This is an unedited, uncorrected chapter.

The final chapter will be available in time for fall.

NOTE: Figures and tables appear at the end of the chapter.

WEB CHAPTER 10

Unbraced Beams

W10.1 Influence of Various Parameters on Elastic Lateral Buckling of Beams

The elastic lateral-torsional buckling moment of a doubly symmetric I-shaped beam with torsionally simple end conditions, under the action of constant moment in the plane of the web is given by

$$M_{cr}^{o} = \frac{\pi}{L}\sqrt{EI_y GJ}\sqrt{1 + \frac{\pi^2}{L^2}\frac{EC_w}{GJ}} \qquad \text{(W10.1.1)}$$

where M^{o}_{cr} = critical, uniformly distributed major-axis moment

L = distance between lateral supports

E = modulus of elasticity

G = shear modulus

I_y = moment of inertia about the minor axis

J = St. Venant torsional constant

C_w = warping torsional constant

EI_y = minor axis flexural rigidity of the beam cross section

EC_w = warping rigidity of the beam cross section

GJ = torsional rigidity of the beam cross section

W10.1.1 Influence of the Ratio I_y / I_x on Lateral-Torsional Buckling of I-Beams

Equation W10.1.1 is independent of the major axis flexural rigidity EI_x. This is a consequence of employing the assumption that the pre-buckling deflections in the vertical plane are small enough to be neglected in writing the equilibrium equations. Such an assumption is justifiable when the major axis flexural rigidity EI_x is very much greater than the minor axis flexural rigidity EI_y. Including the effect of pre-buckling deflections in the formulation of the problem results in a more precise value of the critical moment [Vacharajittiphan et al., 1974; Vinnakota et al., 1975] as:

$$M_{cr}^{o} = \frac{1}{\sqrt{\eta}} \frac{\pi}{L} \sqrt{EI_y GJ} \sqrt{1 + \frac{\pi^2}{L^2} \frac{EC_w}{GJ}} \qquad \text{(W10.1.2)}$$

where

$$\eta = 1 - \frac{I_y}{I_x} \qquad \text{(W10.1.3)}$$

This correction factor, η, is just less than unity for most beam shapes but may be significantly less than unity for most column shapes. Regardless it is usually neglected in design. As the value of I_y approaches that of I_x, the parameter η tends to zero and the value of M_{cr}^{o} given by Eq. W10.1.2 approaches infinity. Also, for the special case where I_y exceeds I_x, no solution exists indicating that ***lateral-torsional buckling (LTB)*** of doubly symmetric shapes is only possible if: (1) the cross section of the beam possesses different rigidities in the two principal planes; and (2) the applied loading causes bending in the stiffer plane. Thus, lateral-torsional buckling cannot occur if the moment of inertia about the bending axis is equal to or less than the moment of inertia out of plane. That is, for shapes bent about the minor axis, and shapes for which $I_x = I_y$, such as square or circular shapes, the limit state of lateral-torsional buckling is not applicable and yielding controls if the section is compact.

W10.1.2 Influence of the End Support Conditions

Figure W10.1.1: Laterally and torsionally fixed I-beam under uniform moment.

First, let us consider a doubly symmetric I-shaped beam whose ends are free to rotate about the major axis, but are fully restrained against all other deformations at the ends (Fig. W10.1.1). The boundary conditions are therefore

$$\begin{aligned} &v = 0; \quad v'' = 0 && \text{at } z = 0 && \text{and } z = L \\ &u = 0; \quad u' = 0 && \text{at } z = 0 && \text{and } z = L \\ &\phi = 0; \quad \phi' = 0 && \text{at } z = 0 && \text{and } z = L \end{aligned} \tag{W10.1.4}$$

The beam is subjected to pure bending by moments M^o at each end. The lateral-torsional buckling strength of the torsionally fixed beam considered can be shown to be [Chajes, 1974], for example):

$$M_{crF}^{o} = \frac{\pi}{L/2} \sqrt{EI_y \, GJ} \sqrt{1 + \frac{\pi^2}{(L/2)^2} \frac{EC_w}{GJ}} \tag{W10.1.5}$$

The simple beam with torsionally simply supported end conditions considered in Section 10.2, and the simple beam with torsionally fixed end conditions considered in this section, represent a set of limits between which most actual torsional restraint end conditions lie (for a simple beam). Comparing Eq. W10.1.5 with Eq. W10.1.1, we observe that the critical moment of the torsionally fixed beam can be anywhere from two to four times as large as the critical moment of the torsionally simple beam. Thus, if the warping stiffness is negligible compared to the St. Venant stiffness, the strength of the fixed beam is twice that of the hinged beam. However, if the St. Venant stiffness is negligible compared to the warping stiffness, the strength is increased by a factor of four in going from hinged to fixed conditions. Note that the lateral bending strength and warping strength vary inversely with the length of the member, whereas the St. Venant torsional strength does not. Also, the warping and lateral bending strengths are affected by changes in the boundary conditions whereas the St. Venant torsional strength is not.

The influence of other than simple lateral-torsional boundary conditions on the elastic buckling of doubly symmetric beams loaded by uniform moment may conveniently be included using the formula:

$$M^{o}_{cr\,R} = \sqrt{\frac{\pi^2 E I_y}{(K_y L)^2}} \sqrt{GJ + \frac{\pi^2 E C_w}{(K_z L)^2}} \qquad \text{(W10.1.6)}$$

Here, the coefficients K_y and K_z are effective length factors which account for the boundary conditions of the lateral deflection, u, and rotation, ϕ, respectively, on the critical moment of the I-beam. Thus, for torsionally fixed boundary conditions considered in Fig. W10.1.1 we observe from Eqs. W10.1.5 and W10.1.1 that $K_y = 0.5$ and $K_z = 0.5$. Several idealized end restraint conditions for lateral buckling of beams are shown in Fig. W10.1.2. Values for K_y and K_z for a number of boundary conditions are given in Table W10.1.1 [Vlassov, 1961; Galambos, 1968].

Figure W10.1.2: Idealized end restraint conditions for lateral buckling of beams.

TABLE W10.1.1: Effective Length Factors (K_y and K_z) for Lateral-Torsional

Buckling [Galambos, 1968]

Accurate evaluation of the degree of restraint provided by the actual connections in practice is difficult. In view of the imprecise nature of our knowledge of the degree of torsional restraint provided, Eq. W10.1.6 is simplified by assuming that

$$K_y = K_z = K_b \tag{W10.1.7}$$

where K_b is the ***effective length factor of the beam***. The effective length may be defined as the length of a torsionally simple beam, of similar section subjected to uniform moment, which will have the same elastic critical moment as the beam considered. Thus, Eq. W10.1.6 may be rewritten as:

$$M^o_{crR} = \frac{\pi}{K_b L} \sqrt{EI_y} \sqrt{GJ + \frac{\pi^2}{(K_b L)^2} EC_w} \tag{W10.1.8}$$

The following conservative values for K_b are sometimes used [BSI 1969]:

- Ends unrestrained against lateral bending $K_b = 1.00$
- Ends partially restrained against lateral bending $K_b = 0.85$
- Ends practically fixed against lateral bending $K_b = 0.70$

One reason for the conservatism of the recommended values is that, as in the case of columns (see Section 8.5), practical beam end supports are unlikely ever to be capable of providing complete fixity against rotation and warping.

W10.1.3 Influence of the Lateral Supports

As discussed qualitatively in Chapter 9, the stability of a beam against lateral-torsional buckling may be increased by providing intermediate lateral supports. Such lateral supports induce higher modes of buckling of the compression flange-column.

Let us consider a flexurally and torsionally simply supported beam of length L. The middle section of the beam is prevented from rotating with respect to the shear center axis of the beam by providing a lateral brace at midlength (Fig. W10.1.3*a* and *b*). Assuming that the beam is subjected to pure bending in the *yz* plane, the critical value of the bending moment can be again obtained from Eq. 10.2.20. However, the presence of the lateral bracing induces an inflection point in the lateral deflection curve of the buckled beam, as shown in the plan view of the center line of the compression flange of the buckled beam in Fig. W10.1.3*c*. The integration constants A_1, A_2, A_3 and A_4 may now be determined from the boundary conditions:

(W10.1.9)

$\phi = 0$ and $\phi'' = 0$ at $z = 0$ due to torsionally simple end conditions

$\phi = 0$ and $\phi'' = 0$ at $z = L/2$ due to anti-symmetry of the buckled shape

Figure W10.1.3: Lateral buckling of a simple beam with lateral support at midspan.

The eigenvalues of the differential equation are now given by:

$$\Omega(L/2) = n\pi \qquad \text{(W10.1.10)}$$

The lowest critical moment corresponds to the lowest value of the integer n (i.e., for $n = 1$), and is given by:

$$M^o_{cr} = \frac{\pi}{(L/2)}\sqrt{EI_y}\sqrt{GJ + \frac{\pi^2}{(L/2)^2}EC_w} \quad \text{(W10.1.11)}$$

By defining ***the unbraced length L_b of the beam*** as the distance between two adjacent lateral braces, the above equation may be rewritten as:

$$M^o_{crB} = \frac{\pi}{L_b}\sqrt{EI_y}\sqrt{GJ + \frac{\pi^2}{L_b^2}EC_w} \quad \text{(W10.1.12)}$$

Thus, if the compression flange of a simply supported I-beam subjected to uniform bending is supported at intermediate points by equally spaced lateral braces, the unbraced length L_b shall be used in the determination of the lateral buckling load of the beam from Eq. W10.1.12.

W10.1.4 Influence of the Transverse Loads

Let us reconsider the flexurally and torsionally simply supported beam studied in Section 10.2 and determine its lateral buckling load if the member is bent by a central concentrated load Q instead of being subjected to pure moment (Fig. W10.1.4*a* and *b*). The point of application of the load coincides with the shear center of the section (midheight of the shape for the symmetric I-shape considered).

Figure W10.1.4: Lateral buckling of a simple beam under central concentrated load.

Figure W10.1.4 shows the manner in which the load Q moves as the member deforms. Prior to buckling ($Q < Q_{cr}$), as the member bends in the vertical plane, Q moves from position 1 to 2. Then as buckling occurs at $Q = Q_{cr}$, the load moves from position 2 to 3. The movement from position 2 to 3, at buckling under load Q_{cr}, consists of a horizontal displacement u_C, additional vertical displacement v_C, and a rotation ϕ_C. The line of action of the load remains vertical during this movement. The subscript c denotes the fact that the displacements being considered are at midspan (point C).

The critical value of the load can be shown to be [Chajes, 1974]:

$$Q_{cr} = \frac{4\pi^2}{L^2}\sqrt{\frac{3}{\pi^2+6}EI_y GJ}\sqrt{1+\frac{\pi^2}{L^2}\frac{EC_w}{GJ}} \qquad \text{(W10.1.13)}$$

The critical value of the bending moment at the center of the span (Fig. W10.1.6*c*) is therefore

$$\begin{aligned} M_{Q\,cr} &= \frac{Q_{cr}L}{4} \approx 1.36\left[\frac{\pi}{L}\sqrt{EI_y}\sqrt{GJ+\frac{\pi^2}{L^2}EC_w}\right] \\ &= C_b M_{cr}^o \end{aligned} \qquad \text{(W10.1.14)}$$

where M_{cr}^o is the lateral buckling load of the same beam under uniform moment (Eq. W10.1.1). C_b is known as the ***moment modification factor*** or ***beam bending coefficient***, and accounts for the effect of nonuniform distribution of major axis bending moment on the critical value of the bending moment. It is equal to 1 for uniform bending moment, and 1.36 for a concentrated load at midspan. The elastic flexural-torsional buckling of simply supported beams with other loading conditions has also been investigated. The factor C_b has been found to be equal to 1.13 for a uniformly distributed load and 1.04 for two equal concentrated loads at the third points [Clark and Hill, 1962].

Figure W10.1.5: Lateral buckling of a beam under nonuniform moment.

As we have seen in Chapter 9, the compression flange of a beam acts as a pseudo-column, that tends to buckle sideways between points of lateral support, resulting in the phenomenon of lateral- torsional buckling. The bending moment diagram over the unbraced length determines the amount and distribution of axial load to which the compression flange-column is subjected. It experiences zero axial force at points of zero bending moment; the axial force increases as the point of maximum moment in the beam is approached. If a beam is subjected to pure bending moment, its flange-column experiences a constant axial compressive force along its length. By way of contrast, for a simply supported beam with a single concentrated load placed off-center of the span (Fig. W10.1.5*a*), the flange-column will have zero axial force at its ends and will be subjected to uniform increments of axial force, transferred from the web to the flange, as we move towards the load point D (Fig. W10.1.5*d*). As a result, the maximum axial force P_f of the flange-column occurs at the load point. Now, the critical axial load for a column loaded by distributed axial forces in the manner of a beam compression flange will always be greater than that for a column of equal dimensions and supports loaded entirely at its ends. It follows that lateral-torsional buckling is triggered more readily in segments of the beam where the bending moment is uniform than in segments with nonuniform distributions of moment. This may be expressed by the relation:

$$M_{cr} = C_b M_{cr}^o \tag{W10.1.15}$$

where C_b is a ***moment modification factor*** used to adjust the flexural-torsional buckling strength for situations in which variations in the moment diagram occur within the unbraced length. The coefficient C_b has been found to be virtually independent of all factors other than the shape of the moment diagram over the unbraced length, L_b. Empirical relations to calculate the coefficient C_b for various beam support and loading conditions have been discussed in Section 10.4.3.

W10.1.5 Beams Under General Shear Center Loading

The critical moment for doubly symmetric I-shaped beams, when such members are loaded by end couples in the plane of the web, and/or by transverse loads applied at the shear center axis in the plane of the web, may be closely approximated, with the help of Eqs. W10.1.8 and 15, by:

$$M_{cr} = C_b \left(\frac{\pi}{K_b L_b} \right) \sqrt{E I_y G J} \sqrt{1 + \frac{\pi^2}{(K_b L_b)^2} \frac{E C_w}{G J}} \qquad \text{(W10.1.16)}$$

Here C_b, as just alluded to, is a coefficient that depends on the load distribution and end conditions, and K_b is the notional effective length factor of the beam. An alternate form of this relation is given by:

$$M_{cr} = C_b \left(\frac{\pi}{K_b L_b} \right)^2 \sqrt{E I_y E C_w} \sqrt{1 + \frac{(K_b L_b)^2}{\pi^2} \frac{G J}{E C_w}} \qquad \text{(W10.1.17)}$$

For hot-rolled sections of normal proportions, the two terms under the second root of Eq. W10.1.16 are of comparable magnitude. The value of the second radical in Eq. W10.1.16 approaches unity in the case of short beams and girders of very shallow or thick-walled sections, indicating that the St. Venant torsional resistance is dominant. On the other hand, for long beams and girders of deep or thin-walled sections, the second radical in Eq. W10.1.17 approaches unity and the resistance of the compression flange to buckling largely governs.

W10.1.6 Influence of Continuity in the Major- and Minor-Axis Planes of the Beam

Figure W10.1.6: Laterally and vertically continuous beams.

Quite often beams are provided with more than two lateral supports (braces), making the compression flange of the beam continuous in the lateral plane. The lateral-torsional buckling of such beams often involves interaction between adjacent segments. The discussion below is valid for single span beams divided into a number of segments by a series of lateral restraints, such as the beam shown in Fig. W10.1.6*a,* and also for beams continuous in the vertical plane over several spans and laterally braced at the support points only, such as the beam shown in Fig. W10.1.6*b*. If both the lengths of the individual segments and the patterns of moments within them are similar, then the interaction between adjacent segments may be slight or nonexistent. However, if the adjacent segments are shorter than the segment being considered, and/or the adjacent segments are less severely loaded, then the adjacent segments may provide significant restraint to the segment under consideration.

The simplest and most conservative solution to this problem is the one first proposed by Salvadori [1951] which ignores the effects of lateral continuity between adjacent segments. Lateral buckling moment for each unbraced segment is determined using Eq. W10.1.16 taking K_b = 1.0, and including the modification factor C_b corresponding to the moment diagram shape. The elastic critical moment of each segment so determined is then used to evaluate a corresponding beam load set, and the lowest of these is taken as the elastic critical load set. The result is then a conservative lower bound to the elastic lateral buckling load because each unbraced segment is assumed to be laterally and torsionally simply supported. Note that Salvadori's method is equivalent to assuming a buckled shape for LTB in which the points of inflection are located at the points of lateral restraint. Hence, in this method, $L_{b1} = L_{b2} = L/3$ for the laterally continuous beam shown in Fig. W10.1.6*a,* and $L_{b1} = L_{b2} = L$ for the vertically and laterally continuous beam shown in Fig. W10.1.6*b*.

Accounting for the end restraint provided by the adjacent segments on the critical segment at buckling can substantially increase the buckling load [Nethercot, 1983]. A procedure to account for this beneficial effect is given in the SSRC Guide [Galambos, 1998].

W10.1.7 Influence of the Level of Application of Transverse Load

Figure W10.1.7: Effect of position of load on lateral buckling.

Figure W10.1.8: Examples of top and bottom flange loading of beams.

When a beam is subjected to a system of transverse loads, its lateral-torsional buckling strength depends not only on the distribution of the loads along the longitudinal axis but also on the level of application of such loading relative to the shear center axis of the beam cross section. In Section W10.1.4, it was assumed that the central concentrated load is applied at the beam shear center axis. If, however, the transverse load is placed on the top flange of the beam (load Q_T in Fig. W10.1.7) there is an additional destabilizing torsional moment, $Q_T(d/2)\phi$ that develops as soon as the loaded cross section is twisted. Here d is the depth of the section and ϕ is the rotation of the section at the load point. As a consequence the lateral buckling load will be reduced. An example is a crane girder where the crane wheel loads are applied to the crane rail resting on the top of the crane girder (Fig. W10.1.8*a*). Conversely, if the transverse load is suspended from the bottom flange (load Q_B in Fig. W10.1.7), there is a stabilizing torsional moment, $Q_B(d/2)\phi$ that increases the lateral buckling load. An example is a runway beam with the hoist suspended from the bottom flange of the beam (Fig. W10.1.8*b*), where the loads are free to move sideway as the beam buckles.

The elastic lateral buckling strength of a doubly symmetric I-beam that takes into account the influence of the level of application of the transverse loads may be written as [Galambos, 1968; Clark and Hill, 1962]: (W10.1.18)

$$M_{cr} = C_b \frac{\pi}{(K_b L)} \sqrt{E I_y G J} \left[\sqrt{1 + \frac{\pi^2}{(K_b L)^2} \frac{E C_w}{G J} \left(1 + C_\ell^2\right)} \mp C_\ell \frac{\pi}{(K_b L)} \sqrt{\frac{E C_w}{G J}} \right]$$

The coefficient C_ℓ represents the stabilizing/destabilizing effect that occurs if the loads are applied at the bottom flange or top flange of the beam, respectively. The minus sign in the bracketed term is to be used for loads on the top flange and the plus sign for loads on the bottom flange. It can be seen from this equation that the effect of the level of application of the transverse loads is most significant in situations where warping effects are predominant (i.e., for deep, beam-type sections of short span rather than for shallow, column-type sections of long span). C_ℓ is zero if the load is applied at the centroid of the beam, and Eq. W10.1.18 reduces to Eq. W10.1.16. Similarly, C_ℓ is zero if the beam is loaded by end moments only. For simply supported beams, $C_\ell = 0.55$ for a concentrated load at midspan and 0.45 for a uniformly distributed load. Values of C_b, C_ℓ and K_b for a wide variety of support and loading conditions from Clark and Hill [1962] are given in Table W10.1.2. If the transverse loading is applied in such a fashion that twisting of the loaded cross sections is prevented, then the actual level of application of the loading will have no effect.

TABLE W10.1.2: Values of C_b and C_ℓ in Eq. W10.1.18 [Adapted from Clark and Hill, 1962].

EXAMPLE W10.1.1 **Influence of Level of Loading**

A W24×104 of A588 Gr 50 steel is to be used as a simply supported beam for a span of 36 ft with lateral supports at the ends only. Calculate the elastic lateral buckling load if a single concentrated load at midspan is applied at: (a) top flange; (b) shear center; and (c) bottom flange.

Solution

Data

From LRFDM Table 1-1: W-Shapes, for a W24×104

d = 24.1 in.; t_w = 0.500 in.

b_f = 12.8 in.; t_f = 0.750 in.

A = 30.6 in.^{2}; S_x = 258 in.^{3}; I_y = 259 in.^{4}

Also, for A588 Gr 50 rolled steel W-shapes:

E = 29,000 ksi; G = 11,200 ksi; F_y = 50 ksi

F_r = 10 ksi; $F_{yr} = F_y - F_r = 50 - 10 = 40$ ksi

The torsional constants may be calculated as follows:

$$J = \sum \frac{1}{3} b t^3 = 2\left(\frac{1}{3}\right)(12.8)(0.750)^3 + \frac{1}{3}(24.1 - 2\times 0.750)(0.500)^3$$

$$= 3.60 + 0.940 = 4.54 \text{ in.}^4$$

$$C_w = I_y \frac{(d - t_f)^2}{4} = 259\left(\frac{23.35^2}{4}\right) = 35{,}300 \text{ in.}^6$$

$$\sqrt{\frac{E\,C_w}{G\,J}} = \sqrt{\frac{(29{,}000)(35{,}300)}{(11{,}200)(4.54)}} = 142$$

Alternatively the torsional constants can be obtained from the LRFDM Table 1-25: W-Shapes Torsional Properties. Tabulated values for the coefficients J, C_w and $\sqrt{\dfrac{E\,C_w}{G\,J}}$ are 4.72 in.^{4}; 35,200 in.^{6} and 139, respectively.

Use, the more exact tabulated values for calculations.

For a flexurally and torsionally simply supported beam $K_b = 1$ and Eq. W10.1.18 for the lateral buckling strength reduces to

$$M_{cr} = C_b \frac{\pi}{L}\sqrt{EI_y GJ}\left[\sqrt{1 + \frac{\pi^2}{L^2}\frac{EC_w}{GJ}\left(1 + C_\ell^2\right)} \mp C_\ell \frac{\pi}{L}\sqrt{\frac{EC_w}{GJ}}\right]$$

For the simply supported beam, of length $L = 36$ ft, we obtain:

$$\frac{\pi}{L}\sqrt{EI_y GJ} = \frac{\pi}{36\,(12)}\sqrt{29{,}000\,(259)\,(11{,}200)\,(4.72)}$$

$$= 4{,}580 \text{ in.-kips} = 382 \text{ ft-kips}$$

$$\frac{\pi}{L}\sqrt{\frac{EC_w}{GJ}} = \frac{\pi}{36\,(12)}(139) = 1.01$$

For a central concentrated load on a simply supported beam, with no brace at load point, we have $C_b = 1.3$ from Table W10.1.2. So

$$M_{cr} = 1.3\,(382)\left[\sqrt{1 + 1.01^2\,(1 + C_\ell^2)} \mp 1.01\,C_\ell\right]$$

$$= 497\left[\sqrt{2.02 + 1.02\,C_\ell^2} \mp 1.01\,C_\ell\right]$$

a. Top flange loading

For top flange loading, $C_\ell = 0.55$ and we have to use the negative sign in the above relation giving:

$$M_{crT} = 497\left[\sqrt{2.02 + 1.02\,(0.55)^2} - 1.01 \times 0.55\right]$$

$$= 497\,(0.97) = 482 \text{ ft-kips}$$

Critical top flange load,

$$Q_{crT} = \frac{4}{L}M = \frac{4}{36}(482) = 53.6 \text{ kips} \qquad \text{(Ans.)}$$

$$f_{\max} = \frac{M_{\max}}{S_x} = \frac{482\,(12)}{258} = 22.4 \text{ ksi} < F_{yr} = 40 \text{ ksi}$$

b. Shear center loading

For shear center loading, $C_\ell = 0$ and we obtain

$$M_{cr} = 497\left[\sqrt{2.02}\right] = 706 \text{ ft-kips}$$

$$Q_{cro} = \frac{4\,(706)}{36} = 78.4 \text{ kips} \qquad \text{(Ans.)}$$

$$f_{\max} = \frac{706\,(12)}{258} = 32.8 \text{ ksi} < F_{yr} = 40 \text{ ksi, as assumed.}$$

c. Bottom flange loading

For bottom flange loading, $C_\ell = 0.55$, and we have to use the positive sign, resulting in:

$$M_{crB} = 497\left[\sqrt{2.02 + 1.02\,(0.55)^2} + 1.01 \times 0.55\right]$$
$$= 497\,(2.08) = 1030 \text{ ft-kips}$$

$$Q_{crB} = \frac{4\,(1030)}{36} = 114 \text{ kips} \qquad \text{(Ans.)}$$

$$f_{\max} = \frac{1030\,(12)}{258} = 48.0 \text{ ksi} > F_{yr} = 40 \text{ ksi}$$

Since f_{max} for case (c) exceeds the reduced yield stress F_{yr}, the assumption of elastic behavior is not valid for bottom flange loading.

Remark

This example shows the influence of level of loading on the lateral buckling strength of unbraced beams.

W10.2 Lateral Torsional Buckling of Cantilever Beams

Figure W10.2.1: Lateral buckling of a cantilever beam.

A flexural member AB which, in the plane of loading, is built-in at one end ($v_A = 0$; $v_A' = 0$) and free at the other end (v_B, v_B' unrestrained) is known as a cantilever. Such a member is restrained against out-of-plane deformation at the support ($u_A = 0$; $u_A' = 0$; $\phi_A = 0$; $\phi'_A = 0$); however any such deformation at the free end is permitted (u_B, u_B', ϕ_B, ϕ_B' unrestrained). The absence of out-of-plane restraints at the free end of cantilevers drastically changes the buckling mode and buckling strength of such beams as compared to the simply supported beams considered in earlier sections. In a cantilever subjected to concentrated or distributed gravity loads, it is the tension flange which moves farther during buckling, while it is the compression flange in the case of simple beams (Fig. W10.2.1). Cantilevers with end moments that cause uniform bending are rare in practice as most bending actions are due to loads rather than moments. The most severe loading condition for a cantilever normally corresponds to a point load acting at the tip. Nethercot [1973] has shown that for design applications a simple method using the effective length factor, K_b (in Eq. W10.1.16), is satisfactory to determine the critical moment $M_{cr,\,\mathrm{Ca}}$ of a cantilever beam. Thus,

$$M_{cr\,\mathrm{Ca}} = C_b \left(\frac{\pi}{K_b L_b} \right) \sqrt{E I_y G J} \sqrt{1 + \frac{\pi^2}{(K_b L_b)^2} \frac{E C_w}{G J}} \qquad \text{(W10.2.1)}$$

Figure W10.2.2 gives recommended values of effective lengths $K_b L$ for a number of cantilever beams. The table is applicable to both end load and uniformly distributed load cases. It is seen from the table that prevention of twist at the tip is more effective than prevention of lateral deflection. The table is valid for both end load and uniformly distributed load cases.

Figure W10.2.2: Effective lengths $K_b L$, for cantilever beams [Nethercot, 1983].

At first glance the use of a value of K_b less than 1.0 for the basic cantilever may seem incorrect. This can be explained as follows: In a simply supported beam under transverse loads, the maximum compression in the top flange occurs somewhere near the middle of the span, where the beam is free to buckle sideways. On the other hand, the maximum compression in the bottom flange of a cantilever beam is at the support, where the beam is fully restrained against buckling. The cantilever is free to deflect laterally at points of low bending stress, whereas the simple beam is free to deflect laterally at points of high bending stress. It follows that the cantilever has more resistance to lateral buckling than a simple span of equal length, and a conservative value for the effective unbraced length of a properly braced cantilever can be taken as the actual length of the beam.

Cantilever spans will also occur in practice as the overhanging span (say, BC) of a beam (ABC) otherwise continuous in its loaded plane ($v_A = 0$; $v_B = 0$; $v_B'{}_{\text{left}} = v_B'{}_{\text{right}}$, continuity; and v_C, v_C' unrestrained). Nethercot [1973] determined that for such cantilevered end spans, the conditions of lateral restraint at the fulcrum (the most outward point of the vertical at the support nearest the free end) are more significant than the conditions of lateral restraint at the root of the support. More specifically, failure to effectively prevent twist at the fulcrum results in greatly increased K_b values (Fig. W10.2.2).

W10.3 Flexural Strength of Compact Beams Other Than I - Shapes (Bars, Rectangular HSS, C-, Singly-Symmetric I-, and T-Shapes)

I-sections are the most commonly used shapes for beams as they are easier to produce, and more importantly, are much easier to connect to other members. However, in situations where the beam is to be used in a laterally unsupported state over long spans such as crane booms, use of box sections having high torsional stiffness (and hence, high flexural-torsional buckling strength), instead of I-shapes results in economy. Also, channels are often used as beams (purlins, girts, eave struts, lintels, trimmers and headers) for stairwells, lift shafts and other openings.

The beams considered in Sections W10.1 to W10.2 are doubly symmetric beams of uniform cross section. Singly symmetric shapes are sometimes used in structures for a variety of reasons. For example, to provide increased lateral resistance to the beam compression flange in situations where lateral bracing is impossible or expensive. An example is a crane runway beam. Another example is a steel-concrete composite beam, where the concrete slab acts as a very large compression flange, allowing the steel top-flange to be just large enough to accommodate the shear connectors. Here, the small flange is in compression prior to placing of the concrete and the steel beam may be unusually susceptible to lateral-torsional buckling during construction stage. The manufacturers of metal building systems use singly symmetric sections extensively in their effort to optimize rigid frame design.

Behavior and design of these members will be considered in the following sections.

W10.3.1 Rectangular Bars and Rectangular HSS

For a rectangular section the warping constant $C_w = 0$, and so from Eq. W10.1.16, we obtain the critical buckling strength of a beam with a rectangular cross section as:

$$M_{cr} = C_b \frac{\pi}{K_b L_b} \sqrt{E I_y G J} \qquad \text{(W10.3.1)}$$

For a solid rectangular cross section of usual structural proportions ($d > 2t$, where d is the depth and t is the thickness), a sufficiently accurate formula for the torsional constant J is:

$$J = \frac{dt^3}{3} \qquad \text{(W10.3.2)}$$

The extreme fiber stress at which buckling occurs is obtained by dividing the critical moment (Eq. W10.3.1) by the section modulus, $S_x = I_x/(d/2)$ where $I_x = \frac{1}{12} t d^3$. Thus:

$$f_{cr} = \frac{C_b}{K_b} \frac{\pi \sqrt{GE}}{(L_b/t)} \sqrt{\frac{I_y}{I_x}} \qquad \text{(W10.3.3)}$$

indicating that lateral buckling is more likely to occur in beams that are relatively deep, narrow, and /or long.

By letting $I_y = Ar_y^2$, $E = 29{,}000$ ksi and $G = 11{,}200$ ksi in Eq. W10.3.1, the lateral-buckling moment for a solid rectangular bar can be rewritten in the form:

$$M_{cr} = \frac{C_b \sqrt{JA}}{(K_b L_b / r_y)} (56{,}620) \approx \frac{57{,}000\, C_b \sqrt{JA}}{(K_b L_b / r_y)} \qquad \text{(W10.3.4)}$$

Equation W10.3.1 for the rectangular cross section can also be used to evaluate the elastic critical moment for rectangular hollow structural shapes and box girders of rectangular closed cross section, bent about the major axis. The following approximate expression for the torsion constant J of thin-walled box sections, based on Bredt's theory [Roark, 1965] can be used in Eq. W10.3.1:

$$J = \frac{4A_o^2}{\int_s \frac{ds}{dt}} = \frac{2b_o^2 d_o^2}{(b_o/t_f) + (d_o/t_w)} \tag{W10.3.5}$$

where A_o is the enclosed area within the middle planes of the plates making up the box perimeter. By letting $I_y = Ar_y^2$, $E =$ 29,000 ksi and $G =$ 11,200 ksi in Eq. W10.3.1, the lateral buckling moment of a rectangular box girder can be written as

$$M_{cr} = \frac{57{,}000\, C_b \sqrt{JA}}{(K_b L_b / r_y)} \tag{W10.3.6}$$

This is Eq. F1-14 of the LRFDS, for $K_b = 1$. The critical elastic lateral buckling stress of box girders of usual proportions is far above the yield stress. Failure will therefore usually be by inelastic lateral buckling or by local buckling.

For solid rectangular bars and box sections, the limiting laterally unbraced length L_p, for full plastic bending capacity under uniform moment, is given by LRFDS Eq. F1-5 as:

$$L_p = \frac{0.13 r_y E}{M_{px}} \sqrt{JA} \tag{W10.3.7}$$

with

$$M_{px} = \min\,[Z_x F_y;\;\; 1.5\, S_x F_y] \tag{W10.3.8}$$

where A = cross-sectional area

J = torsion constant

Also, for solid rectangular bars and box sections, the limiting laterally unbraced length, L_r, for inelastic lateral-torsional buckling, is given by LRFDS Eq. F1-10 as:

$$L_r = \frac{2 r_y E}{M_{rx}} \sqrt{JA} \tag{W10.3.9}$$

where

$$M_{rx} = S_x F_y \tag{W10.3.10}$$

The design flexural strength of solid rectangular bars and box sections, bent about the major axis is given by:

$$M_{dx} = \phi_b M_{nx} = \min [\, \phi_b Z_x F_y \,;\; \phi_b (1.5)\, S_x F_y \,] \quad \text{for } L_b \leq L_p \tag{W10.3.11a}$$

$$= \min [\, C_b M^o_{dI} \,;\; \phi_b M_{px} \,] \quad \text{for } L_p < L_b \leq L_r \tag{W10.3.11b}$$

$$= \min [\, C_b M^o_{dE} \,;\; \phi_b M_{px} \,] \quad \text{for } L_b > L_r \tag{W10.3.11c}$$

where

$$M^o_{dI} = \phi_b M_{px} - (\phi_b M_{px} - \phi_b M_{rx}) \frac{(L_b - L_p)}{(L_r - L_p)} \tag{W10.3.12}$$

$$M^o_{dE} = \phi_b \frac{57{,}000 \sqrt{JA}}{L_b / r_y} \tag{W10.3.13}$$

W10.3.2 Channels

For rolled channels used as beams, LRFD Specification permits design strength to be calculated by equations for doubly symmetric beams, given in Sections 10.4.1 and 2. The effect of eccentricity of load should be considered in the strength evaluation if the loads are not applied through the shear center.

EXAMPLE W10.3.1 **Plate and HSS as Beams**

A 572 Grade 50 steel beam is to be used as a simply supported beam on a span of 20 ft with lateral supports at the end only. Calculate the elastic lateral buckling moment and the maximum bending stress, if the section is:

a. 1 in.× 20 in. rectangular section bent about its major axis.

b. W21×68 shape.

c. HSS20×4×½.

Solution

Span, L = 20 ft; Unbraced length, L_b = 20 ft

Uniform moment over unbraced length. So, $C_b = 1.0$

a. Rectangle

b = 1.00 in.; d = 20.0 in.

$A = 20.0\,(1.00) = 20.0 \text{ in.}^2$; $S_x = \frac{1}{6}(1.00)\,(20.0)^2 = 33.3 \text{ in.}^3$

$I_y = \frac{1}{12}(20.0)\,(1.00)^3 = 1.67 \text{ in.}^4$; $J = \frac{1}{3}(20.0)\,(1.00)^3 = 6.67 \text{ in.}^4$

For a simply supported beam having a rectangular cross section and laterally supported at the ends only (Eq. W10.3.1):

$$M_{cr}^{o} = \frac{\pi}{L}\sqrt{EI_y GJ} = \frac{\pi}{20\,(12)}\sqrt{29{,}000\,(1.67)\,(11{,}200)\,(6.67)}$$

$$= 787 \text{ in.-kips} = 65.6 \text{ ft-kips}$$

$$f_{\max} = \frac{M_x}{S_x} = \frac{787}{32.3} = 24.4 \text{ ksi} < F_y = 50 \text{ ksi}$$

Elastic lateral buckling moment, $M^o_{cr} = 65.6$ ft-kips (Ans.)

b. W21×68

From LRFDM Tables 1-1: $A = 20.0$ in.^{2}; $S_x = 140$ in.^{3}; $I_y = 64.7$ in.^{4}

From LRFDM Table 5-3: $L_r = 17.3$ ft for a W21×68 of Grade 50 steel.

As $L_b = 20$ ft is greater than L_r, strength of the beam is limited by elastic lateral buckling.

From LRFDM Table 1-25, for a W21×68:

$$J = 2.45 \text{ in.}^4; \quad C_w = 6760 \text{ in.}^6; \quad \sqrt{\frac{EC_w}{GJ}} = 84.5$$

For a simply supported I-beam bent about its major axis, and laterally supported at the ends only (Eq. W10.1.1):

$$M^o_{cr} = \frac{\pi}{L}\sqrt{EI_y GJ}\sqrt{1 + \frac{\pi^2}{L^2}\frac{EC_w}{GJ}}$$

$$= \frac{\pi}{20\,(12)}\sqrt{29{,}000\,(64.7)\,(11{,}200)\,(2.45)}\sqrt{1 + \left(\frac{\pi}{20\,(12)}\right)^2 (84.5)^2}$$

$$= 4400 \text{ in.-kips} = 367 \text{ ft-kips}$$

Alternatively, from LRFDM Table 5-5, for a W21×68 of Grade 50 steel and $L_b = 20$ ft,

$\phi_b M_{cr}^o = 332$ ft-kips, resulting in $M_{cr}^o = 369$ ft-kips. (Ans.)

$f_{max} = (369 \times 12) / 140 = 31.6$ ksi $< F_{yr} = 40$ ksi as assumed.

c. HSS 20×4×½

Nominal wall thickness = ½ in. ; Design thickness = 0.465 in.

$$A = 20\,(4) - 19.07\,(3.07) = 21.1 \text{ in.}^2$$

$$I_y = \frac{1}{12}(20.0)\,(4.00)^3 - \frac{1}{12}(19.1)\,(3.07)^3 = 60.7 \text{ in.}^4$$

$$J = \frac{2\,b_o^2\,h_o^2}{\dfrac{b_o}{t_o} + \dfrac{h_o}{t_o}} = \frac{2\,(19.5)^2\,(3.54)^2}{\dfrac{19.5}{0.465} + \dfrac{3.54}{0.465}} = 192 \text{ in.}^4$$

Alternatively, from LRFDM Table 1-11, for a HSS 20×4×½ :

$A = 20.9 \text{ in.}^2$; $\quad I_y = 58.7 \text{ in.}^4$

$S_x = 83.8 \text{ in.}^3$; $\quad J = 195 \text{ in.}^4$

Use tabular values.

For a torsionally simply supported beam having a rectangular tube section

$$M^o_{cr} = \frac{\pi}{L}\sqrt{EI_y GJ} = \frac{\pi}{20\,(12)}\sqrt{29{,}000\,(58.7)\,(11{,}200)\,(195)}$$

$$= 25{,}200 \text{ in.-kips} = 2100 \text{ ft-kips} \qquad \text{(Ans.)}$$

$$f_{\max} = \frac{25{,}200}{83.8} = 301 \text{ ksi} \quad >> \quad F_y = 50 \text{ ksi}$$

Remarks

1. By using approximately the same amount of material in a more efficient fashion, (i.e., an I-shape versus a solid rectangle), the beam's moment carrying capacity is increased more than fivefold, although its collapse is still governed by elastic lateral buckling. This increase stems entirely from the increase in I_y (due to the presence of the flange) since the value of J is actually decreased (because the component plates are thinner), and warping effects are quite small.

2. Also, by more efficiently using about the same amount of material in the rectangular HSS as that in the W-shape, the beam's moment capacity is further increased. This increase is primarily due to the increase in J (for a closed section as compared to an open section).

EXAMPLE W10.3.2 Channel as Beam

A C10×20 channel of A36 steel is used as a simply supported, uniformly loaded beam. The span length is 20 ft and there is lateral support at the ends and midspan only. If the ratio of live load to dead load is 2.0, determine the service dead load and live load capabilities.

Figure WX10.3.2

Solution

a. Data

From LRFDM Table 1-5: C-Shapes Dimensions and Properties, for a C10×20:

$$A = 5.87 \text{ in.}^2; \qquad Z_x = 19.4 \text{ in.}^3; \qquad S_x = 15.8 \text{ in.}^3$$

$$b_f = 2.74 \text{ in.}; \qquad t_f = 0.436 \text{ in.}; \qquad t_w = 0.379 \text{ in.}$$

$$I_y = 2.80 \text{ in.}^4; \qquad r_y = 0.690 \text{ in.}; \qquad T = 8.0 \text{ in.}$$

Also, from LRFDM Table 1-29: C-Shapes Flexural-Torsional Properties, for a C10×20:

$$J = 0.368 \text{ in.}^4; \qquad C_w = 56.9 \text{ in.}^6$$

b. Limit state of plate local buckling

For A36 steel, $F_y = 36$ ksi:

$$\lambda_{pf} = 0.38\sqrt{\frac{E}{F_y}} = 0.38\sqrt{\frac{29{,}000}{36}} = 10.8$$

$$\lambda_{pw} = 3.76\sqrt{\frac{E}{F_y}} = 3.76\sqrt{\frac{29{,}000}{36}} = 107$$

For the C10×20 channel section

$$\lambda_f = \frac{b_f}{t_f} = \frac{2.74}{0.436} = 6.28; \qquad \lambda_w = \frac{h}{t_w} \approx \frac{T}{t_w} = \frac{8.00}{0.379} = 21.1$$

As $\lambda_f < \lambda_{pf}$ and $\lambda_w < \lambda_{pw}$, the section is compact and plate local buckling will not control the strength of the shape.

c. Limit state of lateral buckling

$$F_y - F_r = 36 - 10 = 26.0 \text{ ksi}$$

$$\phi_b M_{px} = \phi_b Z_x F_y = 0.90\,(19.4)\,(36) = 629 \text{ in.-kips} = 52.4 \text{ ft-kips}$$

$$\phi_b M_{rx} = \phi_b S_x (F_y - F_r) = 0.90\,(15.8)\,(26.0) = 370 \text{ in.-kips} = 30.8 \text{ ft-kips}$$

$$L_p = 1.76\, r_y\sqrt{\frac{E}{F_y}} = 1.76\,(0.69)\sqrt{\frac{29{,}000}{36}} = 34.5 \text{ in.} = 2.87 \text{ ft}$$

$$L_r = \frac{r_y X_1}{(F_y - F_r)} \sqrt{1 + \sqrt{1 + X_2 (F_y - F_r)^2}}$$

where

$$X_1 = \frac{\pi}{S_x} \sqrt{\frac{EGJA}{2}} = \frac{\pi}{15.8} \sqrt{\frac{29{,}000\,(11{,}200)\,(0.368)\,(5.87)}{2}} = 3{,}724 \text{ ksi}$$

$$X_2 = 4 \frac{C_w}{I_y} \left(\frac{S_x}{GJ} \right)^2 = \frac{4\,(56.9)}{2.80} \left(\frac{15.8}{(11{,}200)\,(0.368)} \right)^2 = 0.001195$$

Alternatively, the values of X_1 and X_2 can be read from the LRFDM Table 1-29: C-Shapes Flexural-Torsional Properties, as 3720 and 1190×10^{-6}, respectively.

$$L_r = \frac{0.690\,(3720)}{26.0} \sqrt{1 + \sqrt{1 + 0.001190\,(26.0)^2}} = 151 \text{ in.} = 12.6 \text{ ft}$$

Alternatively, for the C10×20 with F_y = 36 ksi, the value of L_p, L_r, $\phi_b M_{px}$, and $\phi_b M_{rx}$ can be obtained from LRFDM Table 5-9: C-Shapes - Maximum Total Factored Uniformly Distributed Load (2.87 ft, 12.6 ft, 52.4 ft-kips, and 30.8 ft-kips, respectively).

As, (L_p = 2.87 ft) < (L_b = 10 ft) < (L_r = 12.6 ft), the design bending strength for the segment L_b under uniform moment is obtained as:

$$M_d^o = \phi_b M_{px} - \frac{(\phi_b M_{px} - \phi_b M_{rx})}{(L_r - L_p)} (L_b - L_p)$$

$$= 52.4 - \frac{(52.4 - 30.8)}{(12.6 - 2.87)} (10 - 2.87) = 36.6 \text{ ft-kips}$$

Alternatively, the design moment for a C10×20 beam, for L_b = 10 ft, C_b = 1.0 and F_y = 50 ksi, can be read from Beam Design Plots for C-shapes (LRFDM Table 5-11) as

36.8 ft-kips.

For a simply supported beam with lateral supports at the ends and at midspan, under uniformly distributed load, $C_b = 1.30$ from Fig. 10.4.1, or from LRFDM Table 5-1.

The design strength of the given beam segment is:

$$M_d \;=\; \min\,[C_b\, M^o{}_d\,;\;\; \phi_b M_{px}] \;=\; \min\,[1.3(36.6);\;\; 52.4] \;=\; 47.6 \text{ ft-kips}$$

If q_u is the factored distributed load:

$$\frac{q_u L^2}{8} \;=\; 47.6 \quad \rightarrow \quad q_u \;=\; \frac{47.6\,(8)}{20^2} \;=\; 0.952 \text{ klf}$$

As the ratio of live load to dead load is 2.0, we can write:

$$1.2D + \;1.6L \;=\; 1.2D + \;1.6(2D) \;=\; q_u \;=\; 0.952$$

$$\rightarrow \quad D \;=\; 0.216 \text{ klf} \quad \rightarrow \quad L \;=\; 0.216\,(2) \;=\; 0.432 \text{ klf} \qquad \text{(Ans.)}$$

W10.3.3 Singly Symmetric I-Shapes

For doubly symmetric I-shapes considered in Sections W10.1 and W10.2, the shear center coincides with the center of gravity. For singly symmetric beams, such as unequal flanged I-s, the shear center S and the centroid G do not coincide. When such a singly symmetric I-beam loaded in its plane of symmetry twists during lateral buckling, the non-coincidence of S and G results in normal stresses at a section exerting a torque about the shear center. This torque may be shown to be [Galambos, 1968]:

$$M_T \;=\; \bar{K}\,\phi' \;=\; \left[\int_A f\rho^2\, dA\right]\phi' \;=\; M_x\,\beta_x\,\phi' \qquad \text{(W10.3.14)}$$

where f is the normal bending stress at any fiber of the cross section located at a distance ρ from the shear center S; β_x is the monosymmetry coefficient of the cross section; M_x is the bending moment associated with stress, f; and M_T is the resulting torque.

In a doubly symmetric beam such as a I-shape, the destabilizing torque developed by the compressive bending stresses is exactly balanced by the restoring torque due to the tensile bending stresses, and β_x equals zero. In a singly symmetric beam, on the other hand, there is an imbalance between these two torque components that is dominated by the stresses in the smaller flange (which will always be the flange farthest from the shear center). Thus, when the smaller flange is in compression there is a decrease in the effective torsional rigidity (M_T is negative), while the reverse is true (M_T is positive) when the smaller flange is in tension. As a result, the resistance to lateral buckling is increased when the larger flange is in compression, and decreased when the smaller flange is in compression.

Lateral buckling of singly symmetric I-beams has been studied by Anderson and Trahair [1972], Kitpornchai and Wong-Chung [1987], Kitipornchai et al. [1986], Nethercot [1973], Wang and Kitipornchai [1986].

The action of the torque M_T can be thought of as changing the effective torsional rigidity of the section from GJ to $(GJ + M_x \beta_x)$. For a singly symmetric I-beam under uniform moment M^o, the differential Eqs. 10.2.4 therefore become:

$$E I_x v'' + M^o = 0 \qquad \text{(W10.3.15}a\text{)}$$

$$E I_y u'' + M^o \phi = 0 \qquad \text{(W10.3.15}b\text{)}$$

$$EC_w \phi''' - (GJ + \overline{K})\phi' + M^o u' = 0 \qquad \text{(W10.3.15}c\text{)}$$

The first two of these equations are identical with Eqs. 10.2.4a and b. Note that, $f = \dfrac{M^o y}{I_x}$ and

$\rho^2 = (x - x_o)^2 + (y - y_o)^2$, where x_o and y_o are the coordinates of the shear center (with respect to the centroid). If we limit our discussion to shapes which exhibit symmetry about the plane of the web, $\rho^2 = x^2 + (y - y_o)^2$.

An expression for $\overline{K}$ may now be obtained from Eq. W10.3.14 as:

$$\overline{K} = \int \frac{M^o}{I_x} y \left\{(x^2 + y^2) - 2yy_o + y_o^2\right\} dA = M^o \beta_x \qquad \text{(W10.3.16)}$$

The general expression for the monosymmetry coefficient β_x may be written as

$$\beta_x = \left[\frac{1}{I_x} \int (x^2 + y^2) y dA\right] - 2y_o \qquad \text{(W10.3.17)}$$

where integration is over the entire cross-sectional area A. Negative values for y_o result when the larger flange is in compression (for these sections the shear center lies between the centroid and compression flange). The value of $\beta_x = 0$ for a doubly symmetric shape, such as an I-shape. An explicit formula for β_x for a general I-shaped singly symmetric beams is given by Kitipornchai and Trahair [1980] as:

$$\beta_x = 0.9h \left(2\frac{I_{yc}}{I_y} - 1\right)\left[1 - \left(\frac{I_y}{I_x}\right)^2\right] \qquad \text{(W10.3.18)}$$

where h is the distance between the centers of areas of the two flanges, I_{yc} is the minor axis moment of inertia of the compression flange, and I_y is the minor axis moment of inertia of the whole cross section.

Equation W10.3.15*c* may be rewritten as:

$$E\,C_w\,\phi''' \;-\; (GJ + M^o\beta_x)\,\phi' \;+\; M^o u' \;=\; 0 \qquad \text{(W10.3.19)}$$

The effect of monosymmetry is to cause the beam's effective torsional rigidity to be increased when the larger flange is in compression or decreased when the smaller flange is in compression (β_x is positive for the former situation and negative for the latter). The more critical loading case will therefore occur when the smaller flange is in compression. Because the neutral axis will be farther away from the smaller flange, it is also the flange with the greater stress.

For a singly symmetric I-shape, we have:

$$C_w \;=\; h^2\,\frac{I_{yc}\,I_{yt}}{I_y} \qquad \text{(W10.3.20}a\text{)}$$

where I_{yt} is the minor axis moment of inertia of the tension flange. Also, the torsional constant J, of a singly symmetric I-section may be calculated from:

$$J \;=\; \frac{1}{3}b_c t_c^3 \;+\; \frac{1}{3}b_t t_t^3 \;+\; \frac{1}{3}h_w t_w^3 \qquad \text{(W10.3.20}b\text{)}$$

Proceeding as in Section 10.2.1, we obtain (W10.3.21)

$$M_{cr}^o \;=\; \frac{\pi}{L}\sqrt{EI_y GJ}\left[\left(\frac{\beta_x}{2}\frac{\pi}{L}\sqrt{\frac{EI_y}{GJ}}\right) \;+\; \sqrt{\left\{\frac{\beta_x}{2}\frac{\pi}{L}\sqrt{\frac{EI_y}{GJ}}\right\}^2 \;+\; \left\{1 \;+\; \frac{\pi^2}{L^2}\frac{EC_w}{GJ}\right\}}\;\right]$$

When the moment gradient factor C_b and effective length factors for the unbraced length L_b are included, we obtain the general solution for the critical moment of a singly symmetric beam as

[Galambos, 1968]: (W10.3.22)

$$M_{cr} = C_b \frac{\pi}{K_y L_b} \sqrt{E I_y G J} \left[\left(\frac{\beta_x}{2} \right) \left(\frac{\pi}{K_y L_b} \right) \sqrt{\frac{E I_y}{G J}} + \sqrt{ \left\{ \frac{\beta_x}{2} \left(\frac{\pi}{K_y L_b} \right) \frac{E I_y}{G J} \right\}^2 + \left\{ 1 + \frac{\pi^2}{(K_z L_b)^2} \frac{E C_w}{G J} \right\} } \right]$$

By letting $K_y = K_z = K_b$, the above equation can be written as [Trahair, 1977]:

$$M_{cr} = C_b \frac{\pi \sqrt{E I_y G J}}{K_b L_b} \left[B_1 + \sqrt{B_1^2 + 1 + B_2} \right] \qquad \text{(W10.3.23)}$$

where

$$B_1 = \frac{\beta_x}{2} \frac{\pi}{K_b L_b} \sqrt{\frac{E I_y}{G J}} \qquad \text{(W10.3.24)}$$

$$B_2 = \frac{\pi^2}{(K_b L_b)^2} \frac{E C_w}{G J} \qquad \text{(W10.3.25)}$$

by letting $K_b = 1$ and substituting $E = 29{,}000$ ksi, $G = 11{,}200$ ksi, we obtain the following approximate relations for the lateral buckling moment of singly symmetric I-beams:

$$M_{cr} = C_b \frac{57{,}000}{L_b} \sqrt{I_y J} \left[B_1 + \sqrt{B_1^2 + 1 + B_2} \right] \qquad \text{(W10.3.26)}$$

where

$$B_1 = 2.25 \left[2 \left(\frac{I_{yc}}{I_y} \right) - 1 \right] \left[\frac{h}{L_b} \sqrt{\frac{I_y}{J}} \right] \qquad \text{(W10.3.27)}$$

$$B_2 = 25 \left(1 - \frac{I_{yc}}{I_y} \right) \frac{I_{yc}}{J} \left(\frac{h}{L_b} \right)^2 \qquad \text{(W10.3.28)}$$

$$C_b = 1.0 \quad \text{when} \quad \frac{I_{yc}}{I_y} < 0.1 \quad \text{and} \quad \frac{I_{yc}}{I_y} > 0.9$$

$$= \frac{12.5\, M_{\max}}{2.5\, M_{\max} + 3\, M_A + 4\, M_B + 3\, M_C} \quad \text{otherwise} \qquad \text{(W10.3.29)}$$

W10.3.4 Tees and Double Angles as Beams

A T-section beam loaded in the plane of its web (moment about the x-axis) will not exhibit lateral instability if r_x is less than r_y for the section. A significant number of rolled steel tees are in this category. However, when a tee beam is bent in the plane of its web (stem) and $r_x > r_y$, the limit state of lateral-torsional buckling must be considered in design. Lateral buckling strength of singly symmetric T-beams has been studied by Galambos [1968], Kitipornchai and Trahair [1980] and Ellifritt et al. [1993]. The relations derived for a singly symmetric I-shape for determining the lateral buckling moment are also valid for a tee-shaped section or a double angle section. However, for these latter sections, $C_w = 0$ and consequently $B_2 = 0$ from Eq. W10.3.25. Also, $I_{yc} = I_y$ if the flange is in compression and $I_{yc} = 0$ if the stem is in compression. Consequently, from Eq. W10.3.18 we obtain:

$$\beta_x = +0.9h \quad \text{if the flange is in compression}$$

$$= -0.9h \quad \text{if the stem is in compression} \tag{W10.3.30}$$

$$B_1 = \pm 2.3 \frac{h}{L_b} \sqrt{\frac{I_y}{J}} = \pm 2.3 \frac{d}{L_b} \sqrt{\frac{I_y}{J}} \tag{W10.3.31}$$

resulting in:

$$M_{cr} = \frac{57{,}000 \sqrt{I_y J}}{L_b} \left[B_1 + \sqrt{1 + B_1^2} \right] \tag{W10.3.32}$$

Equation W10.3.32 is Eq. F1-15 of the LRFDS into which the values of E (= 29,000 ksi) and G (= 11,200 ksi) have been substituted. The plus sign in the expression (Eq. W10.3.25) for B_1 applies when the flange is in compression and the minus sign applies when the stem is in compression. It should be noted that the lateral buckling strength of a tee with the stem in compression is only about one-fourth of the capacity when the stem is in tension. The expression for C_b used for I-shaped beams (Eq. 10.3.29) is unconservative for tee beams when the stem is in compression. Also, when tee beams are bent in double curvature, the portion with the stem in compression may control the lateral- torsional buckling strength even though these moments may be smaller relative to those in the other portions of the unbraced length. Since the buckling strength is sensitive to the moment diagram, the LRFD Specification uses a conservative value of $C_b = 1$ for tee sections. If the tip of the stem is in compression anywhere along the unbraced length, use of the negative value for B_1 is recommended. Finally, in situations where the stem is intended to be in tension, end connection details should be designed to minimize any end restraining moments which might cause the stem to experience compression.

EXAMPLE W10.3.3 Tee Shape Used as Beam

A WT10.5×22 of A36 steel was used as a simply supported, uniformly loaded beam. The span length is 20 ft and there is lateral support at the ends and midspan only. If the ratio of live load to dead load is 2.0, determine the service dead load and live load capabilities. Consider separately the cases when the flange and then the stem are in compression.

Figure WX10.3.3

Solution

a. Data

From the LRFDM Table 1-8: WT-Shapes Properties and Dimensions, we have for a WT 10.5×22:

A = 6.49 in.^{2}; S_x = 9.68 in.^{3}; y = 2.98 in.

d = 10.3 in.; t_w = 0.350 in.

$b_f = 6.50$ in.; $t_f = 0.450$ in.

$I_x = 71.1$ in.^{4}; $r_x = 3.31$ in.; $Z_x = 18.8$ in.^{3}

$I_y = 10.3$ in.^{4}; $r_y = 1.26$ in.

So

$$S_{x\,\text{flange}} = \frac{71.1}{2.98} = 23.9\ \text{in.}^3; \qquad S_{x\,\text{stem}} = \frac{71.1}{(10.3 - 2.98)} = 9.71\ \text{in.}^3$$

$$\lambda_f = \frac{b_f}{2t_f} = 7.22; \qquad \lambda_w = \frac{h}{t_w} = 26.8$$

From the LRFDM Table 1-32: WT-Shapes Flexural-Torsional Properties, we have for a WT10.5×22 : $J = 0.383$ in.^{4}; $C_w = 1.40$ in.^{6}

b. Flange in compression (Fig. WX10.3.3*b*)

$$\lambda_{pf} = 0.38\sqrt{\frac{E}{F_y}} = 0.38\sqrt{\frac{29{,}000}{36}} = 10.8$$

As the web is in tension and as $(\lambda_f = 7.22) < (\lambda_{pf} = 10.8)$, the section is compact for this orientation of the section.

$$L_p = 1.76\,r_y\sqrt{\frac{E}{F_y}} = \frac{1.76\,(1.26)}{12}\sqrt{\frac{29{,}000}{36}} = 5.25\ \text{ft}$$

$$M_{px} = Z_x F_y = \frac{18.8\,(36)}{12} = 56.4\ \text{ft-kips}$$

$$M_y = \min\,[S_{x\,\text{flange}}\,(F_y - F_r);\ S_{x\,\text{stem}}\,F_y] = \min\,[23.9\,(36 - 10);\ 9.71(36)]$$

$$= \min\,[621.4;\ 349.6] = 349.6\ \text{in.-kips} = 29.1\ \text{ft-kips}$$

Note that we assumed that the flange tips contains a residual compressive stress of F_r and that there will be no significant residual stress at the tip of the web.

$$\phi_b M_{px} = 0.90\,(56.4) = 50.8 \text{ ft-kips}$$

$$\phi_b M_{rx} = 0.90\,(29.1) = 26.2 \text{ ft-kips}$$

We note that since $r_x > r_y$, the limit state of lateral-torsional buckling must be considered. Thus, we will calculate the limiting value of $L_b = L_r$, at which the elastic lateral buckling moment, M_{cr} equals M_r. We have:

$$M_{cr} = \pi \frac{\sqrt{EI_y GJ}}{L_b}\left[B + \sqrt{1 + B^2}\right]$$

with

$$B = \pm\, 2.3 \frac{d}{L_b}\sqrt{\frac{I_y}{J}}$$

where we have omitted the subscript as given in Eq. W10.3.31 since T-sections do not warp, and thus C_w and consequently B_2, as given in Eq. W10.3.25 are zero. For the WT 10.5×22 under consideration, we obtain, with L_b expressed in units of feet:

$$B = \pm \frac{2.3\,(10.3)}{12\,L_b}\sqrt{\frac{10.3}{0.383}} = \pm \frac{10.2}{L_b}$$

The positive sign is to be used when the flange is in compression. Also, with M_{cr} expressed in ft-kips, we obtain:

$$M_{cr} = \frac{\pi\sqrt{29{,}000\,(10.3)\,(11{,}200)\,(0.383)}}{12\,(12)\,L_b}\left[B + \sqrt{1 + B^2}\right]$$

$$= \frac{780.9}{L_b}\left[\frac{10.2}{L_b} + \sqrt{1 + \left(\frac{10.2}{L_b}\right)^2}\right]$$

For $L_b = L_r$, M_{cr} equals M_{rx}. We therefore have:

$$\frac{780.9}{L_r}\left[\frac{10.2}{L_r} + \sqrt{1 + \left(\frac{10.2}{L_r}\right)^2}\right] = 29.1 \quad \rightarrow \quad L_r = 35.6 \text{ ft}$$

As mentioned earlier, since the buckling strength of T-beams is sensitive to the moment diagram, C_b has been conservatively taken as 1.0 in LRFDS Eq. F1-15. Also, we note that LRFDS Secton F1.2*c* limits the design bending strength of T-sections – in which the stem is in tension – to $\phi_b(1.5\ M_y)$.

$$M_d = \min\,[M^o_d;\ \ \phi_b(1.5\ M_y)]$$

Since $L_p < L_b < L_r$,

$$M^o_d = \phi_b M_{px} - \frac{(\phi_b M_{px} - \phi_b M_{rx})}{(L_r - L_p)}(L_b - L_p)$$

$$= 50.8 - \frac{(50.8 - 26.2)}{(35.6 - 5.25)}(10.0 - 5.25) = 46.9 \text{ ft-kips}$$

$$M_d = \min\,[46.9;\ \ 0.90(1.5)\,(29.1)] = 39.3 \text{ ft-kips}$$

If q_u is the factored uniformly distributed load on the beam, we have

$$\frac{q_u L^2}{8} = M_d \quad \rightarrow \quad \frac{q_u (20.0)^2}{8} = 39.3 \quad \rightarrow \quad q_u = 0.786 \text{ klf}$$

As the live load to dead load ratio is 2.0, we can write

$$1.2D + 1.6L = 1.2D + 1.6\,(2D) = q_u = 0.786$$

$$\rightarrow \quad D = 0.179 \text{ klf} \quad \text{and} \quad L = 0.358 \text{ klf} \qquad \text{(Ans.)}$$

c. Stem in compression (Fig. WX10.3.3*c*)

When the stem of the T-section is in compression, such as could occur under suction loads due to wind, LRFD Formula F1-15 used in Case I above is still valid. However, the parameter B must be assigned a negative value. Also, the values of M_y and M_r have to be checked; remember that we have assumed no residual stress to exist at the tip of the stem. Thus we have:

$$M_y = \min [S_{x\,\text{flange}} F_y ;\ S_{x\,\text{stem}} (F_y - 0)] = \min [23.9(36);\ 9.71(36)]$$

$$= \min [842.4;\ 349.6] = 349.6 \text{ in-kips} = 29.1 \text{ ft-kips}$$

Similarly, we find that $M_r = 29.1$ ft-kips. Thus we see that our assumption of zero residual stress at the tip of the stem results in no change to the values for M_y or M_r. Thus, we have:

$$M_{cr} = \frac{780.9}{L_b}\left[-\frac{10.2}{L_b} + \sqrt{1 + \left(\frac{10.2}{L_b}\right)^2}\,\right] = 29.1 \text{ ft-kips}$$

$$\rightarrow L_r = 13.1 \text{ ft}$$

$$M_d^o = 47.5 - \frac{(47.5 - 26.2)}{(13.1 - 5.25)}(10.0 - 5.25) = 34.6 \text{ ft-kips}$$

$$M_d = \min [M^o{}_d;\ \phi_b (1.0)\, M_y]$$

We note that in the above expression for M_d that LRFDS Section F1-2c limits the design strength of T-sections – in which the stem is in compression – to $\phi_b (1.0)\, M_y$.

$$M_d = \min [34.6;\ 0.9(1.0)(29.1)] = \min [34.6;\ 26.2] = 26.2 \text{ ft-kips}$$

If q_u is the factored uniformly distributed load on the beam, we have

$$\frac{q_u L^2}{8} = M_d \rightarrow \frac{q_u (20.0)^2}{8} = 26.2 \rightarrow q_u = 0.524 \text{ klf}$$

As the live load to dead load ratio is 2.0, we can write

$$1.2\,D + 1.6\,L = 1.2\,D + 1.6\,(2\,D) = q_u = 0.524$$

$$\rightarrow D = 0.119 \text{ klf}, \quad \text{and} \quad L = 0.238 \text{ klf} \qquad \text{(Ans.)}$$

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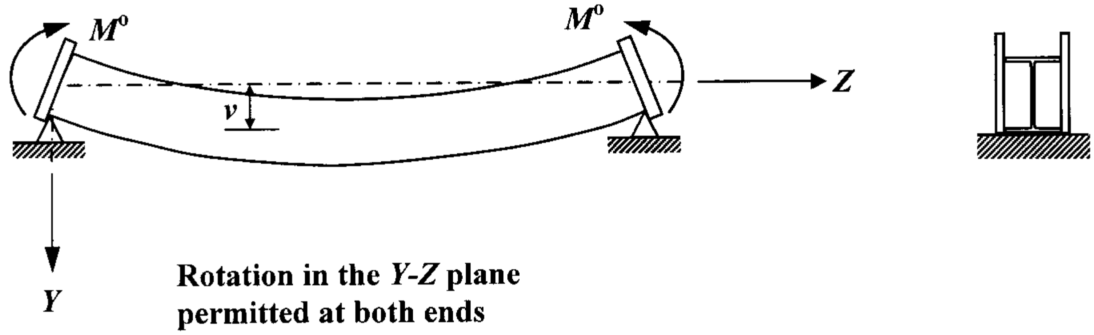
Table W10.1.1 **Effective length factors (K_y and K_z) for Lateral-Torsional Buckling** TO COME SHORTLY.

Table W10.1.2 **Values of C_b and C_1 in Eq. W10.1.18** TO COME SHORTLY.



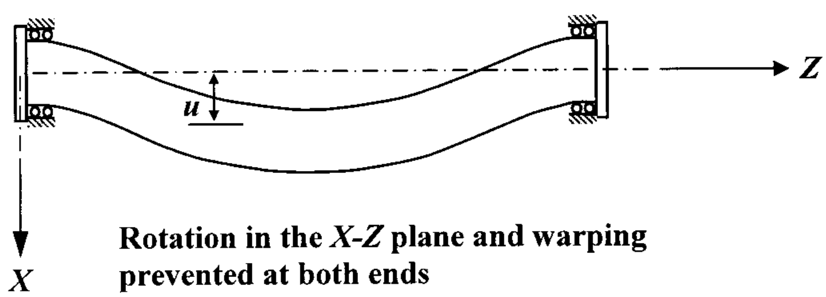


Figure W10.1.1: **Laterally and torsionally fixed I-beam under uniform moment.**

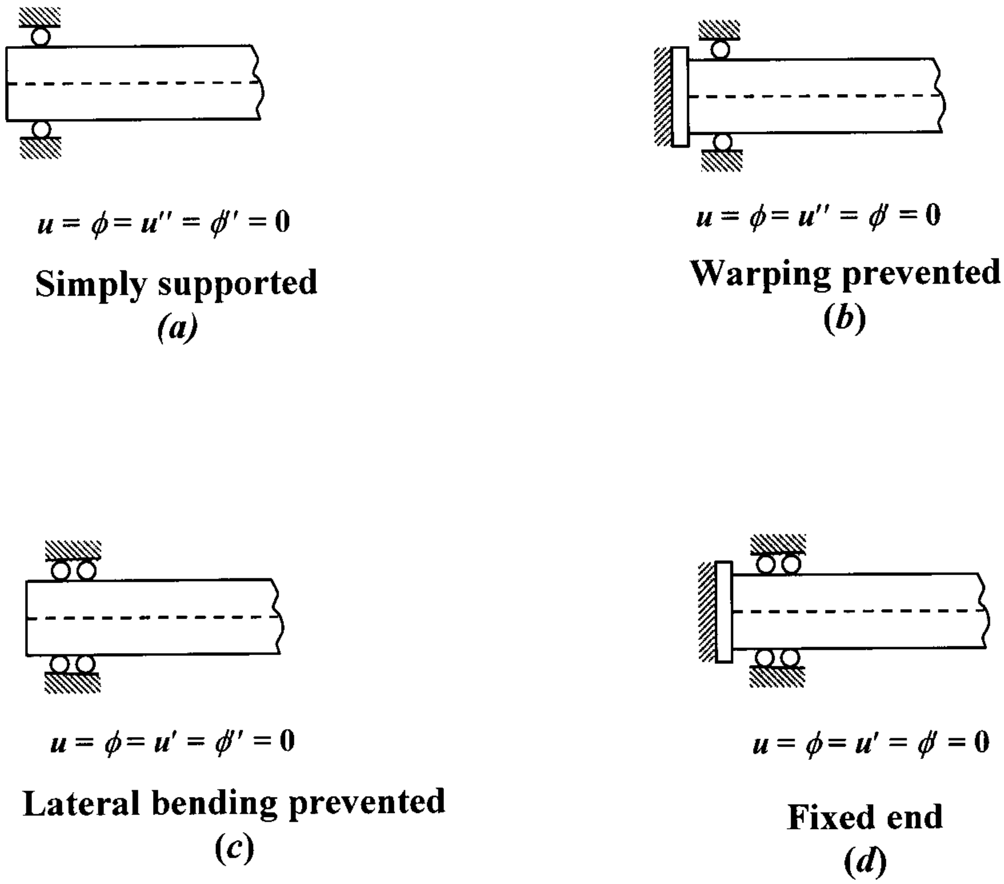
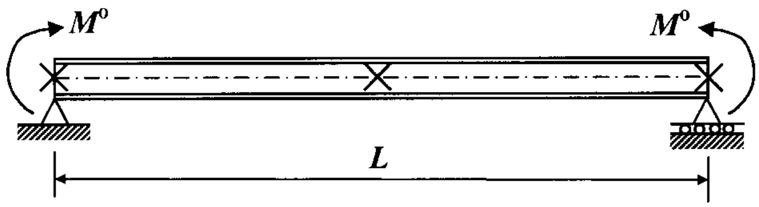
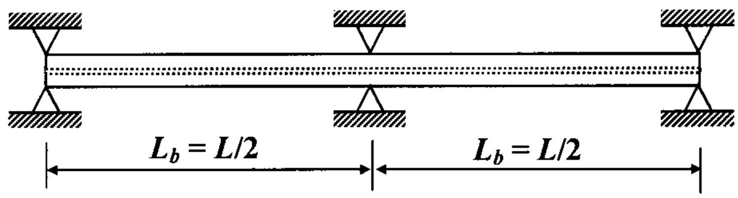


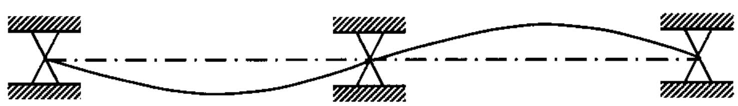
Figure W10.1.2: **Idealized end restraint conditions for lateral buckling of beams.**



Vertical supports
(*a*)



Lateral supports
(*b*)



Plan view – center line of the buckled compression flange
(*c*)

Figure W10.1.3: Lateral buckling of a simple beam with lateral support at mid-span.

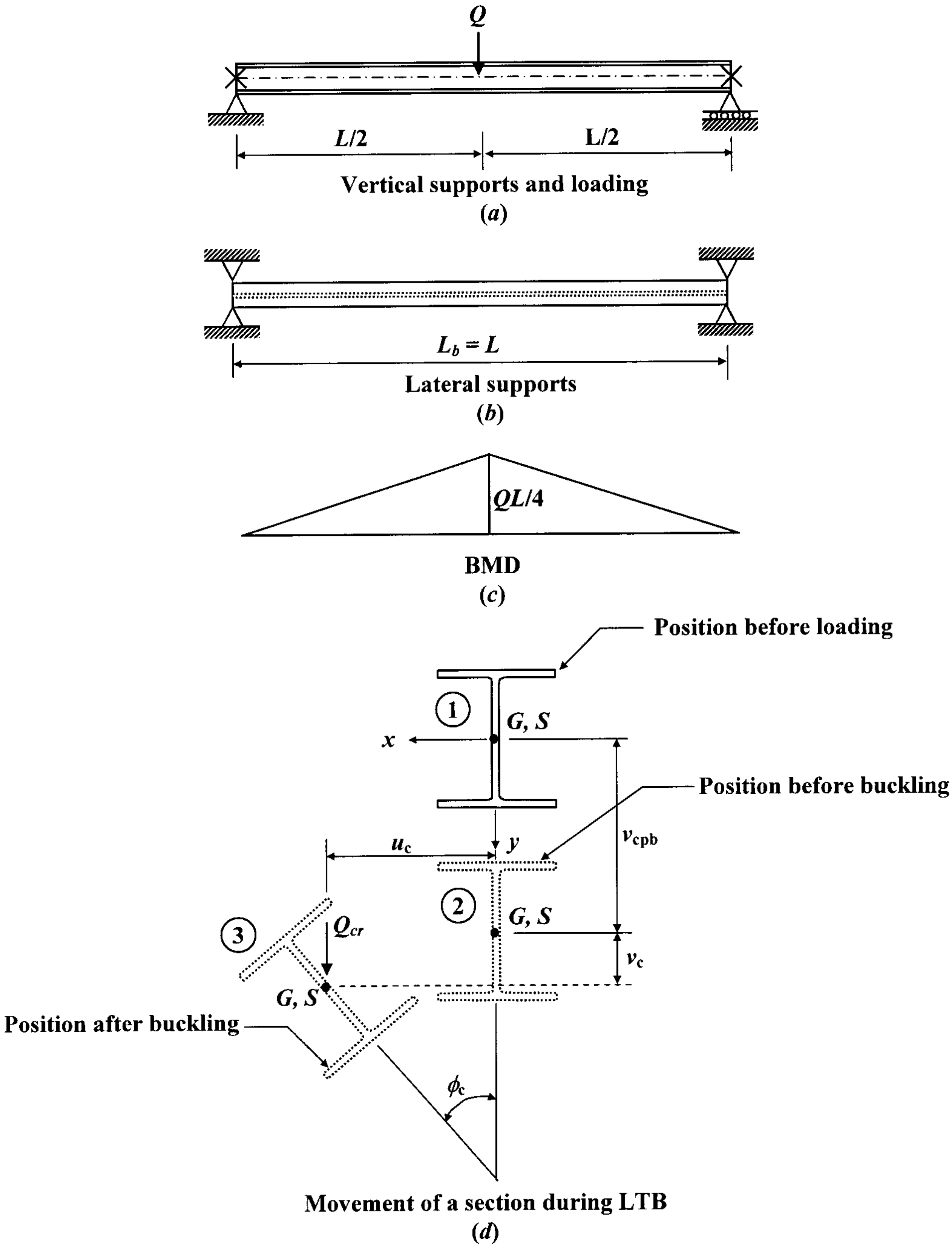
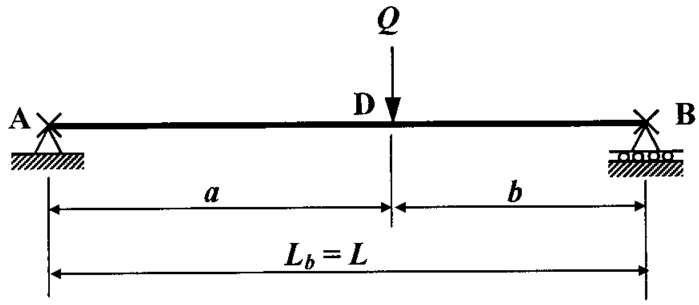
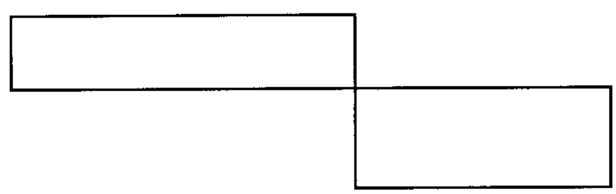


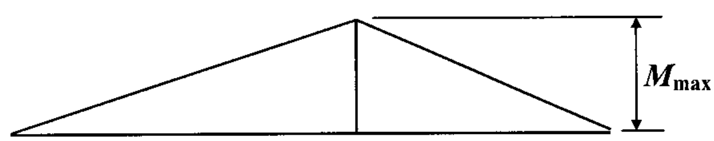
Figure W10.1.4: **Lateral buckling of a simple beam under central concentrated load.**



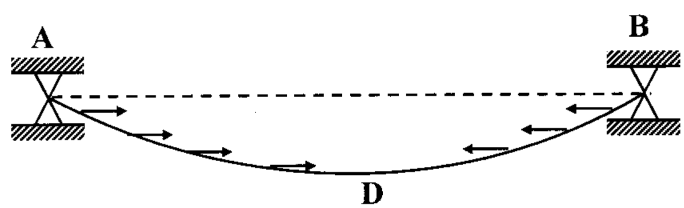
Vertical supports and loading
(*a*)



SFD
(b)

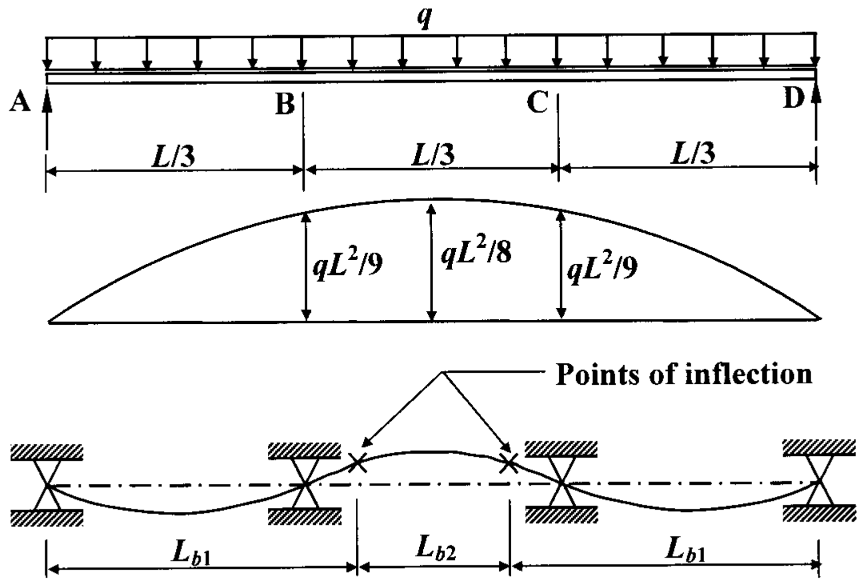


BMD
(*c*)

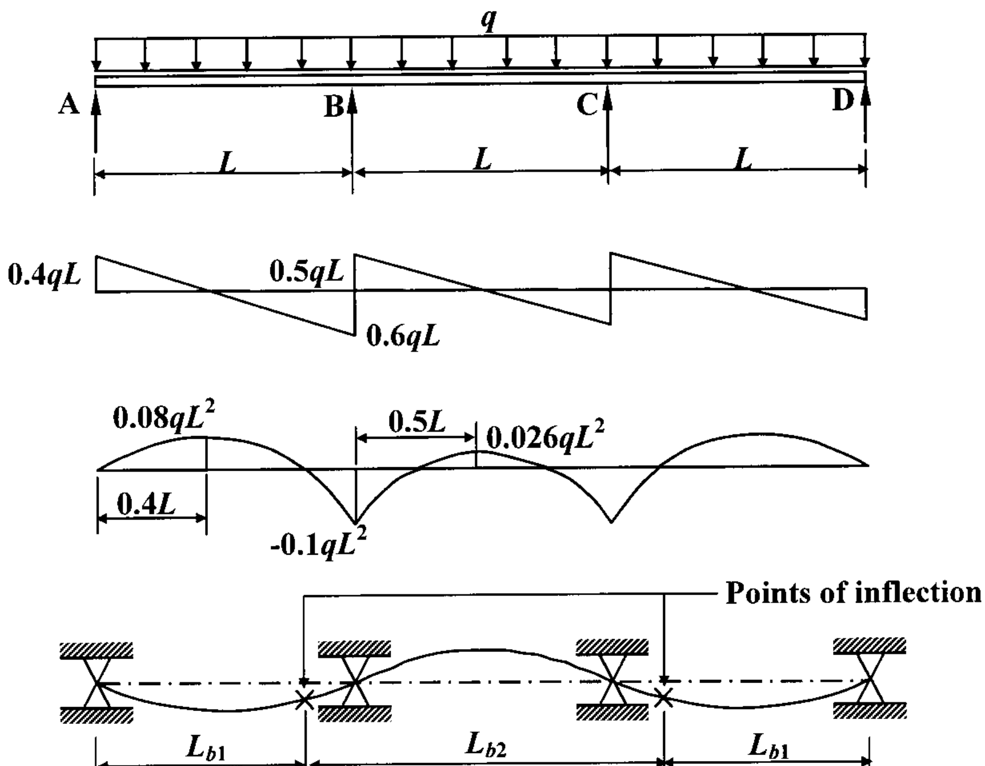


Plan view of laterally buckled compression flange
(*d*)

Figure W10.1.5: **Lateral buckling of a beam under non-uniform moment.**



Laterally continuous simple beam
(*a*)



Laterally and vertically continuous beam
(*b*)

Figure W10.1.6: Laterally and vertically continuous beams.

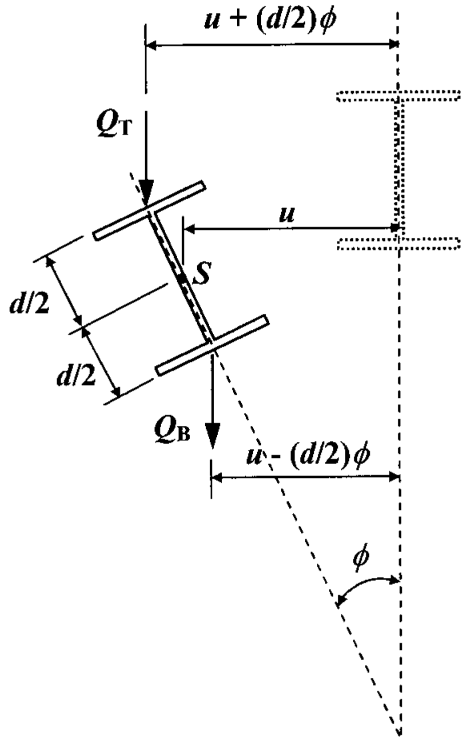


Figure W10.1.7: Effect of position of load on lateral buckling.

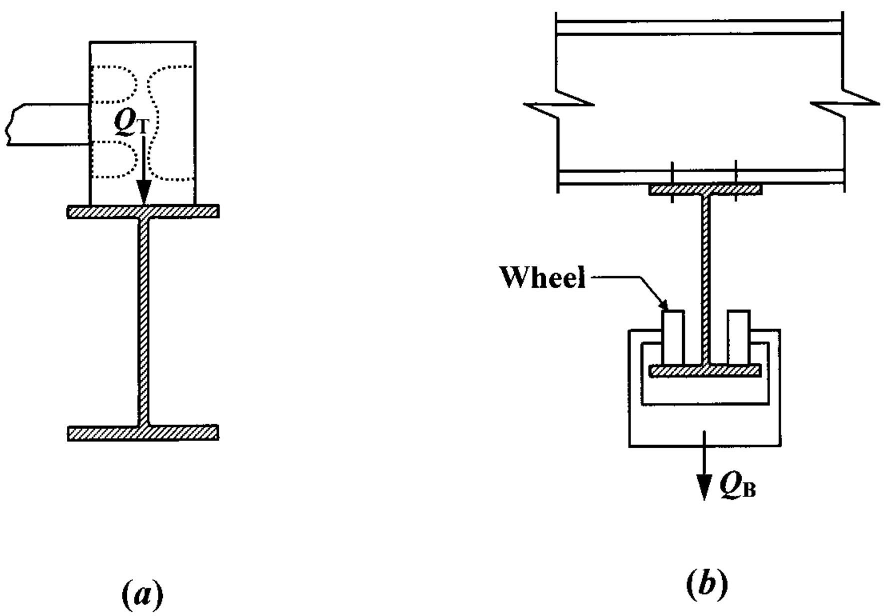


Figure W10.1.8: **Examples of top and bottom flange loading of beams.**

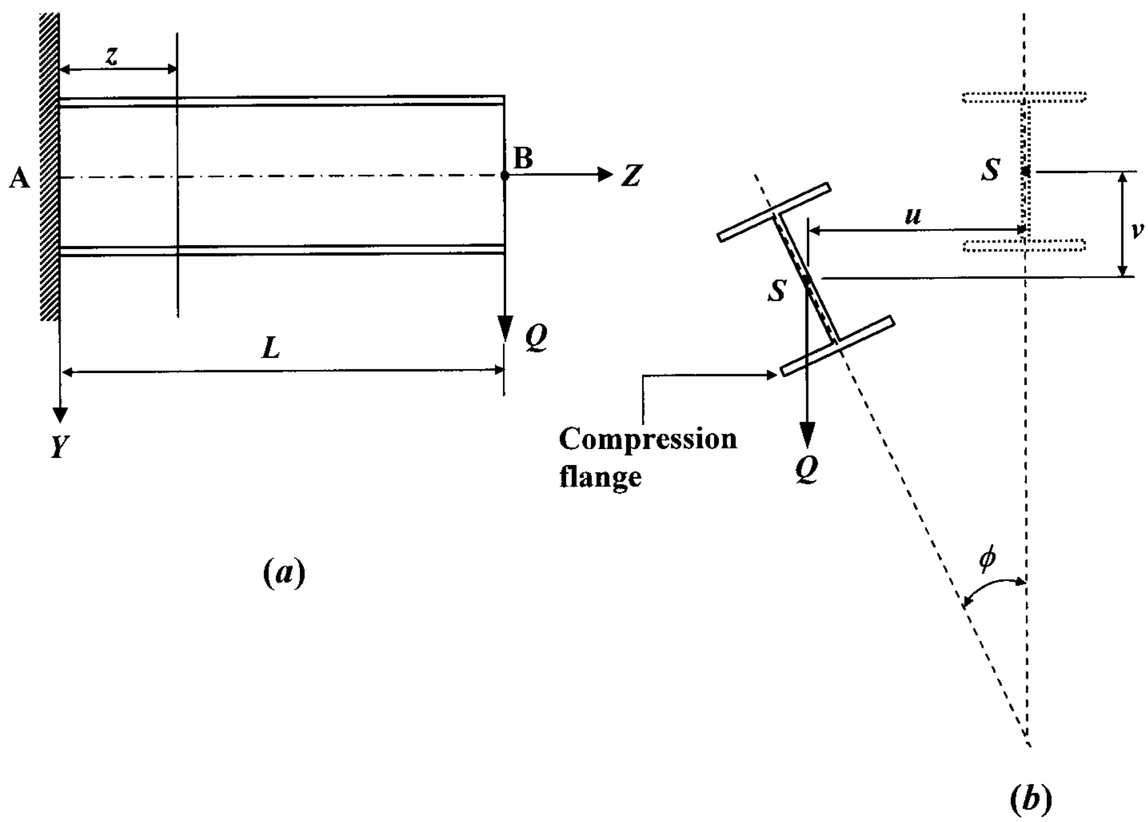
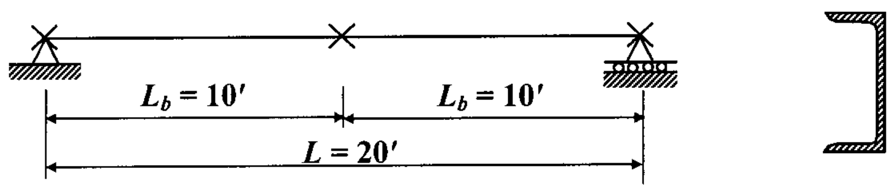


Figure W10.2.1: Lateral buckling of a cantilever beam.

Figure W10.2.2 **Effective lengths K_bL, for cantilever beams** TO COME SHORTLY.



(*a*)

Figure WX10.3.2

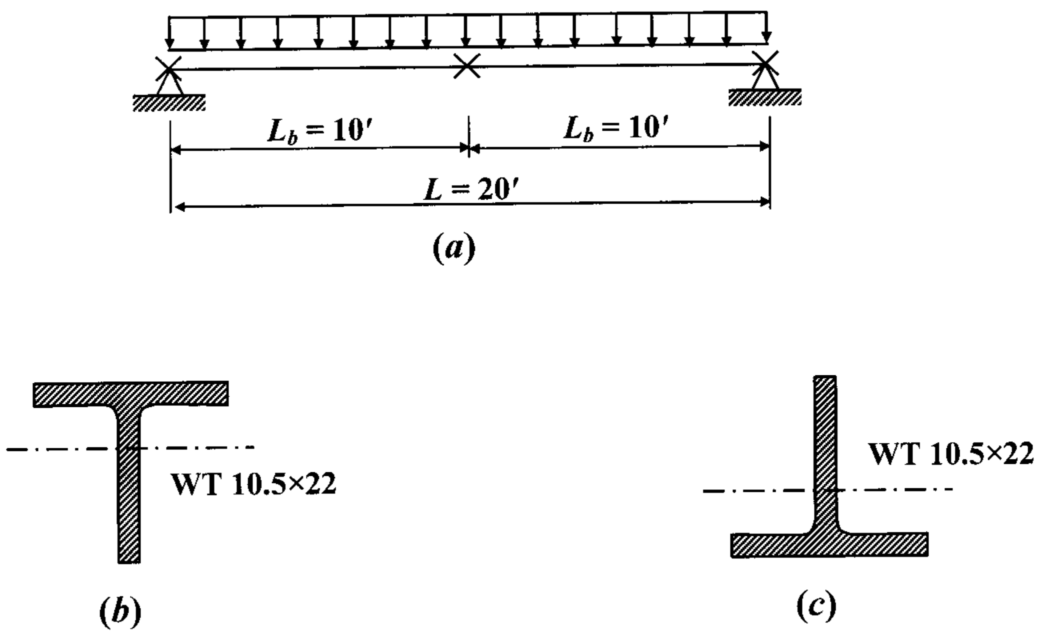


Figure WX10.3.3