This is an unedited, uncorrected chapter.

The final chapter will be available in time for fall.

NOTE: Figures and tables appear at the end of the chapter.

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WEB CHAPTER W11

Members Under Combined Forces

W11.1 Strength of an I-section Under Axial Compression and Uni-Axial Bending

Figure W11.1.1: Stress distributions in an I-section under axial compression P and major axis moment M.

Let us consider a wide flange section subjected to an axial load P and a bending moment M about the major axis of the cross section. Fig. W11.1.1 shows the stress distribution in the section at various stages of deformation as the moment is increased gradually. Due to the axial force, yielding on the compression side precedes that on the tension side. Eventually plastification on the entire section occurs, but since part of the area resists the axial force, the stress block no longer contains equal compression and tension areas (as was the case with pure moments). The plastic neutral axis shifts by a distance y_o from the axis of symmetry, xx. For low values of axial compression, the PNA will be in the web, while for high values of axial loads, the PNA will be in a flange.

Figure W11.1.2: Reduced plastic moment M_{pc} of an I-section under axial compression P, PNA in the web.

Figure W11.1.3: Reduced plastic moment M_{pc} of an I-section under axial compression P, PNA in the flange.

When the PNA is in the web as shown in Fig. W11.1.2*a*, the total stress distribution (sketch i) may be divided into two parts: a part that is associated with the axial load (sketch ii) and a part that corresponds with the bending moment (sketch iii). The stresses shown in Fig. W11.1.2*b* have as their resultant the axial force,

$$P = \left(2y_o t_w\right) F_y \tag{W11.1.1}$$

where F_y is the yield stress, y_o is the distance from the midheight to the PNA, and t_w is the thickness of the web. It is seen from the stress distribution shown in Fig. *b*-iii that the fully plastic moment of the section is reduced by an amount equal to the plastic moment of the central portion of the web of depth $2y_o$ and width, t_w. The reduced plastic moment is therefore given by:

$$M_{pc} = M_p - \left[\frac{1}{4} t_w (2y_o)^2\right] F_y = \left[Z_x - t_w y_o^2\right] F_y \tag{W11.1.2}$$

where Z_x is the plastic section modulus of the wide flange section for bending about its major axis. By substituting the value of y_o obtained from Eq. W11.1.1 into W11.1.2, the reduced plastic moment may be expressed as a function of the axial force P, as:

$$M_{pc} = M_p - \frac{P^2}{4 t_w F_y} \qquad \text{for } P \leq P_w \tag{W11.1.3}$$

where P_w represents the yield load of the web, given by:

$$P_w = t_w(d - 2t_f)F_y \tag{W11.1.4}$$

By dividing both sides of Eq. W11.1.3 by M_p, we obtain:

$$\frac{M_{pc}}{M_p} = 1 - \frac{A^2}{4t_w Z_x}\left(\frac{P}{P_y}\right)^2 \quad \text{for } 0 \leq \frac{P}{P_y} \leq \frac{P_w}{P_y} \tag{W11.1.5}$$

By a similar reasoning, the following relations could be obtained for a fully plastified, compressed and bent I-shape, when the PNA is in the flange instead of the web (Fig. W11.1.3):

$$P = \left[A - b_f(d - 2y_o)\right]F_y \tag{W11.1.6}$$

$$M_{pc} = \left[\frac{d^2}{4} - y_o^2\right]b_f F_y \tag{W11.1.7}$$

from which

$$M_{pc} = \frac{P_y}{2}\left[d\left(1 - \frac{P}{P_y}\right) - \frac{A}{2b_f}\left(1 - \frac{P}{P_y}\right)^2\right] \quad \text{for } \frac{P_w}{P_y} \leq \frac{P}{P_y} < 1 \tag{W11.1.8}$$

and in non-dimensional form

$$\frac{M_{pc}}{M_p} = \left(\frac{P_y d}{2M_p}\right)\left[\left(1 - \frac{P}{P_y}\right) - \left(\frac{A}{2b_f d}\right)\left(1 - \frac{P}{P_y}\right)^2\right] \tag{W11.1.9}$$

In order to account for the presence of axial load on the flexural strength of wide flange sections bent about their major axis, the relations W11.1.5 and 9 may be replaced by the approximate linear relations [ASCE, 1971]:

$$\frac{M_{pcx}}{M_{px}} = 1 \qquad \text{for } 0 < \frac{P}{P_y} \leq 0.15 \tag{W11.1.10a}$$

$$\frac{M_{pcx}}{M_{px}} = 1.18\left(1 - \frac{P}{P_y}\right) \qquad \text{for } 0.15 < \frac{P}{P_y} \leq 1.0 \tag{W11.1.10b}$$

or, alternatively by:

$$\frac{P}{P_y} + \frac{1}{1.18}\frac{M_{pcx}}{M_{px}} \leq 1.0 \; ; \qquad \frac{M_{pcx}}{M_{px}} \leq 1.0 \tag{W11.1.11}$$

Similar procedures could be used to determine the reduced plastic moment M_{pcy} of a wide flange section subjected to bending about its minor axis in the presence of an axial load P. These relations may again be simplified to [Lehigh Notes, 1965]:

$$\frac{M_{pcy}}{M_{py}} = 1.19\left[1 - \left(\frac{P}{P_y}\right)^2\right] \leq 1.0 \tag{W11.1.12}$$

W11.2 Strength of an I-section Under Axial Compression and Bi-axial Bending

The maximum strength of a short beam-column segment subjected to compression combined with biaxial bending occurs when the entire section is fully plastic or yielded. When the position of the plastic neutral axis is known or assumed, the magnitude of the resultant axial force P, major axis bending moment M_x and minor axis moment M_y, which result in the full plastification of the cross section can be calculated directly using the three equations of equilibrium:

$$P = \int_A f\, dA; \quad M_x = \int_A f y\, dA; \quad M_y = \int_A f x\, dA \qquad \text{(W11.2.1)}$$

where f is the normal stress in the yielded fibers and equals F_y or $-F_y$, depending on whether the element considered is plastified in compression or tension, respectively (When the position of the neutral axis is not known, for example, when the stress resultants P, M_x and M_y are prescribed, the equations of equilibrium can be solved only by trial and error). These calculations result in a function which defines an interaction surface, known as the ***limit yield surface*** or simply ***limit surface***. Fig. W11.2.1 shows a typical maximum strength interaction limit surface for steel H-beam-column sections. The intersection of the limit surface with each axis (namely, points A, B, C) represent the capacity of the member when it is subjected to loading of one type only. Also, the intersection of the limit surface with the coordinate-axis-planes (namely, curves OADB, OAEC, OBFC) represent the limit curves under the relevant restricted loading condition ($M_x = 0$; $M_y = 0$; $P = 0$, respectively). A point in this three dimensional generalized-stress space represents, for a given I-shape having known values of yield load P_y and fully plastic moments M_{px} and M_{py} about strong and weak axes of bending of the cross section respectively, a certain combination of biaxial bending moment and axial load. If the point lies within the convex surface OABC, then the combination is one that can be carried by the section. A point on the yield surface represents a combination of axial load and bending moments that just cause the section to become fully plastic. A point outside the limit surface represents an impossible state. Since the capacity of the column of zero length is independent of residual stresses and initial imperfections, the surface OABC is uniquely determined for elastic perfectly plastic sections which are not subject to local buckling.

Figure W11.2.1: Interaction surfaces for a steel section.

Using the expressions derived for a rectangular area and a superposition technic, Chen and Atsuta [1977] developed interaction curves for wide-flange, box, channel, tee, angle and circular sections. Interaction curves for a W12×31 and a WT 15×66 are shown in Fig. W11.2.2. The largest loop of the curves represents the interaction curve with $P/P_y = 0$. When the axial load ratio increases, the loop becomes smaller and smaller. We also observe that interaction of moments about the orthogonal axes is not linear. On the contrary, the interaction curve resembles more closely the quadrant of a circle.

Figure W11.2.2: Examples of interaction curves for an I-section and a T-section [Chen and Atsuta, 1977].

Tebedge and Chen [1974] proposed the following interaction equation to determine the strength of wide-flange sections under axial load and biaxial bending.

$$\left[\frac{M_x}{M_{pcx}}\right]^{\zeta} + \left[\frac{M_y}{M_{pcy}}\right]^{\zeta} = 1.0 \qquad \text{(W11.2.2)}$$

where M_x and M_y are the applied moments, while M_{pcx} and M_{pcy} are the plastic moment capacities of the I-shape about the respective axis, reduced for the presence of axial load. M_{pcx} may be determined from Eq. W11.1.10. Similarly, M_{pcy} may be determined from Eq. W11.1.12. Exponent ζ in Eq.

W11.2.12, for compact I-shapes in which the flange width is not less than 0.5 of the depth of the section is given by [Tebedge and Chen, 1974]:

$$\zeta = 1.6 - \frac{(P/P_y)}{2[\ell n(P/P_y)]} \tag{W11.2.3}$$

where ℓn is the natural logarithm.

W11.3 Elastic Behavior of a Beam-Column with End Moments M_A and M_B

Figure W11.3.1: Beam-column with end-moments M_A, M_B.

Consider a simply supported beam-column of length L subjected to end-moments M_A and M_B at the left and right ends of the member in-addition to an axial force P, shown in Fig. W11.3.1a. The member is of uniform section and the applied forces bend the member in a plane of symmetry. We will assume that the deformations remain small, the material follows Hooke's law, and that the member is laterally braced, if necessary, so that the member can only bend in the vertical plane. Using the free body diagram of a segment of beam-column of length z from the right-end, taken as origin, the external moment acting on the cut section may be written, with the help of Fig. W11.3.1b, as:

$$M = Pu + M_B - \left(M_B - M_A\right)\frac{z}{L} \tag{W11.3.1}$$

Equating this to the internal moment expressed in terms of curvature and rearranging, we have:

$$EIu'' + Pu = -M_B + (M_B - M_A)\frac{z}{L} \tag{W11.3.2}$$

Using the notation $\alpha^2 = (P/EI)$, we can write:

$$u'' + \alpha^2 u = -\frac{M_B}{EI} + \frac{(M_B - M_A)}{EI}\frac{z}{L} \tag{W11.3.3}$$

The solution of this equation is:

$$u = C_1 \sin\alpha z + C_2 \cos\alpha z - \frac{M_B}{EI\alpha^2} + \frac{(M_B - M_A)}{EI\alpha^2}\frac{z}{L} \tag{W11.3.4}$$

The constants C_1 and C_2 can be determined by using the boundary conditions

$$u_{(0)} = 0\,; \quad u_{(L)} = 0 \tag{W11.3.5}$$

From the first boundary condition, we obtain

$$C_2 = \frac{M_B}{EI\alpha^2} \tag{W11.3.6}$$

and from the second boundary condition, we obtain,

$$C_1 = \frac{-(M_B\cos\phi - M_A)}{EI\alpha^2\sin\phi} \tag{W11.3.7}$$

wherein

$$\phi^2 = \alpha^2 L^2 = \frac{PL^2}{EI} = \pi^2 \frac{P}{P_E} \tag{W11.3.8}$$

and P_E is the Euler load of the pin-ended member of length L with the buckling axis taken as the bending axis. With the help of these relations, Eq. W11.3.4 for the deflection of the beam-column can be written as: (W11.3.9)

$$u = \frac{-\left(M_{\mathrm{B}}\cos\phi - M_{\mathrm{A}}\right)}{EI\alpha^2\sin\phi}\sin\alpha z + \frac{M_{\mathrm{B}}}{EI\alpha^2}\cos\alpha z - \frac{M_{\mathrm{B}}}{EI\alpha^2} + \frac{\left(M_{\mathrm{B}} - M_{\mathrm{A}}\right)}{EI\alpha^2}\frac{z}{L}$$

from which, differentiating with respect to z three times, we obtain:

$$u' = \frac{-\left(M_{\mathrm{B}}\cos\phi - M_{\mathrm{A}}\right)}{EI\alpha\sin\phi}\cos\alpha z - \frac{M_{\mathrm{B}}}{EI\alpha}\sin\alpha z + \frac{\left(M_{\mathrm{B}} - M_{\mathrm{A}}\right)}{EI\alpha^2 L} \tag{W11.3.10}$$

$$u'' = \frac{\left(M_{\mathrm{B}}\cos\phi - M_{\mathrm{A}}\right)}{EI\sin\phi}\sin\alpha z - \frac{M_{\mathrm{B}}}{EI}\cos\alpha z \tag{W11.3.11}$$

$$u''' = \frac{\alpha\left(M_{\mathrm{B}}\cos\phi - M_{\mathrm{A}}\right)}{EI\sin\phi}\cos\alpha z + \frac{\alpha M_{\mathrm{B}}}{EI}\sin\alpha z \tag{W11.3.12}$$

To locate the section $z = z^*$ where the maximum second-order moment occurs, we set the shear force ($-EI\,u'''$) equal to zero. We thus obtain, with the help of Eq. W11.3.12:

$$\tan\alpha z^* = \frac{\left(M_{\mathrm{A}} - M_{\mathrm{B}}\cos\phi\right)}{M_{\mathrm{B}}\sin\phi} \tag{W11.3.13}$$

The solution of this equation may result in values of z^* which are positive or negative.

$$z^* \text{ is positive when } \frac{M_A}{M_B} \geq \cos\phi\text{, and negative when } \frac{M_A}{M_B} < \cos\phi \qquad \text{(W11.3.14)}$$

When the value of z^* is positive the maximum second-order moment occurs within the length of the beam (Fig. W11.3.1*c*). A negative value for z^* means that the relationship expressed by Eq. W11.3.11 has no maximum within the region $0 < z^* < L$ as the maximum ordinate of the curve occurs outside the span (Fig. W11.3.1*d*). In this case, the maximum second-order moment in the beam is the end moment M_B itself. The transition between the two cases occurs when $z^* = 0$. This happens when

$$\frac{M_A}{M_B} = \cos\phi = \cos\left(\pi\sqrt{\frac{P}{P_E}}\right)$$

The maximum second-order moment is obtained by writing the value of $\sin \alpha z^*$ and $\cos \alpha z^*$ derived from Eq. W11.3.13 (see Fig. W11.3.1*e*), in the expression for $M = -EI u''$ where u'' is given by the Eq. W11.3.11. Thus

$$M^*_{max} = \left[\frac{(M_B\cos\phi - M_A)^2}{\sin\phi} + M_B^2\sin\phi\right]\frac{1}{\sqrt{(M_A - M_B\cos\phi)^2 + M_B^2\sin^2\phi}}$$

$$= M_B\sqrt{\frac{1 - 2(M_A/M_B)\cos\phi + (M_A/M_B)^2}{\sin^2\phi}} \qquad \text{(W11.3.15)}$$

This relation is for a beam-column bent in single curvature as shown in Fig. W11.3.1*a*. For a member bent in double curvature, the expression for maximum moment may be obtained by replacing M_A by $-M_A$ in Eq. W11.3.15 resulting in:

$$M^*_{\max} = M_B \left[\sqrt{\frac{1 + 2(M_A/M_B)\cos\phi + (M_A/M_B)^2}{\sin^2\phi}} \right] \tag{W11.3.16}$$

Just as in the case of beam-columns bent in single curvature, the maximum bending moment given by Eq. W11.3.16 for a beam-column bent in reverse curvature may occur at the member end rather than within the member. Thus, to check the validity of Eq.W11.3.16 one should evaluate z^* from the equation

$$\tan\alpha z^* = -\frac{(M_B\cos\phi + M_A)}{M_B\sin\phi} \tag{W11.3.17}$$

If the calculated value of z^* falls outside the range $0 \leq z^* \leq L$, the maximum second-order moment is the member end-moment M_B itself. Note that Eq. W11.3.17 is the same as Eq. W11.3.13, except that M_A has been replaced by $-M_A$. Eqs. W11.3.15 and W11.3.16 can be combined into:

$$M^*_{\max} = |M_B| \left[\sqrt{\frac{1 + 2r_M\cos\phi + r_M^2}{\sin^2\phi}} \right] \tag{W11.3.18a}$$

where

$$r_M = \pm\frac{|M_A|}{|M_B|} \tag{W11.3.18b}$$

Here, the moment ratio r_M is to be assigned a positive sign if the beam-column is bent in double-curvature and a negative sign if it is bent in single-curvature. The absolute value for the larger of the two end moments, namely of M_B, is used in Eq. W11.3.18*a* because we are interested only in the magnitude, not the direction of $M^*_{\max}$.

Figure W11.3.2: Beam-column under uniform moment.

A particular case for a beam-column subjected to end moments is the member under symmetric single curvature bending, i.e., with $M_A = M_B = M^o$ as shown in Fig. W11.3.2. In this case, the maximum second-order moment occurs at midspan. Its magnitude is obtained by setting $r_M = -1$ in Eq. W11.3.18*a* as:

$$M^*_{max} = M^o \sqrt{\frac{2(1 - \cos\phi)}{\sin^2\phi}} = M^o \sec\frac{\phi}{2} \tag{W11.3.19}$$

The maximum deflection δ^o also occurs at the center of the member. By letting $z = L/2$, and $M_A = M_B = M^o$ in Eq. W11.3.9, we obtain:

$$\delta^o = \frac{M^o}{P}\left[\sec\frac{\phi}{2} - 1\right] \tag{W11.3.20}$$

The end slope of the beam-column deflection curve, under end-moments producing symmetric single curvature bending, is obtained by letting $M_A = M_B = M^o$ and $z = 0$ in Eq. W11.3.10, as:

$$\theta^o_A = \frac{M^o}{EI\alpha}\frac{(1 - \cos\phi)}{\sin\phi} \tag{W11.3.21}$$

W11.4 Elastic Behavior of Beam-Columns with Lateral Loads

Figure W11.4.1: Beam-column with central concentrated load.

Let us consider a simply supported beam-column of length L that is simultaneously acted on by a central concentrated load Q and axial loads P, as shown in Fig. W11.4.1a. A free-body diagram of a segment of the beam-column of length z from the left support is shown in Fig. W11.4.1b. The external moment acting at the cut section is:

$$M_{\text{ext}} = \frac{Qz}{2} + Pu \tag{W11.4.1}$$

The internal bending moment M_{int} is related to the curvature u'' by the linear relationship

$$M_{\text{int}} = -EIu'' \tag{W11.4.2}$$

where EI is the flexural rigidity of the beam about the bending axis. For equilibrium, the external moment must equal the internal moment. Therefore, by equating Eq. W11.4.2 to Eq. W11.4.1 and rearranging, we obtain:

$$EIu'' + Pu = -\frac{Qz}{2} \tag{W11.4.3}$$

or

$$u'' + \alpha^2 u = -\frac{Qz}{2EI} \tag{W11.4.4}$$

where

$$\alpha^2 = \frac{P}{EI} \tag{W11.4.5}$$

The general solution to Eq. W11.4.4 is

$$u = C_1 \sin \alpha z + C_2 \cos \alpha z - \frac{Qz}{2P} \tag{W11.4.6}$$

where C_1 and C_2 are the integration constants. The boundary condition $u = 0$ at $z = 0$ leads to

$$C_2 = 0 \tag{W11.4.7}$$

Due to the symmetry of the structure and the loading, the slope of the deflected shape is zero at midspan. That is, $u' = 0$ at $z = L/2$, which when substituted in the derivative of the expression for u in Eq. W11.4.6, results in:

$$C_1 = \frac{Q}{2P} \frac{1}{\alpha \cos\left(\alpha \frac{L}{2}\right)} \tag{W11.4.8}$$

Substitution of the values of C_1 and C_2 from Eqs. W11.4.8 and W11.4.7, in Eq. W11.4.6 gives

$$u = \frac{Q}{2P\alpha}\left[\frac{\sin \alpha z}{\cos\left(\alpha \frac{L}{2}\right)} - \alpha z\right] \tag{W11.4.9}$$

By differentiating this relation and rearranging we obtain the slope and bending moment along the length of the member as

$$u' = \frac{Q}{2P}\left[\frac{\cos \alpha z}{\cos\left(\frac{\alpha L}{2}\right)} - 1\right] \tag{W11.4.10a}$$

$$M = -EIu'' = \frac{Q}{2\alpha} \frac{\sin \alpha z}{\cos\left(\frac{\alpha L}{2}\right)} \tag{W11.4.10b}$$

To simplify the results that follow, we introduce the additional notation:

$$\mu \;=\; \frac{\alpha L}{2} \;=\; \frac{L}{2}\sqrt{\frac{P}{EI}} \;=\; \frac{\pi}{2}\sqrt{\frac{P}{P_E}} \qquad \text{(W11.4.11)}$$

where P_E is the elastic buckling load of the pin-ended column about the plane of bending.

To find the slope of the deflection curve at the end of the beam-column, we substitute $z = 0$ into Eq. W11.4.10 a, which gives:

$$\theta_A \;=\; \left(\frac{QL^2}{16EI}\right)\left[\frac{2(1 - \cos\mu)}{\mu^2\cos\mu}\right] \qquad \text{(W11.4.12)}$$

where the first term on the right-hand side represents the slope that would exist if the concentrated lateral load Q were acting alone. Also, by using Eq. W11.4.10*b*, we obtain the maximum second-order bending moment, which occurs at the center, as

$$M^*_{\max} \;=\; M_C \;=\; \left(\frac{QL}{4}\right)\left[\frac{\tan\mu}{\mu}\right] \qquad \text{(W11.4.13)}$$

where $(Q\,L/4)$ is the maximum bending moment that would exist if the concentrated lateral load Q were acting alone (i.e., the maximum first-order moment).

Figure W11.4.2: Beam-column with uniformly distributed transverse load.

Next let us consider the case of a simply supported member bent by a uniformly distributed lateral load q and a set of axial forces P, as shown in Fig. W11.4.2*a*. As before, we consider the equilibrium of a segment of length z in the deformed position (Fig. W11.4.2*b*). Equating the internal moment to the external moment, we obtain, after rearranging

$$u'' + \alpha^2 u = \frac{q}{2EI} z^2 - \frac{qL}{2EI} z \qquad \text{(W11.4.14)}$$

Using the boundary conditions $u_{(0)} = 0$; $u_{(L)} = 0$, we obtain:

$$u = \frac{q}{EI\alpha^4}\left[\tan\mu \sin\alpha z + \cos\alpha z - 1\right] - \frac{q}{2EI\alpha^2} z(L - z) \qquad \text{(W11.4.15}a\text{)}$$

$$M = \frac{qL^2}{4\mu^2}\left[\tan\mu \sin\left(\frac{2z}{L}\mu\right) + \cos\left(\frac{2z}{L}\mu\right) - 1\right] \qquad \text{(W11.4.15}b\text{)}$$

The maximum second-order moment for the pin-ended beam-column with uniformly distributed lateral load q and axial load P is obtained by letting $z = L/2$ in Eq. W11.4.15*b* as:

$$M^*_{\max} = M_C = \left(\frac{qL^2}{8}\right)\left[\frac{2}{\mu^2}(\sec\mu - 1)\right] \qquad \text{(W11.4.16)}$$

where the term $(qL^2/8)$ represents the maximum bending moment that would exist if the uniform lateral load q were acting alone (i.e., the maximum first-order moment).

Figure W11.4.3: Superposition of loading cases for a beam-column.

The principle of superposition is valid for beam-columns, in elastic domain, provided that the same axial force P is included in each subsolution. For example, the beam-column shown in Fig. W11.4.3a can be studied by superimposing the solution of the member acting as a beam-column with end-moments M^o only (Fig. W11.4.3b), on the solution of the same member acting as a beam-column with the central concentrated load Q only (Fig. W11.4.3c). In both cases, the axial load P must be included since primary moment is increased in both cases by Pu.

The principle of superposition may be used to determine the end moments of a transversely loaded beam-column with one or both of its ends fixed. The unknown fixing moments M^F are determined from the fact that to satisfy the continuity condition of zero slope(s) at the built-in end(s), the algebraic sum of the rotation at the end(s) produced by the transverse load and that produced by the end-moment(s) must be zero.

Figure W11.4.4: Beam-columns fixed at one or both ends.

For a fixed-ended beam-column with a central concentrated lateral load Q (Fig. W11.4.4a), we obtain

$$M_{\max}^{*-} = M_A^F = M_B^F = -\left(\frac{QL}{8}\right)\left[\frac{2(1 - \cos\mu)}{\mu \sin\mu}\right] \qquad \text{(W11.4.17)}$$

$$M_{\max}^{*+} = M_C = \left(\frac{QL}{8}\right)\left[\frac{2(1 - \cos\mu)}{\mu \sin\mu}\right] \qquad \text{(W11.4.18)}$$

Similarly, for a fixed-ended beam-column with a uniformly distributed lateral load q (Fig. W11.4.4b), we obtain:

$$M^{*-}_{\max} = M^F_A = M^F_B = -\left(\frac{qL^2}{12}\right)\left[\frac{3(\tan\mu - \mu)}{\mu^2 \tan\mu}\right] \qquad \text{(W11.4.19)}$$

$$M^{*+}_{\max} = M_C = \left(\frac{qL^2}{24}\right)\left[\frac{6(\mu - \sin\mu)}{\mu^2 \sin\mu}\right] \qquad \text{(W11.4.20)}$$

For a beam-column fixed at one end, simply supported at the other end and subjected to a uniformly distributed load q (Fig. W11.4.4*c*), we get

$$M^{*-}_{\max} = M^F_B = \left(\frac{qL^2}{8}\right)\left[\frac{8(\tan^2\mu - \mu\tan\mu)}{2\mu(\tan 2\mu - 4\mu^2)}\right] \qquad \text{(W11.4.21)}$$

The maximum positive bending moment which occurs off-center will not be critical.

Note that all these relations for maximum second-order moments, $M^*_{\max}$, could be rewritten in the format

$$M^*_{\max} = A_f\, M_{\max} \qquad \text{(W11.4.22)}$$

where $M_{\max}$ is the corresponding maximum first-order moment and A_f is a moment amplification factor due to the presence of axial load P. It is seen that the amplification factor is a function of the axial load ratio P/P_e, type of the transverse load, and the support conditions.

W11.5 Equivalent Moment Factors C_m for Beam-Columns with Lateral Loads

Let us reconsider the simply supported beam-column with concentrated later load Q studied in

Section W11.4. The deflection of this member is given by Eq. W11.4.9. The maximum deflection of this member occurs at mid-span and is given, by letting $z = L/2$ in Eq. W11.4.9 and rearranging as:

$$u_{\max} = \delta = \frac{Q}{2P\alpha}\left[\tan\mu - \mu\right] \qquad \text{(W11.5.1)}$$

where μ is defined in Eq. W11.4.11. Multiplying and dividing the expression in Eq. W11.5.1 by $\dfrac{L^3}{24EI}$ gives

$$\delta = \frac{QL^3}{48EI}\,\frac{24EI}{\alpha PL^3}\left(\tan\mu - \mu\right) = \left(\frac{QL^3}{48EI}\right)\left[\frac{3}{\mu^3}\left(\tan\mu - \mu\right)\right] \qquad \text{(W11.5.2)}$$

The left-hand factor in this relation is the deflection that would exist if the transverse load Q were acting by itself (i.e., $P = 0$) and hence, the primary deflection at the center of the beam. Accordingly, we introduce the notation

$$\delta_o = \frac{QL^3}{48EI} \qquad \text{(W11.5.3)}$$

and rewrite Eq. W11.5.2 as

$$\delta = \delta_o\left[\frac{3}{\mu^3}\left(\tan\mu - \mu\right)\right] \qquad \text{(W11.5.4)}$$

To simplify the expression for δ, we use the power series expansion for $\tan\mu$, namely

$$\tan\mu = \mu + \frac{\mu^3}{3} + \frac{2}{15}\mu^5 + \frac{17}{315}\mu^7 + \ldots \tag{W11.5.5}$$

Substitution of this series for tan μ in Eq. W11.5.4 gives

$$\delta = \delta_o\left[1 + \frac{2}{5}\mu^2 + \frac{17}{105}\mu^4 + \ldots\right] \tag{W11.5.6}$$

which, with the help of Eq. W11.4.11 can be written as

$$\delta = \delta_o\left[1 + 0.984\frac{P}{P_E} + 0.998\left(\frac{P}{P_E}\right)^2 + \ldots\right] \tag{W11.5.7}$$

or very nearly

$$\delta = \delta_o\left[1 + \frac{P}{P_E} + \left(\frac{P}{P_E}\right)^2 + \ldots\right] \tag{W11.5.8}$$

Since the sum of the geometric series inside the brackets is $\left[1 - \frac{P}{P_E}\right]^{-1}$, Eq. W11.5.8 reduces to

$$\delta = \delta_o\frac{1}{\left(1 - \frac{P}{P_E}\right)} \tag{W11.5.9}$$

The maximum second-order moment in the beam-column occurs under the load. Its value, with the help of Eqs. W11.4.1, W11.5.3 and W11.5.9 could be written as (W11.5.10)

$$M^*_{\max} = \frac{QL}{4} + P\delta = \frac{QL}{4} + \frac{PQL^3}{48EI}\frac{1}{\left(1 - \frac{P}{P_E}\right)} = \frac{QL}{4}\left[1 + \frac{PL^2}{12EI}\frac{1}{\left(1 - \frac{P}{P_E}\right)}\right]$$

Simplifying the term inside the bracket gives

$$M^*_{\max} = \frac{QL}{4}\left[1 + 0.82\frac{P}{P_E}\frac{1}{\left(1 - \frac{P}{P_E}\right)}\right] = \left(\frac{QL}{4}\right)\frac{\left[1 - 0.18\frac{P}{P_E}\right]}{\left[1 - \frac{P}{P_E}\right]} \qquad \text{(W11.5.11)}$$

The factor (QL /4) in Eq. W11.5.11 is the maximum moment that would exist in the member if no axial force were present. That is

$$\frac{QL}{4} = M_{\max} \qquad \text{(W11.5.12)}$$

the maximum 1st order moment in the pin-ended beam-column under a single concentrated transverse load Q at the center. Equation W11.5.11 can be rewritten as:

$$M^*_{\max} = \frac{\left[1 - 0.18\frac{P}{P_E}\right]}{\left[1 - \frac{P}{P_E}\right]} M_{\max} \qquad \text{(W11.5.13)}$$

For design purposes, the exact relations for maximum second-order moment(s) for transversely loaded pinned or fixed-ended beam-columns, obtained by the differential equation approach described in Sections W11.3 and W11.4, could be approximated by the single formula:

$$M^*_{\max} = B_1 M_{\max} = \frac{C_m}{\left(1 - \frac{P}{P_e}\right)} M_{\max} \qquad \text{(W11.5.14)}$$

where B_1 is a moment magnification factor, P is the axial load in the beam-column and P_e is the elastic critical load of the beam-column about the bending axis, consistent with the end support conditions. From Eqs. W11.5.13 and W11.5.14, we observe that for a simply supported beam-column under a central concentrated load

$$C_m \approx \left[1 - 0.2\frac{P}{P_e}\right]; \qquad P_e = P_E = \frac{\pi^2 EI}{L^2} \tag{W11.5.15}$$

This is the value for C_m given in Table W11.5.1. Values of C_m for several common cases of lateral loading for beam-columns simply supported at their ends are summarized in Table W11.5.1, and for beam-columns fixed at one or both its ends in Table W11.5.2. For additional information see [Iwankiw, 1984]. In lieu of using the equations and the tables mentioned above, $C_m = 1.0$ can be used conservatively for laterally loaded beam-columns with unrestrained ends, and 0.85 for members with restrained ends.

TABLE W11.5.1: Moment Reduction Factor C_m for Beam-Columns with Pinned-Ends and Transverse Loading

TABLE W11.5.2: Moment Reduction Factor C_m for Beam-Columns with One or Both Ends Restrained Subjected to Transverse Loads

W11.6 Elastic Lateral-Torsional Buckling of Beam-Columns of Singly or Doubly Symmetric I-Sections

Let us consider a perfectly straight beam-column of a uniform, mono-symmetric I-shape and of length L. The ends of the beam-column are assumed to be flexurally and torsionally simply supported and free to warp. The member is bent about its major axis by equal and opposite end moments M^{o} and loaded by an axial force P, with the applied moments acting in the plane of symmetry (yz plane) as shown in Fig. W11.6.1.

Figure W11.6.1: Lateral-torsional buckling of a singly symmetric beam-column.

The three differential equations of equilibrium for lateral-torsional buckling of beams developed in Section 10.2, can be extended to include the effect of a centrally applied compression force P by adding to the components of M^{o} the second order moments due to P. Thus, the differential equilibrium equations governing the elastic behavior of the beam-column are given by (see [Galambos, 1968]:

$$EI_x v'' + Pv = -M^o \qquad \text{(W11.6.1}a\text{)}$$

$$EI_y u'' + Pu + (M^o + Py_o)\,\phi = 0 \qquad \text{(W11.6.1}b\text{)}$$

$$EC_w \phi''' - (GJ + \bar{K})\phi' + (M^o + Py_o)\,u' = 0 \qquad \text{(W11.6.1}c\text{)}$$

where EI_x is the bending stiffness of the section about the x axis; EI_y, the bending stiffness about the y axis; EC_w, the warping stiffness and GJ, the St. Venant torsional stiffness. The term $\bar{K}\phi'$ represents the torque exerted by transverse components of the longitudinal bending stresses which develop when the beam is twisted (also known as Wagner effect). We have:

$$\bar{K} = \int \rho^2 f\,dA = \int f\left\{(x - x_o)^2 + (y - y_o)^2\right\} dA \qquad \text{(W11.6.2)}$$

in which ρ is the distance of a point (x, y) in the cross section from the shear center (x_o, y_o) and f is the longitudinal stress at that point. In the case of mono-symmetric I-shape under uniform moment considered here $x_o = 0$, while the moments M^o and the axial load P develop stresses

$$f = -\frac{P}{A} + \frac{M^o}{I_x} y \tag{W11.6.3}$$

resulting in

$$\bar{K} = -P\bar{r}_o^2 + M^o \beta_x \tag{W11.6.4}$$

with

$$\beta_x = \left[\frac{1}{I_x}\left\{\int_A x^2 y\,dA + \int_A y^3 dA\right\} - 2y_o\right] \tag{W11.6.5}$$

$$\bar{r}_o^2 = I_o/A = \left[\int\left\{(x - x_o)^2 + (y - y_o)^2\right\}dA\right]/A \tag{W11.6.6}$$

Here, β_x is a mono-symmetry parameter of the cross section and I_o is the polar moment of inertia of the section about the shear center. The action of the torque $\beta_x M^o \phi'$ is equivalent to changing the effective torsional rigidity of the section from GJ to $(GJ + M^o \beta_x)$. In doubly symmetric beams the disturbing torque exerted by the compressive bending stresses is exactly balanced by the restoring torque due to the tensile stresses, resulting in a value of zero for β_x.

The first equilibrium equation (Eq. W11.6.1*a*) is nonhomogeneous and independent of the other two. When integrated it gives the deflected shape of the beam-column in the plane of symmetry which is also the plane of loading. The second and third relations are coupled and homogeneous. They describe the critical condition when lateral-torsional buckling of the beam-column takes place.

Differentiating Eqs. W11.6.1*a* and *b* twice and Eq. W11.6.1*c* once, we obtain:

$$EI_x v^{IV} + P v'' = 0 \tag{W11.6.7a}$$

$$EI_y u^{IV} + P u'' + (M^o + Py_o)\phi'' = 0 \tag{W11.6.7b}$$

$$EC_w \phi^{IV} - (GJ + \bar{K})\phi'' - \bar{K}'\phi' + (M^o + Py_o)u'' = 0 \tag{W11.6.7c}$$

Noting that, for uniform members in elastic domain, $\bar{K}$ is a constant along the length $(\bar{K}' = 0)$, the two differential equations that describe the lateral torsional buckling of a mono-symmetric beam-column under axial load P and uniform moment M^o may now be written as:

$$EI_y u^{IV} + Pu'' + (M^o + Py_o)\phi'' = 0 \tag{W11.6.8a}$$

$$EC_w \phi^{IV} - (GJ - P\bar{r}_o^2 + M^o\beta_x)\phi'' + (M^o + Py_o)u'' = 0 \tag{W11.6.8b}$$

For the simply supported boundary conditions considered here, namely,

$$\begin{aligned} u_{(0)} &= 0\,; \qquad u''_{(0)} = 0\,; \qquad \phi_{(0)} = 0 \\ u_{(L)} &= 0\,; \qquad u''_{(L)} = 0\,; \qquad \phi_{(L)} = 0 \end{aligned} \tag{W11.6.9}$$

the differential equations W11.6.8*a* and *b* have the solution:

$$u = A_1 \sin\frac{\pi z}{L}; \qquad \phi = A_2 \sin\frac{\pi z}{L} \tag{W11.6.10}$$

Substituting these deflection functions and their derivatives, into the governing differential equations W11.6.8*a* and *b* results in:

$$\left[\left(\frac{\pi^2 EI_y}{L^2} - P\right)A_1 - (M^o + Py_o)A_2\right]\left(\frac{\pi^2}{L^2}\sin\frac{\pi z}{L}\right) = 0 \qquad \text{(W11.6.11)}$$

$$\left[-(M^o + Py_o)A_1 + \left(\frac{\pi^2}{L^2}EC_w + GJ - P\bar{r}_o^2 + M^o\beta_x\right)A_2\right]\left(\frac{\pi^2}{L^2}\sin\frac{\pi z}{L}\right) = 0$$

The condition sin $(\pi z/L) = 0$ indicates that the beam-column continues to bend in the plane of applied moment only, as $u = 0$ and $\phi = 0$ for all z, from Eq. W11.6.10. Hence, non-trivial solutions for u and ϕ are only possible when the determinant of the coefficient matrix for A_1 and A_2 becomes equal to zero. That is:

$$\begin{vmatrix} (P_{Ey} - P) & -(M^o + Py_o) \\ -(M^o + Py_o) & [\bar{r}_o^2(P_{Ez} - P) + M^o\beta_x] \end{vmatrix} = 0 \qquad \text{(W11.6.12)}$$

The lateral-torsional buckling condition for the mono-symmetric beam-column is therefore given by the characteristic equation:

$$(P_{Ey} - P)[\bar{r}_o^2(P_{Ez} - P) + M^o\beta_x] = (M^o + Py_o)^2 \qquad \text{(W11.6.13)}$$

wherein

$$P_{Ey} = \frac{\pi^2 EI_y}{L^2}; \qquad P_{Ez} = \frac{1}{\bar{r}_o^2}\left[\frac{\pi^2 EC_w}{L^2} + GJ\right] \qquad \text{(W11.6.14)}$$

Here, P_{Ey} represents the elastic flexural buckling load about the weak axis and P_{Ez} represents the elastic torsional buckling load of a flexurally and torsionally simply supported member. For any given beam-column, Eq. W11.6.13 can be solved for the critical combination of P and M^o.

For doubly symmetric sections the mono-symmetry parameter, $\beta_x = 0$, and the distance of the shear center from the center of gravity of the section, $y_o = 0$ and therefore Eq. W11.6.13 simplifies to:

$$(P_{Ey} - P)(P_{Ez} - P) = \frac{1}{\bar{r}_o^2}(M^o)^2 \qquad \text{(W11.6.15)}$$

Solving for M^o, the elastic critical moment for a flexurally and torsionally simply supported beam-column of a doubly-symmetric I-shape subjected simultaneously to an axial load P is obtained as:

$$M^o = M^o_{ccrE} = \sqrt{\frac{(I_x + I_y)}{A}(P_{Ey} - P)(P_{Ez} - P)} \qquad \text{(W11.6.16)}$$

The elastic critical moment for a flexurally and torsionally simply supported beam of a doubly-symmetric I-shape, given by Eq. 10.2.34, may now be rewritten with the help of Eq. W11.6.14 as:

$$M^o_{crE} = \sqrt{\frac{\pi^2 EI_y}{L^2}\left(\frac{\pi^2}{L^2}EC_w + GJ\right)} = \sqrt{\left(\frac{I_x + I_y}{A}\right)P_{Ey}P_{Ez}} \qquad \text{(W11.6.17)}$$

The critical moment for the doubly symmetric I-shaped beam-column, given by Eq. W11.6.16, may now be rewritten in the non-dimensional form:

$$\frac{M^o_{ccrE}}{M^o_{crE}} = \sqrt{\left(1 - \frac{P}{P_{Ey}}\right)\left(1 - \frac{P}{P_{Ez}}\right)} \qquad \text{(W11.6.18)}$$

The results obtained above, for lateral-torsional buckling of a flexurally and torsionally simply supported beam-column under uniform, major axis moment may be generalized to account for other end conditions by introducing appropriate effective length factors. Thus, Eq. W11.6.13 may be rewritten as:

$$(P_{ey} - P)\,[\bar{r}_o^2(P_{ez} - P) + M^o\beta_x] = K_{13}^2(M^o + Py_o)^2 \tag{W11.6.19}$$

with

$$P_{ey} = \frac{\pi^2 EI_y}{(K_y L)^2}; \qquad P_{ez} = \frac{1}{\bar{r}_o^2}\left[\frac{\pi^2 EC_w}{(K_z L)^2} + GJ\right] \tag{W11.6.20}$$

and K_{13} is a coefficient depending on the boundary conditions. Values of K_{13} are given by Vlassov [1961] and Galambos [1968] for various combinations of support conditions (see Table W10.1.1). K_{13} may be conservatively taken as 1.0 for all cases.

If an approximate allowance is made for the effects of in-plane deflection, as suggested by Chen and Atsuta [1977], Eq. W11.6.18 becomes

$$\frac{M_{ccrE}^o}{M_{crE}^o} = \left[\left(1 - \frac{P}{P_{ex}}\right)\left(1 - \frac{P}{P_{ey}}\right)\left(1 - \frac{P}{P_{ez}}\right)\right]^{\frac{1}{2}} \tag{W11.6.21}$$

wherein P_{ex} is the elastic buckling load for major axis flexural buckling. Noting that for I-shapes $P_{ez} > P_{ey}$,

$$\left(1 - \frac{P}{P_{ez}}\right) > \left(1 - \frac{P}{P_{ey}}\right)\left(1 - \frac{P}{P_{ex}}\right) \tag{W11.6.22}$$

enables Eq. W11.6.21 to be simplified to:

$$\frac{P}{P_{ey}} + \frac{M^o}{M_{crE}^o\left(1 - \dfrac{P}{P_{ex}}\right)} \leq 1.0 \tag{W11.6.23}$$

Note that the parameters P_{ey} and M^{o}_{crE} appearing in the denominator relate to out-of-plane failure.

W11.7 Planar Strength of Steel Beam-Columns

Ketter, Kaminsky and Beedle [1955] studied analytically and experimentally the carrying capacity of pin-ended beam-columns loaded eccentrically. Their analysis included, for the first time, the influence of residual stresses on the planar strength of steel beam columns. Galambos and Ketter [1961] presented a rational approach, using the Newmark method of numerical integration, to determine the carrying capacity of pin ended wide flange beam-columns. They gave interaction curves relating the axial load and end moments, as a function of slenderness ratio of the beam-column. Bijlaard, Fisher and Winter [1953] reported buckling tests on columns elastically restrained and eccentrically loaded. They showed clearly that, in elastically restrained columns, a portion of the applied end-moment is resisted by the end restraints so that the column resists only the remaining portion. The fraction of the total end moment carried by the column decreases with increasing load, both in the elastic and plastic range.

The concept of column deflection curves, originally developed by Von Karman [1910] was extended by Ojalvo [1960] to study the planar strength of restrained beam-columns, wherein the restraints have non-linear moment-rotation response. Vinnakota developed in 1967 a numerical approach using the method of transfer matrices to determine the planar strength of rotationally and directionally restrained, initially straight, beam-columns [Vinnakota, 1967; Vinnakota and Badoux, 1970; Vinnakota and Badoux, 1971]. The influence of residual stresses is included and the restraints are assumed to be elastic-perfectly plastic. The beam-column may be subjected to end moments and/or lateral loads.

Figure W11.7.1: Ultimate strength interaction curves for beam-columns under uniform moment [Galambos and Ketter, 1961].

Interaction curves for strong axis bending of wide flange section beam-columns, including the effect of residual stresses, have been developed by Galambos and Ketter [1961]. Figure W11.7.1 gives the curves for a loading condition in which two equal end-moments cause the column to bend in the symmetric single curvature form (the most severe loading case). The calculations were based on a steel with a yield stress of 33 ksi and a W8×31 shape having a symmetric residual stress pattern (linear variation between flange tip and web-to-flange junction in the flange, and uniform tensile residual stress in the web). The maximum residual compressive stress in the flange tips equals $0.3F_y$, where F_y is the yield stress of the material. The interaction curves show the relation between axial load P (non-dimensionalized by the yield load P_y) and the end-moments M^o_x (non-dimensionalized by the plastic moment M_{px}) for a given slenderness ratio L/r_x about the major axis. From the Fig. W11.7.1 we observe that:

- When $P = 0$, the member is a beam and can support a moment equal to M_{px} corresponding to complete yielding of a cross section of the column. Thus, curves corresponding to different slenderness ratios converge to a point.
- When $M = 0$, the member is a column which can carry a load equal to its own critical load (tangent modulus load). Thus, intersections of the interaction curves with the vertical axis depend on slenderness ratio L/r_x.

- For a given value of P, the member for which $L/r = 0$ can carry considerably more moment than the member with $L/r_x = 120$, indicating that short members are stronger than long members.
- Up to $L/r_x = 60$ the interaction curves are nearly straight lines. For higher slenderness ratios the curves sag downward, thus indicating the greater influence of secondary moments due to deflection ($P\delta$ effect).

The validity of the interaction curves shown in Fig. W11.7.1 has been verified by extensive experiments [Van Kuren and Galambos, 1964] and the correlation between the experimental results and the theoretical predictions was found to be good. Interaction curves have also been developed for other end-moment ratios and for beam-columns subjected to lateral loads [Lu and Kamalvand, 1968; Chen 1970, 1971].

Figure W11.7.2: Moment-rotation curves for beam-columns [Lehigh, 1965].

The method of column deflection curves (CDC-s) developed by Ojalvo [1960] permits to determine the end moment versus end rotation characteristics of a beam-column in the plane of moments for any axial load and any end-moment ratio, r_M [Ojalvo and Fukumoto, 1962]. The M-θ curve represents the equilibrium configuration of the beam-column under selected values of P and L/r. M-θ curves for strong axis bending of wide-flange shapes for several values of L/r_x are reproduced in Fig. W11.7.2*a* for axial load ratio of 0.6 and moment ratio $r_M = -1.0$, and in Fig. W11.7.2*b* for axial load ratio of 0.6 and moment ratio $r_M = 0$. The ordinates have been non-dimensionalized by M_{pcx} the plastic moment reduced by the presence of axial load, P. The cut-off points for the curves (dashed line) represent inception of local buckling. We observe that for a given P the maximum strength decreases as the member length increases. Additional information on CDC-s and M-θ curves is available in [Lehigh, 1965].

The ultimate strengths (P, M_{cx}) of an I-section beam-column bent about its major axis by equal or unequal end moments and/or transverse loading between points of support which fail in the plane of the applied moments, can be closely approximated by the interaction formulae:

$$\frac{P}{P_{crx}} + \frac{C_{mx}}{\left(1 - \frac{P}{P_{ex}}\right)} \frac{M_{cx}}{M_{px}} \leq 1.0 \qquad \text{(W11.7.1}a\text{)}$$

$$\frac{M_{cx}}{M_{pcx}} \leq 1.0 \qquad \text{(W11.7.1}b\text{)}$$

Here, P_{crx} is the axial load producing failure in the absence of any applied moments (i.e., beam-column with $M = 0$) computed for major axis buckling. P_{ex} is the elastic buckling load of the concentrically loaded column. M_{cx} is the maximum applied major axis moment. C_{mx} is a moment reduction factor discussed in Section 11.3. Equation W11.7.1a translates conditions of failure resulting from instability due to excessive bending occurring within the member. However, when a beam-column is bent by moments producing plastic hinges at one or both ends, it is necessary to limit such terminal moments to within M_{pc}, the plastic hinge moment modified to include the effect of axial load P. This is expressed by Eq. W11.7.1b.

For beam-columns bent about their minor axis, the corresponding interaction formulae are:

$$\frac{P}{P_{cry}} + \frac{C_{my}}{\left(1 - \dfrac{P}{P_{ey}}\right)} \frac{M_{cy}}{M_{py}} \leq 1.0 \qquad \text{(W11.7.2}a\text{)}$$

$$\frac{M_{cy}}{M_{pcy}} \leq 1.0 \qquad \text{(W11.7.2}b\text{)}$$

W11.8 Inelastic Lateral-Torsional Buckling of Steel Beam-Columns

In 1956, Campus and Massonnet [1956] presented a tangent modulus solution for doubly-symmetric I-section beam-columns with equal end-moments applied in the plane of the web producing symmetric single curvature bending. Also presented in the report were the test results of ninety-two obliquely loaded as-rolled beam-columns that failed by lateral-torsional buckling. Figure W11.8.1*a* shows the distribution of plastic zones (shown shaded) at three levels of loading for an I-section bent about its major axis; while the Fig. W11.8.1*b* shows the corresponding elastic core that defines the stiffness of the section. Note that the elastic core is no more doubly-symmetric. The study showed that the shift in the position of the shear center with plasticity has considerable influence on the torsional response of these columns. Theoretical solutions for inelastic lateral torsional buckling of beam-columns have been developed by Galambos [1959], Fukumoto [1963] and Galambos, Adams and Fukumoto [1965]. Fukumoto and Galambos [1966] presented a solution to a more general case of beam-columns subjected to unequal end-moments where the degree of yielding varies along the length of the member. Their calculations included the influence of residual stresses.

Figure W11.8.1: Influence of plastification on distribution of elastic core of steel beam-column sections.

As mentioned earlier, real beam-columns will have initial out-of-plane imperfections and initial twist. In addition, the beam-column loads may have accidental eccentricity. When such a beam-column is loaded into the plastic range, the shear center will not only shift along the web as is assumed in the above studies, but across it. This further increases the effect of torsion. Thus, the maximum strength of initially crooked beam-columns $M^o_{cs\,\max}$ could be lower than the lateral buckling load $M^o_{ccr\,1}$ of the corresponding idealized beam-column (Fig. 11.6.1). Vinnakota [1977*a*, *b*] presented such a general approach to study the inelastic spatial stability of laterally unsupported I-beams and evaluate the load $M^o_{cs\,\max}$.

In the case of laterally unsupported beam-columns of I-shaped sections bent in the plane of the web, Campus and Massonnet [1956] have found that the following modified interaction formula is quite satisfactory for failures in both elastic range and inelastic range.

$$\frac{P}{P_{cr}} + \frac{C_{mx} M_{cx}}{\left(1 - \dfrac{P}{P_{ex}}\right) M^o_{cr}} \leq 1.0 \qquad \text{(W11.8.1)}$$

which is similar to Eq. W11.7.1*a* except that P_{cr} now is the smaller of P_{crx} and P_{cry}, and M^o_{cr} reflects the possibility of lateral-torsional buckling of the member acting as a beam.

Figure W11.8.2: Comparison of Eq. W11.8.1 with numerical data of Vinnakota for lateral-torsional strength of beam-columns [Galmbos, 1988].

Campus and Massonnet [1956] found that Eq. W11.8.1 agrees well with their tests on obliquely loaded as rolled beam-columns. Hill and Clark [1951] showed that the interaction Eq. W11.8.1 shows good agreement with the test results for eccentrically loaded I-shaped aluminum columns that fail by lateral-torsional buckling. Validity of the interaction equation W11.8.1 for beam-columns for the limit state of lateral-torsional buckling was confirmed in Galambos [1988] as shown in Fig. W11.8.2. Here, the straight line represents Eq. W11.8.1 with the left hand side equated to 1.0. The points shown are obtained by using, in the left hand side of Eq. W11.8.1, the values of $M^o_{cs\,\max}$ for zero axial load, values of P_{cr} for zero moment, and the values of (P, M_{cx}) corresponding to the limit state of flexural-torsional stability of beam-columns of different slenderness ratios, obtained by Vinnakota [1977*b*].

W11.9 Strength of Biaxially Bent Steel Beam-Columns

Goodier [1942] presented the governing differential equations of equilibrium for open section beam-column subjected to an axial compression with equal end eccentricities at both ends. The differential equations describing the spatial behavior of beam-columns were derived by Timoshenko [1945], Vlassov [1961] and Galambos [1968] based on geometrical considerations and by Bleich [1952] based on energy principles.

Gent and Milner [1968] have shown that if beams framing into a column remain elastic under increasing load, while the column stiffness deteriorates, the column moments will diminish. This is called the phenomenon of ***shedding down of moments***. There is a transition from the column restraining the beam to the beam restraining the column, which in certain circumstances enables the column load to approach its full yield capacity. This is equally true of biaxial loading.

Vinnakota and associates working at the Swiss Federal Institute of Technology, Lausanne, Switzerland [1973, 1974, 1976, 1977*a* and *b*] reported solutions for elasticially restrained beam-columns under biaxial bending and torsion, using finite difference methods. Thus, rotationally restrained columns were considered by Vinnakota and Aoshima [1974*a*], while rotationally and directionally restrained columns were studied by Vinnakota and Aysto [1974] and Vinnakota and Aoshima [1974*b*]. Numerical examples that show the influence of restraints on the strength of biaxially bent columns and the phenomenon of moment shedding for biaxially loaded beam-columns were given by Vinnakota [1974]. All the beam-columns considered in the studies mentioned above were initially straight and untwisted and loaded at their ends only. Vinnakota [1976] developed a general method to study the spatial behavior of laterally loaded I-section beam-columns. The study included the influence of initial crookedness. The restrained beam-column considered in these studies is shown in Fig. W11.9.1. As indicated in Fig. W11.9.1*b*, the origin of the coordinate system is taken at an arbitrary point C of the cross-section (not the center of gravity, G). Further, the pole of the sectorial areas is taken as an arbitrary point O of the cross section (not the shear center, S). Using this approach the influence of the shift in the center of gravity and the shear center of the elastic core of partially plastified sections on the formulation of the equilibrium equations can be included automatically. A parametric study to judge the influence of residual stresses and/or geometrical imperfections on the maximum strength of biaxially bent isolated columns was given by Vinnakota [1976].

Figure W11.9.1: Restrained biaxially loaded beam column [Vinnakota 1977*b*].

A large series of buckling tests (about 100) on compressed and biaxially bent I-columns were conducted in 1977 and 1978 by Massonnet at Liege University. Vinnakota using a modified theory given by Vinnakota and Aoshima [1974*b*] predicted in 1975, the anticipated results of the then proposed tests, using nominal section dimensions, yield stress, residual stresses and geometrical imperfections. The first series of test results, presented by Vinnakota and Anslijn [1977] indicated good agreement with these predicted values. The complete test data was published by Anslijn [1983].

Based on the cross sectional properties of W8×31 of $F_y =$ 36 ksi steel, with linear distribution of residual stresses due to rolling, and compressive residual stress at flange tip (f_{rc}) of $0.33F_y$, families of numerical interaction relations for biaxially bent beam-columns are presented by Tebedge and Chen [1974]. The values are given in non-dimensional form between the end moments M_x/M_{px} and M_y/M_{py} for various major axis slenderness ratios L/r_x ranging from 0 to 100 and for axial load ratios P/P_y from 0 to 0.9.

Figure W11.9.2: Maximum strength interaction surface for a column subjected to biaxial bending.

Figure W11.9.2 shows schematically a typical maximum strength interaction surface for a beam-column of particular length and cross section. Each axis represents the capacity of the member when it is subjected to loading of one type only, while the curves represent the combination of two types of loading, namely, (P, M_x); (P, M_y); (M_x, M_y). The surface formed by connecting the three curves represents the interaction of axial load and biaxial bending, (P, M_x, M_y). From Fig. W11.9.2, observe that if a member is fully loaded, under axial load and bending about one axis, then there is no spare capacity to accept moment about the other axis. However as the loading decreases slightly below maximum, capacity rapidly develops to carry bending about the other axis. Based on these findings and in parallel with their proposal for the case of short columns (Eq. W11.2.2), Tebedge and Chen [1974] proposed the nonlinear interaction formula for long beam-columns, namely,

$$\left(\frac{M_x}{M_{ncx}}\right)^{\eta} + \left(\frac{M_y}{M_{ncy}}\right)^{\eta} \leq 1.0 \tag{W11.9.1}$$

The values M_{ncx} and M_{ncy}, respectively, are the maximum uniform single curvature moment which can be resisted by the member about the respective axis in the presence of the axial load, but in the absence of the other moment. The value of these moments may be determined by transposing Eq. W11.8.1 and W11.7.2a as follows:

$$M_{ncx} = M_{nx}\left[1 - \frac{P}{P_{cr}}\right]\left[1 - \frac{P}{P_{ex}}\right] \tag{W11.9.2a}$$

$$M_{ncy} = M_{ny}\left[1 - \frac{P}{P_{cr}}\right]\left[1 - \frac{P}{P_{ey}}\right] \tag{W11.9.2b}$$

Normally, M_{ny} will be equal to M_{py} but M_{nx} may need to be reduced below M_{px} for lateral-torsional buckling. The following values for η are based on the studies by Ross and Chen [1976]:

$$\eta = \max\left(0.4 + \frac{P}{P_y} + \frac{b_f}{d};\ 1.0\right) \quad \text{for} \quad \frac{b_f}{d} \geq 0.3 \tag{W11.9.3a}$$

$$= 1.0 \quad \text{for} \quad \frac{b_f}{d} < 0.3 \tag{W11.9.3b}$$

W11.10 Amplification Factor, B_2, Based on the Story Stiffness Concept

The $P\Delta$ moment amplification factor B_2 for multi story frame (expressed by LRFDS Formula C1-4) was developed based on the story stiffness concept [Rosenblueth, 1965; Cheong-Siat-Moy, 1972] assuming that each story behaves independently of other stories.

When lateral forces $\sum H$ act on a frame (Fig. W11.10.1*a*), the frame will deflect laterally until an equilibrium position is reached. The corresponding first-order deflection (lateral deflection calculated on the basis of the original geometry of the frame) is denoted as Δ_o. Referring to Fig. W11.10.1*b* first-order moment-equilibrium of the *i* th column of the story requires:

$$(M_{\ell t1} + M_{\ell t2})_i = hV_i \tag{W11.10.1}$$

or, considering all the n columns of the story (Fig. W11.10.1*c*):

$$\sum_{i=1}^{n} \left(M_{\ell t1} + M_{\ell t2}\right)_i = h\sum_{i=1}^{n} V_i = h\sum_{j=1}^{m} H_j \tag{W11.10.2}$$

where $M_{\ell t1}$ and $M_{\ell t2}$ are the first-order moments in the *i*-th column of the story considered, resulting from a sway analysis of the unbraced frame under factored lateral loads.

If in addition to $\sum H_j$, vertical forces $\sum P_i$ are acting on the frame, these axial forces in the columns will interact with the lateral displacement Δ_o caused by $\sum H$ (Fig. W11.10.1*d*), to drift the frame further until a new equilibrium position is reached. The lateral deflection that corresponds to the new equilibrium position is denoted by Δ^* (Fig. W11.10.1*e*). The vertical forces $\sum P_i$ interact with the lateral displacement of the frame (the $P\Delta$ effect) and result in an increase in drift and an increase in overturning moment. Since the additional deflection and overturning moment have deleterious effects on the stiffness and stability of the frame, they should be included in the design.

Figure W11.10.1: B_2 factor for an unbraced frame.

The moment equilibrium of the ith column in the displaced position may now be expressed as (Fig. W11.10.1*f*):

$$M^*_{\ell t1} + M^*_{\ell t2} = h V^*_i + P_i \Delta^* \tag{W11.10.3}$$

and for all the columns in the story, we obtain: (W11.10.4)

$$\sum_{i=1}^{n} \left(M^*_{\ell t1} + M^*_{\ell t2} \right)_i = h \sum_{i=1}^{n} V^*_i + \Delta^* \sum_{i=1}^{n} P_i = h \sum_{j=1}^{m} H_j + \Delta^* \sum_{i=1}^{n} P_i$$

This relation could be rewritten as:

$$\sum_{i=1}^{n} B_2 \left(M_{\ell t1} + M_{\ell t2} \right)_i = h \sum_{j=1}^{m} H_j + \Delta^* \sum_{i=1}^{n} P_i \tag{W11.10.5}$$

where B_2 is the magnification factor. Comparing Eq. W11.10.5 with Eq. W11.10.2 gives:

$$B_2 = \frac{h\sum H_j + \Delta^* \sum P_i}{h\sum H_j} = \frac{\sum H_j + \dfrac{\Delta^*}{h}\sum P_i}{\sum H_j} \qquad \text{(W11.10.6)}$$

The numerator represents the equivalent magnified lateral load.

The sway stiffness of a story can be defined as:

$$S = \frac{\text{horizontal force}}{\text{lateral displacement}} \qquad \text{(W11.10.7)}$$

which is valid for first order analysis. Thus

$$\Delta_o = \frac{1}{S}\sum_{j=1}^{m} H_j \qquad \text{(W11.10.8)}$$

and using the equivalent magnified lateral load

$$\Delta^* = \frac{1}{S}\left[\sum_{j=1}^{m} H_j + \frac{\Delta^*}{h}\sum_{i=1}^{n} P_i\right] \qquad \text{(W11.10.9)}$$

Eliminating S from Eqs. W11.10.8 and W11.10.9 results in:

$$\frac{\Delta^*}{\Delta_o} = \frac{\sum H_j + \dfrac{\Delta^*}{h}\sum P_i}{\sum H_j} \qquad \text{(W11.10.10)}$$

from which solving for Δ^* gives

$$\Delta^* = \frac{\Delta_o}{1 - \dfrac{\Delta_o}{h}\dfrac{\sum P_i}{\sum H_j}} = A_f \Delta_o \qquad \text{(W11.10.11)}$$

This equation indicates that the final deflection Δ^* can be estimated from the first-order deflection Δ_o by multiplying the latter by an amplification factor, A_f. Substituting Eq. W11.10.11 into Eq. W11.10.6 gives:

$$B_2 = \frac{1}{1 - \dfrac{\Delta_o}{h} \dfrac{\sum_{i=1}^{n} P_i}{\sum_{j=1}^{m} H_j}} \qquad \text{(W11.10.12)}$$

Even though the deflection Δ_o and the total lateral force $\sum H$ are for factored loads in the derivation, the ratio $\dfrac{\Delta_o}{\sum H}$ for factored loads or $\dfrac{\Delta_{os}}{\sum H_s}$ for service loads will be the same, since the first-order analysis is to be elastic. We therefore have:

$$M_{\ell t}^* = B_2 M_{\ell t} \qquad \text{(W11.10.13)}$$

From Eq. W11.10.13, it can be seen that the second-order moment accounting for the $P\Delta$ effect can be obtained by multiplying the first-order sway moment $M_{\ell t}$ by the $P\Delta$ moment amplification factor, B_2, given by Eq. W11.10.12.

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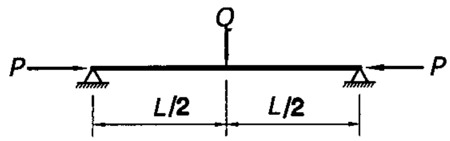
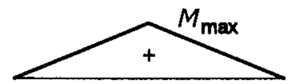
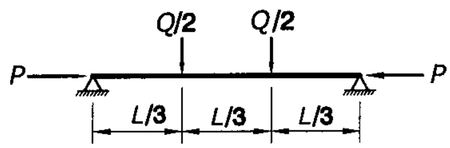
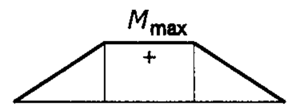
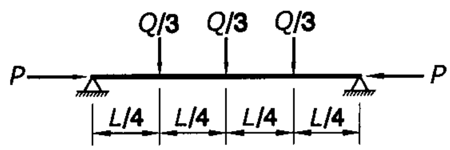
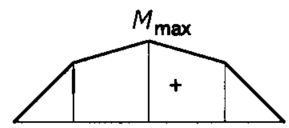
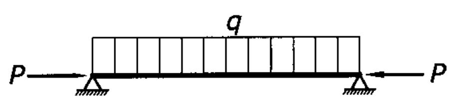
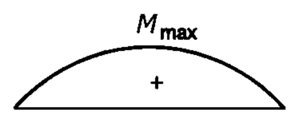
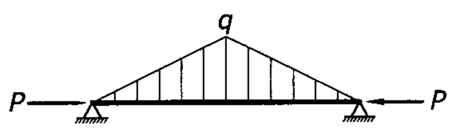
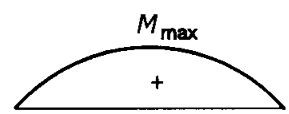
	Case	C_m	Primary Bending Moment
1		$1 - 0.2\alpha$	
2		$1 + 0.051\alpha$	
3		$1 - 0.023\alpha$	
4		1	
5		$1 - 0.013\alpha$	
$\alpha = \frac{P_u}{P_e}$; $P_e = P_E = \pi^2 \frac{EI}{L^2}$			

Table W11.5.1: Moment reduction factor C_m for beam-columns with pinned ends and transverse loading.

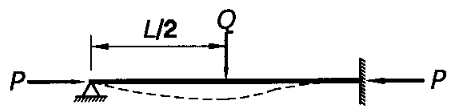
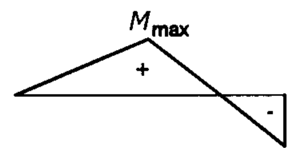
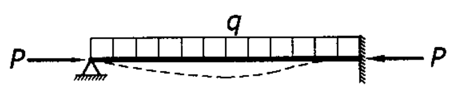
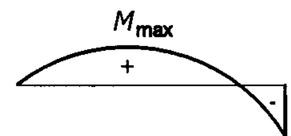
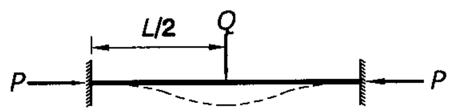
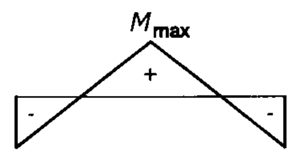
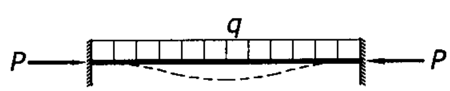
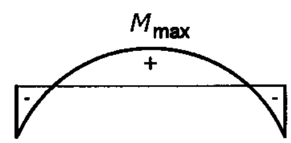
	Case	C_m (positive moment)	C_m (negative moment)	Primary bending moment	K
1		$1 - 0.4\alpha$	$1 - 0.3\alpha$		0.7
2		$1 - 0.3\alpha$	$1 - 0.4\alpha$		0.7
3		$1 - 0.6\alpha$	$1 - 0.2\alpha$		0.5
4		$1 - 0.4\alpha$	$1 - 0.4\alpha$		0.5
$\alpha = \frac{P_u}{P_e}$; $P_e = \frac{\pi^2 E I}{(K L)^2}$					

Table W11.5.2: Moment reduction factor C_m for beam-columns with one or both ends restrained subjected to transverse loads.

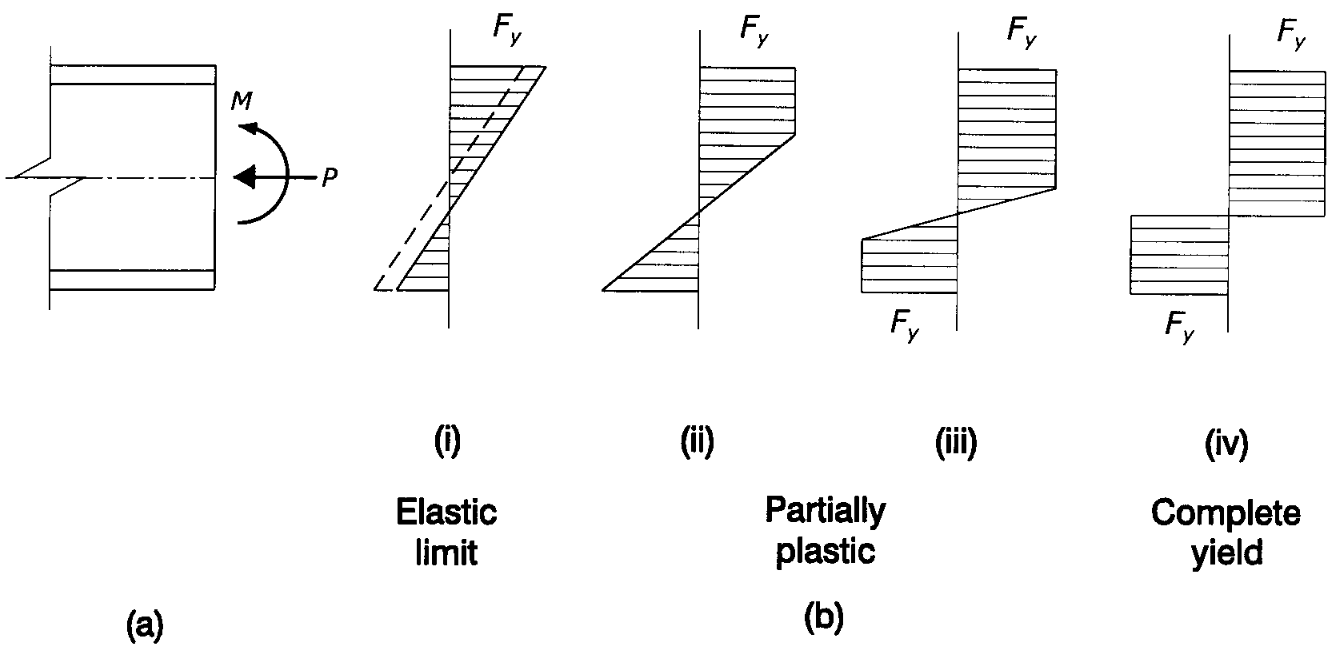


Figure W11.1.1: Stress distributions in an I-section under axial compression P and major axis moment M.

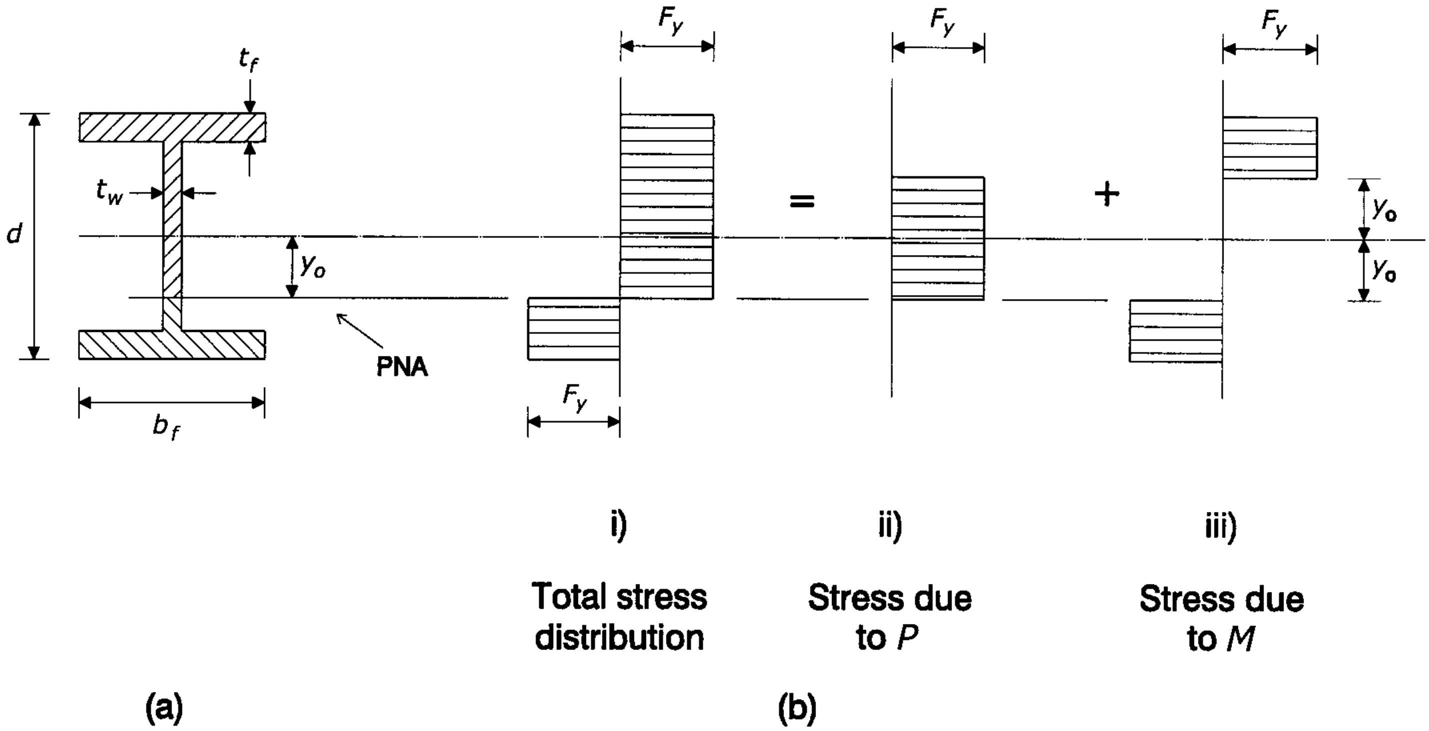


Figure W11.1.2: Reduced plastic moment M_{pc} of an I-section under axial compression P, PNA in the web.

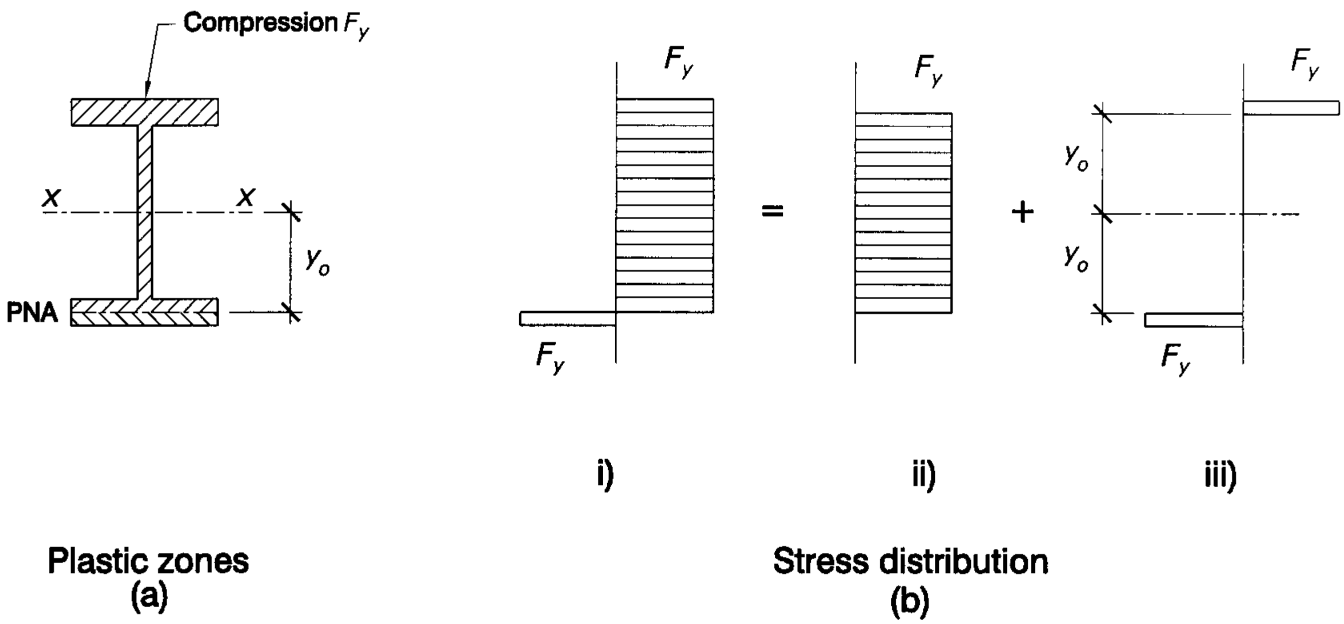


Figure W11.1.3: Reduced plastic moment M_{pc} of an I-section under axial compression P, PNA in the flange.

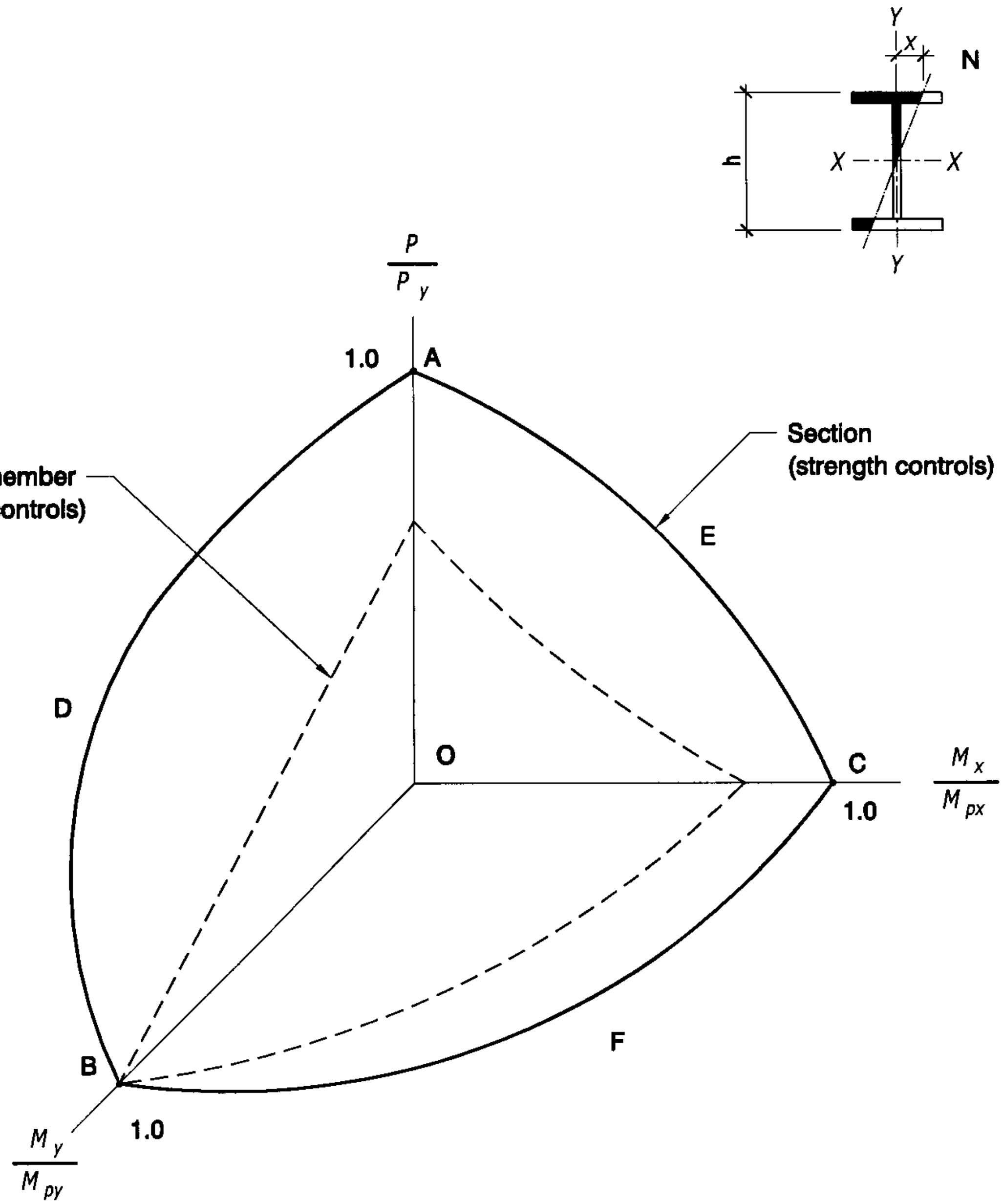


Figure W11.2.1: Interaction surfaces for a steel section.

Figure W11.2.2 **Examples of interaction curves for an I-section and a T-section** TO COME SHORTLY.

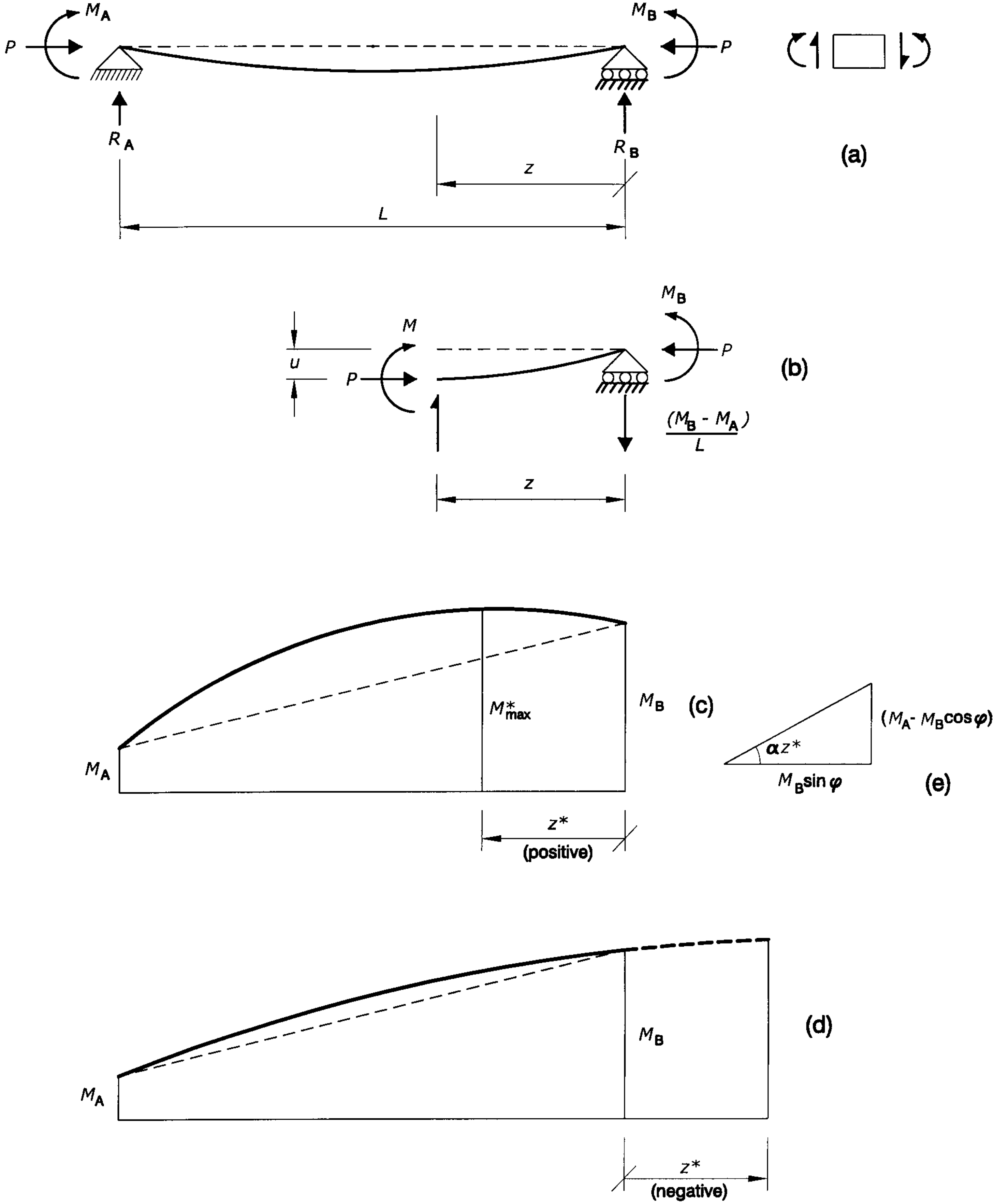


Figure W11.3.1 : Beam-column with end moments M_A, M_B.

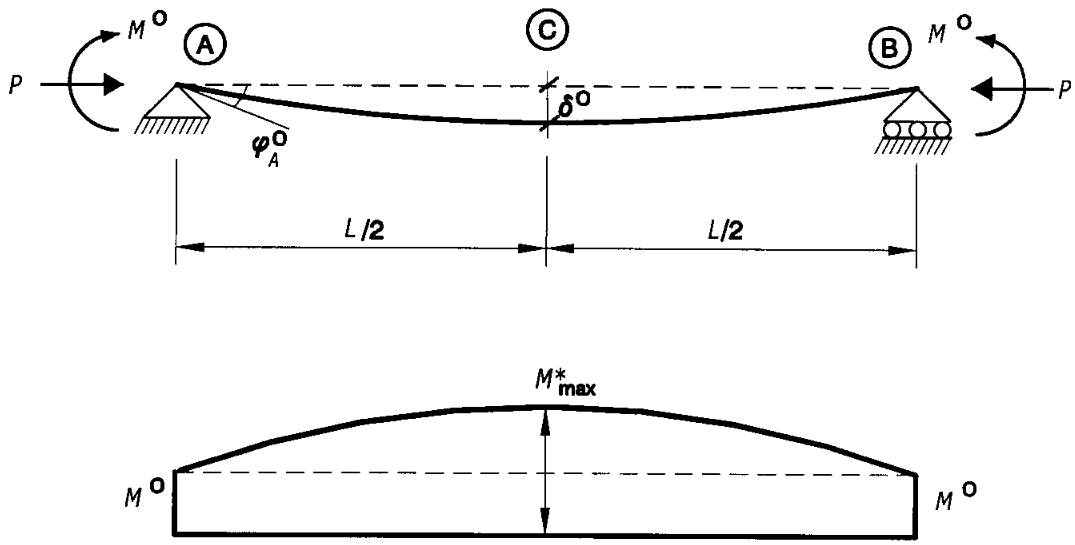


Figure W11.3.2 : Beam-column under uniform moment.

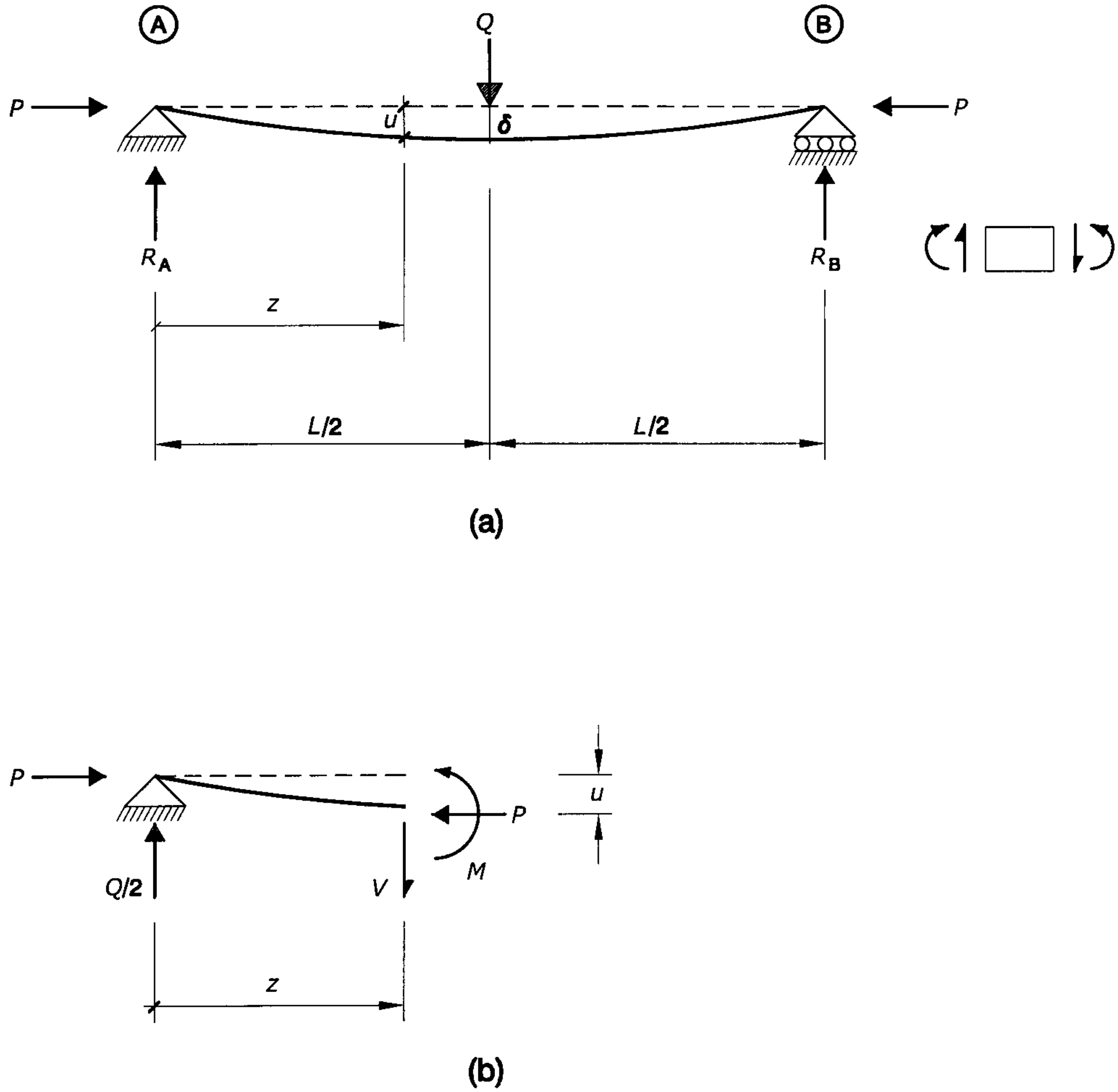
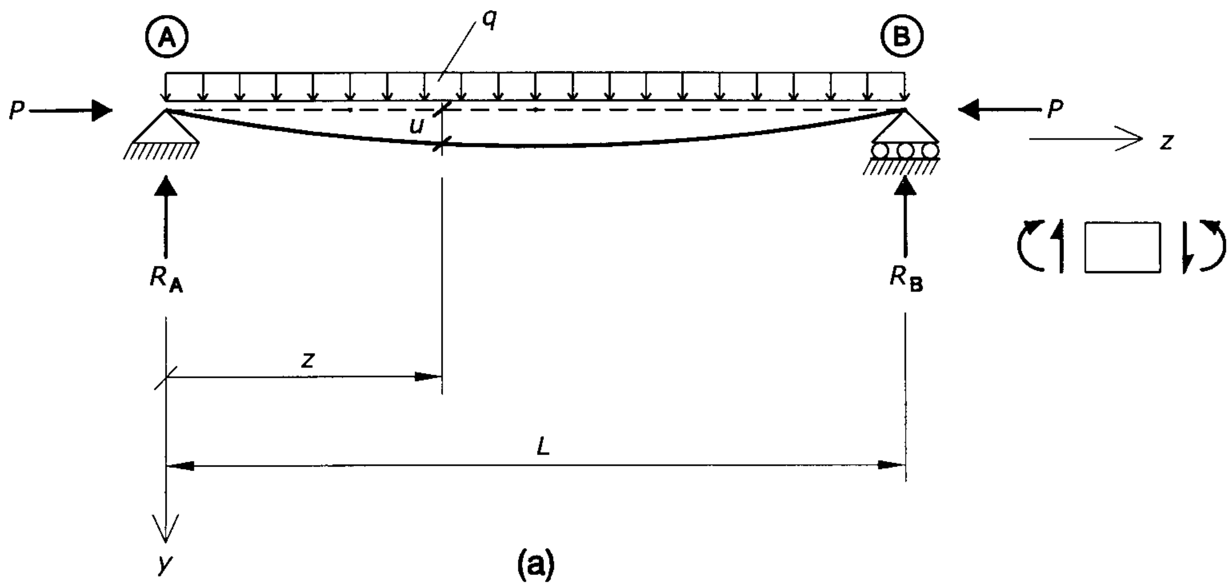
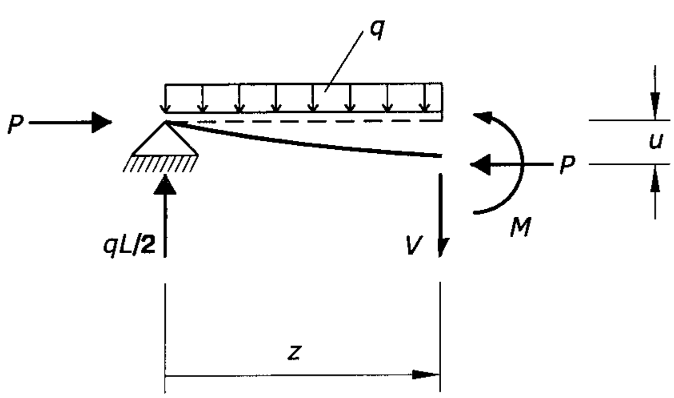


Figure W11.4.1: Beam-column with central concentrated load.



(a)



(b)

Figure W11.4.2: Beam-column with uniformly distributed transverse load.

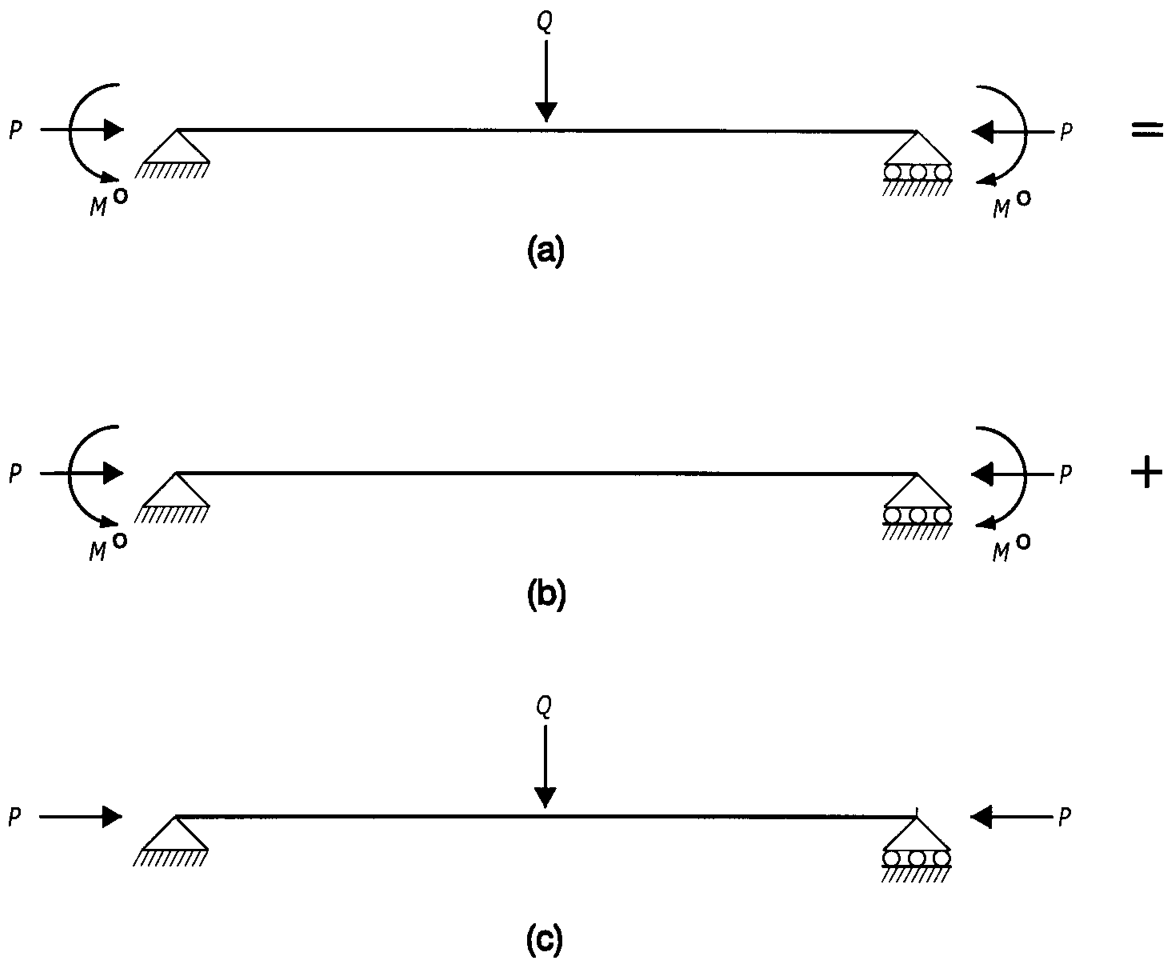


Figure W11.4.3: Superposition of loading cases for a beam-column.

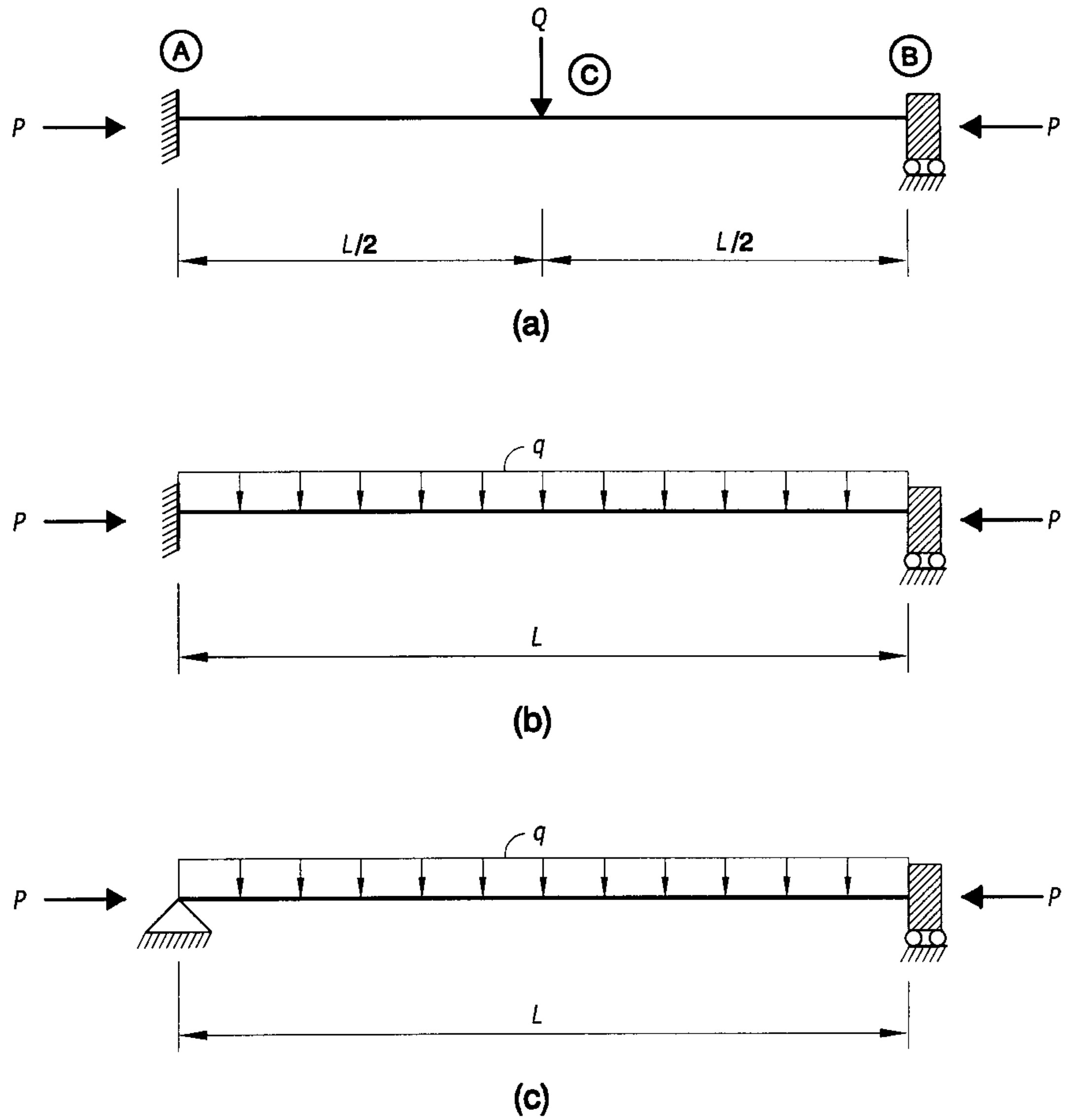


Figure W11.4.4: Beam-columns fixed at one or both ends.

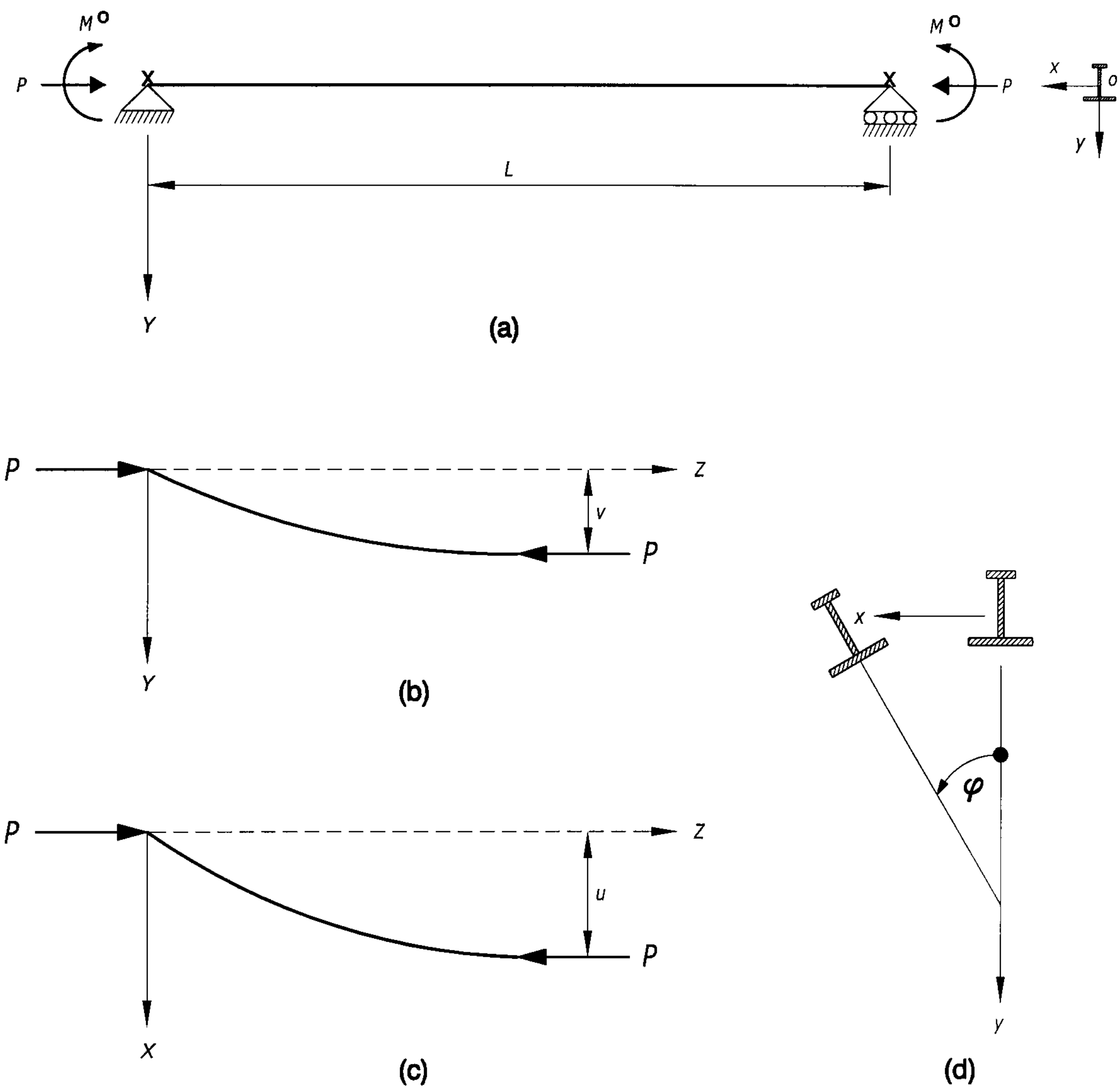
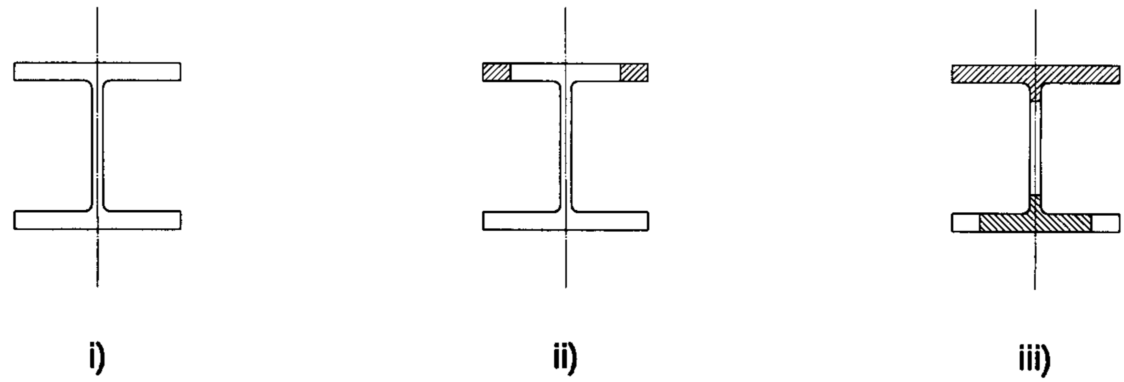


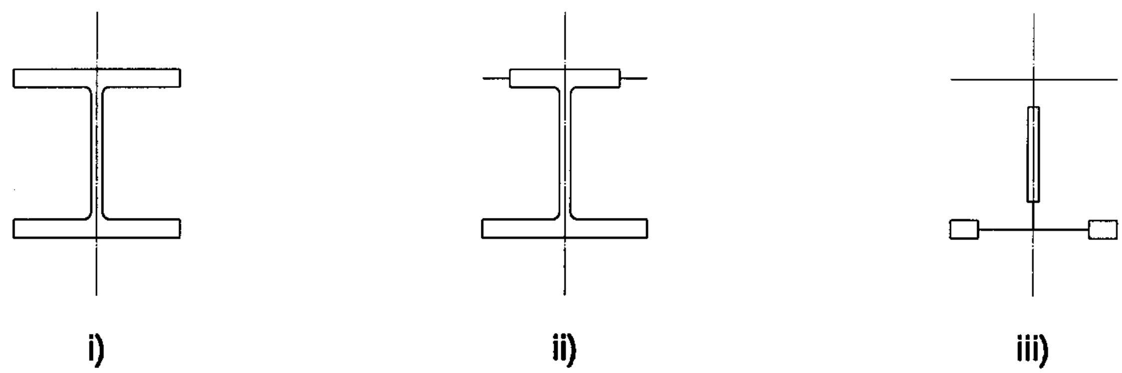
Figure W11.6.1: Lateral torsional buckling of a singly symmetric beam-column.

Figure W11.7.1 **Ultimate strength interaction curves for beam-columns under uniform moment** TO COME SHORTLY.

Figure W11.7.2 **Moment-rotation curves for beam columns**
TO COME SHORTLY.



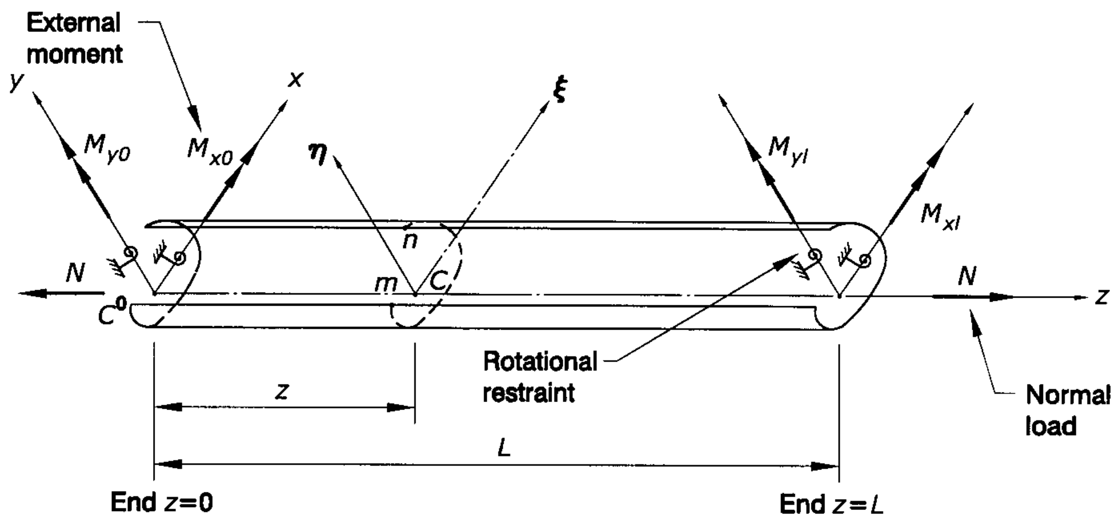
Distribution of plastic zones
(a)



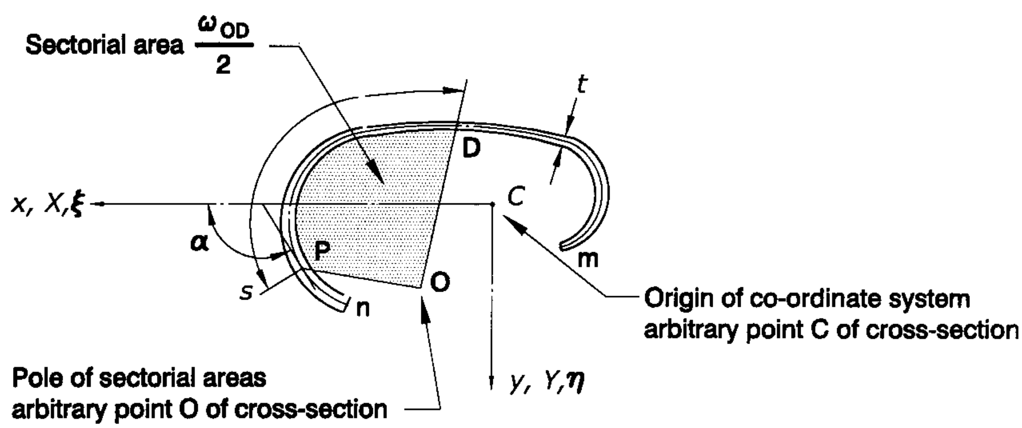
Elastic core
(b)

Figure W11.8.1: Influence of plastification on distribution of elastic core of steel beam-column sections.

Figure W11.8.2 **Comparison of Eq. W11.8.1 with numerical data of Vinnakota for lateral-torsional strength of beam-columns** TO COME SHORTLY.



Beam-column under consideration
(a)



Arbitrary thin-walled open cross-section under consideration
(b)

Figure W11.9.1: Restrained biaxially loaded beam-column [Vinnakota, 1977b].

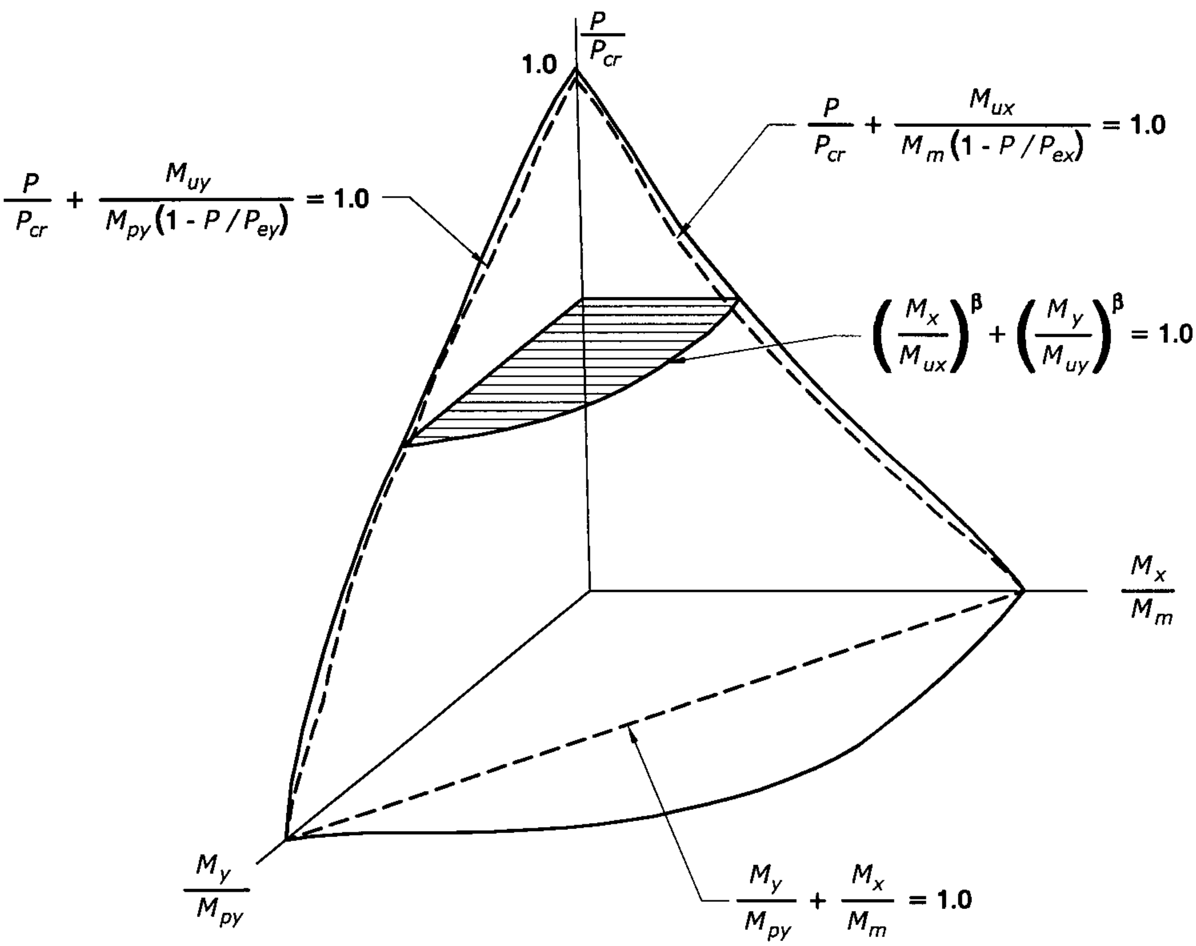
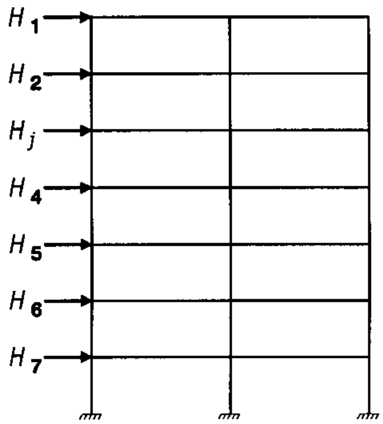
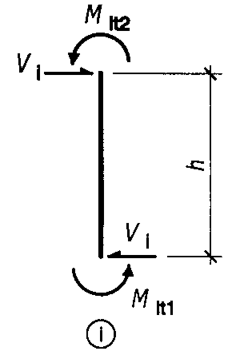


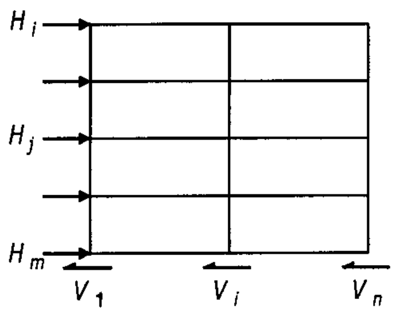
Figure W11.9.2: Maximum strength interaction surface for a column subjected to biaxial bending.



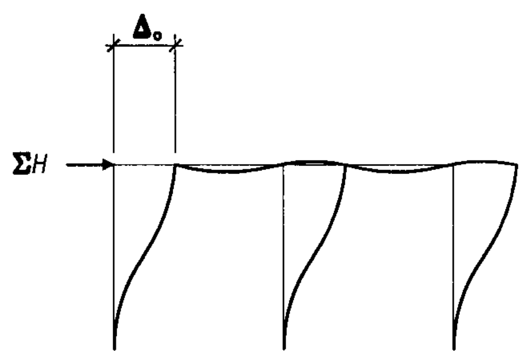
Unbraced frame loads
(a)



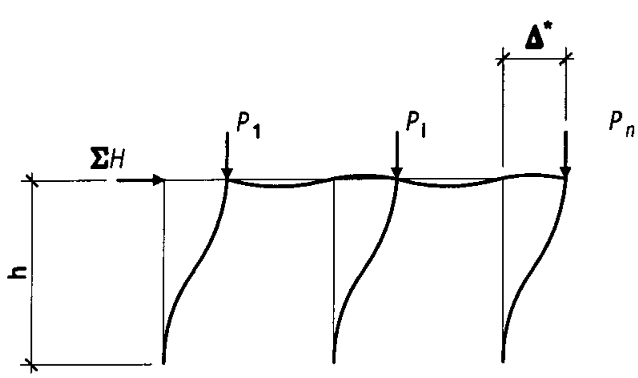
First order forces on ith column
(b)



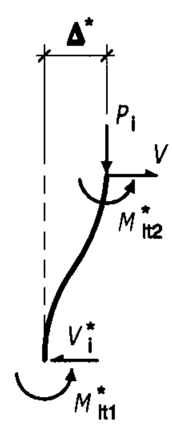
(c)



(d)



(e)



Second order forces on ith column
(f)

Figure W11.10.1: B_2 factor for an unbraced frame.