**e)** What is the approximate maximum revenue?

**f)** Use the accompanying graph to estimate the price at which the revenue is zero.

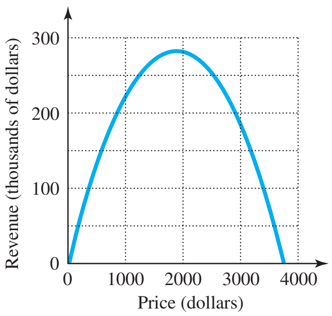


FIGURE FOR EXERCISE 90

**91.** ***Volume of a tank.*** The volume of a fish tank with a square base and height $y$ is $y^3 - 6y^2 + 9y$ cubic inches. Find the length of a side of the square base.

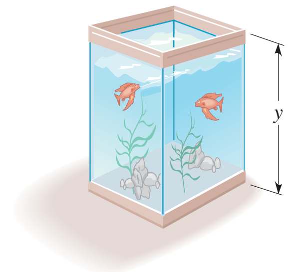


FIGURE FOR EXERCISE 91

## GETTING MORE INVOLVED

**92.** ***Discussion.*** For what real number $k$, does $3x^2 - k$ factor as $3(x - 2)(x + 2)$?

**93.** ***Writing.*** Explain in your own words how to factor a four-term polynomial by grouping.

**94.** ***Writing.*** Explain how you know that $x^2 + 1$ is a prime polynomial.

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# 6.3 FACTORING $ax^2 + bx + c$ WITH $a = 1$

**In this section**

- Factoring $ax^2 + bx + c$ with $a = 1$
- Prime Polynomials
- Factoring with Two Variables
- Factoring Completely

In this section we will factor the type of trinomials that result from multiplying two different binomials. We will do this only for trinomials in which the coefficient of $x^2$, the leading coefficient, is 1. Factoring trinomials with leading coefficient not equal to 1 will be done in Section 6.4.

## Factoring $ax^2 + bx + c$ with $a = 1$

Let's look closely at an example of finding the product of two binomials using the distributive property:

$$\begin{aligned}(x + 2)(x + 3) &= (x + 2)x + (x + 2)3 && \text{Distributive property}\\ &= x^2 + 2x + 3x + 6 && \text{Distributive property}\\ &= x^2 + 5x + 6 && \text{Combine like terms.}\end{aligned}$$

To factor $x^2 + 5x + 6$, we reverse these steps as shown in our first example.

**EXAMPLE 1** **Factoring a trinomial**

Factor.

**a)** $x^2 + 5x + 6$ **b)** $x^2 + 8x + 12$ **c)** $a^2 - 9a + 20$

### Solution

**a)** The coefficient 5 is the sum of two numbers that have a product of 6. The only integers that have a product of 6 and a sum of 5 are 2 and 3. So write $5x$ as $2x + 3x$, then factor by grouping:

$$\begin{aligned}x^2 + 5x + 6 &= x^2 + 2x + 3x + 6 && \text{Replace } 5x \text{ by } 2x + 3x.\\ &= (x^2 + 2x) + (3x + 6) && \text{Group terms together.}\\ &= x(x + 2) + 3(x + 2) && \text{Factor out common factors.}\\ &= (x + 2)(x + 3) && \text{Factor out } x + 2.\end{aligned}$$

**b)** To factor $x^2 + 8x + 12$, we must find two integers that have a product of 12 and a sum of 8. The pairs of integers with a product of 12 are 1 and 12, 2 and 6, and 3 and 4. Only 2 and 6 have a sum of 8. So write $8x$ as $2x + 6x$ and factor by grouping:

$$\begin{aligned} x^2 + 8x + 12 &= x^2 + 2x + 6x + 12 \\ &= (x + 2)x + (x + 2)6 && \text{Factor out the common factors.} \\ &= (x + 2)(x + 6) && \text{Factor out } x + 2. \end{aligned}$$

Check by using FOIL: $(x + 2)(x + 6) = x^2 + 8x + 12$.

**c)** To factor $a^2 - 9a + 20$, we need two integers that have a product of 20 and a sum of $-9$. The integers are $-4$ and $-5$. Now replace $-9a$ by $-4a - 5a$ and factor by grouping:

$$\begin{aligned} a^2 - 9a + 20 &= a^2 - 4a - 5a + 20 && \text{Replace } -9a \text{ by } -4a - 5a. \\ &= a(a - 4) - 5(a - 4) && \text{Factor by grouping.} \\ &= (a - 5)(a - 4) && \text{Factor out } a - 4. \end{aligned}$$

**study tip**

Effective time management will allow adequate time for school, social life, and free time. However, at times you will have to sacrifice to do well.

After sufficient practice factoring trinomials, you may be able to skip most of the steps shown in these examples. For example, to factor $x^2 + x - 6$, simply find a pair of integers with a product of $-6$ and a sum of 1. The integers are 3 and $-2$, so we can write

$$x^2 + x - 6 = (x + 3)(x - 2)$$

and check by using FOIL.

## EXAMPLE 2 Factoring trinomials

Factor.

**a)** $x^2 + 5x + 4$  **b)** $y^2 + 6y - 16$  **c)** $w^2 - 5w - 24$

### Solution

**a)** To get a product of 4 and a sum of 5, use 1 and 4:

$$x^2 + 5x + 4 = (x + 1)(x + 4)$$

Check by using FOIL on $(x + 1)(x + 4)$.

**b)** To get a product of $-16$ we need a positive number and a negative number. To also get a sum of 6, use 8 and $-2$:

$$y^2 + 6y - 16 = (y + 8)(y - 2)$$

Check by using FOIL on $(y + 8)(y - 2)$.

**c)** To get a product of $-24$ and a sum of $-5$, use $-8$ and 3:

$$w^2 - 5w - 24 = (w - 8)(w + 3)$$

Check by using FOIL.

Polynomials are easiest to factor when they are in the form $ax^2 + bx + c$. So if a polynomial can be rewritten into that form, rewrite it before attempting to factor it. In the next example we factor polynomials that need to be rewritten.

## EXAMPLE 3 Factoring trinomials

Factor.

**a)** $2x - 8 + x^2$  **b)** $-36 + t^2 - 9t$

## Solution

**a)** Before factoring, write the trinomial as $x^2 + 2x - 8$. Now, to get a product of $-8$ and a sum of 2, use $-2$ and 4:

$$2x - 8 + x^2 = x^2 + 2x - 8 \quad \text{Write in } ax^2 + bx + c \text{ form.}$$
$$= (x + 4)(x - 2) \quad \text{Factor and check by multiplying.}$$

**b)** Before factoring, write the trinomial as $t^2 - 9t - 36$. Now, to get a product of $-36$ and a sum of $-9$, use $-12$ and 3:

$$-36 + t^2 - 9t = t^2 - 9t - 36 \quad \text{Write in } ax^2 + bx + c \text{ form.}$$
$$= (t - 12)(t + 3) \quad \text{Factor and check by multiplying.}$$

## Prime Polynomials

To factor $x^2 + bx + c$, we try pairs of integers that have a product of $c$ until we find a pair that has a sum of $b$. If there is no such pair of integers, then the polynomial cannot be factored and it is a prime polynomial. Before you can conclude that a polynomial is prime, you must try *all* possibilities.

**EXAMPLE 4** **Prime polynomials**

Factor.

**a)** $x^2 + 7x - 6$ **b)** $x^2 + 9$

## Solution

**a)** Because the last term is $-6$, we want a positive integer and a negative integer that have a product of $-6$ and a sum of 7. Check all possible pairs of integers:

| Product | Sum |
|---|---|
| $-6 = (-1)(6)$ | $-1 + 6 = 5$ |
| $-6 = (1)(-6)$ | $1 + (-6) = -5$ |
| $-6 = (2)(-3)$ | $2 + (-3) = -1$ |
| $-6 = (-2)(3)$ | $-2 + 3 = 1$ |

None of these possible factors of $-6$ have a sum of 7, so we can be certain that $x^2 + 7x - 6$ cannot be factored. It is a prime polynomial.

**b)** Because the $x$-term is missing in $x^2 + 9$, its coefficient is 0. That is, $x^2 + 9 = x^2 + 0x + 9$. So we seek two positive integers or two negative integers that have a product of 9 and a sum of 0. Check all possibilities:

| Product | Sum |
|---|---|
| $9 = (3)(3)$ | $3 + 3 = 6$ |
| $9 = (-3)(-3)$ | $-3 + (-3) = -6$ |
| $9 = (9)(1)$ | $9 + 1 = 10$ |
| $9 = (-9)(-1)$ | $-9 + (-1) = -10$ |

None of these pairs of integers has a sum of 0, so we can conclude that $x^2 + 9$ is a prime polynomial. Note that $x^2 + 9$ does not factor as $(x + 3)^2$ because $(x + 3)^2$ has a middle term: $(x + 3)^2 = x^2 + 6x + 9$.

**study tip**

Stay alert for the entire class period. The first 20 minutes is the easiest and the last 20 minutes the hardest. Some students put down their pencils, fold up their notebooks, and daydream for those last 20 minutes. Don't give in. Recognize when you are losing it and force yourself to stay alert. Think of how much time you will have to spend outside of class figuring out what happened during those last 20 minutes.

**helpful hint**

Don't confuse $a^2 + b^2$ with the difference of two squares $a^2 - b^2$ which is not a prime polynomial:

$$a^2 - b^2 = (a + b)(a - b).$$

The prime polynomial $x^2 + 9$ in Example 4(b) is a sum of two squares. It can be shown that any sum of two squares (in which there are no common factors) is a prime polynomial.

**Sum of Two Squares**

If a sum of two squares, $a^2 + b^2$, has no common factor other than 1, then it is a prime polynomial.

## Factoring with Two Variables

In the next example we factor polynomials that have two variables using the same technique that we used for one variable.

**EXAMPLE 5** **Polynomials with two variables**

Factor.

**a)** $x^2 + 2xy - 8y^2$ **b)** $a^2 - 7ab + 10b^2$

**Solution**

**a)** To get a product of $-8$ and a sum of 2, use 4 and $-2$. To get a product of $-8y^2$ use $4y$ and $-2y$:

$$x^2 + 2xy - 8y^2 = (x + 4y)(x - 2y)$$

Check by multiplying $(x + 4y)(x - 2y)$.

**b)** To get a product of 10 and a sum of $-7$, use $-5$ and $-2$. To get a product of $10b^2$, we use $-5b$ and $-2b$:

$$a^2 - 7ab + 10b^2 = (a - 5b)(a - 2b)$$

Check by multiplying. ■

## Factoring Completely

In Section 6.2 you learned that binomials such as $3x - 5$ (with no common factor) are prime polynomials. In Example 4 of this section we saw a trinomial that is a prime polynomial. There are infinitely many prime trinomials. When factoring a polynomial completely, we could have a factor that is a prime trinomial.

**EXAMPLE 6** **Factoring completely**

Factor each polynomial completely.

**a)** $x^3 - 6x^2 - 16x$ **b)** $4x^3 + 4x^2 + 4x$ **c)** $3wy^2 + 18wy + 27w$

**Solution**

**a)** $x^3 - 6x^2 - 16x = x(x^2 - 6x - 16)$ Factor out the GCF.

$= x(x - 8)(x + 2)$ Factor $x^2 - 6x - 16$.

**b)** First factor out $4x$, the greatest common factor:

$$4x^3 + 4x^2 + 4x = 4x(x^2 + x + 1)$$

To factor $x^2 + x + 1$, we would need two integers with a product of 1 and a sum of 1. Because there are no such integers, $x^2 + x + 1$ is prime, and the factorization is complete.

**c)** $3wy^2 + 18wy + 27w = 3w(y^2 + 6y + 9)$ Factor out the GCF.

$= 3w(y + 3)^2$ Perfect square trinomial ■

## WARM-UPS

**True or false? Answer true if the correct factorization is given and false if the factorization is incorrect. Explain your answer.**

**1.** $x^2 - 6x + 9 = (x - 3)^2$
**2.** $x^2 + 6x + 9 = (x + 3)^2$
**3.** $x^2 + 10x + 9 = (x - 9)(x - 1)$
**4.** $x^2 - 8x - 9 = (x - 8)(x - 9)$
**5.** $x^2 + 8x - 9 = (x + 9)(x - 1)$
**6.** $x^2 + 8x + 9 = (x + 3)^2$
**7.** $x^2 - 10xy + 9y^2 = (x - y)(x - 9y)$
**8.** $x^2 + x + 1 = (x + 1)(x + 1)$
**9.** $x^2 + xy + 20y^2 = (x + 5y)(x - 4y)$
**10.** $x^2 + 1 = (x + 1)(x + 1)$

## 6.3 EXERCISES

***Reading and Writing*** *After reading this section, write out the answers to these questions. Use complete sentences.*

**1.** What types of polynomials did we factor in this section?

**2.** How can you check if you have factored a trinomial correctly?

**3.** How can you determine if $x^2 + bx + c$ is prime?

**4.** How do you factor a sum of two squares?

**5.** When is a polynomial factored completely?

**6.** What should you always look for first when attempting to factor a polynomial completely?

*Factor each trinomial. Write out all of the steps as shown in Example 1.*

**7.** $x^2 + 4x + 3$
**8.** $y^2 + 6y + 5$
**9.** $x^2 + 9x + 18$
**10.** $w^2 + 6w + 8$
**11.** $a^2 - 7a + 12$
**12.** $m^2 - 9m + 14$
**13.** $b^2 - 5b - 6$
**14.** $a^2 + 5a - 6$

*Factor each polynomial. See Examples 2–4. If the polynomial is prime, say so.*

**15.** $y^2 + 7y + 10$
**16.** $x^2 + 8x + 15$
**17.** $a^2 - 6a + 8$
**18.** $b^2 - 8b + 15$
**19.** $m^2 - 10m + 16$
**20.** $m^2 - 17m + 16$
**21.** $w^2 + 9w - 10$
**22.** $m^2 + 6m - 16$
**23.** $w^2 - 8 - 2w$
**24.** $-16 + m^2 - 6m$
**25.** $a^2 - 2a - 12$
**26.** $x^2 + 3x + 3$
**27.** $15m - 16 + m^2$
**28.** $3y + y^2 - 10$
**29.** $a^2 - 4a + 12$
**30.** $y^2 - 6y - 8$
**31.** $z^2 - 25$
**32.** $p^2 - 1$
**33.** $h^2 + 49$
**34.** $q^2 + 4$
**35.** $m^2 + 12m + 20$
**36.** $m^2 + 21m + 20$
**37.** $t^2 - 3t + 10$
**38.** $x^2 - 5x - 3$
**39.** $m^2 - 18 - 17m$

**40.** $h^2 - 36 + 5h$
**41.** $m^2 - 23m + 24$
**42.** $m^2 + 23m + 24$
**43.** $5t - 24 + t^2$
**44.** $t^2 - 24 - 10t$
**45.** $t^2 - 2t - 24$
**46.** $t^2 + 14t + 24$
**47.** $t^2 - 10t - 200$
**48.** $t^2 + 30t + 200$
**49.** $x^2 - 5x - 150$
**50.** $x^2 - 25x + 150$
**51.** $13y + 30 + y^2$
**52.** $18z + 45 + z^2$

*Factor each polynomial. See Example 5.*

**53.** $x^2 + 5ax + 6a^2$
**54.** $a^2 + 7ab + 10b^2$
**55.** $x^2 - 4xy - 12y^2$
**56.** $y^2 + yt - 12t^2$
**57.** $x^2 - 13xy + 12y^2$
**58.** $h^2 - 9hs + 9s^2$
**59.** $x^2 + 4xz - 33z^2$
**60.** $x^2 - 5xs - 24s^2$

*Factor each polynomial completely. Use the methods discussed in Sections 6.1 through 6.3. See Example 6.*

**61.** $w^2 - 8w$
**62.** $x^4 - x^3$
**63.** $2w^2 - 162$
**64.** $6w^4 - 54w^2$
**65.** $x^2w^2 + 9x^2$
**66.** $a^4b + a^2b^3$
**67.** $w^2 - 18w + 81$
**68.** $w^2 + 30w + 81$
**69.** $6w^2 - 12w - 18$
**70.** $9w - w^3$
**71.** $32x^2 - 2x^4$
**72.** $20w^2 + 100w + 40$
**73.** $3w^2 + 27w + 54$
**74.** $w^3 - 3w^2 - 18w$
**75.** $18w^2 + w^3 + 36w$
**76.** $18a^2 + 3a^3 + 36a$
**77.** $8vw^2 + 32vw + 32v$
**78.** $3h^2t + 6ht + 3t$
**79.** $6x^3y + 30x^2y^2 + 36xy^3$
**80.** $3x^3y^2 - 3x^2y^2 + 3xy^2$

*Use factoring to solve each problem.*

**81.** ***Area of a deck.*** A rectangular deck has an area of $x^2 + 6x + 8$ square feet and a width of $x + 2$ feet. Find the length of the deck.

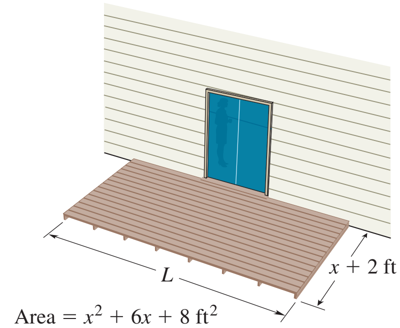


**FIGURE FOR EXERCISE 81**

**82.** ***Area of a sail.*** A triangular sail has an area of $x^2 + 5x + 6$ square meters and a height of $x + 3$ meters. Find the length of the sail's base.

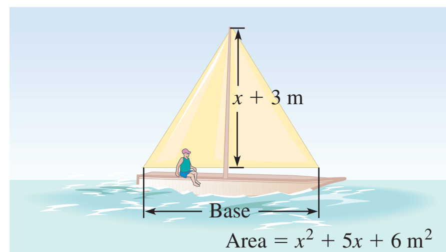


**FIGURE FOR EXERCISE 82**

**83.** ***Volume of a cube.*** Hector designed a cubic box with volume $x^3$ cubic feet. After increasing the dimensions of the bottom, the box has a volume of $x^3 + 8x^2 + 15x$ cubic feet. If each of the dimensions of the bottom was increased by a whole number of feet, then how much was each increase?

**84.** ***Volume of a container.*** A cubic shipping container had a volume of $a^3$ cubic meters. The height was decreased by a whole number of meters and the width was increased by a whole number of meters so that the volume of the container is now $a^3 + 2a^2 - 3a$ cubic meters. By how many meters were the height and width changed?

## GETTING MORE INVOLVED

**85.** ***Discussion.*** Which of the following products is not equivalent to the others. Explain your answer.

**a)** $(2x - 4)(x + 3)$ **b)** $(x - 2)(2x + 6)$
**c)** $2(x - 2)(x + 3)$ **d)** $(2x - 4)(2x + 6)$

**86.** ***Discussion.*** When asked to factor completely a certain polynomial, four students gave the following answers. Only one student gave the correct answer. Which one must it be? Explain your answer.

**a)** $3(x^2 - 2x - 15)$ **b)** $(3x - 5)(5x - 15)$
**c)** $3(x - 5)(x - 3)$ **d)** $(3x - 15)(x - 3)$