chapter

5 Autonomous Equations and Systems

An *autonomous* differential equation is a differential equation in which the independent variable does not appear explicitly. The decay equation $y' = -ky$ and the linear oscillator $ay'' + by' + cy = 0$ are examples. A natural process is autonomous if the rates of change depend only on the *state* of the system, and not on the time or position; such processes are modeled by autonomous differential equations. Autonomous problems are worth special study for two reasons: they are common in nature, and the special properties they have are the basis for some useful techniques. This chapter introduces some of these techniques: the phase line and phase plane, the direction field, trajectory analysis, and nullcline analysis.

The modeling in the chapter focuses on population dynamics, including both single populations and systems of two interacting populations. Several population models are introduced in a unified manner in Section 5.1, and these models are discussed as the methods for their study are developed in later sections.

In Section 5.2, we study the *phase line,* a special case of the slope field, for single autonomous first-order equations.

The remaining sections of Chapter 5 focus on autonomous systems of two first-order equations. The *phase plane* interpretation of such systems is introduced in Section 5.3, with the focus on *trajectories* taken by solutions. The *direction field* is introduced in Section 5.4, as are *equilibrium solutions* and the notion of *stability. Nullcline analysis* is presented in Section 5.5; this useful technique is similar to the isocline technique of Section 2.3.

5.1 Population Models

Mathematical models in the physical sciences are based on physical laws that are supported by a wealth of quantitative data. In the life sciences and social sciences, there is seldom such a clear starting point. Population growth, for example, does not seem to be governed by anything as definite as Newton's laws of motion. Instead, models for population growth tend to follow largely from educated guesses. Nevertheless, such models are able to reproduce the qualitative features of observations and can sometimes even produce accurate quantitative results. Population models begin with assumptions about how the relative rate of change[1] of a population depends on the current size of the population and whatever environmental factors, such as food supply or the presence of natural enemies, are considered relevant.

The simplest differential equation model of population growth is based on the assumption that the relative rate of change of a population is constant.[2] Thus,

$$\frac{1}{p}\frac{dp}{dt} = r,$$

or

$$\frac{dp}{dt} = rp,$$

where r is a positive constant with dimension time^{-1}. This **natural growth** equation is the same as the decay equation that we saw in Chapter 1, except that the relative growth rate is positive rather than negative. As before, solutions have the form

$$p = p_0 e^{rt}.$$

The model therefore predicts that population growth will be exponential.[3]

As simple as it is, the natural growth model is still useful for populations that are not significantly affected by resource limitations or crowding. It also serves as a good starting point for discussing population growth. In particular, any better model for population growth has to include a mechanism to account for limited capacity. In the remainder of Section 5.1, we consider several ways to do this.

MODEL PROBLEM 5.1

Construct population models that include reasonable mechanisms for limiting the growth of a rabbit population because of either limited food or predation by carnivores.

Logistic Growth

The best-known mathematical model of population growth to incorporate a limited capacity is the **logistic growth** model.[4] We assume that there is a limiting population value K that represents the environmental capacity. Instead of assuming a constant relative growth rate r, we assume a relative growth rate that is r when the population is small but decreases to zero when the population reaches the value K. The simplest such model assumes that the relative growth rate is a linear

[1] Section 1.1.

[2] This model is attributed to Thomas Malthus (1766–1834).

[3] Malthus was not so naive as to predict that any natural population would undergo exponential growth for all time. Rather, he recognized that population increase leads to increases in disease and starvation. His point was that it was not possible to end starvation because abundant food would simply allow the population to grow to a point where food became limited again.

[4] In the military, the term *logistics* is used to describe the process of delivering adequate supplies to troops in the field. In our context, the growth is described as "logistic" because it is limited by availability of supplies. This model is attributed to P. F. Verhulst (1804–1849).

function of the population. The relative growth rate is to have the value r when $p = 0$ and the value 0 when $p = K$ (see Fig. 5.1.1); thus we have

$$\frac{1}{p}\frac{dp}{dt} = r\left(1 - \frac{p}{K}\right),$$

or

$$\frac{dp}{dt} = rp\left(1 - \frac{p}{K}\right). \tag{1}$$

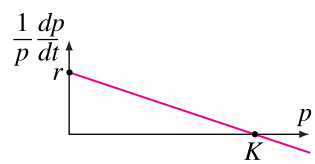

Figure 5.1.1
Relative population growth rate as a function of population in the logistic growth model.

✦ INSTANT EXERCISE 1

Derive the formula $r(1 - p/K)$ from the requirements that the function be linear and yield r for $p = 0$ and 0 for $p = K$.

The **state** of a system is the set of values of the dependent variables in the system. The independent variable t does not appear explicitly in either the natural growth equation or the logistic growth equation; thus, the rates of change depend only on the state of the system. A differential equation or system is **autonomous** if the functions representing the rates of change depend only on the state of the system.

The Lotka–Volterra Model

Another way to limit population growth is to incorporate the effects of competition between populations. Consider populations of rabbits and coyotes living in a grassland. We can think of the coyotes as a mechanism for limiting the population of rabbits. Let $x(t)$ be the rabbit population in a given region, and let $y(t)$ be the coyote population in that region. Both of these functions are unknown, so the model requires two differential equations, each prescribing the growth rate of one of the species. Suppose we assume that the rabbit population exhibits natural growth in the absence of coyotes but decreases in the presence of coyotes. One possibility is

$$\frac{dx}{dt} = rx - sxy, \tag{2}$$

where r is a positive constant with dimension time^{-1} and s is a positive constant with dimension coyotes^{-1}time^{-1}.

Why should we make this choice? In the absence of any fundamental principle or real data, we don't know what to choose, so it makes sense to start with the simplest function that has the right qualitative properties. The function $rx - sxy$ has these properties:

- It is proportional to the rabbit population.
- It assumes that the loss of rabbits due to coyotes is proportional to the product of the rabbit and coyote populations.
- It corresponds to a relative rabbit growth rate that is a linear function of the coyote population.

All these properties seem reasonable. Twice as many rabbits ought to produce a growth rate twice as large. Coyotes eat rabbits some fraction of the time that coyotes and rabbits encounter each other, and it makes sense that the rate at which these encounters occur would be proportional to each population.[5] The encounter rate would be proportional to each population if creatures moved about randomly, as do molecules in a gas. For movement of living creatures, proportionality is only a rough approximation. Nevertheless, linear functions make models simpler, so it makes sense to choose a linear function for a first attempt. We have no way to know in advance whether the choice $rx - sxy$ will produce a successful model. One way to decide this question is to compare the predictions the model makes with observations of real populations.[6] Meanwhile, we should not have much faith in the model, since it is not based on strong evidence.

What about a differential equation to model changes in the coyote population? First we need to make assumptions about the factors most directly responsible for increases or decreases in the number of coyotes. The most logical choice for the cause of coyote population decrease is the starvation that would occur if there were no rabbits. In the absence of rabbits, we might expect that the coyote population would experience a constant relative decay rate, as in a radioactive decay model. The key factor promoting growth in the coyote population is the consumption of rabbits. The differential equation, assuming that these two factors are the only ones needed, is

$$\frac{dy}{dt} = csxy - my, \tag{3}$$

where m is a positive constant with dimension time^{-1} and c is a positive dimensionless constant. The term $csxy$ is the gain in coyote population as a result of the coyote-rabbit interaction; c is the conversion factor of rabbit loss into coyote gain. The constant m represents the per capita mortality rate of the coyotes in the absence of rabbits.

The model given by Equations (2) and (3) is the **Lotka–Volterra**[7] predator-prey model. The specific species that serve as the predator and as the prey are unimportant as long as the designated prey is the principal food source of the designated predator, which in turn is the primary cause of death for the prey. There are also two important assumptions hidden in the model:

- The region is approximately uniform in population densities. Otherwise, it would be necessary to consider spatial variations.
- The region is closed, meaning that predator and prey cannot move in or out of the region.

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[5]The letter s is chosen for this model because the constant that it represents is a measure of both the *search* rate of the predator and the probability of *success* in hunting.

[6]If the model fails this test, it still requires some thought to determine *which* feature of the model is the most likely problem.

[7]The model is named after the mathematicians Alfred J. Lotka and Vito Volterra, who independently proposed it in 1925 and 1926.

✦ INSTANT EXERCISE 2

In the predator-prey model

$$\frac{dp}{dt} = p(2q-3), \qquad \frac{dq}{dt} = q(5-8p),$$

which is the predator and which is the prey? How do you know?

Mathematically, the Lotka–Volterra model is a **coupled** system of two autonomous nonlinear first-order differential equations because the rate of change of each variable depends explicitly on the other variable. If one of the equations in a system can be solved first without considering the other, the system is **uncoupled.**

EXAMPLE 1

The system

$$\frac{dx}{dt} = -ty^2, \qquad \frac{dy}{dt} = y$$

is an uncoupled system. The second equation can be solved immediately, with solution

$$y = Ae^t.$$

The first equation then becomes

$$\frac{dx}{dt} = -A^2te^{2t}.$$

This can be integrated immediately, with the result

$$x = \frac{A^2}{4}(1-2t)e^{2t} + C.$$

✦ INSTANT EXERCISE 3

Integrate the differential equation for x in Example 1 to obtain the final result.

Solution formulas cannot generally be obtained for a coupled system of two nonlinear equations such as the Lotka–Volterra equations. Nevertheless, we will discover methods that can yield a wealth of useful qualitative information about the solutions of such systems.[8] They can also be solved approximately by the rk4 method[9] or other numerical method.

Interacting Populations in General

We could have thought of the rabbit population as being limited by the amount of plant life rather than the number of coyotes. The plants would grow naturally, but be eaten by the rabbits.

[8] This chapter is devoted to such methods. See also Section 6.7.

[9] Section 2.6.

We would then have gotten a predator-prey model in which the rabbits were the predators and the plants were the prey. In this context, the model is often called a consumer-resource model rather than a predator-prey model, and it may be preferable to think of the dependent variables as representing the biomasses of the population, rather than the number of individuals. There are other types of interspecies interactions, and it is interesting to try to construct models for some of these.

Consider the different possibilities for the relative growth rate of a population x to be a linear function of another population y. Assuming that the constants a and b are positive, we have four possibilities:

1. $$\frac{1}{x}\frac{dx}{dt} = a + by,$$
2. $$\frac{1}{x}\frac{dx}{dt} = a - by,$$
3. $$\frac{1}{x}\frac{dx}{dt} = -a + by,$$
4. $$\frac{1}{x}\frac{dx}{dt} = -a - by.$$

Not all these possibilities are reasonable, however. For the population to have a chance to survive, there must be a possibility of a positive growth rate, and this eliminates type 4. There must be some mechanism to limit the population growth, and this requirement eliminates type 1. Linear relative growth rates work only if the terms have opposite signs. In the Lotka–Volterra model, the rabbit equation is of type 2 and the coyote equation is of type 3. Could we get a meaningful model if both equations are of the same type?

Suppose both relative growth rates are of type 2. Each species has a positive growth rate if the other species is absent. Each species has its growth rate decrease in the presence of the other. This is a simple model for competing species. Similarly, suppose both relative growth rates are of type 3. Then each species requires the other for its survival. Clearly this is a simple model of cooperating species. We shall see in the subsequent analysis of these models in Section 5.5 that neither is satisfactory. In both cases, what appears to be a sufficient means of limiting the populations is actually not sufficient. Understanding why the models fail leads to better models for competing or cooperating species.

✦ INSTANT EXERCISE 4

Which of the three types of interacting populations (predator-prey, competing species, cooperating species) could be represented by this system?

$$\frac{da}{dt} = a(3 - 2b), \qquad \frac{db}{dt} = b(1 - 3a).$$

A Complete Family of Examples by Nondimensionalization

Each of the relative growth rate formulas in the interacting species models has two parameters, so our families of simple predator-prey, competing species, and cooperating species models are four-parameter families. Having so many parameters makes it difficult to determine results general enough to apply to the whole family of models. As is typical in mathematical modeling, we can reduce the number of parameters to a reasonable number by nondimensionalization.[10]

Consider the logistic growth equation (1). The dimensions of the parameter K and of the variable p are the same (only a dimensionless number can be subtracted from the dimensionless number 1), so a dimensionless population variable can be defined by

$$P = \frac{p}{K}.$$

The relative growth rate is zero for a population $p = K$ and positive for $p < K$. A small initial population can grow toward K, but cannot exceed K. Thus, K is an upper bound for p. In addition to having the right dimensions, it is a value that is good for comparison. Regardless of what species we are studying with the logistic growth equation or how large the area that the population occupies, $P = 0.5$ will mean that the population is one-half of the upper bound K.

Similarly, note that the relative growth rate is at most r. During a period of maximum growth, we can approximate population changes by

$$\Delta p \approx \frac{dp}{dt}\,\Delta t \approx rp\,\Delta t.$$

Rearranging this relationship gives us

$$\Delta t \approx \frac{1}{r}\,\frac{\Delta p}{p}.$$

This shows that the relative growth rate r has units of inverse time, and it also shows that significant population changes require a time period on the order of $1/r$. Thus, $1/r$ is a good reference value for time. Accordingly, we can choose the variable τ defined by

$$\tau = rt$$

as a dimensionless measure of the time.

It remains to convert the original logistic equation to a dimensionless version by replacing p with P and t with τ. We begin by substituting KP for p:

$$\frac{d(KP)}{dt} = rKP(1 - P).$$

The constant K can be removed from the derivative, yielding

$$\frac{dP}{dt} = rP(1 - P).$$

Now to remove t from the equation, note that P is supposed to be a function of τ rather than t. Applying the chain rule, we have

$$\frac{dP}{dt} = \frac{dP}{d\tau}\,\frac{d\tau}{dt}.$$

The second factor is determined from the equation $\tau = rt$ to be r; hence

$$\frac{dP}{dt} = r\,\frac{dP}{d\tau}.$$

[10]Section 2.1.

This substitution reduces the model to

$$\frac{dP}{d\tau} = P(1-P). \tag{4}$$

Note that the new model is similar to the original, but the parameters K and r have disappeared. Instead of a two-parameter family of equations to analyze, we have one single equation. The results produced by the logistic growth model are the same for all species in that the shape of the graph is the same. The only differences are in the amount of actual time that corresponds to $\tau = 1$ and the actual number of individuals that corresponds to $P = 1$.

There are several different ways to nondimensionalize the two-species interaction models. All result in dimensionless counterparts with only one parameter; for example,

Predator-prey

$$X' = X(1-Y), \qquad Y' = kY(X-1), \tag{5}$$

Competing species

$$X' = X(1-Y), \qquad Y' = kY(1-X), \tag{6}$$

Cooperating species

$$X' = X(Y-1), \qquad Y' = kY(X-1), \tag{7}$$

where derivatives are with respect to a suitable time variable τ and the dimensionless parameter k is suitably defined.

5.1 Exercises

1. Consider the logistic growth model

$$p' = 4p\left(1 - \frac{p}{10}\right).$$

Under what circumstances will the population increase? Under what circumstances will the population decrease?

2. Consider the predator-prey model

$$x' = x(1-2y), \qquad y' = y(x-1).$$

a. Are there any values of x and y for which the populations will remain unchanged? If so, what are they?

b. Suppose $x(0) = 1$ and $y(0) = 1$. Will each of these populations be larger than 1 or smaller than 1 shortly after time 0?

3. Which of the following systems are predator-prey models? Indicate which variable represents the predator and which the prey.

a. $x' = -3x + 2xy, \quad y' = -4y + 3xy$

b. $x' = 3x - 2xy, \quad y' = -4y + 3xy$

c. $x' = 3x - 2xy, \quad y' = 4y - 3xy$

4. One of the systems in Exercise 3 represents a model for two cooperative species. Which one is it and how do you know?

5. One of the systems given below represents a model for two species that are competing for the same resources. Which one is it and how do you know?

   a. $x' = x(1 - x - y), \quad y' = y(3 - 2y + x)$
   b. $x' = x(1 - x - y), \quad y' = y(3 + 2y - x)$
   c. $x' = x(1 - x - y), \quad y' = y(3 - 2y - x)$

6. Consider the general Lotka–Volterra model in Equations (2) and (3). Define the new variables
$$X = \frac{csx}{m}, \qquad Y = \frac{sy}{r}, \qquad \tau = rt.$$
   a. Use the chain rule to obtain a set of differential equations for $X(\tau)$ and $Y(\tau)$.
   b. Define the dimensionless parameter k to obtain the dimensionless model (5).
   c. The dimensionless parameter k represents a ratio of two rates. What are these rates? Do you expect $k > 1$ or $k < 1$?

7. Consider the general competition model
$$\frac{dx}{dt} = rx - sxy, \qquad \frac{dy}{dt} = ay - bxy.$$
Define new dimensionless variables and a dimensionless parameter to obtain the dimensionless model (6).

8. Consider the general cooperation model
$$\frac{dx}{dt} = axy - mx, \qquad \frac{dy}{dt} = bxy - ny.$$
Define new dimensionless variables and a dimensionless parameter to obtain the dimensionless model (7).

9. Suppose the prey in the predator-prey model is limited by logistic growth in the absence of predators. Incorporate this assumption into the predator-prey model.

10. Suppose $y(t)$ is the proportion of people in a city who have heard a particular rumor. Suppose the rate at which the rumor is spread, relative to the population of people who have not yet heard the rumor, is proportional to the proportion of people who have heard the rumor. Derive the differential equation for the spread of the rumor. This model is also used for the spread of technological innovations within a given society.

11. The standard SIS (susceptible, infective, susceptible) epidemic model divides a population into two classes: the infective class (I), consisting of individuals who are capable of transmitting the disease, and the susceptible class (S), consisting of individuals who are not infective, but could become infective. Let $S(t)$ and $I(t)$ be the populations of the susceptible and infective classes, respectively.

   a. Derive a system of differential equations for the SIS model, using the following assumptions:

      1. Changes of classification for any individual occur by only two mechanisms: A susceptible individual can become infected, and an infected individual can recover to become susceptible again.
      2. The rate at which susceptible people become infected is proportional to the susceptible population and to the infective population, with proportionality coefficient r.

3. The rate at which infective people recover is proportional to the infective population, with proportionality coefficient γ.

b. Explain why assumptions 2 and 3 are reasonable.

12. The standard SIR epidemic model divides a population into three classes: the infective class (I), consisting of individuals who are capable of transmitting the disease; the susceptible class (S), consisting of individuals who are not infective but could become infective; and the removed class (R), consisting of individuals who have had the disease and are no longer able to be infected. Let $S(t)$, $I(t)$, and $R(t)$ be the populations of the susceptible, infective, and removed classes, respectively.

a. Derive a system of differential equations for the SIR model, using the following assumptions:

1. Changes of classification for any individual occur by only two mechanisms: A susceptible individual can become infected, and an infected individual can become removed.
2. The rate at which susceptible people become infected is proportional to the susceptible population and to the infective population, with proportionality coefficient r.
3. The rate at which infective people become removed is proportional to the infective population, with proportionality coefficient γ.

b. Explain why assumptions 2 and 3 are reasonable.
c. Show that the sum of the populations in the three classes is constant. Use this fact to obtain a system of equations for the variables S and I.
d. Explain why this model is not suitable for the common cold.

13. In this exercise, we derive a model for a waste treatment process that utilizes bacteria.

a. Let $w(t)$ be the amount of waste in a vessel of volume V, and let $x(t)$ be the population of bacteria in the vessel. Assume that the waste is consumed at the rate kwx, where k is a constant that measures the relative reaction rate of waste per unit amount of bacteria. Let c be a conversion factor for bacteria growth from waste consumption, similar to the parameter c in the Lotka–Volterra predator-prey model. Let m be the relative death rate of the bacteria. Construct the differential equations for the waste and the bacteria.
b. It is possible to remove all the parameters from the differential equation by nondimensionalization, but it is not obvious what to choose for the reference quantities. Let w_r, x_r, and t_r be the as yet undetermined reference quantities for nondimensionalization. Let $W = w/w_r$, $X = x/x_r$, and $\tau = t/t_r$ be the variables in the dimensionless version of the problem. Determine the differential equations for $W(\tau)$ and $X(\tau)$.
c. Observe that the differential equations from part *b* include a total of three terms, each of which has a dimensionless coefficient that includes at least one of the reference quantities. Determine the values of the reference quantities needed to make each of the three dimensionless coefficients equal to 1.

14. One of the first mathematical studies of epidemics was done by Daniel Bernoulli in 1760. Bernoulli's goal was to determine the likely effect of a controversial program to immunize young people against smallpox. The model assumes that survivors of the disease have lifetime immunity. Let $P(t)$ be the population of a cohort of individuals who are the same age, with

$t = 0$ corresponding to their birth year. Let $S(t)$ be the population of susceptible individuals from this cohort. Let $m(t)$ be the relative death rate of the cohort in the absence of smallpox. This relative rate of decrease is not constant because the population is to be tracked long enough that mortality must be considered a function of age. We might expect, for example, that $m(t)$ (at least in 1760) was quite large for infants and decreased as the cohort aged toward adulthood. The populations P and S in Bernoulli's model are described by the differential equations

$$S' = -rS - m(t)S, \qquad P' = -brS - m(t)P,$$

where r is a constant and b is the fraction of smallpox cases that result in death.

a. The first term on the right side of the first equation is the rate at which susceptible individuals become infected. For some diseases, this term would have been $-kIS$, where I is the population of infectives. Explain the assumption we are making about smallpox in choosing $-rS$ rather than $-kIS$.

b. The differential equations in the model are impossible to analyze fully without detailed information about $m(t)$. This information is not needed if our goal is simply to determine $y = S/P$, the proportion of individuals who are still susceptible at time t. Let $z = y^{-1} = P/S$. Use the chain rule to differentiate z; substitute the equations of the Bernoulli model to obtain a differential equation that contains only z, along with the parameters b and r. The equation for z should be similar to Newton's law of cooling (Section 1.1).

c. Solve the differential equation for z. Use the initial condition corresponding to the assumption that babies are never born with smallpox. Obtain a formula for y from the formula for z.

d. Bernoulli estimated $b = r = \frac{1}{8}$ from the limited data available. Given these data, what fraction of 20-year-olds is susceptible to smallpox? At what age does susceptibility to smallpox hold for only 5% of the population?

✦ 5.1 INSTANT EXERCISE SOLUTIONS

1. The most general linear function of p is $f(p) = mp + b$, where m is the slope and b the intercept. Given the requirement $f(0) = r$, we have $b = r$. The additional requirement $f(K) = 0$ yields $0 = f(K) = mK + r$, or $Km = -r$. Thus, $m = -r/K$ and $f(0) = -rp/K + r = r(1 - p/K)$.
2. The predator is p because the interaction term pq contributes to the growth of p and the decrease of q.
3. We have

$$x = \int -A^2 te^{2t}\,dt = -\frac{A^2}{2}\int t(2e^{2t})\,dt = -\frac{A^2}{2}\int t(e^{2t})'\,dt.$$

Integrating by parts, we have

$$x = -\frac{A^2}{2}\left(te^{2t} - \int e^{2t}\,dt\right) = -\frac{A^2}{2}\left(te^{2t} - \frac{1}{2}e^{2t}\right) + C = \frac{A^2}{4}(1 - 2t)e^{2t} + C.$$

4. The interaction term is negative for both species. Since each decreases because of the other, the system could represent competing species.

5.2 The Phase Line

The autonomous first-order differential equation

$$\frac{dy}{dt} = f(y)$$

represents a process for which changes in the quantity of interest depend only on the level of the quantity and not on the time. In such cases, it might be of interest to know what will eventually happen if the process is allowed to continue indefinitely. We will use the term *longtime behavior* to describe the possible behaviors of a process or system left alone indefinitely.

MODEL PROBLEM 5.2

Identify the possible longtime behaviors for solutions of the equation

$$\frac{dy}{dt} = y(1-y),$$

and determine which longtime behavior results from any given initial condition.

Since we want to know how the solution depends on the initial condition, it makes sense to include the initial data as a parameter in the problem. We therefore consider the one-parameter family of problems

$$\frac{dy}{dt} = y(1-y), \qquad y(0) = y_0. \tag{1}$$

Longtime Behavior and the Limitations of Solution Formulas

Separation of variables yields the solution formula for Model Problem 5.2:

$$y = \frac{y_0}{y_0 + (1-y_0)e^{-t}}. \tag{2}$$

✦ INSTANT EXERCISE 1

Derive the solution of the initial-value problem (1).

To understand the solution, we cannot simply graph the function defined by Equation (2), because it is a one-parameter family of functions. We can graph the solution for any particular y_0, but how can we be sure that the graphs for some specific values represent all the possible behaviors? Instead, we might look at the properties of the family.

Observe that the second term in the denominator of the solution gradually disappears as t increases. Indeed,

$$\lim_{t\to\infty} y = \begin{cases} 1 & y_0 \neq 0 \\ 0 & y_0 = 0. \end{cases}$$

The solution of an initial-value problem may not exist for all t. We have established that the limit as $t \to \infty$ of the *function* defined by the solution formula (2) is always 1 (except for the special case $y_0 = 0$). But this does not by itself mean that solutions of Equation (1) approach $y = 1$. As demonstrated in Section 1.2, functions do not always represent the solution of an initial-value problem over their whole domain.

Consider as an example the case $y_0 = 2$. Here, the solution formula is

$$y = \frac{2}{2 - e^{-t}}.$$

This function is continuous for all $t > 0$, so it defines a solution for all $t > 0$, which therefore approaches 1 as $t \to \infty$. Similarly, the case $y_0 = \frac{1}{2}$ gives the solution formula

$$y = \frac{1}{1 + e^{-t}},$$

which is also continuous for $t > 0$ and so defines a solution that approaches 1 as $t \to \infty$. But now suppose we try $y_0 = -\frac{1}{2}$. This time, the solution formula is

$$y = \frac{1}{1 - 3e^{-t}}.$$

This situation is different. The denominator approaches 0 as $t \to \ln 3$. Thus, the solution of the initial-value problem with $y_0 = -\frac{1}{2}$ exists only on the interval $t < \ln 3$. The three solutions we have already found are plotted together in Figure 5.2.1. Clearly the longtime behavior of a solution of Model Problem 5.2 depends in an important way on the initial condition.

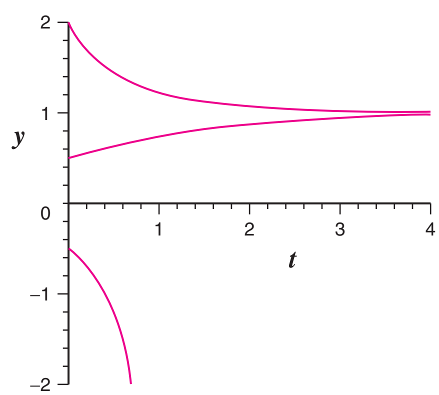

Figure 5.2.1
Some solutions of $y' = y(1 - y)$.

We've seen that solution formulas do not always give a complete picture of the longtime behavior. However, the longtime behavior of an autonomous differential equation can be inferred directly from the differential equation itself, without the need for a solution formula, and we now explore this possibility.

Critical Points and Equilibrium Solutions

The model differential equation is of the form

$$\frac{dy}{dt} = f(y),$$

with

$$f(y) = y(1 - y).$$

The function f represents the rate of change of y as a function of the quantity y. Of particular interest are values of y for which the rate of change is 0. A value of y for which $f(y) = 0$ is called a **critical point** of the differential equation $dy/dt = f(y)$. The function $y(t) = y_c$, with y_c a critical point, is an **equilibrium solution** of the differential equation. The use of the term *critical point* is consistent with its meaning in calculus. In both settings, given a differentiable function, the term refers to points on a graph where the tangent line is horizontal; the difference is that these points are identified by the independent variable in calculus and by the dependent variable for solutions of autonomous differential equations.

The critical points for Model Problem 5.2 are the solutions of

$$y(1 - y) = 0,$$

namely, $y = 0$ and $y = 1$. The functions $y \equiv 0$ and $y \equiv 1$ are equilibrium solutions. The importance of equilibrium solutions stems in part from the consequences of the existence and uniqueness theorem (Theorem 2.4.2). This theorem guarantees a unique solution for $dy/dt = y(1 - y)$ through any point in the (t, y) plane, given that $f(y) = y(1 - y)$ and its derivative are continuous everywhere. This means that solution curves cannot cross in the (t, y) plane. The horizontal lines $y = 0$ and $y = 1$ are solution curves; hence, they divide the (t, y) plane into three distinct regions where there could be significant differences in long-term behavior.

The Slope Field and Stability

A convenient way to study the longtime behavior of solutions of differential equations is to use the slope field.[11] Figure 5.2.2 shows the slope field for Model Problem 5.2, along with the same three solution curves and the equilibrium solutions. The slope field clearly confirms what we have already learned for the cases $y_0 = 2$, $y_0 = \frac{1}{2}$, and $y_0 = -1$. It also gives a complete picture of the longtime behavior of *all* the solutions. Solutions beginning with positive values of y_0 tend toward 1 as $t \to \infty$, and solutions beginning with negative values of y_0 tend toward $-\infty$ in finite time.

Qualitatively, the slope field gives more information than the solution formula because it gives a clear picture of the solution behavior. Solutions that begin near the critical point $y = 1$ tend toward that point as time increases, while solutions beginning near the critical point $y = 0$ tend away from that point as time increases. In this example and in general, some critical points tend to attract nearby solutions, while others repel them. This idea is the basis of the concept of *stability.* The idea is essentially that critical points, and the corresponding equilibrium solutions, are stable if they attract nearby solutions and are unstable if they repel them. Some care is required in the formal definition to give precise meaning to the notions of attraction and repulsion.

[11] Section 2.3.

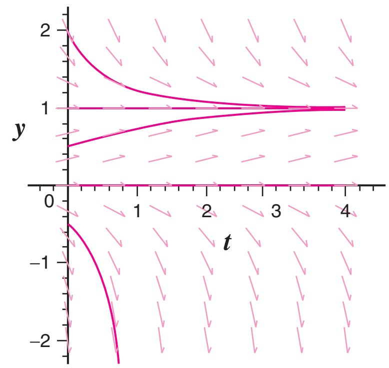

Figure 5.2.2
The slope field for $dy/dt = y(1-y)$, along with some solution curves.

An equilibrium solution[12] $y \equiv y_c$ for a differential equation $y' = f(y)$ is **asymptotically stable** if there is some open interval $(y_c - \epsilon,\ y_c + \epsilon)$ with the property that $\lim_{t\to\infty} y(t) = y_c$ for any solution that is initially in the interval. An equilibrium solution is **unstable** if there is some open interval $(y_c - \epsilon, y_c + \epsilon)$ with the property that $\lim_{t\to\infty} |y(t) - y_c| > \epsilon$ for any solution, other than $y \equiv y_c$, that is initially in the interval.

Consider first the equilibrium solution $y \equiv 1$. If $y_0 > 1$, then the slope of the solution curve through (t, y_0) is

$$f(y_0) = y_0(1 - y_0) < 0.$$

Thus, the solution is decreasing. It must continue to decrease as long as $y > 1$; hence, it must either reach $y = 1$ in finite time or approach $y = 1$ in the limit as $t \to \infty$. The argument is similar for $0 < y_0 < 1$. In summary, we have

$$\lim_{t\to\infty} y = 1 \qquad \text{whenever} \qquad y_0 > 0.$$

Thus, the equilibrium solution $y \equiv 1$ is asymptotically stable.

The equilibrium solution $y \equiv 0$ is unstable by the definition. Solution curves beginning above $y = 0$ move toward $y = 1$, while solutions beginning below $y = 0$ decrease without bound. In either case, solutions cannot stay within a small interval containing the point $y = 0$. The combined stability results completely describe the longtime behavior of the solutions. If $y_0 > 0$, then the solution approaches $y = 1$; if $y_0 < 0$, then the solution decreases without bound.

[12]The terms *critical point* and *equilibrium solution* are almost synonymous, the only distinction being that the critical point is a value of the dependent variable y while an equilibrium solution is a function that takes the same value all the time. Statements about stability can be made using either term; the choice of *equilibrium solution* in this context is a matter of taste.

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The Phase Line

Look again at the slope field in Figure 5.2.2. All the minitangents on a given horizontal line are parallel. This is always true for autonomous equations. In the example, the slope is 0 at all points $(t, 1)$ and $\frac{1}{4}$ at all points $(t, \frac{1}{2})$. Now suppose we illustrated only the minitangents at $t = 0$, as in Figure 5.2.3*a*. With fewer arrows and less horizontal space, this plot includes all the essential information that is in the full slope field.

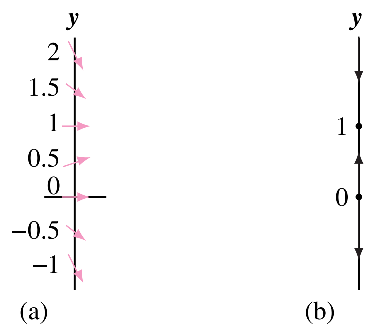

Figure 5.2.3
(*a*) A portion of the slope field for $y' = y(1 - y)$; (*b*) with the phase line.

An even simpler view of the differential equation can be made by placing the arrows directly on the y axis, using them to show only whether the solution curves are increasing or decreasing. This is the **phase line,** as depicted in Figure 5.2.3*b*. All information about the rates of increase or decrease with respect to t is lost in the phase line picture. However, the essential information needed to determine the stability of equilibrium solutions and the longtime behavior of other solutions is efficiently contained in the phase line.

The phase line shows the same properties that we found from the slope field: solution curves beginning below $y = 0$ move off to $-\infty$, solution curves beginning between $y = 0$ and $y = 1$ move up toward $y = 1$, and solution curves beginning above $y = 1$ move down toward $y = 1$. A rough sketch of the solution curves is easily obtained from the phase line.

One advantage of using the phase line is that it is very easy to sketch the phase line from the graph of $f(y)$ for the differential equation $y' = f(y)$. Equilibrium solutions are given by the zeros of f. Regions where f is positive correspond to regions on the phase line where the arrow points up, and regions where f is negative correspond to regions on the phase line where the arrow points down. This interpretation is illustrated in Figure 5.2.4. In this example, the function f vanishes at 0 and 1, and these give the locations of the critical points on the phase line. The function f is positive between 0 and 1; since the differential equation is $y' = f(y)$, this means that y' is positive between $y = 0$ and $y = 1$, and this information gives the direction of the arrowhead between the two critical points on the phase line. The arrowheads in other regions of the phase line are obtained in the same manner. The phase line in Figure 5.2.4 is oriented horizontally to make the connection with $f(y)$ clearer.

✦ INSTANT EXERCISE 2

Sketch the phase line for $y' = y(y - 2)$, and use it to determine the stability of the critical points.

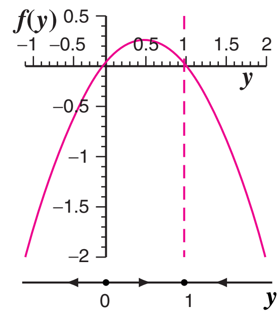

Figure 5.2.4
The function $f(y) = y(1-y)$ and the phase line for $y' = y(1-y)$.

The Logistic Growth Equation

The logistic growth model, from Section 5.1, is

$$\frac{dp}{dt} = rp\left(1 - \frac{p}{K}\right). \tag{3}$$

The dimensionless form of the model, with $P = p/K$, is

$$\frac{dP}{d\tau} = P(1-P),$$

which is exactly the same as Model Problem 5.2. All the results obtained in this section hold for the dimensional logistic growth model (3), with appropriate changes in the variable values. The critical point $p = K$ in Equation (3) corresponds to $y = 1$ in Model Problem 5.2; it is asymptotically stable. The critical point $p = 0$, corresponding to $y = 0$, is unstable. Solution curves beginning with positive values of p tend toward $p = K$, while those beginning with negative values of p, corresponding to initial conditions that are inappropriate for the model, tend toward negative infinity. Of course it is also possible to get these results directly by applying the qualitative methods to the logistic equation (3).

A Final Comment

The solution of an initial-value problem can be thought of as a point that moves along the phase line in time. At any given point in the solution's progress, the phase line shows the value of the dependent variable, but not that of the independent variable. The phase line is only used for autonomous equations $y' = f(y)$. The lack of a time coordinate on the phase line is not very important for an autonomous equation. The current value of y is all that is needed to determine the rate of change of y. We would not want to use the phase line for a general problem $y' = f(t, y)$ because in such a case the current value of y is not sufficient to determine the current rate of change of y.

5.2 Exercises

1. Sketch the function $f(P) = rP(1 - P/K)$, and use it to obtain a sketch of the phase line for the logistic equation. Note that your graphs will be identical to those obtained for Model Problem 5.2 except for the variable values that appear on the axes.

For each of Exercises 2 through 8, (*a*) list the equilibrium solutions, (*b*) sketch y' as a function of y, (*c*) sketch the phase line, and (*d*) determine the stability of all equilibrium solutions. An equilibrium solution is **semistable** if solutions on one side of it tend to approach it while solutions on the other side tend to recede from it.

2. $y' + ky = kS, \quad k > 0, \quad S > 0$
3. $y' = y(y^2 - 4)$
4. $y' = -3y(1 - y)(3 - y)$
5. $y' = 1 - e^{-y}$
6. $y' = \sin y$
7. $y' = -\dfrac{\arctan y}{1 + y^2}$
8. $y' = y^2(1 - y^2)$
9. One of the equations in Exercises 2 through 8 represents a model of a population that can become extinct if it drops below a critical value. Which one is it? What is the critical value? Is the critical value a critical point? If so, what kind is it?
10. Sketch the phase line for the model of rumor spreading from Exercise 10 in Section 5.1. Discuss the behavior of the model.
11. Consider the differential equation $y' = f(y)$. Suppose y_0 is a point satisfying $f(y_0) = 0$ and $f'(y_0) < 0$. Show that y_0 is a stable critical point.
12. The Gompertz model for population growth is

$$p' = kp \ln \frac{M}{p}, \qquad p > 0, \qquad k > 0, \qquad M > 0.$$

   a. Sketch the graph of p' as a function of p, sketch the phase line, and determine the stability of any equilibrium solutions.
   b. Solve the Gompertz equation by utilizing the substitution $y = \ln(M/p)$.
   c. Use the solution from part b to verify the results of part a.

13. The rate at which a drug disseminates into the bloodstream is governed by the differential equation

$$x' = B(A - x),$$

where A and B are positive constants. Find the stable equilibrium value of x. At what time is the concentration one-half of the equilibrium value if there is no drug in the patient's system prior to time 0?

14. Let $x(t)$ be the mass of chemical A in a chemical reactor that initially contains 50 liters (L) of pure water. A solution of A having a concentration 2 kg/L flows into the reactor at a rate of 5 L/min. The solution inside the reactor is kept well mixed and flows out of the reactor at a rate of 5 L/min. The chemical reaction inside the reactor decreases the mass of chemical A at a rate equal to 0.4 times the amount of A present. Determine a differential equation for $x(t)$. Find the equilibrium value and sketch the phase line. Solve the problem to determine $x(t)$.

15. Suppose the population of deer in a forest is governed by a logistic growth equation. Suppose further that the agency charged with managing the deer population grants a limited number of permits to deer hunters. If we assume that the number of deer killed by hunters is proportional to the deer population and the number of hunting permits, we obtain the differential equation

$$\frac{dp}{dt} = rp\left(1 - \frac{p}{K}\right) - Ep,$$

where r and K are as in the logistic equation and E is a parameter with units of time^{-1} that measures the amount of hunting effort and is taken to be proportional to the number of permits issued. This model is called the *Schaefer model;* it is named after the biologist M. B. Schaefer, who applied it to fishery management.

   a. The model is easier to analyze if it is scaled by a suitable change of variables. Define new variables y and τ and a new parameter h by

$$y = \frac{p}{K}, \qquad \tau = rt, \qquad h = \frac{E}{r}.$$

   Note that y is the population relative to the environmental capacity and h is the hunting effort relative to the natural growth rate. Change variables to arrive at the differential equation

$$y' = y(1 - y) - hy,$$

   where the superscript prime represents $d/d\tau$.

   b. Assume $E > r$. What will happen to the deer population?

   c. Show that there is a positive equilibrium value y_e if $E < r$, and use the phase line to show that y_e is stable.

   d. Observe that the value of y_e can be manipulated to achieve any value between 0 and 1 by controlling the number of hunting permits. Sketch a graph of h versus y_e. This graph could be used to determine the appropriate number of permits for any desired equilibrium deer population.

   e. Suppose the goal of the management program is to allow a maximum amount of sustainable deer hunting. The hunting is represented by the term hy in the differential equation, so the goal is to maximize the function $Y(h) = hy_e$. Sketch the graph of the function Y.

   f. Determine the value of h that maximizes Y.

16. Stefan's law of radiative cooling says that the rate of change of temperature T of a body that interacts with a medium of temperature M is given by

$$T' = K(M^4 - T^4).$$

The temperatures T and M must be given on an absolute temperature scale such as the Kelvin or Rankine, and not in Celsius or Fahrenheit.

   a. Sketch the phase line for Stefan's law. Discuss the behavior of an object whose temperature is initially hotter than that of the medium.

   b. Suppose the temperatures T and M are roughly comparable. Factor the polynomial $M^4 - T^4$. The resulting equation can be approximated if it is done carefully. Replace T with M in sums, but not in differences; for example, replace $M^2 + T^2$ by $2M^2$. Show that the resulting approximation leads to Newton's law of cooling.

✦ 5.2 INSTANT EXERCISE SOLUTIONS

1. The differential equation is separable, so we can write it as

$$\frac{1}{y(1-y)}\frac{dy}{dt} = 1$$

and integrate with respect to t to get

$$\int \frac{1}{y(1-y)}\,dy = \int dt.$$

The integral in y can be done by a partial fraction decomposition. We need to find the numerators A and B so that

$$\frac{A}{y} + \frac{B}{1-y} = \frac{1}{y(1-y)}.$$

Multiplying both sides of this equation by the common denominator gives

$$A(1-y) + By = 1,$$

or

$$A + (B-A)y = 1 + 0y.$$

For this equation to hold for all y, we must have $A = 1$ and $B - A = 0$. Thus $A = B = 1$. We may therefore rewrite the integral equation as

$$\int \left(\frac{1}{y} + \frac{1}{1-y}\right) dy = \int dt.$$

Now we integrate:

$$\ln|y| - \ln|1-y| = t + C.$$

The two terms on the left combine to give

$$\ln\left|\frac{y}{1-y}\right| = t + C.$$

Raising both sides over e yields

$$\left|\frac{y}{1-y}\right| = e^{t+C} = e^C e^t.$$

Thus,

$$\frac{y}{1-y} = \pm e^C e^t = Ae^t,$$

where A is an arbitrary constant whose value must be determined from the initial condition. Substituting $y = y_0$ and $t = 0$ gives the result $A = y_0/(1-y_0)$. We now have the solution of the initial-value problem, still in implicit form, as

$$\frac{y}{1-y} = \frac{y_0}{1-y_0}e^t.$$

Multiplying both sides by $1-y$, $1-y_0$, and e^{-t} yields

$$(1-y_0)e^{-t}y = y_0(1-y).$$

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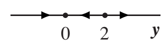
Putting terms with y on the left side and other terms on the right side gives

$$[y_0 + (1 - y_0)e^{-t}]y = y_0,$$

leading to the final answer (2).

2.



The critical point $y = 2$ is unstable while the critical point $y = 0$ is asymptotically stable. The phase line could just as well have been oriented vertically.

5.3 The Phase Plane

The phase line for an autonomous first-order differential equation is a one-dimensional space with the dependent variable on the axis and arrows to indicate whether solutions increase or decrease. The phase line plot displays many qualitative features of the solutions of the differential equation. Similarly, the **phase plane** for an autonomous second-order differential equation $y'' = f(y, y')$ is a two-dimensional space with one axis for the dependent variable y and the other for its derivative y'. A solution can be represented as a parameterized curve in the phase plane with the independent variable as the parameter; these curves are called **trajectories.** The phase plane for an autonomous system of two equations,

$$\frac{dx}{dt} = f(x, y), \qquad \frac{dy}{dt} = g(x, y), \tag{1}$$

is similarly a two-dimensional space with coordinates for the dependent variables x and y. The use of arrows in the phase plane is discussed in Section 5.4; here, we concentrate on the notion of the phase plane and on trajectories.

EXAMPLE 1

The initial-value problem

$$y'' + 2y' + 26y = 0, \qquad y(0) = 1, \qquad y'(0) = 0$$

has solution (Section 3.5, Example 2)

$$y = e^{-t}(\cos 5t + 0.2 \sin 5t).$$

This initial-value problem could represent the movement of a mass on a spring, in which case the unknown function y represents the distance measured from the rest position. We might also be interested in the velocity, given by

$$v = y' = -5.2e^{-t} \sin 5t.$$

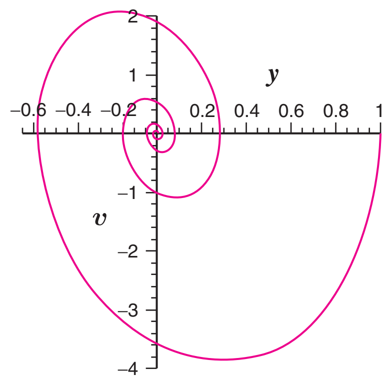


Figure 5.3.1
The solution of Example 1 in the phase plane.

Now suppose we are interested specifically in the relationship between distance and velocity. We could think of the solution as defining a curve with coordinates $(y(t), v(t)) = (e^{-t}(\cos 5t + 0.2 \sin 5t), -5.2e^{-t} \sin 5t)$. By plotting points for various values of t, we obtain the curve shown in Figure 5.3.1. The solution begins at the point (1, 0) and moves around the origin in the clockwise direction. As the velocity becomes more and more negative, the distance decreases from 1 toward 0. Eventually, at the point corresponding to the bottom of the curve, the velocity reaches its most negative value. The distance soon becomes negative, while the velocity becomes less negative. When the point at the left of the curve is reached, the distance is at its most negative value while the velocity is zero, indicating that the mass is momentarily motionless. The remainder of the curve can be similarly interpreted.

✦ INSTANT EXERCISE 1

Let $y(t)$ be the solution of the initial-value problem

$$\frac{d^2y}{dt^2} + y = 0, \qquad y(0) = 1, \qquad \frac{dy}{dt}(0) = 0.$$

Describe how the solution appears in a plot with y on the horizontal axis and $v = y'$ on the vertical axis.

Compare Figure 5.3.1 with Figure 3.5.4. Both graphs show the solution of Example 1 on the time interval $0 < t < 5$, but they show the information in different ways, with different disadvantages. The graph of $y(t)$ in Figure 3.5.4 shows the complete history of the distance y, but it does not show the velocity except by inference. The phase plane graph of Figure 5.3.1 shows the relationship of distance and velocity, but it does not give any indication of the time corresponding to any point on the curve. It does not even give the direction of the motion (clockwise inward spiral or counterclockwise outward spiral) except by inference.

In the case of systems, the phase plane is an alternative to a pair of plots for the two dependent variables against time. The phase plane is useful whenever there is added value to being able to plot the dependent variables in a plane, even though the information about time is lost. Phase plane plots are almost always used when the longtime behavior is the feature of greatest interest.

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MODEL PROBLEM 5.3

Suppose a projectile is fired upward with speed v_0 from a point y_0 above the surface of the moon. Determine the motion of the projectile; in particular, determine whether it will return to the moon.

Since Model Problem 5.3 is about motion, we begin with Newton's second law,

$$F = ma,$$

or

$$\frac{dv}{dt} = \frac{F}{m}.$$

For a flight on the moon, there is no air resistance, so the only force that needs to be included in the model is the force of gravity, which is given by[13]

$$F_g = -mg\frac{R^2}{(R+z)^2},$$

where $z(t)$ is the height of the projectile above the moon's surface, R is the radius of the moon, and g is the gravitational constant at the surface of the moon. Thus we have

$$\frac{dv}{dt} = -\frac{gR^2}{(R+z)^2}.$$

The model is not quite complete because there are two unknown functions, v and z. There are two ways to complete the model. One is to rewrite the left-hand side of the equation as d^2z/dt^2 so as to obtain a second-order differential equation for z. This is generally preferable whenever the second-order equation can be solved, as solution methods for higher-order equations are usually more efficient than solution methods for the corresponding system. The other option is to add the equation $dz/dt = v$ and obtain a system of two autonomous first-order equations. This is preferable when a model is to be studied in the phase plane because some phase plane techniques require the problem to be written as a system. Since our plan is to use the phase plane, we have the initial-value problem

$$\frac{dz}{dt} = v, \qquad \frac{dv}{dt} = -\frac{gR^2}{(R+z)^2}, \qquad z(0) = z_0, \qquad \frac{dz}{dt}(0) = v_0, \tag{2}$$

where z_0 is the height from which the projectile is launched and v_0 is the velocity of the projectile at launch.

Writing a Higher-Order Equation as a System

Suppose we had written the differential equation of Model Problem 5.3 as the second-order differential equation

$$\frac{d^2z}{dt^2} = -\frac{gR^2}{(R+z)^2}.$$

Then we could convert the equation to the system (2) by defining $v = dz/dt$ as a second dependent variable and using dv/dt in place of d^2z/dt^2 in the differential equation. Any second-order

[13] Section 2.1.

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differential equation

$$y'' = f(t, y, y')$$

can be converted to the system

$$y' = v, \qquad v' = f(t, y, v)$$

in this manner. Differential equations of order $n > 2$ can be similarly converted to higher-dimension systems by defining new variables to represent derivatives up to order $n - 1$. Writing higher-order equations as systems is often convenient. Not only does phase space analysis work only with (autonomous) systems, but also numerical packages are usually designed to work only on first-order equations and systems. When the higher-order equation can be solved by the methods of Chapters 3 and 4, it is almost always better to do so than to try to solve the corresponding system.

✦ INSTANT EXERCISE 2

Write the equation

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 3y = 0$$

as an autonomous system of two first-order equations.

Nondimensionalization

The differential equations of the current model have two parameters, R and g. We could find numerical values for these parameters and use them. However, it is a better practice to remove the parameters from the problem by nondimensionalization, as was done with the population models in Section 5.1. Results for the dimensionless version of the model will be correct not only on the moon, but on other small astronomical bodies as well.[14] To perform the nondimensionalization, we need to find or construct a reference length, velocity, and time from the parameters R, g, z_0, and v_0. If possible, these quantities should be representative values for z, v, and t so that the problem is properly scaled.

Both the radius R and the initial height z_0 are lengths. Of these, z_0 is not suitable for a reference, because its value could be zero, while R ought to be representative of the height for projectiles that are moving almost fast enough to escape the moon's gravity. Thus, R is an excellent choice for the reference length. There are two reasonable choices for a reference velocity. The initial velocity v_0 is one. The other can be found from a combination of the parameters R and g. Since R has dimension of length and g has dimension of length per time squared, the product gR has dimension of velocity squared. Thus, $\sqrt{gR}$ is a velocity that is somehow representative of the astronomical body. Given that the goal is to determine whether or not the projectile returns to the surface, the moon-based velocity $\sqrt{gR}$ is the better choice for the reference velocity. Given the length R and the velocity $\sqrt{gR}$, a reasonable choice for the reference time is the time required to move a distance R at velocity $\sqrt{gR}$, and this is $\sqrt{R/g}$.

[14]For a large body with a significant atmosphere, the accuracy of the model is affected by the damping of the atmosphere on the motion.

Define dimensionless variables Z, V, and τ by

$$Z = \frac{z}{R}, \qquad V = \frac{v}{\sqrt{gR}}, \qquad \tau = \frac{\sqrt{g}\,t}{\sqrt{R}}.$$

We first replace z by RZ and v by $\sqrt{gR}\,V$. After simplification, this yields

$$\frac{dZ}{dt} = \sqrt{\frac{g}{R}}\,V, \qquad \frac{dV}{dt} = -\sqrt{\frac{g}{R}}\,\frac{1}{(1+Z)^2}.$$

Now we use the chain rule:

$$\frac{dZ}{dt} = \frac{dZ}{d\tau}\frac{d\tau}{dt} = \sqrt{\frac{g}{R}}\,\frac{dZ}{d\tau},$$

and similarly with V, to get the dimensionless model

$$\frac{dZ}{d\tau} = V, \qquad \frac{dV}{d\tau} = -\frac{1}{(1+Z)^2}, \qquad Z(0) = Z_0, \qquad V(0) = V_0, \tag{3}$$

where $Z_0 = z_0/R$ and $V_0 = v_0/\sqrt{gR}$. The dimensionless model is the same as the original model, except that the distance Z is measured in terms of moon radii rather than meters or miles, the velocity V is measured in terms of $\sqrt{gR}$ rather than meters per second or miles per hour, and the time τ is measured in terms of $\sqrt{R/g}$ rather than seconds. Values for these reference quantities are needed to obtain results in familiar units, but not to determine the behavior of the model.

Trajectories and the Phase Portrait

A solution of an autonomous system (1) is a pair of functions $x(t)$ and $y(t)$, which define a trajectory in the phase plane. We can visualize any nonequilibrium solution as a point that moves along some trajectory. Each trajectory represents a family of solutions, because any solution passing through a point on a given trajectory at some time must stay on that trajectory for all time.[15]

The set of all trajectories is the **phase portrait** for the equation or system. Of course one cannot display the entire phase portrait, with curves through every point in a region. Instead, sketches of the phase portrait show a representative set of trajectories. This is analogous to the distinction in Section 2.3 between the slope field, consisting of infinitely many minitangents, and a sketch of the slope field. Standard terminology is to use the term *phase portrait* to denote a *sketch* of the phase portrait; nevertheless, it is important to realize that a phase portrait sketch includes only some of the trajectories. Good visualization of the behavior of a system depends on making a good choice of which trajectories to display in the phase portrait sketch. In Example 1, just one spiral trajectory is sufficient to illustrate the phase portrait. Adding a second trajectory makes the graph more confusing without adding important information. Systems with simpler or more varied trajectories require a larger and thoughtfully chosen sample of trajectories to make a good phase portrait sketch.

For most systems of practical interest, the trajectories have an important property, which is summarized in the following theorem.

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[15]This is so because the problem is autonomous. No matter when a solution passes through the point (0, 0), for example, it must follow the same path as other solutions that pass through the same point, since the rates of change depend only on the position in the phase plane and not on the time.

Theorem 5.3.1

Uniqueness of Trajectories If the functions $f(x, y)$ and $g(x, y)$ have continuous first derivatives with respect to both x and y on a region in the xy plane, then the system

$$\frac{dx}{dt} = f(x, y), \qquad \frac{dy}{dt} = g(x, y),$$

has exactly one trajectory through each point (x_0, y_0) in the interior of the region.

Theorem 5.3.1 means that trajectories cannot cross each other except possibly at points where the derivatives dx/dt and dy/dt are not both smooth functions.

The Differential Equation for the Trajectories

It is sometimes convenient to determine a differential equation relating the dependent variables of a system. In Equations (3), suppose we think of V as a function of Z. Since Z is in turn a function of τ, the chain rule gives us

$$\frac{dV}{d\tau} = \frac{dV}{dZ}\frac{dZ}{d\tau}.$$

Substitution from Equations (3) yields a first-order differential equation for V as a function of Z. Along with this differential equation, we have an initial condition that identifies V_0 as the velocity that goes with the height Z_0. Thus, the trajectories are determined by the initial-value problem

$$V\frac{dV}{dZ} = -\frac{1}{(1+Z)^2}, \qquad V(Z_0) = V_0. \tag{4}$$

The differential equation for $V(Z)$ is separable. Integrating both sides with respect to Z and applying the initial condition yields the solution formula

$$V^2 = V_0^2 + \frac{2}{1+Z} - \frac{2}{1+Z_0}. \tag{5}$$

✦ INSTANT EXERCISE 3

Confirm the solution (5) by solving the problem (4) for the trajectories.

We can now sketch the phase portrait by plotting some of the trajectories. The result is shown in Figure 5.3.2. Note that there appear to be two different types of trajectories. Assuming an initial height of 0, corresponding to a launch from the moon's surface, the trajectories that start at $V = 1.2$ and below clearly indicate that projectiles fired at those speeds will reach a maximum height and then fall back to the surface. The trajectories that start at higher initial velocities appear to indicate that projectiles fired at those speeds continue to move away from the surface and eventually escape the moon's gravity. Separating these two types of trajectories is the critical trajectory for which V approaches zero as Z approaches infinity. This happens when $V_0^2 = 2$, as can be seen by taking a limit of the trajectory equation as $V \to 0$ and $Z \to \infty$, given $Z_0 = 0$. To interpret this result, we restate it in terms of the original variables of the problem. The critical

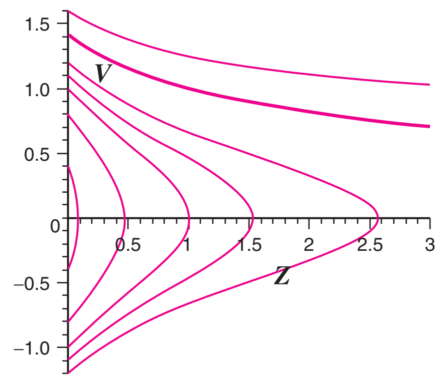

Figure 5.3.2
The phase portrait for Model Problem 5.3.

velocity is $V = \sqrt{2}$, or

$$\frac{v}{\sqrt{gR}} = \sqrt{2}.$$

The parameter

$$v_e = \sqrt{2gR}$$

for any astronomical body is called the **escape velocity.**

Some Conclusions

Note that some information is lost in the phase portrait. One sees the path taken by the solution but cannot see the speed with which the solution progresses along the path. Often the end of the path is approached only in the limit $t \to \infty$. One can also plot separate graphs of the two functions $x(t)$ and $y(t)$. These **time series graphs** are very useful, but they are often difficult to obtain because two-dimensional systems are hard to solve. Much of the important information about solutions of systems can be obtained from the phase plane in the same way that much of the important information about solutions of single first-order equations can be obtained from the phase line. Just as it is easier to sketch the phase line than to solve a scalar differential equation, so is it easier to sketch phase portraits than to solve a system of differential equations. Symbolic solution of autonomous linear systems is the subject of Chapter 6.

Throughout this chapter, we consider methods for obtaining the phase portrait of a two-dimensional system (either a system of two first-order equations or a single second-order equation). We have seen two of these in this section.

- If we have solution formulas for the system, we can sketch in the phase plane the curves represented parametrically by those solution formulas.
- If the differential equation for the solution curves can be solved symbolically, we can sketch the solutions in the phase plane.

Neither of these methods works very often. Solution formulas are generally available only for second-order linear equations with constant coefficients, as in Example 1. We can always determine a differential equation for the trajectories, but we can only solve the differential equation for the trajectories when it is of a type, such as separable, for which solutions can be found.

EXAMPLE 2

Consider the system

$$X' = X(1 - Y), \qquad Y' = Y(X - 1),$$

corresponding to the Lotka–Volterra model from Section 5.1. It is possible to derive a differential equation for the trajectories of the system. Think of Y as a function of X, which in turn is a function of t. By the chain rule,

$$Y' = \frac{dY}{dX} X'.$$

Substituting the differential equations into this equation yields

$$Y(X - 1) = X(1 - Y)\frac{dY}{dX}.$$

This differential equation is separable and leads to the integral form

$$\int \frac{1 - Y}{Y}\, dY = \int \frac{X - 1}{X}\, dX.$$

Integration yields

$$\ln Y - Y = X - \ln X + C,$$

or

$$Y + X + C = \ln XY.$$

This implicit solution formula cannot be solved explicitly for Y, nor does it have an obvious parameterization. Nevertheless, the trajectories given by this equation can be plotted using a computer algebra system. Figure 5.3.3 illustrates some of these trajectories.

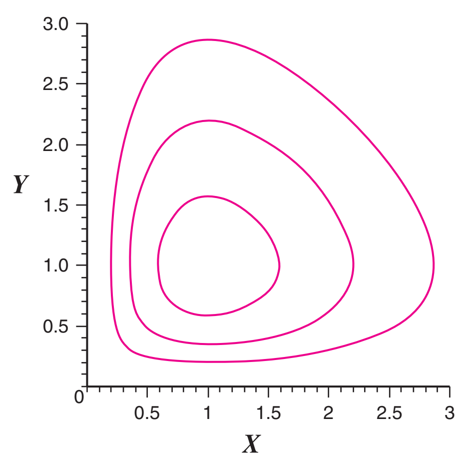

Figure 5.3.3
Trajectories for $X' = X(1 - Y)$, $Y' = Y(X - 1)$.

✦ INSTANT EXERCISE 4

Use the differential equations of Example 2 to determine the direction of solutions traveling on the trajectories of Figure 5.3.3.

Equations of the Form $y'' = f(y)$ In general, a system of two autonomous equations has the form (1). The differential equation for the trajectories is then

$$\frac{dy}{dx} = \frac{g(x, y)}{f(x, y)},$$

and it is unlikely for f and g to be such that the result is separable. The situation is a little better for second-order equations

$$\frac{d^2y}{dt^2} = f\left(y, \frac{dy}{dt}\right).$$

Setting $v = dy/dt$ always yields a system $y' = v$, $v' = f$, so the differential equation for trajectories is

$$v\,\frac{dv}{dy} = f(y, v).$$

✦ INSTANT EXERCISE 5

Derive the differential equation for the trajectories of the differential equation

$$\frac{d^2y}{dt^2} = f\left(y, \frac{dy}{dt}\right).$$

The trajectory equation is separable whenever f depends only on y. It is always possible to sketch a phase portrait by solving the differential equation for the trajectories when the original problem, like Model Problem 5.3, has the form

$$\frac{d^2y}{dt^2} = f(y).$$

EXAMPLE 3

For the differential equation $y'' = e^{-y}$, the trajectories are given by the equation

$$v\,\frac{dv}{dy} = e^{-y}.$$

Integrating this equation yields

$$\int 2v\,dv = \int 2e^{-y}\,dy,$$

or

$$v^2 = C - 2e^{-y}.$$

For any initial conditions $y(0) = y_0$ and $y'(0) = v_0$, the constant C is $C = v_0^2 + 2e^{-y_0}$. Thus,

$$v = \pm\sqrt{v_0^2 + 2(e^{-y_0} - e^{-y})}.$$

Some of the trajectories are illustrated in Figure 5.3.4. The direction of travel along the trajectories must be inferred from the system of equations. For any system derived from a second-order equation, one of the equations in the system is $dy/dt = v$. Hence, the sign of v gives the sign of dy/dt. Trajectories in the upper half-plane show movement to the right, and trajectories in the lower half-plane show movement to the left.

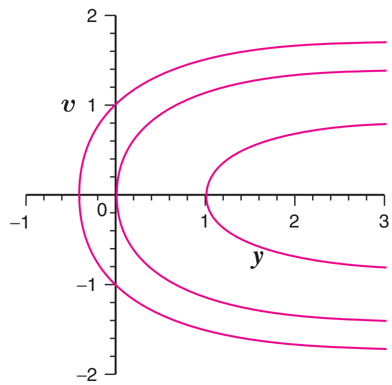

Figure 5.3.4
Trajectories for $y'' = e^{-y}$.

In the final two sections of this chapter, we will develop more general methods for sketching phase portraits.

5.3 Exercises

In Exercises 1 through 6, find and solve the differential equation for the trajectories and sketch the phase portrait.

1. $x' = -xy, \quad y' = -x$
2. $x' = y(1 + x^2), \quad y' = 2xy$
3. $x' = 3y^2, \quad y' = e^x$

T 4. $x' = e^y, \quad y' = e^{-x}$

5. $y'' + y^2 = 0$

T 6. $y'' = \dfrac{1}{1 + y^2}$

T 7. Determine the differential equation for the trajectories of the simple competition model (6) in Section 5.1 with $k = 1$, find an implicit solution formula, and plot some trajectories.

T 8. Determine the differential equation for the trajectories of the simple cooperation model (7) in Section 5.1 with $k = 1$, find an implicit solution formula, and plot some trajectories.

T 9. Plot some trajectories of the Lotka–Volterra model (5) in Section 5.1 with $k = 0.2$. Compare with Figure 5.3.3 and explain the significance of the parameter k.

T 10. Consider the system

$$x' = 2y, \qquad y' = x - y.$$

a. Determine the differential equation for the trajectories.
b. Let $v = y/x$. Determine the differential equation for the trajectories in the xv plane.
c. Solve the differential equation for the trajectories in the xv plane. Then replace v by y/x to obtain the trajectories of the original system. *Note:* With proper simplification, the result should have the form $(x - By)(x + y)^2 = C$, for some constant B.
d. Plot the phase portrait of the system.

F. W. Lanchester in 1916 proposed a model to predict the results of combat based on the size, efficiency, and character of the opposing forces. The simplest version of the Lanchester model,

$$x' = -y, \qquad y' = -x,$$

assumes that the forces are of equal quality but unequal size. Exercises 11 through 14 concern the Lanchester model and its application to a battle of historical significance.

11. *a.* Explain the assumptions the Lanchester model makes about how losses in combat occur.
b. Find and solve the differential equation for the trajectories.
c. If the larger force is initially twice the size of the smaller force and the battle lasts until the smaller force is destroyed, what fraction of the larger force survives?
d. The British navy won a significant victory over the French navy at the battle of Trafalgar in 1805. The battle began with 27 British ships and 33 French ships, with the ships roughly comparable in combat strength. In a typical naval battle of the time, the ships of each fleet formed two parallel lines so that they could fire cannons from their long sides at the enemy ships. They kept firing until one side conceded the fight. Suppose the battle of Trafalgar had followed standard naval procedure and continued until one fleet was completely destroyed. Who would have won the battle, and how many of their ships would have survived?
e. Sketch the phase portrait for the simplified Lanchester model, including the trajectory that would have been followed in the battle analyzed in part *d*.

12. Suppose the British commander at Trafalgar, Admiral Lord Nelson, had on his staff a mathematician familiar with the Lanchester model. (Ignore the fact that the battle occurred 111 years before the model was published.) Could this mathematician have devised a winning strategy for the British? One idea would be to divide the battle up into two sub-battles, one favoring each side. Suppose the British navy had divided into a large group and a small group and had managed to arrange the situation so that the larger British group engaged 17 French ships while the smaller British group engaged 16 French ships. Show that both sides have the same number of survivors if the British groups have 23 and 4 ships, respectively.

13. Do equal numbers of survivors from the sub-battles of Exercise 12 guarantee equal chances in the overall battle? This problem analyzes this question.
    a. Use the Lanchester equations to derive a second-order differential equation for $x(t)$, the number of ships on one side of the battle.
    b. Solve the equation from part a, and evaluate the constants, using the initial data $x(0) = x_0$ and $y(0) = y_0$.
    c. Assuming $x_0 < y_0$, show that the time at which the battle ends is given by

$$t = \operatorname{arctanh}\frac{x_0}{y_0} = \frac{1}{2}\ln\frac{y_0 + x_0}{y_0 - x_0},$$

    where tanh is the hyperbolic tangent function defined by $\tanh u = (e^u - e^{-u})/(e^u + e^{-u})$ and arctanh is the inverse of the hyperbolic tangent function.
    d. Use the result from part c to determine the time that each sub-battle of Exercise 12 ends. If the British had managed to arrange the two sub-battles, what would have happened at Trafalgar?

14. The British strategy in the battle of Trafalgar was indeed to divide the battle into two unequal portions, but they managed to do so in a way that actually did yield an advantage.
    a. Suppose the British managed to arrange a sub-battle of 20 British ships against 12 French ships and a second sub-battle of 7 British ships against 21 French ships. Use the results from Exercises 11 and 13c to determine the number of British survivors for the first battle and the time required for its completion.
    b. Assume that the British admiral ordered the 7 ships in the second sub-battle to try to prolong the fighting with a lot of maneuvering. This would correspond to a change in the model

$$F' = -aB, \qquad B' = -aF, \qquad F(0) = 21, \qquad B(0) = 7,$$

    where $a < 1$ is a factor that measures the amount of stalling by the British. Solve the model for this sub-battle.
    c. Show that the number of survivors is not changed by a but that the total time for the battle is changed by a. In particular, show that

$$t = \frac{1}{2a}\ln 2,$$

    for the second sub-battle.
    d. Suppose $a = 0.2$. Which sub-battle ends first? At the time when the first sub-battle ends, approximately how many ships are left for both sides, counting both sub-battles?

T 15. Consider the SIR model of Exercise 12, Section 5.1. This model has three dependent variables, S, I, and R, but R can be determined after S and I by using the fact that the sum of the three variables is constant. Thus, the system can be thought of as a system with two differential equations for the unknowns S and I. Determine the differential equation for the trajectories in the SI plane, and solve it. Explain why epidemics start only if $rS_0 > \gamma$, where S_0 is the initial number of susceptibles. (Note that an epidemic requires a large number of susceptibles but does not require many infectives. This explains why initial exposure of a few individuals to European diseases caused severe epidemics in Native American populations.)

T 16. The equation for motion of a pendulum (suitably scaled) is

$$\theta'' + \sin\theta = 0,$$

where θ is the angle of the pendulum measured from the vertical and derivatives are with respect to time t. (See Exercise 15 of Section 3.1.) Let $\omega = \theta'$ (ω is the **angular velocity**).

a. Determine the equation for the trajectories in the $\theta\omega$ phase plane.
b. Solve the equation for the trajectories.
c. Suppose the pendulum has angular velocity $\omega = \omega_0$ when in the vertical position. Plot the trajectories for a variety of values of ω_0. You should see two distinct types of trajectories.
d. Explain the behavior of the pendulum for each of the two types of trajectories.

17. The simplest model for skydiving must consider two stages: a fast fall with the parachute still closed and a slow fall with the parachute open. Let m be the mass of the skydiver and v_I the maximum safe-impact speed. (The velocity at impact is then $-v_I$.)

a. A skydiving rule of thumb holds that the largest distance a person can fall and safely land on his/her feet is 3 m. Write down a model for a fall from a height of 3 m, assuming constant gravitational force and no damping. Determine the differential equation for the solution curves and solve it. You will need to use the initial conditions appropriate for a free fall from a height of 3 m. Use the solution to determine the velocity at impact, and hence the maximum safe-impact speed.
b. Write down a model for a fall with a closed parachute, assuming constant gravitational force and linear damping, with the damping coefficient $b = 15$ kg/s. Determine the terminal closed-chute velocity v_{cc} for a person of mass 75 kg. (The terminal velocity for a falling object is the stable equilibrium velocity of the differential equation. Note that the terminal velocity is negative in a coordinate system where up is positive.)
c. Skydivers choose a parachute size that yields a damping coefficient of about 1.4 times their mass (in mass units per second). Write down a model for a fall with an open parachute, assuming constant gravitational force and linear damping, with damping coefficient $b = 1.4m$. Determine the terminal open-chute velocity v_{oc}.
d. The damping coefficient $b = 1.4m$ is achieved by selecting the appropriate parachute for each jumper. Explain why this coefficient is a good choice.
e. Determine the differential equation for the solution curves for the open-chute stage. Solve it and plot the phase portrait.
f. Determine the particular trajectory for which the jumper will hit the ground at the maximum safe-impact speed. Now determine the minimum height at which the parachute can safely be opened. You may assume that the jumper is moving with velocity v_{cc} at the moment the chute is opened.

T 18. Consider the dimensionless waste treatment model

$$W' = -WX, \qquad X' = WX - X,$$

where W is the amount of a waste chemical in a given volume of water and X is the population of the bacteria in the water (see Exercise 13 of Section 5.1).

a. Determine the differential equation for the trajectories in the WX plane.
b. Solve the differential equation for the trajectories with initial data $X = X_0$ at $W = W_0$.
c. Plot trajectories in the region $0 < W < 3$, $0 < X < 1$. Describe the behavior of the treatment system, paying particular attention to the direction in which the solutions move along the trajectories. In particular, discuss the difference between trajectories whose W intercept is less than 1 and those whose W intercept is greater than 1.

T 19. Suppose a hoop is placed in a vertical plane so that it just touches the ground. The hoop is then spun at a constant speed with a vertical axis of rotation. (See Fig. 5.3.5.) Suppose further that the hoop is grooved, so that a small bead can slide up or down the hoop as it is spun. Common sense says that the bead will always slide to the bottom of the hoop, as it would if the hoop were not spinning. In this exercise we investigate this prediction.

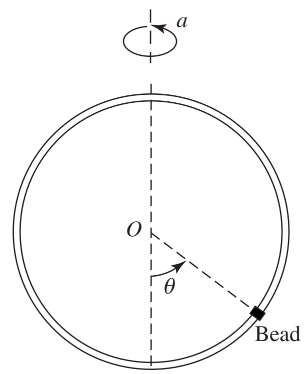

Figure 5.3.5
A bead on a rotating hoop.

Let θ be the angle formed by a vertical ray through the center of the hoop and a ray that connects the center of the hoop to the bead. Thus, $\theta = 0$ corresponds to the position of the bead at the bottom of the hoop. It can be shown that, in the absence of damping forces, the changes in the angle θ are modeled by the differential equation

$$\theta'' = (a^2 \cos\theta - 1)\sin\theta,$$

where the parameter a is proportional to the rotation speed of the hoop. (Note that the case $a = 0$ corresponds to the pendulum model of Exercise 16.)

a. Let $\omega = \theta'$. Determine the differential equation for the trajectories in the $\theta\omega$ phase plane.
b. Solve the differential equation for the trajectories.
c. Plot the phase portrait for the model for the case where $a = 1/\sqrt{2}$.
d. Repeat part c, but for the case $a = \sqrt{2}$.
e. Suggest an explanation for the differences between the behavior observed in parts c and d.

✦ 5.3 INSTANT EXERCISE SOLUTIONS

1. The solution is $y = \cos t$; hence, $v = y' = -\sin t$. By inspection, $y^2 + v^2 = 1$. The solution follows a circle in the yv plane, beginning at $(1, 0)$. The solution moves clockwise; to see this, note that $v'(0) = -\cos 0 = -1$. The solution moves down from the initial point.

2. Let $v = dy/dt$. Then $dv/dt = d^2y/dt^2 = -3y - 2\,dy/dt = -3y - 2v$. The system is

$$\frac{dy}{dt} = v, \qquad \frac{dv}{dt} = -3y - 2v.$$

3. We have

$$2V\,dV = -\frac{2}{(1+Z)^2}\,dZ = -2u^{-2}\,du,$$

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where $u = 1 + Z$. Integration yields $V^2 = 2u^{-1} + C = 2/(1+Z) + C$. The constant is evaluated from the initial condition: $C = V_0^2 - 2/(1+Z_0)$. The solution follows from the substitution of C into the general solution formula.

4. The solutions move counterclockwise. This can be determined by examining the system of equations in a number of ways. Consider, for example, points with $Y = 1$ and $X < 1$. The equation $Y' = Y(X-1)$ shows that $Y' < 0$ at these points. Hence, the direction of solutions to the left of the plot in Figure 5.3.3 is primarily downward, corresponding to counterclockwise motion.
5. With $v = y'$, the equation reduces to the first-order equation $v' = y'' = f(y, v)$. Thus, $dv/dy = v'/y' = f(y, v)/v$ or $v(dv/dy) = f(y, v)$.

5.4 The Direction Field and Critical Points

Recall that the slope field[16] of a scalar differential equation

$$\frac{dy}{dt} = f(t, y)$$

is a set of minitangents in the ty plane, each minitangent a line segment centered at some point (t, y) and having the slope dy/dt given by the differential equation. The minitangent at a particular point therefore indicates the direction of a solution curve passing through that point. The slope field can be generated by a computer without having to solve the equation. It can then be used to determine characteristics of the solutions. Autonomous two-dimensional systems can be studied by a similar graphical method.

MODEL PROBLEM 5.4

Describe the trajectories of the systems

$$x' = -2x + 2y, \qquad y' = 2x - 5y,$$

and

$$x' = -2x + 2y, \qquad y' = 2x + y.$$

The Direction Field

The **direction field** for an autonomous system of two equations is the set of arrows in the phase plane that point in the direction taken by a trajectory through the point at its center.

The direction field for an autonomous system is a lot like the slope field of a single nonautonomous differential equation. Both indicate the direction on a graph corresponding to the progress of solutions. The difference is in the coordinates on the axes. The slope field shows the changes in the single dependent variable compared to changes in the independent variable. The direction field shows the path in the two-dimensional phase space that solutions take as time advances. It is not necessary to use arrows to indicate directions in a slope field, because the forward

[16] Section 2.3.

direction for the independent variable is necessarily to the right. The independent variable does not appear on an axis in a direction field, so the direction of forward movement must be indicated with arrows.

The systems of Model Problem 5.4 can be thought of as pairs of formulas that indicate the rates of change of the variables on the axes of a direction field plot. Consider the points that lie on the x axis. For these points, both systems prescribe the rates of change as $x' = -2x$ and $y' = 2x$. All these points have values of x' and y' that are equal in magnitude but opposite in sign. Since both coordinates are changing at the same rate, the direction of motion is along the line $y = -x$. The actual rates of change of position with respect to time are different for each of these points. We could indicate the overall rate of change by using different lengths for direction field arrows. In this example, the arrow at the point (1, 0) could be of length 2 and the arrow at (2, 0) of length 4. A plot in which the lengths of the arrows are relative to the magnitudes of the rate of change is a **vector field.** Vector fields are used sometimes in fluid flow, but they are not often used with other systems of differential equations because the plots can be visually confusing. Direction fields indicate only the direction of motion, not the rate at which the motion occurs.[17]

To sketch the direction field for a system of two autonomous differential equations, one can compute values of x' and y' at points on a grid and then compute the arrow slope from

$$\frac{dy}{dx} = \frac{y'}{x'} \tag{1}$$

It is not practical to do this by hand, but it is easy to use a computer algebra system to sketch direction fields. The direction field for the first system in the model problem appears in Figure 5.4.1. As we computed by hand, the slopes of the arrows at points on the x axis are all -1. Note that the arrowheads point toward the upper left for $x > 0$ and toward the lower right for $x < 0$.

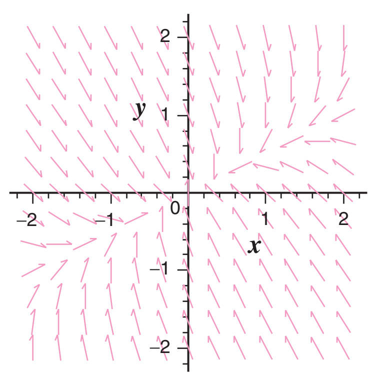

Figure 5.4.1
The direction field for $x' = -2x + 2y$, $y' = 2x - 5y$.

The arrows in the direction field are tangent to the trajectories. Thus, we can use the direction field to generate a sketch of the trajectories. Figures 5.4.2 and 5.4.3 show the phase portraits, including the direction fields, for both systems of Model Problem 5.4.

[17]Some fluid flow maps use the width of the direction field arrows to indicate the speed of the flow, but this practice does not appear to have caught on in mathematics.

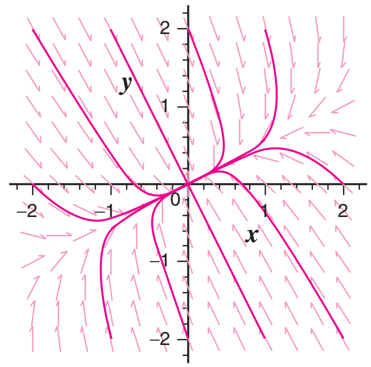

Figure 5.4.2
The phase portrait for $x' = -2x + 2y$, $y' = 2x - 5y$.

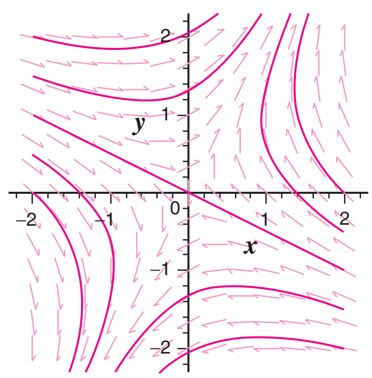

Figure 5.4.3
The phase portrait for $x' = -2x + 2y$, $y' = 2x + y$.

Critical Points and Equilibrium Solutions

Like single autonomous first-order equations, autonomous systems have equilibrium solutions for any values of the dependent variables that make the derivatives zero. A **critical point** for a system $x' = f(x, y)$, $y' = g(x, y)$ is a solution of the algebraic equations $f(x, y) = 0$, $g(x, y) = 0$. The corresponding constant solution is an **equilibrium solution.**

The concept of stability for scalar equations holds equally well for systems, with some modifications. Some care is needed in the definitions. Let (x_c, y_c) be a critical point. For any point (x, y), define the distance $d(x, y)$ by $d(x, y) = \sqrt{(x - x_c)^2 + (y - y_c)^2}$. The distance d allows us to be precise in describing the intuitive ideas of "staying near" a critical point and "approaching" a critical point.

- An equilibrium solution is *asymptotically stable* if all trajectories that come within a distance δ of the critical point enter the critical point in forward time. Formally, an equilibrium solution is **asymptotically stable** if there is a positive number δ small enough that $\lim_{t\to\infty} d(x, y) = 0$ for all initial points (x_0, y_0) for which $d(x_0, y_0) < \delta$.

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- An equilibrium solution is *stable* if solution curves that begin close enough to the critical point remain close to it. Formally, an equilibrium solution is **stable** if for any positive number ϵ, there exists a positive number δ small enough that $d(x, y) < \epsilon$ for all initial points (x_0, y_0) for which $d(x_0, y_0) < \delta$.
- An equilibrium solution is **unstable** if it is not stable. The failure of stability occurs when some solution curves that start arbitrarily close to the critical point move away from it.

Note that the requirement for asymptotic stability is stricter than that for mere stability.

For both systems of Model Problem 5.4, the origin is the only critical point. In Figure 5.4.2, the origin appears to be asymptotically stable. The origin in the system of Figure 5.4.3 is a little more difficult to characterize. Most trajectories seem to be moving toward the origin for a while before turning away. The trajectories beginning at the points $(2, -1)$ and $(-2, 1)$, however, appear to head directly into the origin. The origin in Figure 5.4.3 is an example of a *saddle point.* This concept will be more carefully defined in Chapter 6. For now, it is sufficient to think of a saddle point as a critical point for a two-dimensional system that is unstable in spite of having a pair of trajectories that enter it in forward time.

So far, we have considered only systems with just one critical point. The same concepts apply to systems with more than one critical point.

EXAMPLE 1

Consider the predator-prey model

$$X' = X(1 - Y), \qquad Y' = 0.4Y(X - 1).$$

Critical points satisfy $X(1 - Y) = 0$ and $Y(X - 1) = 0$. The first equation requires either $X = 0$ or $Y = 1$. If $X = 0$, then the second equation requires $Y = 0$; if $Y = 1$, then the second equation is satisfied only for $X = 1$. There are two critical points: (0, 0) and (1, 1). The phase portrait, shown in Figure 5.4.4, suggests that the equilibrium solution $x = 0$, $y = 0$ is unstable while the equilibrium solution $x = 1$, $y = 1$ is stable, but not asymptotically stable. The origin appears to be a saddle point because a trajectory enters in along the Y axis. A critical point is a **center** if the nearby trajectories are a set of closed concentric curves; it appears that (1, 1) is a center.

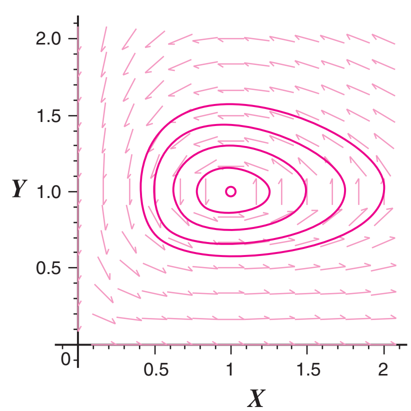

Figure 5.4.4
The phase portrait for $X' = X(1 - Y)$, $Y' = 0.4Y(X - 1)$.

Note the tentative language used in Example 1. Definite conclusions cannot generally be drawn from graphs, particularly graphs generated by numerical methods. A computer-generated phase portrait often provides strong clues for the classification of equilibrium points, but symbolic methods are necessary to confirm the results suggested by the phase portrait.

Separatrices

Systems with multiple critical points often have trajectories that serve to separate regions with different qualitative behavior. These special trajectories are called **separatrices.**

EXAMPLE 2

Consider the competition model

$$X' = X(1 - Y), \qquad Y' = 2Y(1 - X).$$

As in Example 1, there are two critical points: (0, 0) and (1, 1). The phase portrait is shown in Figure 5.4.5. The critical point (1, 1) appears to be a saddle point because most trajectories move away from it, but there is a pair that seems to enter it. The equilibrium solution $x = 0$, $y = 0$ is unstable. All the trajectories are unbounded, but tend to approach one of the axes as time increases. The two trajectories that enter the saddle point divide the first quadrant between the region where trajectories approach the X axis and the region where trajectories approach the Y axis. These two trajectories are separatrices.

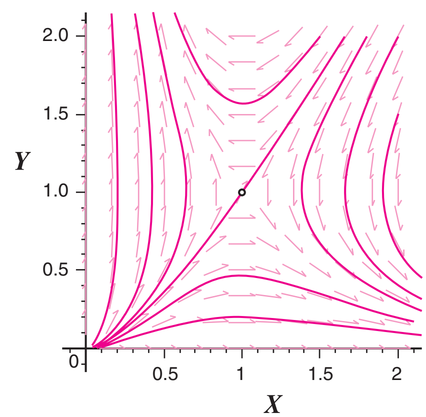

Figure 5.4.5
The phase portrait for $X' = X(1 - Y)$, $Y' = 2Y(1 - X)$.

There are several possible longtime behaviors for a two-dimensional system of autonomous equations.

- Trajectories can terminate in an asymptotically stable equilibrium state.
- Trajectories can approach a closed (stable) trajectory, as in the Lotka–Volterra model (Example 1), where all trajectories are closed.
- Trajectories can be unbounded, as in the simple competition model (Example 2).

Sometimes a system has more than one possible behavior as $t \to \infty$. In these cases, there are portions of the phase plane where the trajectories have one longtime behavior and other portions

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that have different longtime behavior, and the curves that are the boundaries of these regions are the separatrices.

Separatrices are important because they determine the ultimate fate of a system. In the simple competition model, one species wins and the other loses because the longtime behavior is for one population to increase indefinitely while the other one appears to vanish. Which one wins depends on the initial condition. Initial points to the upper left of the separatrices are on trajectories that ultimately move up and to the left; thus, Y increases and X decreases. Note that the curve at the top of the figure shows an initial decrease in both populations; however, eventually the Y population reaches a minimum value and increases thereafter. Similarly, initial points to the lower right of the separatrices are on trajectories that ultimately move down and to the right, with Y decreasing toward zero and X apparently increasing without bound.

Recapitulation

The examples in this section have illustrated a variety of possibilities for the behavior of autonomous systems. All critical points can be classified by stability. The classification of a critical point can usually be obtained from a computer-generated sketch of the phase portrait, but not with complete reliability. There is also an analytical (symbolic) method of classifying critical points that is the focus of Chapter 6. Saddle points are particularly noteworthy because they have a pair of trajectories that terminate at them even though most trajectories move away from them. Centers are particularly noteworthy because trajectories near them are closed curves. Generally, solutions evolve toward stable equilibrium solutions or closed curves, or are unbounded. Where more than one of these ultimate behaviors is possible, the regions corresponding to each such long-term behavior are separated by trajectories called separatrices.

5.4 Exercises

In Exercises 1 through 4, determine the critical points for the given system.

1. $r' = 3r - 2cr, \quad c' = cr - 4c$
2. $u' = 3u - 2v - 5, \quad v' = u + v - 5$
3. $x' = \sin y, \quad y' = y - x$
4. $x' = y(x^2 + y^2 - 1), \quad y' = -x(x^2 + y^2 - 1)$

In Exercises 5 through 10, use a computer or calculator to sketch the phase portrait; determine whether the equilibrium solution $x = 0$, $y = 0$ is stable or unstable; and determine whether the origin is a saddle point or a center.

T 5. $x' = -x + y, \quad y' = x + y$

T 6. $x' = x - 2y, \quad y' = -2x$

T 7. $x' = -2y, \quad y' = x - 3y$

T 8. $x' = 3x - 2y, \quad y' = 2x - 2y$

T 9. $x' = 2x - 5y, \quad y' = x - 2y$

T 10. $x' = -x + y, \quad y' = -5x + 3y$

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In Exercises 11 through 14, (a) determine the critical points, (b) use a computer or calculator to sketch the phase portrait, (c) determine whether each equilibrium solution is stable or unstable, (d) identify the saddle point, and (e) plot the separatrices. (*Hint:* to plot a separatrix, choose a point that is near the saddle point and appears to lie close to the separatrix. Run the plot backward in time to get a curve that is approximately the separatrix.)

T 11. $x' = x^2 - y, \quad y' = -x + y$

T 12. $x' = y, \quad y' = x - y - x^3$

T 13. $x' = 2y - x, \quad y' = x(2x + y - 5)$

T 14. $x' = x + y, \quad y' = 1 - y^2$

T 15. The growth of a vapor bubble during the boiling process is governed by the differential equation

$$\frac{2}{3} r r'' + (r')^2 = 1 - \frac{1}{r},$$

where r is the radius of the bubble. Define a new variable v by $v = r'$. Write the bubble growth equation as a system, and determine the critical point(s) for this system. Use a computer or calculator to sketch the phase portrait. Determine the stability of the equilibrium solution(s) and identify any saddle points or centers. Note that a bubble radius cannot be negative, so only the half-plane $r \geq 0$ is of interest.

5.5 Qualitative Analysis

Qualitative accuracy means getting the general features right; *quantitative* accuracy means getting agreement to within some satisfactory error tolerance. Both are important. The work we did in Section 5.4 to classify equilibrium solutions was quantitative, as it was based on computer-generated phase portrait sketches. One drawback of quantitative approximation is that it can only yield information about the system for one set of parameter values at a time. Often it is important to determine the longtime behavior of solutions and how that behavior depends on the parameter values. This is the realm of **qualitative analysis.** The phase line of Section 5.2 is an example of a qualitative technique. Nullcline analysis, which is the subject of this section, is a two-dimensional analog of the phase line technique. Sometimes, qualitative analysis is enough to determine the longtime behavior of a system. Other times, it is necessary to use the symbolic quantitative methods that are the subject of Chapter 6.

MODEL PROBLEM 5.5

Use the information given in the differential equations

$$x' = 2y, \qquad y' = x - y$$

to classify the equilibrium solution $x = 0$, $y = 0$ without having to solve the system or rely on numerical approximations.

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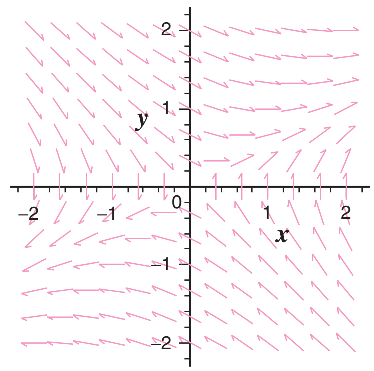

Figure 5.5.1
The direction field for $x' = 2y$, $y' = x - y$.

A plot of the direction field for Model Problem 5.5 appears in Figure 5.5.1. Notice that all the arrows on the x axis point in a vertical direction. This is confirmed by the differential equation for x: $x' = 0$ when $y = 0$. Curves consisting of points at which x' is always zero or points at which y' is always zero are useful tools for qualitative analysis.

A **nullcline** is a curve in the phase plane on which the arrows all point in a direction parallel to a coordinate axis. In the xy plane, for example, an x nullcline is a curve on which $x' = 0$ and a y nullcline is a curve on which $y' = 0$.

The first differential equation in Model Problem 5.5 is $x' = 2y$; thus, $x' = 0$ precisely when $y = 0$. The line $y = 0$ is an x nullcline for the system. Note in the figure that all the arrows in the upper half-plane show x increasing (because the x component of the vector is positive), while all arrows in the lower half-plane show x decreasing. This illustrates an important property of nullclines:

The x nullclines divide the xy phase plane into regions, with each region consisting of points where x is increasing or points where x is decreasing. Similarly, regions set off by y nullclines consist of points where y is increasing or points where y is decreasing.

Note that we can determine which regions have x increasing and which have x decreasing simply by looking at the differential equation for x. We have $x' > 0$ whenever $2y > 0$, and this is in the upper half-plane. A schematic diagram showing the result of the analysis of the equation $x' = 2y$ appears in Figure 5.5.2*a*. Some vertical minitangents appear on the line $y = 0$, which is the x nullcline. Above the nullcline is an arrow indicating that x is increasing, and below it is an arrow indicating that x is decreasing.

Figure 5.5.2*b* shows a similar schematic analysis of the equation $y' = x - y$. The y nullcline is the line $y = x$, on which $y' = 0$. The horizontal minitangents indicate that this line is a y nullcline.

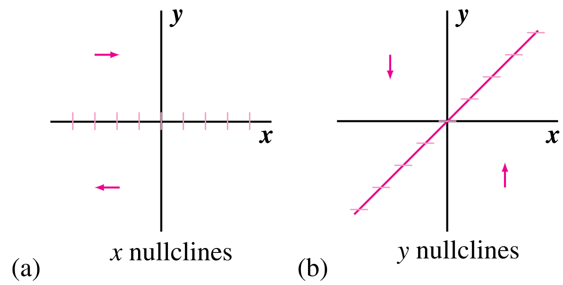

Figure 5.5.2
The individual nullclines for the system $x' = 2y$, $y' = x - y$.

The region $x > y$ lies to the lower right of the nullcline, and this region is marked with an upward arrow to indicate that this is a region where y is increasing. The region on the other side of the nullcline is marked with a downward arrow because $y' = x - y < 0$ for points in that region. Note that the details regarding both nullclines can also be identified in the computer-generated direction field of Figure 5.5.1. All the arrows on the line $y = x$ are indeed horizontal, all arrows above and to the left of this line show y decreasing, and all arrows below and to the right of the line show y increasing.

✦ INSTANT EXERCISE 1

Sketch the individual nullclines (as in Figure 5.5.2) for the system

$$x' = y - x, \qquad y' = -x.$$

Note that the sketches in Figure 5.5.2 do not show arrowheads on the nullcline minitangents. Each sketch was prepared using the information from just one of the differential equations. Information from the x' equation cannot be used to determine whether y is increasing or decreasing on the x nullclines. This information can only be found by studying the differential equations together.

Nullcline Diagrams

Each of the sketches in Figure 5.5.2 is of some value by itself, but a lot more information is obtained by combining the two into a full nullcline diagram. Figure 5.5.3 shows the result. The minitangents on the y nullcline $y = x$ now have arrowheads—these point to the right for the portion above the x axis and to the left for the portion below the x axis. These details follow from the information provided by the x nullclines. Similarly, the minitangents on the x nullcline have arrowheads determined by the analysis of the y nullcline. There is one point where the nullclines intersect. This is the critical point (0, 0). Since critical points are points where neither variable is changing, they are always at the intersection of opposite nullclines. Taken together, the nullclines of Model Problem 5.5 divide the phase plane into four regions, labeled A, B, C, and D in the figure. All points in region A are below the y nullcline and above the x nullcline. Based on our analysis of the nullclines, this means that both x and y are increasing in region A. The symbol consisting of

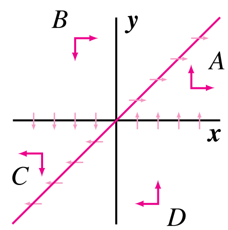

Figure 5.5.3
The nullcline diagram for the system $x' = 2y$, $y' = x - y$.

two connected arrows pointing up and to the right indicates the general trend of the direction field for region A. Similarly, there are symbols in each of regions B, C, and D that indicate the trends of the direction field. The directions of the arrowheads on the nullclines are provided by the trends in the adjoining regions. For example, the nullcline $y = x$ in the first quadrant has horizontal arrows. Both regions A and B, which are adjacent to this nullcline, indicate a general trend to the right; hence the arrows on this portion of the nullcline point to the right. Below the x axis, the arrows on the same nullcline point to the left.

✦ INSTANT EXERCISE 2

Sketch the complete nullcline diagram for the system

$$x' = y - x, \qquad y' = -x.$$

Using Nullclines to Study Critical Points

It is helpful to look at the nullclines in Figure 5.5.3 as providing information about longtime behavior in the same manner as the phase line, but generalized to two dimensions. This analysis is based on the observation that solution curves can only cross a nullcline in the indicated direction.

Consider the behavior of the trajectory through some point in region A. Any solution in region A has both x and y increasing. Stated differently, trajectories in region A can only move up and to the right. What is more important is that they cannot leave region A. The arrows on the boundaries between A and B and between A and D show that solutions can cross these boundaries only to enter region A. Thus, trajectories that begin in A or enter A must thereafter stay in A. The same property holds for region C; trajectories that begin in C must stay in C. Solutions in regions A and C always move away from the origin. This demonstrates that the equilibrium solution $x = 0$, $y = 0$ is unstable.

Now consider regions B and D. Some trajectories in each of these regions move into A, while others move into C. From this information, we can argue that there must be some trajectory that enters the origin. Consider a thought experiment. Mark out all the points in D that are on trajectories that enter A in one color, and mark all the points in D whose trajectories enter C in a different color. There cannot be any white space between these two regions, nor can there be any overlap, because (Theorem 5.3.1) there is a unique trajectory through every point, no matter how

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close the point is to the origin. Thus, the two regions must have a common boundary that passes through the origin. This boundary is itself a trajectory; indeed, it is a separatrix because it divides points that go into region C from points that go into region A.

Nullcline diagrams can sometimes confirm stability as well as instability, and there are also systems for which no conclusions can be drawn from the nullcline diagram alone.

EXAMPLE 1

Figure 5.5.4 shows the nullclines for

$$x' = -2x + 2y, \qquad y' = 2x - 5y.$$

As in Model Problem 5.5, trajectories in region A or region C must remain in that region. This time, those trajectories move toward the origin, suggesting that the equilibrium solution at the origin is asymptotically stable. To confirm this claim, it is necessary to examine regions B and D. Suppose there is a trajectory in one of these regions that does not approach the origin. Such a trajectory cannot remain in region B or D, but must instead enter either A or C. However, trajectories in A and C move toward the origin. Ultimately, trajectories that begin in any of the four regions must go to the origin, so the origin is indeed asymptotically stable. The phase portrait for this system appears in Figure 5.4.2; trajectories in the phase portrait illustrate the predictions made by nullcline analysis.

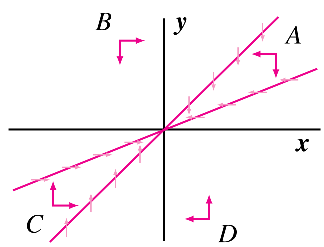


Figure 5.5.4
The nullcline diagram for the system $x' = -2x + 2y$, $y' = 2x - 5y$.

EXAMPLE 2

Figure 5.5.5 shows the nullclines for

$$x' = -2y, \qquad y' = x - 3y.$$

In this case there seems to be a general flow of solution curves from A to B to C to D and so on, which suggests that the solution curves rotate around the origin, but does not offer any evidence concerning stability. The phase portrait in Figure 5.5.6 reveals that the origin is asymptotically stable. Solution curves do not spiral around the origin, but follow paths confined to one or two regions. For example, solution curves beginning in the second quadrant move through region C into region D and then proceed to the origin without leaving region D. The nullclines alone do not give enough information to distinguish this behavior from trajectories that spiral around the origin, such as in Figure 5.3.1, or trajectories that are closed curves, as in Figure 5.3.3.

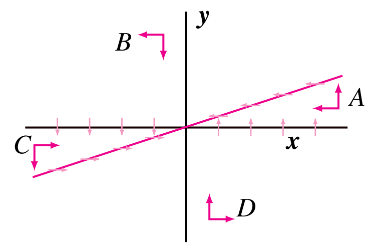


Figure 5.5.5
The nullcline diagram for the system $x' = -2y$, $y' = x - 3y$.

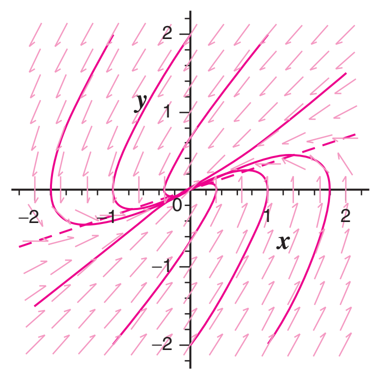


Figure 5.5.6
The phase portrait for $x' = -2y$, $y' = x - 3y$, along with the nullcline $x = 3y$ (dashed).

✦ INSTANT EXERCISE 3

What conclusions, if any, can be drawn from the nullcline diagram for the system

$$x' = y - x, \qquad y' = -x?$$

Nullcline Analysis of Some Population Models

Consider now the three basic models for interacting populations that were developed in Section 5.1. These are the basic predator-prey model

$$X' = X(1 - Y), \qquad Y' = kY(X - 1), \tag{1}$$

the basic competition model

$$X' = X(1 - Y), \qquad Y' = kY(1 - X), \tag{2}$$

and the basic cooperation model

$$X' = X(Y - 1), \qquad Y' = kY(X - 1). \tag{3}$$

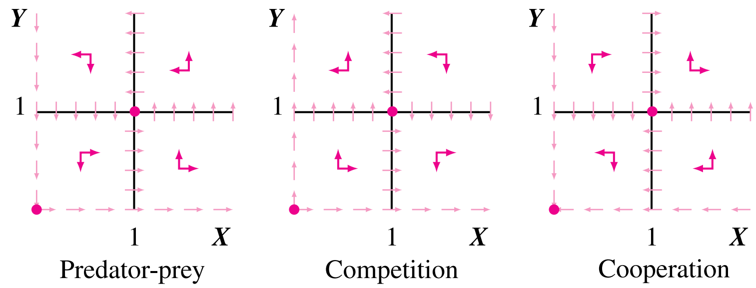


Figure 5.5.7
Nullcline diagrams for the three basic models of interacting populations.

These models show, among other things, the power of abstraction. The nondimensionalization process reduced the number of parameters from four to one, and now we get additional benefits from the change of variables. In all three models, the X nullclines consist of the lines $X = 0$ and $Y = 1$, and the Y nullclines are the lines $Y = 0$ and $X = 1$. The parameter k does not affect the nullclines at all. The nullclines for the three models are shown in Figure 5.5.7. The only difference among the three models is the direction of the arrowheads.

We now use the nullcline diagrams to study the three basic models. In particular, we want to know what happens to the solutions as time increases. The insight of Malthus that populations cannot grow unchecked leads us to require that, in any reasonable model, the solution curves be bounded.

The Basic Predator-Prey Model The nullcline diagram for the basic predator-prey model shows that the origin is a saddle point. This is so because the X axis is a solution that leaves the origin, while the Y axis is a solution that enters the origin. The nullcline diagram does not give enough information to classify the equilibrium point at (1, 1), nor does it guarantee that the solution curves are bounded. Consider, for example, the region $X, Y > 1$. The nullcline diagram shows that X is decreasing in this region while Y is increasing. It appears that the solution curves will cross over the line $X = 1$ and enter into a region where Y is decreasing as well as X. But this is not the only picture consistent with the nullcline diagram. It could be that X remains larger than 1 while Y increases without bound, as would happen if the point (1, 1) were a source.

Of course we have more information than just the nullcline diagram. We also have the computer-generated phase portrait for the case $k = 0.4$, as shown in Figure 5.4.4. This picture appears to confirm that the point (1, 1) is a center and that the solution curves are bounded. This is strong evidence, but it is limited to a single value of the parameter k. The solution of the differential equation for the trajectories as in Example 2 of Section 5.3, can be used to confirm that solutions are bounded.

One final issue to explore is the significance of the parameter k. Figure 5.5.8 shows the phase portraits for $k = 1$ and $k = 0.5$. The common nullcline diagram guarantees that these systems have similar phase portraits, although there are some quantitative differences.

The Basic Competition Model The nullcline diagram for the competition model clearly indicates that the equilibrium point (1, 1) is a saddle point, with the same argument as in Model Problem 5.5. The origin is unstable also. Both equilibrium solutions are saddles, so all the trajectories

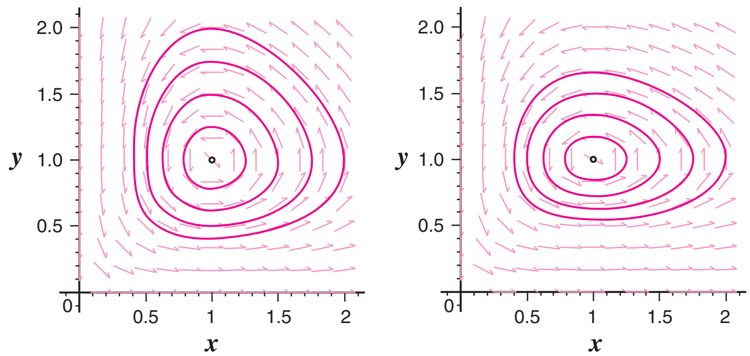


Figure 5.5.8
The phase portrait for the predator-prey model with $k = 0$ (left) and $k = 0.5$ (right).

must be unbounded. Whichever species loses the competition dies out, but then the other grows without bound. This model is too simple to be a useful model for competing species. It can be improved by adding logistic terms to the differential equations. Such models are considered in the exercises.

The Basic Cooperation Model The nullcline diagram for the cooperation model indicates that the point (1, 1) is again a saddle point. The origin, however, appears to be asymptotically stable. Solutions that begin with $X, Y < 1$ move inevitably toward the origin, indicating that the two species die out. Alternatively, solutions that begin with $X, Y > 1$ clearly increase without bound. As in the competition model, the cooperation model requires some additional features in order to work for all starting population levels. The model works only for starting levels too small to sustain the populations and for short-term behavior with larger starting populations.

A Technical Note

Phase portrait plots show good numerical approximations of some example trajectories. Nullclines indicate features that all trajectories in a region must have. Conclusions drawn from nullclines do not rest on numerical approximations, nor are they deduced from examples. They are demonstrated by evidence taken directly from the system of differential equations.

5.5 Exercises

For Exercises 1 through 8, sketch the nullclines for each variable separately, combine them to form a nullcline diagram, and draw what conclusions you can about the nature of the critical points. If there are any separatrices, indicate in which region(s) they lie in. Compare the results with a computer-generated phase portrait.

1. $x' = -x + y, \quad y' = x + y$ (Section 5.4, Exercise 5)
2. $x' = x - 2y, \quad y' = -2x$ (Section 5.4, Exercise 6)

3. $u' = 3u - 2v - 5, \quad v' = u + v - 5$
4. $x' = 3x - 2y, \quad y' = 2x - 2y$ (Section 5.4, Exercise 8)
5. $x' = x^2 - y, \quad y' = -x + y$ (Section 5.4, Exercise 11)
6. $x' = y, \quad y' = x - y - x^3$ (Section 5.4, Exercise 12)
7. $x' = 2y - x, \quad y' = x(2x + y - 5)$ (Section 5.4, Exercise 13)
8. $x' = x + y, \quad y' = 1 - y^2$ (Section 5.4, Exercise 14)
9. Prepare a nullcline diagram for the bubble growth equation

$$\frac{2}{3} r r'' + (r')^2 = 1 - \frac{1}{r},$$

from Section 5.4, Exercise 15. Just consider the half-plane $r \geq 0$ corresponding to the physical interpretation of the model. Discuss what the nullcline diagram says about the growth of bubbles of different initial size.

10. Consider the Lotka–Volterra predator-prey model

$$X' = X(1 - Y), \qquad Y' = kY(X - 1).$$

The first study done with this model involved a curious observation made in the 1920s. Italian fishermen at that time caught sharks as well as the more desirable fish. Prior to World War I, about 10 to 12% of the catch consisted of sharks. During World War I, the percentage of sharks rose to as high as 30%. By the mid-1920s the percentage was back down to about 15%. It was obvious that there was much less fishing during the war, but it was not at all clear why that should affect the proportions of the species.

Assume that the model applies to the populations of sharks and food fish during the war, when there was little or no commercial fishing. After the war, the resumption of fishing changes the model by adding an extra term to each equation. Assuming that the rate of fish caught is proportional to the population, the new model is

$$X' = X(1 - Y) - aX, \qquad Y' = kY(X - 1) - bkY,$$

where a and b are positive parameters of fairly small magnitude. Note that the rate of decrease of the shark population (Y) from fishing is written as $-bkY$ rather than $-bY$ so that the parameter k does not affect the nullclines.

a. Sketch the nullcline diagram for the revised model, and compare it with the predator-prey model from Figure 5.5.7. Does the Lotka–Volterra equation adequately explain the observations?

b. The reintroduction of fishing after World War I increased the ratio of food fish to sharks by a factor of 3. Assuming $a = b$, how large must these parameters be in order to change the equilibrium population ratio by this much?

c. The Lotka–Volterra model has also been applied to other problems. Consider a population of rabbits and coyotes, and suppose the local chicken farmers want to hunt coyotes to reduce losses of chickens. Use the Lotka–Volterra model with selective hunting of predators ($a = 0$). What does the model predict will happen to the coyote and rabbit populations if b is increased in an attempt to remove the coyotes? Does this model accurately predict the hunting of predators to extinction that has occurred in many areas?[18]

11. Perhaps the competitive population model would be improved by adding logistic growth. Consider the model

$$X' = X\left(1 - Y - \frac{X}{a}\right), \qquad Y' = Y\left(1 - X - \frac{Y}{b}\right),$$

where $a > 1$ and $b > 1$.

a. Prepare a nullcline diagram. What does the model predict will happen? Discuss whether this model is reasonable as a simple model for competing species.

b. Reconsider the model with $a < 1$ and $b < 1$. What does the model predict will happen? Does the model work for competing species in this case?

T 12. Suppose the Lotka–Volterra model is modified by assuming that the population of prey, in the absence of predators, is governed by a logistic growth model. With this change, the revised model is

$$X' = X\left(1 - \frac{X}{M} - Y\right), \qquad Y' = kY(X - 1).$$

a. Prepare a nullcline diagram for this model, assuming $M > 1$.

b. Plot the phase portrait, using $k = 1$ and $M = 5$.

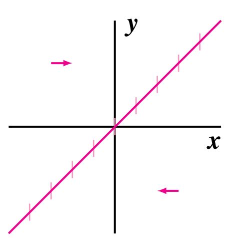
c. If we add hunting of predators to the model, we change the Y equation to

$$Y' = kY(X - 1 - b).$$

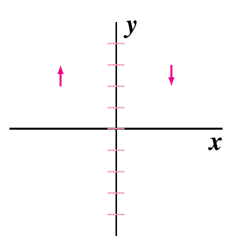
What does the model predict will happen to the coyote and rabbit populations if b is increased in an attempt to remove the coyotes? Compare with Exercise 10.

✦ 5.5 INSTANT EXERCISE SOLUTIONS

1.



x nullclines



y nullclines

[18]The failure of the Lotka–Volterra model for selective hunting of predators is not a refutation of the model. It is, rather, a warning that models must be fitted to the phenomena to be modeled. The model works well for the shark problem, but it is not sufficient for the coyote problem. It is up to the applied mathematician to determine, through thought experiments and examination of real data, whether a given model is appropriate for a given problem.

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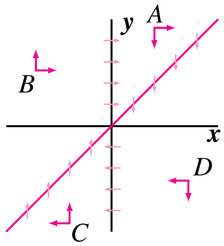
2.



3. No conclusions can be drawn without additional information.

CASE STUDY 5

A Self-Limiting Population

There are many situations in which population growth is limited by environmental factors caused by the population itself. A simpler example would be the population of microorganisms in a closed environment, such as the population of yeast in a commercial fermentation process. Like all other living creatures, yeast produces waste products. In a closed environment, the waste products accumulate, making the environment less hospitable to the yeast.

The Model

We begin with a conceptual model for the problem. Our conceptual model includes a number of assumptions:

1. The microorganisms are distributed uniformly throughout a closed container of fixed volume. Otherwise we would have to consider spatial variations as well as evolution in time.
2. The only relevant quantities in the model are the population $p(t)$ of microorganisms and the amount $w(t)$ of toxic waste material. We are ignoring the possibility that the system could be affected by such things as temperature changes.
3. The rate of waste production is proportional to the population, with rate constant k.
4. In the absence of waste, the population would undergo logistic growth with maximum population M and initial relative growth rate r.
5. The effect of the waste is to kill microorganisms at some rate dependent on both the population and the amount of waste. A reasonable guess is that the death rate relative to the population should be proportional to the amount of waste, with b the constant of proportionality. Intuitively, it makes sense that doubling the amount of waste should double the death rate.
6. Initially, there is a population p_0 that is relatively small compared with the capacity M, and there is no waste.

From these assumptions, we get a pair of differential equations, with initial conditions, for w and p:

$$\frac{dw}{dt} = kp, \qquad w(0) = 0, \tag{1}$$

$$\frac{dp}{dt} = rp\left(1 - \frac{p}{M}\right) - bpw, \qquad p(0) = p_0. \tag{2}$$

The first differential equation follows from assumption 3, while the second is a combination of assumptions 4 and 5.

In its current form, the model has five parameters: the rate constants k, r, and b; the population capacity M; and the initial population p_0. It is very difficult to understand a model with this many parameters. Following the procedure of Section 5.1, we can convert the model to dimensionless form. Using the dimensionless variables

$$P = \frac{p}{M}, \qquad W = \frac{bw}{r}, \qquad \tau = rt,$$

the model takes a simpler form (see Exercise 1).

Self-Limiting Population—Dimensionless Model

Determine the behavior of the system

$$W' = KP, \qquad W(0) = 0, \tag{3}$$

$$P' = P(1 - P - W), \qquad P(0) = P_0, \tag{4}$$

where $K > 0$ is a dimensionless parameter. Of particular interest is the behavior of the system when P_0 is very small.

The dimensionless version of the model has only two parameters, one of which is considerably restricted in its possible values. In this form, the model is much easier to analyze.

Qualitative Analysis

The nullclines for the dimensionless system are obtained in the usual way, by setting each of the derivatives equal to zero. The line

$$P = 0$$

is both a W nullcline and a P nullcline. This means that all points on the W axis are critical points. When the critical points are connected like this, we cannot think of stability in the usual way. The standard concept of stability applies only to isolated critical points. The only other nullcline is the line

$$P + W = 1,$$

which is a P nullcline. The full nullcline diagram appears in Figure C5.1, including only the realistic portion in the first quadrant. Note that neither K nor P_0 appears in the equations for the nullclines; thus, the model has a unique nullcline diagram. The dots along the W axis indicate that all the points on that line are critical points.

The nullcline diagram gives us the general trend of trajectories. All trajectories progress to the right, so we can obtain all of them by considering solutions that begin on the P axis. Those beginning above $P = 1$ move down and to the right, corresponding to decreasing population and

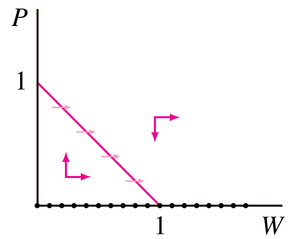


Figure C5.1
The nullcline diagram for $W' = KP$, $P' = P(1 - P - W)$.

increasing waste right from the beginning. Those beginning below $P = 1$ move up and to the right until they reach the line $P + W = 1$, after which they also move down and to the right. The waste increases in all cases. If the starting population is small, it will increase for a time, but eventually it reaches a maximum value and decreases thereafter. The population must approach zero as $t \to \infty$, but it is not clear where on the W axis the trajectories terminate.

The Differential Equation for the Trajectories

Following the usual procedure, we can write the differential equation for the trajectories as

$$\frac{dP}{dW} = \frac{1 - P - W}{K}. \tag{5}$$

This equation is not separable, but it can be solved by a simple change of variables. Let $Z(W)$ be defined by

$$Z = P + W.$$

It can be shown (see Exercise 2) that Z satisfies the differential equation

$$\frac{dZ}{dW} = \frac{1 + K - Z}{K}. \tag{6}$$

In the form (6), the differential equation for the trajectories is autonomous and therefore separable. Before we write the solution of the equation, it is well to note that we can use the phase line to study the autonomous equation for the trajectories. The phase line sketch appears in Figure C5.2.

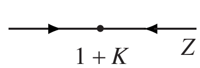


Figure C5.2
The phase line for $dZ/dW = (1 + K - Z)/K$.

If the independent variable were time, the implication of the phase line sketch would be that Z approaches $1 + K$ as time increases without bound. However, the variable W does not increase to infinity, so the best we can say is that Z always changes toward $1 + K$. This is useful information for the trajectories in the WP plane. Suppose Z is initially less than $1 + K$. For the entire life of the population, Z must then remain less than $1 + K$, as indicated by the phase line sketch. Translated to the actual variables of the problem, it means that any trajectories beginning at a point $(0, P_0)$, with $P_0 < 1 + K$, must always satisfy

$$P + W < 1 + K. \tag{7}$$

This result also tells us something about the ultimate fate of the population. The waste W continues to increase, but $P + W$ is bounded by $1 + K$. Eventually, P reaches 0, at which point we have

$$W < 1 + K.$$

Those trajectories that begin at $(0, P_0)$, with $P_0 < 1 + K$ intersect the W axis at a point to the left of $1 + K$.

A Formula for the Trajectories

The solution of the differential equation for the trajectories is

$$Z = 1 + K + Ae^{-W/K}.$$

Since $Z = P + W$, this gives us the result

$$P = 1 + K - W + Ae^{-W/K}. \tag{8}$$

The phase portrait, unlike the nullclines, does depend on the value of the parameter K. We continue now with $K = 0.5$ chosen as an example. With this value for K, the equation for the trajectories is

$$P = 1.5 - W + Ae^{-2W}. \tag{9}$$

The phase portrait appears in Figure C5.3 along with the limiting line $P + W = 1.5$ and the nullcline $P + W = 1$.

Discussion

The problem statement was deliberately open-ended, with the instructions being to "determine the behavior of the system." The analysis used a variety of tools, including nullclines and the differential equation for the trajectories. Analysis of the latter equation required the phase line and symbolic quantitative techniques to obtain the formula for the trajectories. Using a variety of methods has the obvious advantage of giving the largest number of results, but it also has the less obvious advantage of confirming results by consistency. Observe in the phase portrait that the trajectories do indeed cross the nullcline with horizontal slope, and they are indeed bounded above by the inequality $P + W < 1 + K$.

Note that we never solved the original problem of determining the population and waste as functions of time. However, by making use of the autonomy of the model, we have obtained a deep understanding of the history of the self-limiting population represented by the model. The only thing we don't know from our investigation is how rapidly the predicted changes occur.

We might also wonder about the implication of our study for real populations, such as the population of humans on earth. Fortunately, this study says little about the human population. While we create wastes that harm our environment, we live in an environment that is large enough

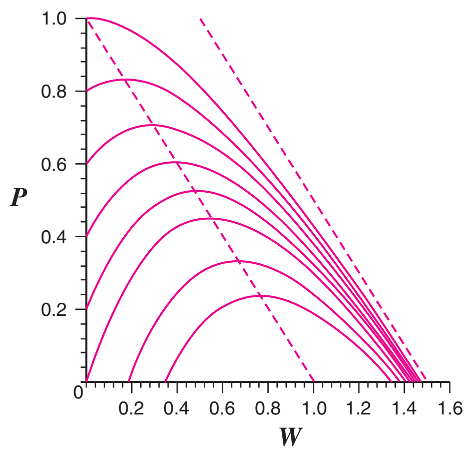


Figure C5.3
The phase portrait for the dimensionless model with $K = 0.5$, along with the nullcline $P + W = 1$ (dashed) and the limiting line $P + W = 1 + K$ (dotted).

and rich enough to have mechanisms for eliminating waste. A more realistic study of human populations would have to include a natural mechanism that decreases the waste.

Case Study 5 Exercises

1. Nondimensionalize the models (1) and (2), using dimensionless variables

$$P = \frac{p}{M}, \qquad W = \frac{bw}{r}, \qquad \tau = rt.$$

Define dimensionless parameters K and P_0 so that you get the dimensionless models (3) and (4). What does K represent?

2. Derive Equation (6) for $Z(W)$.

3. Derive Equation (8) for the trajectories in the WP plane by solving the differential equation (6) for the trajectories in the WZ plane. Note that the equation can be written in the same form as that for Newton's law of cooling from Section 1.1.

T 4. Complete the phase portrait of Figure C5.3 with some trajectories above the line $P + W = 1 + K$.

T 5. Prepare phase portraits with $K = 0.25$ and $K = 1$. Discuss the significance of the parameter K.

6. Results obtained with one tool often suggest investigations to be done with another. For example, we might be interested in knowing the largest population value achieved when the starting population is small. This point can be found in the phase portrait as the intersection of the nullcline with the trajectory that passes through the origin. It can also be found symbolically. Let $K = 0.5$. Set the formula for the trajectory equal to that for the nullcline. Find the waste level at which the maximum population occurs and the corresponding population. How does

the actual maximum population for this case compare with the theoretical limit (that where there is no waste)?

7. Suppose we alter the model to allow for the gradual decrease of waste due to natural environmental renewal. This change results in a new dimensionless model

$$W' = KP - QKW, \qquad P' = P(1 - P - W).$$

a. Prepare a nullcline diagram for this model. Can any conclusions be drawn from the nullcline sketch regarding the correct classification of the equilibrium points? Does it matter what values are chosen for K and Q?

b. Take $K = 0.5$ and $Q = 0.5$. Plot the phase portrait for the system. Describe what happens to populations of various initial sizes, assuming there is initially no waste.