Appendix A2. Picard Iteration

The following first order initial value problem

$$y'(t) = f(t, y(t)) \ , \ y(0) = 0$$

can be reformulated as the an integral equation

$$y(t) = \int_0^t f(s, y(s)) \, ds$$

See Ledder, Appendix A.2.

Assume that $f(t, y)$ and its y partial are continuous on a rectangle containing the point (0,0). Then starting with an initial "guess" for a solution

$$y_0(t) = g(t)$$

the recursive relation

$$y_n(t) = \int_0^t f(s, y_{n-1}(s)) \, ds$$

generates a sequence of functions converging to the solution. These functions are called the Picard iterates generated by $g(t)$.

Mathematica can generate some of these iterates, use an iteratively defined process.

The following example is taken from Ledder, see Chapter A.2, Exercise 2.

<u>Example</u> Obtain the first three Picard iterates for the following IVP, generated by the function $g(t) = 0$.

$$y'(t) = t + t^2 \, y(t), \ y(0) = 0$$

Sketch their graphs and the graph of the exact solution.

Start with the definition of the function f.

```
f[t_,y_] := t + t^2*y
```

Use it to define the differential equation.

```
DE = y'[t] == f[t,y[t]]
```
$$y'[t] == t + t^2 \, y[t]$$

Define the expression Y[0,t] to be 0. Then define Y[n,t] recursively as shown below.

```
Y[0,t_] := 0
Y[n_,t_] := Integrate[f[s,Y[n-1,s]], {s,0,t}]
```

Here are the first three Picard iterates.

```
Table[Y[n,t], {n,1,3} ]
```

$$\left\{ \frac{t^2}{2}, \ \frac{t^2}{2} + \frac{t^5}{10}, \ \frac{t^2}{2} + \frac{t^5}{10} + \frac{t^8}{80} \right\}$$
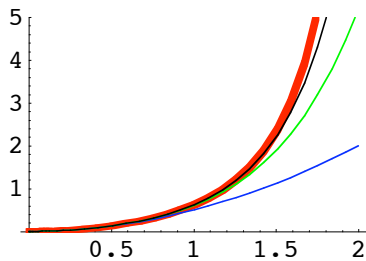
The solution formula is obtained next. It is not pretty.

```
soln = DSolve[ {DE, y[0]==0}, y[t], t ]
```

$$\left\{ \left\{ y[t] \to -\frac{1}{3} \ e^{\frac{t^3}{3}} \left(t^2 \ \text{ExpIntegralE}\left[\frac{1}{3}, \ \frac{t^3}{3}\right] - 3^{2/3} \ \text{Gamma}\left[\frac{2}{3}\right] \right) \right\} \right\}$$

The plot of the solution, red and thick, and the three iterates, blue, green, black:

```
Show[ Plot[ y[t]/.soln, {t,0,2},
            PlotStyle->{RGBColor[1,0,0],Thickness[0.02]} ],
      Plot[ Y[1,t], {t,0,2}, PlotStyle->{RGBColor[0,0,1]} ],
      Plot[ Y[2,t], {t,0,2}, PlotStyle->{RGBColor[0,1,0]} ],
      Plot[ Y[3,t], {t,0,2} ], PlotRange->{0,5} ]
```


The next output reveals what appears to be a bug in *Mathematica*. The Picard iterates are the partial sums for the Taylor series for the solution yet **Series** generates a series approximation to the solution that does not converge to the correct solution curve.

```
Series[ y[t]/.soln[[1]], {t,0,8} ];
Normal[%]
```

$$\frac{t^{10/3}}{2 \ 3^{2/3}} + \frac{t^{19/3}}{10 \ 3^{2/3}} + \frac{\text{Gamma}\left[\frac{2}{3}\right]}{3^{1/3}} - \frac{1}{3} \ t^{4/3} \ \text{Gamma}\left[\frac{2}{3}\right] +$$

$$\frac{t^3 \ \text{Gamma}\left[\frac{2}{3}\right]}{3 \ 3^{1/3}} - \frac{1}{9} \ t^{13/3} \ \text{Gamma}\left[\frac{2}{3}\right] + \frac{t^6 \ \text{Gamma}\left[\frac{2}{3}\right]}{18 \ 3^{1/3}} - \frac{1}{54} \ t^{22/3} \ \text{Gamma}\left[\frac{2}{3}\right]$$

```
Plot[ %, {t,0,2}, PlotRange->{0,5} ]
```