4.1 GRAPHING LINES IN THE COORDINATE PLANE

In this section

- Ordered Pairs
- The Rectangular Coordinate System
- Plotting Points
- Graphing a Linear Equation
- Graphing a Line Using Intercepts
- Applications

In Chapter 1 you learned to graph numbers on a number line. We also used number lines to illustrate the solution to inequalities in Chapter 3. In this section you will learn to graph pairs of numbers in a coordinate system made up of a pair of number lines. We will use this coordinate system to illustrate the solution to equations and inequalities in two variables.

Ordered Pairs

The equation $y = 2x - 1$ is an equation in two variables. This equation is satisfied if we choose a value for x and a value for y that make it true. If we choose $x = 2$ and $y = 3$, then $y = 2x - 1$ becomes

$$\underset{\uparrow y}{3} = 2(\underset{\uparrow x}{2}) - 1.$$

$$3 = 3$$

Because the last statement is true, we say that the pair of numbers 2 and 3 **satisfies the equation** or is a **solution to the equation.** We use the **ordered pair** (2, 3) to represent $x = 2$ and $y = 3$. The format is always to write the value for x first and the value for y second. The first number of the ordered pair is called the ***x*-coordinate,** and the second number is called the ***y*-coordinate.** Note that the ordered pair (3, 2) does not satisfy the equation $y = 2x - 1$, because for $x = 3$ and $y = 2$ we have

$$2 \neq 2(3) - 1.$$

The ordered pair (2, 3) is a solution to $y = 2x - 1$. We can find as many solutions as we please by simply choosing any value for x or y and then using the equation to find the other coordinate of the ordered pair.

helpful hint

In this chapter you will be doing a lot of graphing. Using graph paper will help you understand the concepts and help you recognize errors. For your convenience, a page of graph paper can be found at the end of this chapter. Make as many copies of it as you wish.

EXAMPLE 1 **Finding solutions to an equation**

Each of the ordered pairs below is missing one coordinate. Complete each ordered pair so that it satisfies the equation $y = -3x + 4$.

a) (2,) **b)** (, −5) **c)** (0,)

Solution

a) The x-coordinate of (2,) is 2. Let $x = 2$ in the equation $y = -3x + 4$:

$$\begin{aligned} y &= -3 \cdot 2 + 4 \\ &= -6 + 4 \\ &= -2 \end{aligned}$$

The ordered pair (2, −2) satisfies the equation.

b) The y-coordinate of (, −5) is −5. Let $y = -5$ in the equation $y = -3x + 4$:

$$\begin{aligned} -5 &= -3x + 4 \\ -9 &= -3x \\ 3 &= x \end{aligned}$$

The ordered pair (3, −5) satisfies the equation.

c) Replace x by 0 in the equation $y = -3x + 4$:

$$\begin{aligned} y &= -3 \cdot 0 + 4 \\ &= 4 \end{aligned}$$

So (0, 4) satisfies the equation.

The Rectangular Coordinate System

We use the **rectangular** (or **Cartesian**) **coordinate system** to get a visual image of ordered pairs of real numbers. The rectangular coordinate system consists of two number lines drawn at a right angle to one another, intersecting at zero on each number line, as shown in Fig. 4.1. The plane containing these number lines is called the **coordinate plane.** On the horizontal number line the positive numbers are to the right of zero, and on the vertical number line the positive numbers are above zero.

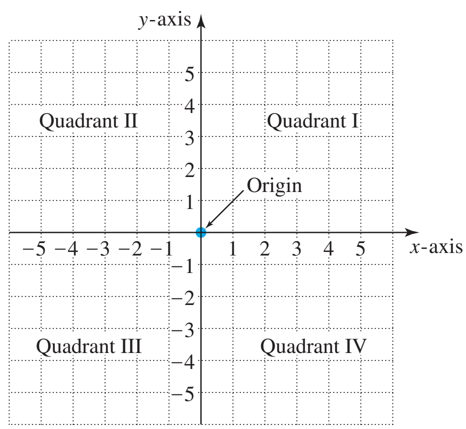


FIGURE 4.1

The horizontal number line is called the ***x*-axis,** and the vertical number line is called the ***y*-axis.** The point at which they intersect is called the **origin.** The two number lines divide the plane into four regions called **quadrants.** They are numbered as shown in Fig. 4.1. The quadrants do not include any points on the axes.

Plotting Points

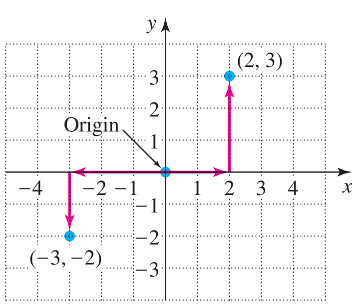


FIGURE 4.2

Just as every real number corresponds to a point on the number line, *every pair of real numbers corresponds to a point in the rectangular coordinate system.* For example, the point corresponding to the pair (2, 3) is found by starting at the origin and moving two units to the right and then three units up. The point corresponding to the pair $(-3, -2)$ is found by starting at the origin and moving three units to the left and then two units down. Both of these points are shown in Fig. 4.2.

When we locate a point in the rectangular coordinate system, we are **plotting** or **graphing** the point. Because ordered pairs of numbers correspond to points in the coordinate plane, we frequently refer to an ordered pair as a point.

EXAMPLE 2 **Plotting points**

Plot the points (2, 5), (−1, 4), (−3, −4), and (3, −2).

Solution

To locate (2, 5), start at the origin, move two units to the right, and then move up five units. To locate (−1, 4), start at the origin, move one unit to the left, and then move up four units. All four points are shown in Fig. 4.3. ■

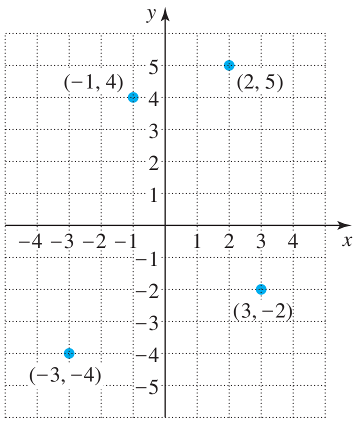
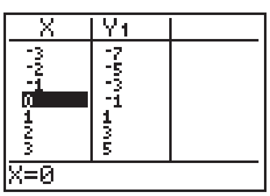


FIGURE 4.3

calculator close-up

You can make a table of values for x and y with a graphing calculator. Enter the equation $y = 2x - 1$ using Y= and then press TABLE.



Graphing a Linear Equation

The **graph** of an equation is an illustration in the coordinate plane that shows all of the ordered pairs that satisfy the equation. When we draw the graph, we are **graphing the equation.**

Consider again the equation $y = 2x - 1$. The following table shows some ordered pairs that satisfy this equation.

x	−3	−2	−1	0	1	2	3
$y = 2x - 1$	−7	−5	−3	−1	1	3	5

The ordered pairs in this table are graphed in Fig. 4.4. Notice that the points lie in a straight line. If we choose any real number for x and find the point (x, y) that satisfies $y = 2x - 1$, we get another point along this line. Likewise, any point along this line satisfies the equation. So the graph of $y = 2x - 1$ is the straight line shown in Fig. 4.5. Because it is not possible to actually show all of the line, the arrows on the ends of the line indicate that it goes on indefinitely in both directions. The equation $y = 2x - 1$ is an example of a linear equation in two variables.

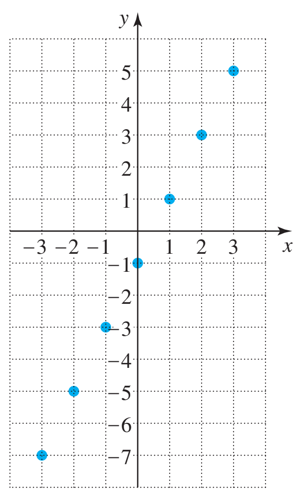


FIGURE 4.4

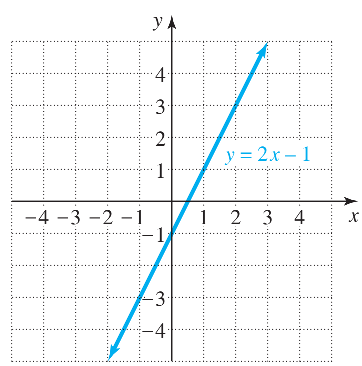


FIGURE 4.5

Linear Equation in Two Variables

A **linear equation in two variables** is an equation that can be written in the form

$$Ax + By = C,$$

where A, B, and C are real numbers, with A and B not both equal to zero.

Equations such as

$$x - y = 5, \quad y = 2x + 3, \quad 2x - 5y - 9 = 0, \quad \text{and} \quad x = 8$$

are linear equations because they could all be rewritten in the form $Ax + By = C$. The graph of any linear equation is a straight line.

EXAMPLE 3

Graphing an equation

Graph the equation $3x + y = 2$. Plot at least five points.

Solution

It is easier to make a table of ordered pairs if the equation is solved for y:

$$y = -3x + 2$$

Next, arbitrarily select values for x and then calculate the corresponding value for y. The following table of values shows five ordered pairs that satisfy the equation:

x	-2	-1	0	1	2
$y = -3x + 2$	8	5	2	-1	-4

The graph of the line through these points is shown in Fig. 4.6.

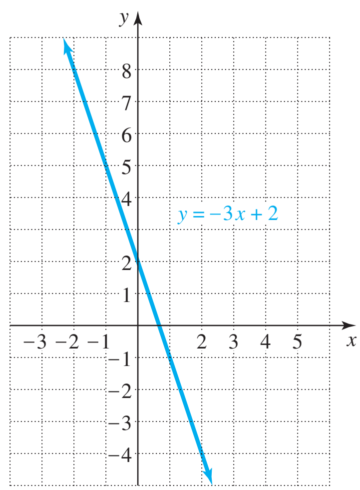
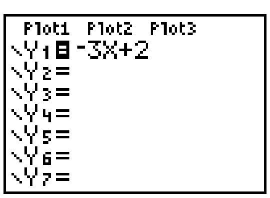


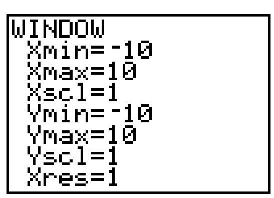
FIGURE 4.6

calculator close-up

To graph $y = -3x + 2$, enter the equation using the Y= key:

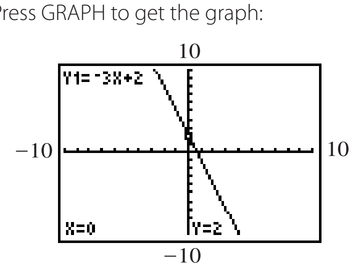


Next, set the viewing window (WINDOW) to get the desired view of the graph. Xmin and Xmax indicate the minimum and maximum x-values used for the graph; likewise for Ymin and Ymax. Xscl and Yscl



(scale) give the distance between tick marks on the respective axes.

Press GRAPH to get the graph:



Even though the graph is not really "straight," it is consistent with the graph of $y = -3x + 2$ in Fig. 4.6.

EXAMPLE 4

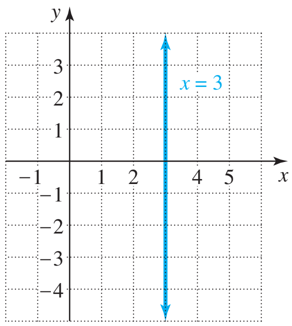


FIGURE 4.7

A vertical line

Graph the equation $x + 0 \cdot y = 3$. Plot at least five points.

Solution

If we choose a value of 3 for x, then we can choose any number for y, since y is multiplied by 0. The equation $x + 0 \cdot y = 3$ is usually written simply as $x = 3$. The following table shows five ordered pairs that satisfy the equation:

$x = 3$	3	3	3	3	3
y	−2	−1	0	1	2

Figure 4.7 shows the line through these points. ■

calculator close-up

You cannot graph the vertical line $x = 3$ on most graphing calculators. The only equations that can be graphed are ones in which y is written in terms of x.

CAUTION If an equation such as $x = 3$ is discussed in the context of equations in two variables, then we assume that it is a simplified form of $x + 0 \cdot y = 3$, and there are infinitely many ordered pairs that satisfy the equation. If the equation $x = 3$ is discussed in the context of equations in a single variable, then $x = 3$ has only one solution, 3.

All of the equations we have considered so far have involved single-digit numbers. If an equation involves large numbers, then we must change the scale on the x-axis, the y-axis, or both to accommodate the numbers involved. The change of scale is arbitrary, and the graph will look different for different scales.

EXAMPLE 5

Adjusting the scale

Graph the equation $y = 20x + 500$. Plot at least five points.

Solution

The following table shows five ordered pairs that satisfy the equation.

x	−20	−10	0	10	20
$y = 20x + 500$	100	300	500	700	900

To fit these points onto a graph, we change the scale on the x-axis to let each division represent 10 units and change the scale on the y-axis to let each division represent 200 units. The graph is shown in Fig. 4.8.

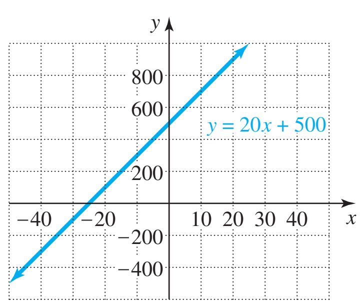


FIGURE 4.8 ■

Graphing a Line Using Intercepts

We know that the graph of a linear equation is a straight line. Because it takes only two points to determine a line, we can graph a linear equation using only two points. The two points that are the easiest to locate are usually the points where the line crosses the axes. The point where the graph crosses the x-axis is the ***x*-intercept,** and the point where the graph crosses the y-axis is the ***y*-intercept.**

EXAMPLE 6

Graphing a line using intercepts

Graph the equation $2x - 3y = 6$ by using the x- and y-intercepts.

helpful hint

You can find the intercepts for $2x - 3y = 6$ using the *cover-up method.* Cover up $-3y$ with your pencil, then solve $2x = 6$ mentally to get $x = 3$ and an x-intercept of (3, 0). Now cover up $2x$ and solve $-3y = 6$ to get $y = -2$ and a y-intercept of (0, −2).

Solution

To find the x-intercept, let $y = 0$ in the equation $2x - 3y = 6$:

$$2x - 3 \cdot 0 = 6$$
$$2x = 6$$
$$x = 3$$

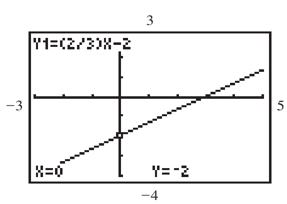
The x-intercept is (3, 0). To find the y-intercept, let $x = 0$ in $2x - 3y = 6$:

$$2 \cdot 0 - 3y = 6$$
$$-3y = 6$$
$$y = -2$$

The y-intercept is (0, −2). Locate the intercepts and draw a line through them as shown in Fig. 4.9. To check, find one additional point that satisfies the equation, say (6, 2), and see whether the line goes through that point.

calculator close-up

To check the result in Example 6, graph $y = (2/3)x - 2$:



Since the calculator graph appears to be the same as the graph in Fig. 4.9, it supports the conclusion that Fig. 4.9 is correct.

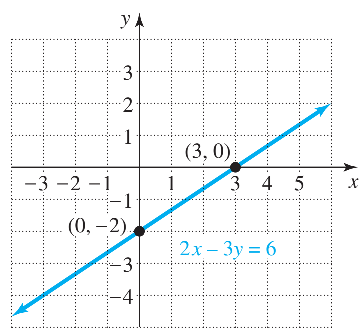


FIGURE 4.9

Applications

Linear equations occur in many real-life situations. If the cost of plans for a house is \$475 for one copy plus \$30 for each additional copy, then $C = 475 + 30x$, where x is the number of additional copies. If you have \$1000 budgeted for landscaping with trees at \$50 each and bushes at \$20 each, then $50t + 20b = 1000$, where t is the number of trees and b is the number of bushes. In the next example we see a linear equation that models ticket demand.

EXAMPLE 7

Ticket demand

The demand for tickets to see the Ice Gators play hockey can be modeled by the equation $d = 8000 - 100p$, where d is the number of tickets sold and p is the price per ticket in dollars.

a) How many tickets will be sold at $20 per ticket?

b) Find the intercepts and interpret them.

c) Graph the linear equation.

d) What happens to the demand as the price increases?

Solution

a) If tickets are $20 each, then $d = 8000 - 100 \cdot 20 = 6000$. So at $20 per ticket, the demand will be 6000 tickets.

b) Replace d with 0 in the equation $d = 8000 - 100p$ and solve for p:

$$0 = 8000 - 100p$$
$$100p = 8000 \quad \text{Add } 100p \text{ to each side.}$$
$$p = 80 \quad \text{Divide each side by 100.}$$

If $p = 0$, then $d = 8000 - 100 \cdot 0 = 8000$. So the intercepts are (0, 8000) and (80, 0). If the tickets are free, the demand will be 8000 tickets. At $80 per ticket, no tickets will be sold.

c) Graph the line using the intercepts (0, 8000) and (80, 0) as shown in Fig. 4.10.

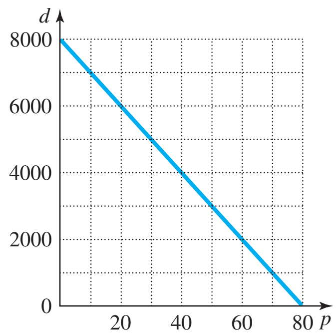


FIGURE 4.10

d) When the tickets are free, the demand is high. As the price increases, the demand goes down. At $80 per ticket, there will be no demand.

MATH AT WORK

$x^2 + (x + 1)^2 = 5^2$

BIOMECHANIST

Christopher J. Edington, manager of the Biomechanics Laboratory at Converse, Inc., is a specialist in studying human movements from the hip down. In the past he has worked with diabetics, helping to educate them about the role of shoes and stress points in the shoes and their relationship to preventing foot injuries. More recently, he has helped to design and run tests for Converse's new athletic and leisure shoes. The latest development is a new basketball shoe that combines the lightweight characteristic of a running shoe with the support and durability of a standard basketball sneaker.

Information on how the foot strikes the ground, the length of contact time, and movements of the foot, knee, and hip can be recorded by using high-speed video equipment. This information is then used to evaluate the performance and design requirements of a lightweight, flexible, and well-fitting shoe. To meet the requirements of a good basketball shoe, Mr. Edington helped design and test the "React" shock-absorbing technology that is in Converse's latest sneakers.

In Exercise 84 of this section you will see the motion of a runner's heel as Mr. Edington does.

WARM-UPS

True or false? Explain your answer.

1. The point (2, 4) satisfies the equation $2y - 3x = -8$.
2. If (1, 5) satisfies an equation, then (5, 1) also satisfies the equation.
3. The origin is in quadrant I.
4. The point (4, 0) is on the y-axis.
5. The graph of $x + 0 \cdot y = 9$ is the same as the graph of $x = 9$.
6. The graph of $x = -5$ is a vertical line.
7. The graph of $0 \cdot x + y = 6$ is a horizontal line.
8. The y-intercept for the line $x + 2y = 5$ is (5, 0).
9. The point $(5, -3)$ is in quadrant II.
10. The point $(-349, 0)$ is on the x-axis.

4.1 EXERCISES

Reading and Writing *After reading this section, write out the answers to these questions. Use complete sentences.*

1. What is an ordered pair?
2. What is the rectangular coordinate system?
3. What name is given to the point of intersection of the x-axis and the y-axis?
4. What is the graph of an equation?
5. What is a linear equation in two variables?
6. What are intercepts?

Complete each ordered pair so that it satisfies the given equation. See Example 1.

7. $y = 3x + 9$: (0,), (, 24), (2,)
8. $y = 2x + 5$: (8,), (−1,), (, −1)
9. $y = -3x - 7$: (0,), (−4,), (, −1)
10. $y = -5x - 3$: (, 2), (−3,), (0,)
11. $y = -12x + 5$: (0,), (10,), (, 17)
12. $y = 18x + 200$: (1,), (−10,), (, 200)
13. $2x - 3y = 6$: (3,), (, −2), (12,)
14. $3x + 5y = 0$: (−5,), (, −3), (10,)
15. $0 \cdot y + x = 5$: (, −3), (, 5), (, 0)
16. $0 \cdot x + y = -6$: (3,), (−1,), (4,)

Plot the points on a rectangular coordinate system. See Example 2.

17. (1, 5)
18. (4, 3)
19. (−2, 1)
20. (−3, 5)
21. $\left(3, -\frac{1}{2}\right)$
22. $\left(2, -\frac{1}{3}\right)$
23. (−2, −4)
24. (−3, −5)
25. (0, 3)
26. (0, −2)
27. (−3, 0)
28. (5, 0)
29. $(\pi, 1)$
30. $(-2, \pi)$
31. (1.4, 4)
32. (−3, 0.4)

Graph each equation. Plot at least five points for each equation. Use graph paper. See Examples 3 and 4. If you have a graphing calculator, use it to check your graphs.

33. $y = x + 1$

34. $y = x - 1$

35. $y = 2x + 1$

36. $y = 3x - 1$

37. $y = 3x - 2$

38. $y = 2x + 3$

39. $y = x$

40. $y = -x$

41. $y = 1 - x$

42. $y = 2 - x$

43. $y = -2x + 3$

44. $y = -3x + 2$

45. $y = -3$

46. $y = 2$

47. $x = 2$

48. $x = -4$

49. $2x + y = 5$

50. $3x + y = 5$

51. $x + 2y = 4$

52. $x - 2y = 6$

53. $x - 3y = 6$

54. $x + 4y = 5$

55. $y = 0.36x + 0.4$

56. $y = 0.27x - 0.42$

For each point, name the quadrant in which it lies or the axis on which it lies.

57. $(-3, 45)$

58. $(-33, 47)$

59. $(-3, 0)$

60. $(0, -9)$

61. $(-2.36, -5)$

62. $(89.6, 0)$

63. $(3.4, 8.8)$

64. $(4.1, 44)$

65. $\left(-\frac{1}{2}, 50\right)$

66. $\left(-6, -\frac{1}{2}\right)$

67. $(0, -99)$

68. $(\pi, 0)$

Graph each equation. Plot at least five points for each equation. Use graph paper. See Example 5. If you have a graphing calculator, use it to check your graphs.

69. $y = x + 1200$

70. $y = 2x - 3000$

71. $y = 50x - 2000$

72. $y = -300x + 4500$

73. $y = -400x + 2000$

74. $y = 500x + 3$

Graph each equation using the x- and y-intercepts. See Example 6. Use a third point to check.

75. $3x + 2y = 6$

76. $2x + y = 6$

77. $x - 4y = 4$

78. $-2x + y = 4$

79. $y = \frac{3}{4}x - 9$

80. $y = -\frac{1}{2}x + 5$

81. $\frac{1}{2}x + \frac{1}{4}y = 1$

82. $\frac{1}{3}x - \frac{1}{2}y = 3$

Solve each problem. See Example 7.

83. ***Percentage of full benefit.*** The age at which you retire affects your Social Security benefits. The accompanying graph gives the percentage of full benefit for each age from 62 through 70, based on current legislation and retirement after the year 2005 (Source: Social Security Administration). What percentage of full benefit does a person receive if that person retires at age 63? At what age will a retiree receive the full benefit? For what ages do you receive more than the full benefit?

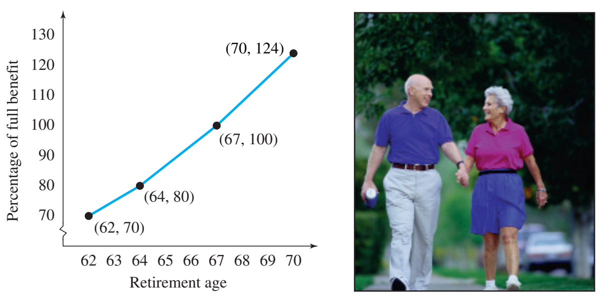


FIGURE FOR EXERCISE 83

84. ***Heel motion.*** When designing running shoes, Chris Edington studies the motion of a runner's foot. The following data gives the coordinates of the heel (in centimeters) at intervals of 0.05 millisecond during one cycle of level treadmill running at 3.8 meters per second (*Sagittal Plane Kinematics, Milliron and Cavanagh*):

(31.7, 5.7), (48.0, 5.7), (68.3, 5.8), (88.9, 6.9),
(107.2, 13.3), (119.4, 24.7), (127.2, 37.8),
(125.7, 52.0), (116.1, 60.2), (102.2, 59.5)
(88.7, 50.2), (73.9, 35.8), (52.6, 20.6),
(29.6, 10.7), (22.4, 5.9).

Graph these ordered pairs to see the heel motion.

85. ***Medicaid spending.*** The payment in billions by Medicaid (health care for the poor) can be modeled by the equation

$$P = 12.6n + 81.3,$$

where n is the number of years since 1990 (Health Care Financing Administration, www.hcfa.gov).

a) What amount was paid out by Medicaid in 1995?
b) In what year will the payment reach \$220 billion?
c) Graph the equation for n ranging from 0 through 20.

86. ***Women on the board.*** The percentage of companies with at least one woman on the board is growing steadily (*Forbes,* February 10, 1997). The percentage can be approximated with the linear equation

$$p = 2n + 36,$$

where n is the number of years since 1980.

a) Find and interpret the p-intercept for the line.
b) Find and interpret the n-intercept.
c) Graph the line for n ranging from 0 through 20.
d) If this trend continues, then in what year would you expect to find nearly all companies having at least one woman on the board?

87. ***Hazards of depth.*** The table on page 169 shows the depth below sea level and atmospheric pressure (*Encyclopedia of Sports Science,* 1997). The equation

$$A = 0.03d + 1$$

expresses the atmospheric pressure in terms of the depth d.

a) Find the atmospheric pressure at the depth where nitrogen narcosis begins.
b) Find the maximum depth for intermediate divers.
c) Graph the equation for d ranging from 0 to 250 feet.

Depth (ft)	Atmospheric pressure (atm)	Comments
21	1.63	Bends are a danger
60	2.8	Maximum for beginners
100		Nitrogen narcosis begins
	4.9	Maximum for intermediate
200	7.0	Severe nitrogen narcosis
250	8.5	Extremely dangerous depth

FIGURE FOR EXERCISE 87

88. ***Demand equation.*** Helen's Health Foods usually sells 400 cans of ProPac Muscle Punch per week when the price is \$5 per can. After experimenting with prices for some time, Helen has determined that the weekly demand can be found by using the equation

$$d = 600 - 40p,$$

where d is the number of cans and p is the price per can.

a) Will Helen sell more or less Muscle Punch if she raises her price from \$5?

b) What happens to her sales every time she raises her price by \$1?

c) Graph the equation.

d) What is the maximum price that she can charge and still sell at least one can?

89. ***Advertising blitz.*** Furniture City in Toronto had \$24,000 to spend on advertising a year-end clearance sale. A 30-second radio ad costs \$300, and a 30-second local television ad costs \$400. To model this situation, the advertising manager wrote the equation $300x + 400y = 24{,}000$. What do x and y represent? Graph the equation. How many solutions are there to the equation, given that the number of ads of each type must be a whole number?

90. ***Material allocation.*** A tent maker had 4500 square yards of nylon tent material available. It takes 45 square yards of nylon to make an 8×10 tent and 50 square yards to make a 9×12 tent. To model this situation, the manager wrote the equation $45x + 50y = 4500$. What do x and y represent? Graph the equation. How many solutions are there to the equation, given that the number of tents of each type must be a whole number?

GRAPHING CALCULATOR EXERCISES

Graph each straight line on your graphing calculator using a viewing window that shows both intercepts. Answers may vary.

91. $2x + 3y = 1200$

92. $3x - 700y = 2100$

93. $200x - 300y = 6$

95. $y = 300x - 1$

94. $300x + 5y = 20$

96. $y = 300x - 6000$

4.2 SLOPE

In this section

- Slope Concepts
- Slope Using Coordinates
- Graphing a Line Given a Point and Its Slope
- Parallel Lines
- Perpendicular Lines
- Interpreting Slope

In Section 4.1 you learned that the graph of a linear equation is a straight line. In this section, we will continue our study of lines in the coordinate plane.

Slope Concepts

If a highway rises 6 feet in a horizontal run of 100 feet, then the grade is $\frac{6}{100}$ or 6%. See Fig. 4.11. The grade of a road is a measurement of the steepness of the road. It is the rate at which the road is going upward.

The steepness of a line is called the **slope** of the line and it is measured like the grade of a road. As you move from (1, 1) to (4, 3) in Fig. 4.12 on page 171 the x-coordinate increases by 3 and the y-coordinate increases by 2. The line rises 2 units in a horizontal run of 3 units. So the slope of the line is $\frac{2}{3}$. The slope is the rate at which the y-coordinate is increasing. It increases 2 units for every 3-unit increase in x or it increases $\frac{2}{3}$ of a unit for every 1-unit increase in x. In general, we have the following definition of slope.

FIGURE 4.11

Slope

$$\text{Slope} = \frac{\text{change in } y\text{-coordinate}}{\text{change in } x\text{-coordinate}}$$