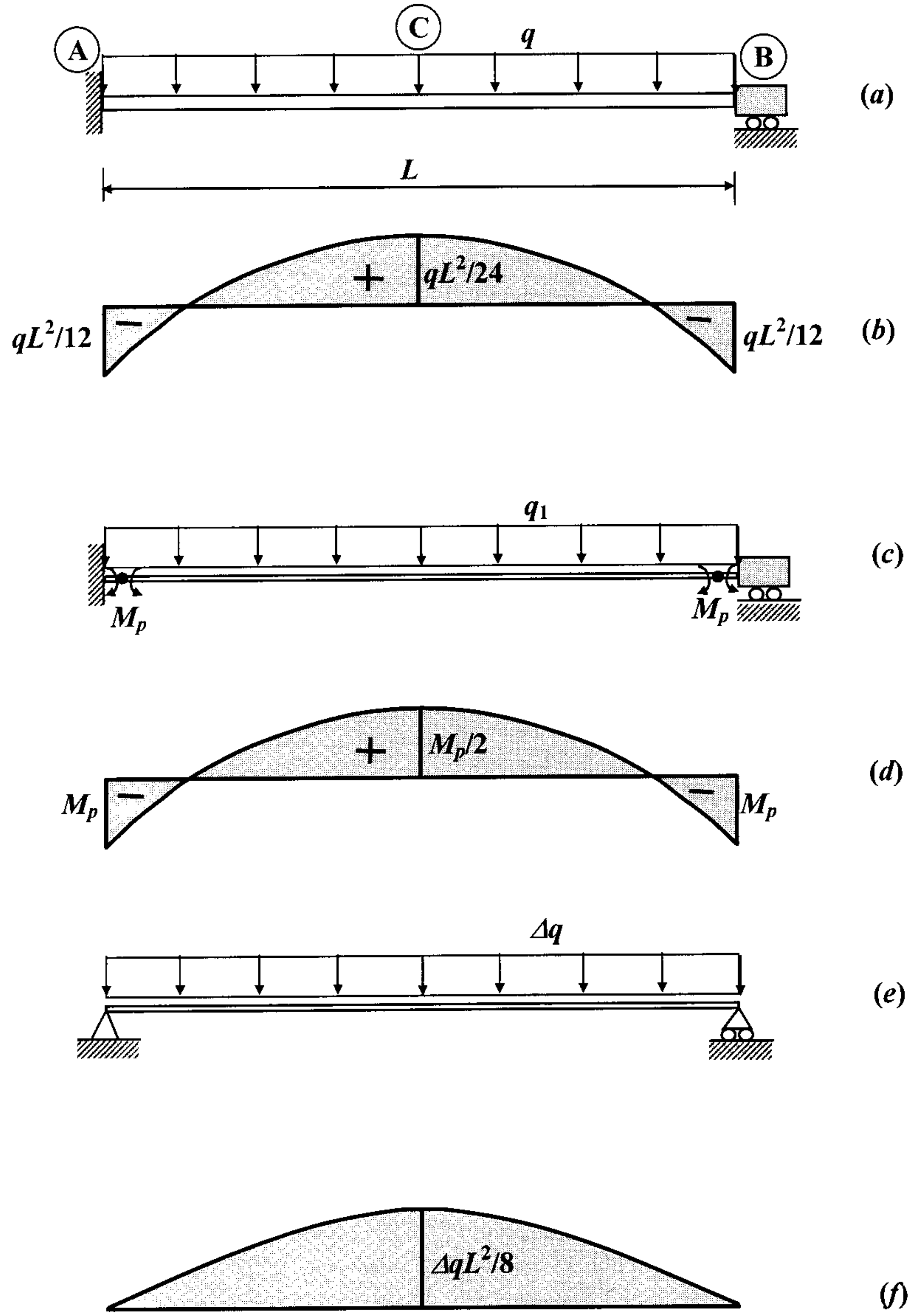
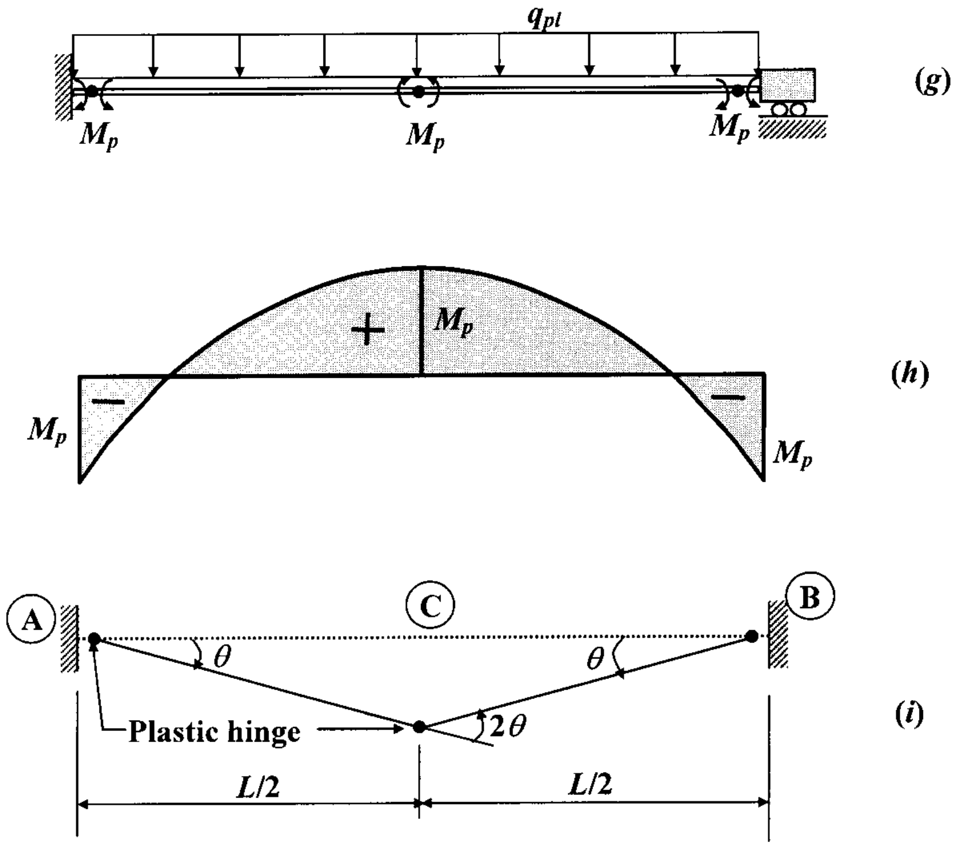


Figure W9.1.2: Behavior of a fixed-ended beam under uniformly distributed load.



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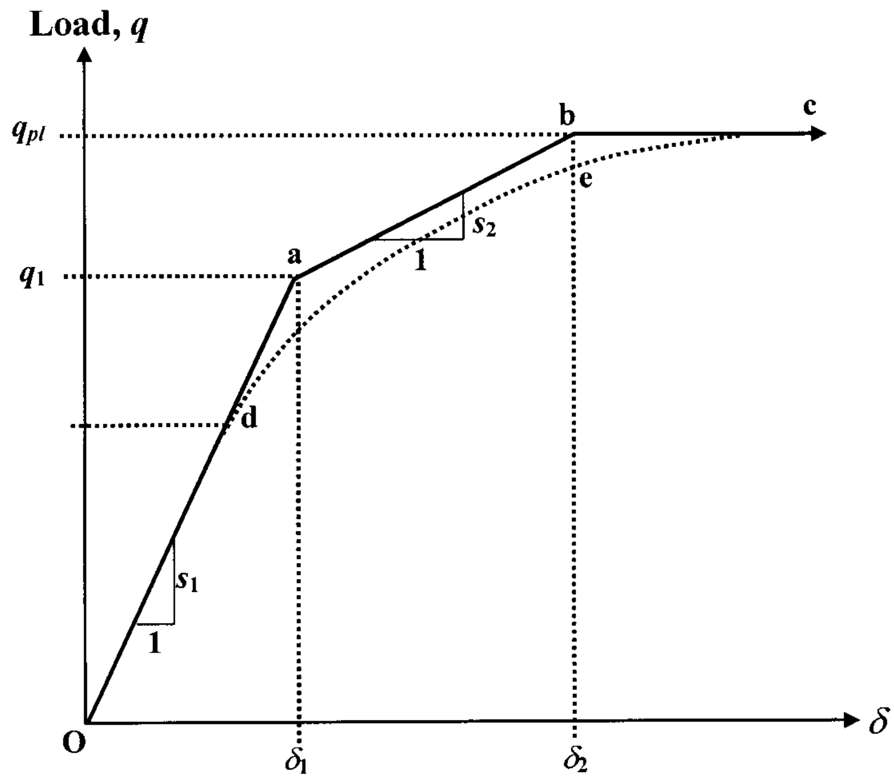


Figure W9.1.3: Load-deflection relation for fixed-ended beam under a uniformly distributed load.

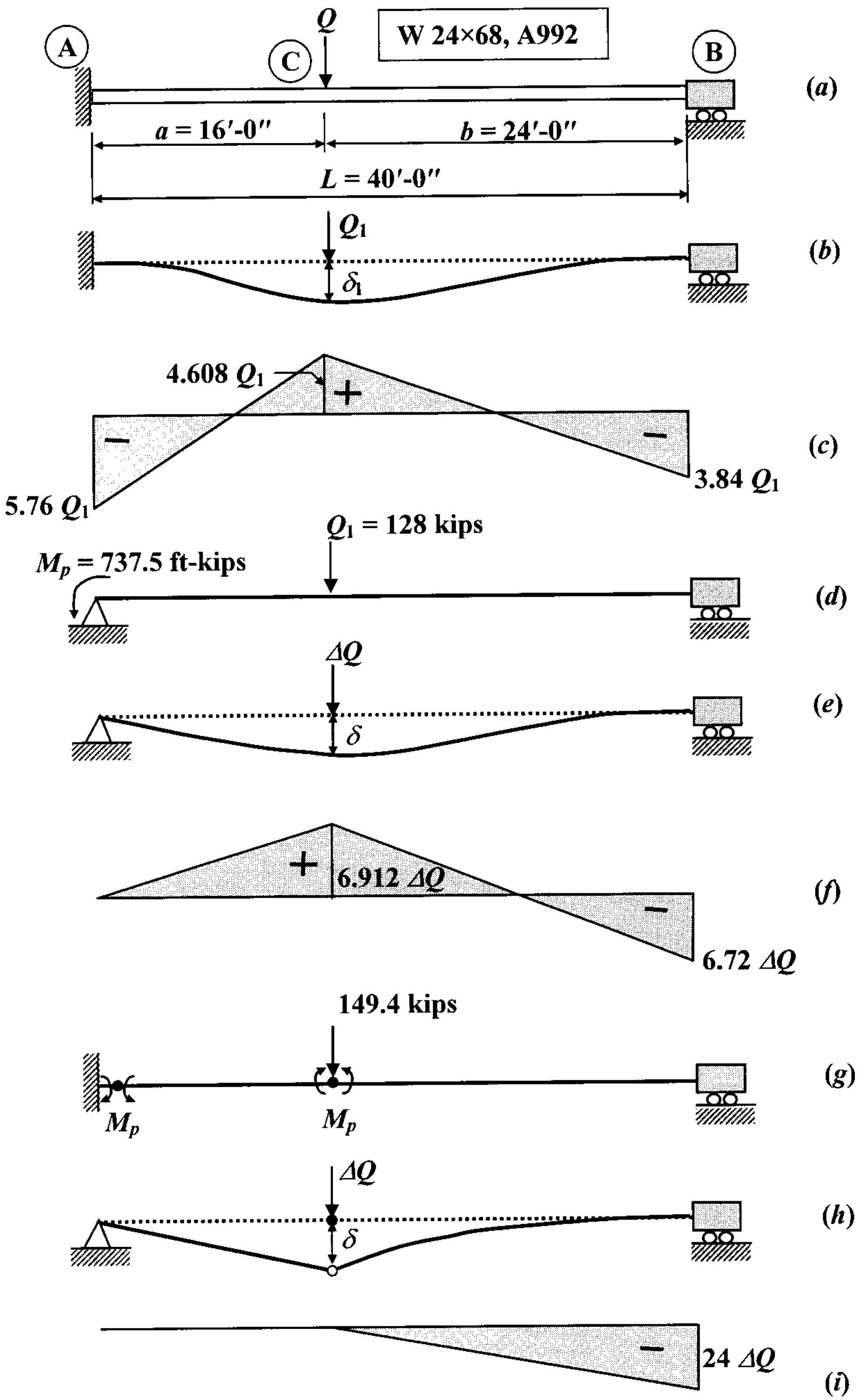


Figure WX9.1.1: **Behavior of a fixed-ended beam under a concentrated load.**

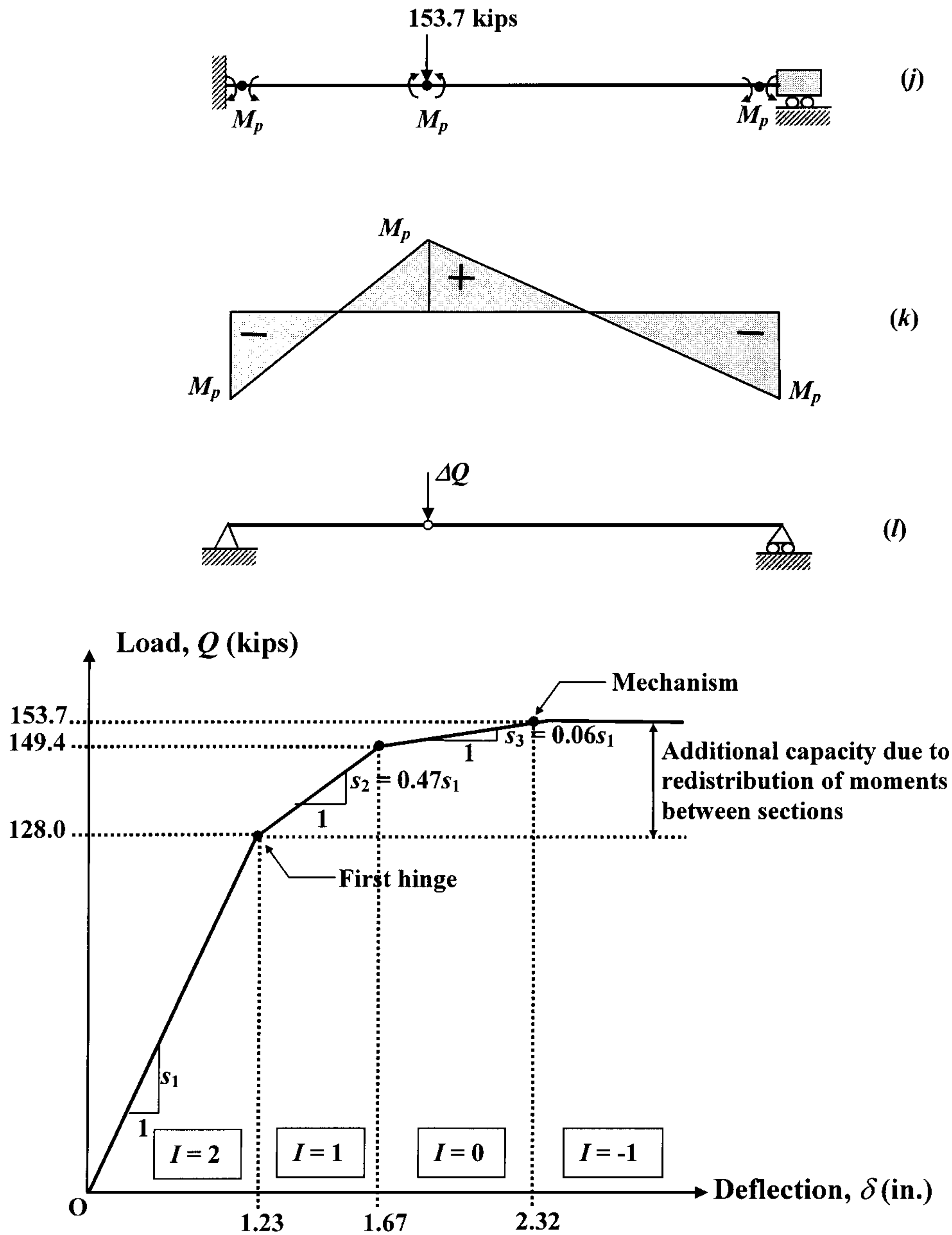
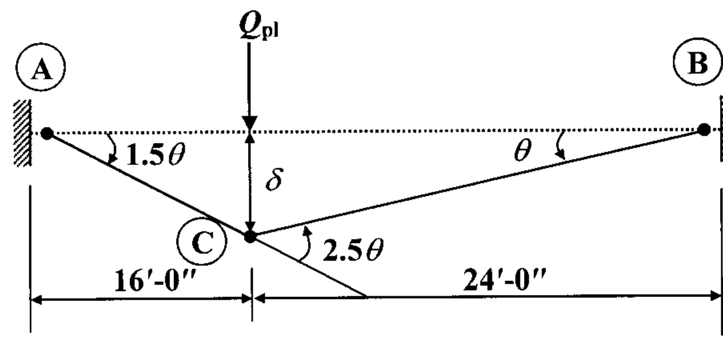
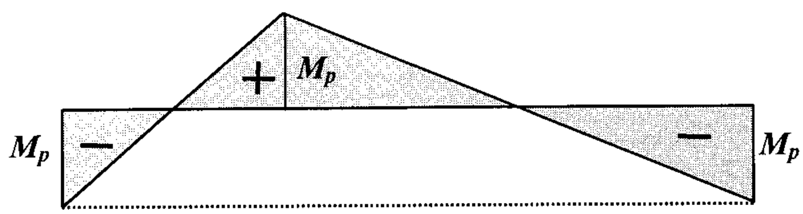


Figure WX9.1.1: (Continued.)

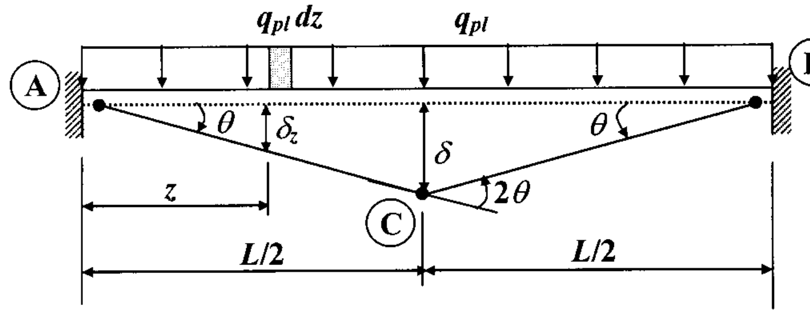


(*a*)

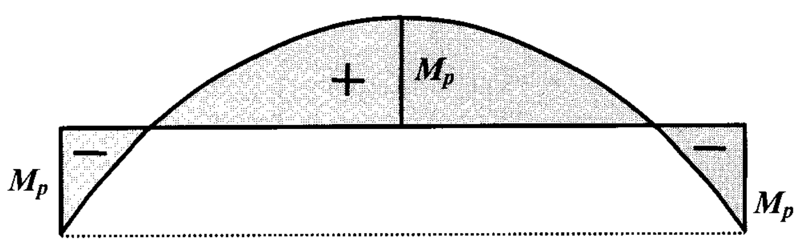


(*b*)

Figure: WX9.1.2



(*a*)



(*b*)

Figure: WX9.1.3

Figure W9.2.1 **Longspan and deep longspan steel joists**
TO COME SHORTLY.

Figure W9.2.2 **LRFD load table and weight table for steel joists**
TO COME SHORTLY.

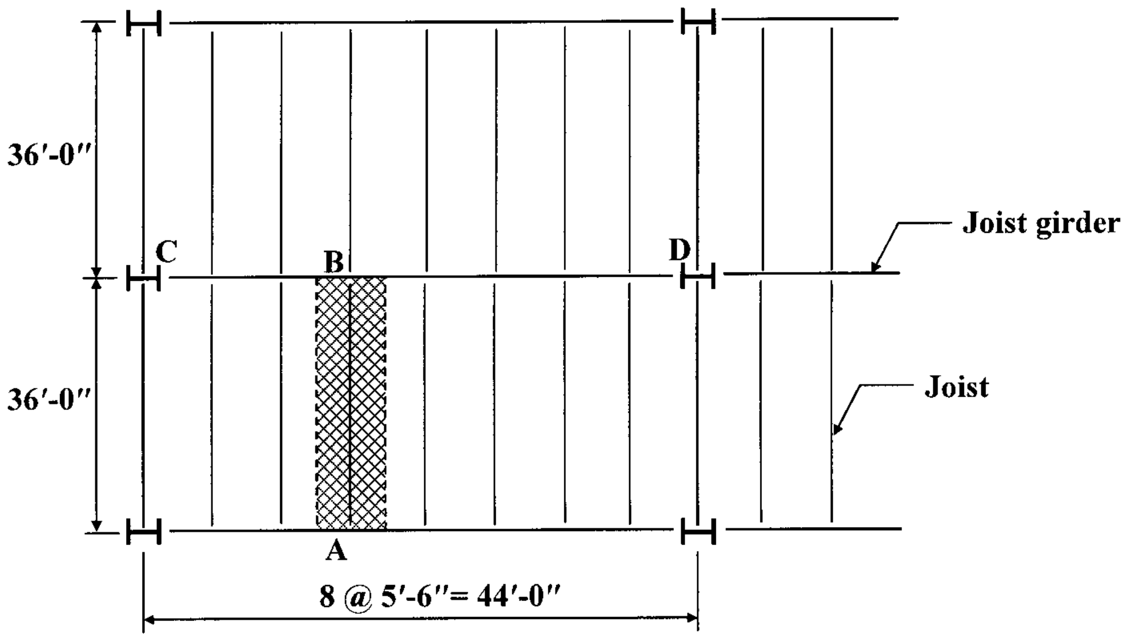


Figure: WX9.2.1