7.2 MULTIPLICATION AND DIVISION

In this section

- Multiplication of Rational Numbers
- Multiplication of Rational Expressions
- Division of Rational Numbers
- Division of Rational Expressions
- Applications

In Section 7.1 you learned to reduce rational expressions in the same way that we reduce rational numbers. In this section we will multiply and divide rational expressions using the same procedures that we use for rational numbers.

Multiplication of Rational Numbers

Two rational numbers are multiplied by multiplying their numerators and multiplying their denominators.

Multiplication of Rational Numbers

If $b \neq 0$ and $d \neq 0$, then

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

EXAMPLE 1 **Multiplying rational numbers**

Find the product $\frac{6}{7} \cdot \frac{14}{15}$.

Solution

The product is found by multiplying the numerators and multiplying the denominators:

$$\frac{6}{7} \cdot \frac{14}{15} = \frac{84}{105}$$

$$= \frac{21 \cdot 4}{21 \cdot 5} \quad \text{Factor the numerator and denominator.}$$

$$= \frac{4}{5} \quad \text{Divide out the GCF 21.}$$

helpful hint

Did you know that the line separating the numerator and denominator in a fraction is called the *vinculum*?

The reducing that we did after multiplying is easier to do before multiplying. First factor all terms, reduce, and then multiply:

$$\frac{6}{7} \cdot \frac{14}{15} = \frac{2 \cdot \cancel{3}}{\cancel{7}} \cdot \frac{2 \cdot \cancel{7}}{\cancel{3} \cdot 5}$$

$$= \frac{4}{5}$$

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Multiplication of Rational Expressions

We multiply rational expressions in the same way we multiply rational numbers. As with rational numbers, we can factor, reduce, and then multiply.

EXAMPLE 2

Multiplying rational expressions

Find the indicated products.

a) $\frac{9x}{5y} \cdot \frac{10y}{3xy}$ **b)** $\frac{-8xy^4}{3z^3} \cdot \frac{15z}{2x^5y^3}$

Solution

a) $\frac{9x}{5y} \cdot \frac{10y}{3xy} = \frac{3 \cdot 3x}{5y} \cdot \frac{2 \cdot 5y}{3xy}$ Factor.

$= \frac{6}{y}$

b) $\frac{-8xy^4}{3z^3} \cdot \frac{15z}{2x^5y^3} = \frac{-2 \cdot 2 \cdot 2xy^4}{3z^3} \cdot \frac{3 \cdot 5z}{2x^5y^3}$ Factor.

$= \frac{-20xy^4z}{z^3x^5y^3}$ Reduce.

$= \frac{-20y}{z^2x^4}$ Quotient rule

EXAMPLE 3

Multiplying rational expressions

Find the indicated products.

a) $\frac{2x - 2y}{4} \cdot \frac{2x}{x^2 - y^2}$

b) $\frac{x^2 + 7x + 12}{2x + 6} \cdot \frac{x}{x^2 - 16}$

c) $\frac{a + b}{6a} \cdot \frac{8a^2}{a^2 + 2ab + b^2}$

Solution

a) $\frac{2x - 2y}{4} \cdot \frac{2x}{x^2 - y^2} = \frac{2(x - y)}{2 \cdot 2} \cdot \frac{2 \cdot x}{(x - y)(x + y)}$ Factor.

$= \frac{x}{x + y}$ Reduce.

b) $\frac{x^2 + 7x + 12}{2x + 6} \cdot \frac{x}{x^2 - 16} = \frac{(x + 3)(x + 4)}{2(x + 3)} \cdot \frac{x}{(x - 4)(x + 4)}$ Factor.

$= \frac{x}{2(x - 4)}$ Reduce.

$= \frac{x}{2x - 8}$

c) $\frac{a + b}{6a} \cdot \frac{8a^2}{a^2 + 2ab + b^2} = \frac{a + b}{2 \cdot 3a} \cdot \frac{2 \cdot 4a^2}{(a + b)^2}$ Factor.

$= \frac{4a}{3(a + b)}$ Reduce.

$= \frac{4a}{3a + 3b}$

study tip

We are all creatures of habit. When you find a place in which you study successfully, stick with it. Using the same place for studying will help you to concentrate and associate the place with good studying.

Division of Rational Numbers

Division of rational numbers can be accomplished by multiplying by the reciprocal of the divisor.

Division of Rational Numbers

If $b \neq 0$, $c \neq 0$, and $d \neq 0$, then

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}.$$

EXAMPLE 4 **Dividing rational numbers**

Find each quotient.

a) $5 \div \frac{1}{2}$ **b)** $\frac{6}{7} \div \frac{3}{14}$

Solution

a) $5 \div \frac{1}{2} = 5 \cdot 2 = 10$

b) $\frac{6}{7} \div \frac{3}{14} = \frac{6}{7} \cdot \frac{14}{3} = \frac{2 \cdot \cancel{3}}{\cancel{7}} \cdot \frac{2 \cdot \cancel{7}}{\cancel{3}} = 4$

Division of Rational Expressions

We divide rational expressions in the same way we divide rational numbers: Invert the divisor and multiply.

EXAMPLE 5 **Dividing rational expressions**

Find each quotient.

a) $\frac{5}{3x} \div \frac{5}{6x}$ **b)** $\frac{x^7}{2} \div (2x^2)$

c) $\frac{4 - x^2}{x^2 + x} \div \frac{x - 2}{x^2 - 1}$

Solution

a) $\frac{5}{3x} \div \frac{5}{6x} = \frac{5}{3x} \cdot \frac{6x}{5}$ Invert the divisor and multiply.

$= \frac{\cancel{5}}{\cancel{3x}} \cdot \frac{2 \cdot \cancel{3x}}{\cancel{5}}$ Factor.

$= 2$ Divide out the common factors.

b) $\frac{x^7}{2} \div (2x^2) = \frac{x^7}{2} \cdot \frac{1}{2x^2}$ Invert and multiply.

$= \frac{x^5}{4}$ Quotient rule

helpful hint

A doctor told a nurse to give a patient half of the usual dose of a certain medicine. The nurse figured, "dividing in half means dividing by 1/2 which means multiply by 2." So the patient got four times the prescribed amount and died (true story). There is a big difference between dividing a quantity in half and dividing by one-half.

c) $$\frac{4-x^2}{x^2+x} \div \frac{x-2}{x^2-1} = \frac{4-x^2}{x^2+x} \cdot \frac{x^2-1}{x-2}$$ Invert and multiply.

$$= \frac{\overset{-1}{\cancel{(2-x)}}(2+x)}{x\cancel{(x+1)}} \cdot \frac{\cancel{(x+1)}(x-1)}{\cancel{x-2}}$$ Factor.

$$= \frac{-1(2+x)(x-1)}{x}$$ $\frac{2-x}{x-2} = -1$

$$= \frac{-1(x^2+x-2)}{x}$$ Simplify.

$$= \frac{-x^2-x+2}{x}$$

We sometimes write division of rational expressions using the fraction bar. For example, we can write

$$\frac{a+b}{3} \div \frac{1}{6} \quad \text{as} \quad \frac{\frac{a+b}{3}}{\frac{1}{6}}.$$

No matter how division is expressed, we invert the divisor and multiply.

EXAMPLE 6 **Division expressed with a fraction bar**

Find each quotient.

a) $\dfrac{\frac{a+b}{3}}{\frac{1}{6}}$ b) $\dfrac{\frac{x^2-1}{2}}{\frac{x-1}{3}}$

c) $\dfrac{\frac{a^2+5}{3}}{2}$

Solution

a) $$\frac{\frac{a+b}{3}}{\frac{1}{6}} = \frac{a+b}{3} \div \frac{1}{6}$$ Rewrite as division.

$$= \frac{a+b}{3} \cdot \frac{6}{1}$$ Invert and multiply.

$$= \frac{a+b}{\cancel{3}} \cdot \frac{2 \cdot \cancel{3}}{1}$$ Factor.

$$= (a+b)2$$ Reduce.

$$= 2a + 2b$$

b) $$\frac{\frac{x^2-1}{2}}{\frac{x-1}{3}} = \frac{x^2-1}{2} \div \frac{x-1}{3}$$ Rewrite as division.

$$= \frac{x^2-1}{2} \cdot \frac{3}{x-1}$$ Invert and multiply.

$$= \frac{\cancel{(x-1)}(x+1)}{2} \cdot \frac{3}{\cancel{x-1}}$$ Factor.

$$= \frac{3x+3}{2}$$ Reduce.

c) $$\frac{\frac{a^2+5}{3}}{2} = \frac{a^2+5}{3} \div 2$$ Rewrite as division.

$$= \frac{a^2+5}{3} \cdot \frac{1}{2}$$

$$= \frac{a^2+5}{6}$$

Applications

We saw in the last section that rational expressions can be used to represent rates. Note that there are several ways to write rates. For example, miles per hour is written mph, mi/hr, or $\frac{\text{mi}}{\text{hr}}$. The last way is best when doing operations with rates because it helps us reconcile our answers. Notice how hours "cancels" when we multiply miles per hour and hours in the next example, giving an answer in miles, as it should be.

EXAMPLE 7 **Writing rational expressions**

Answer each question with a rational expression.

a) Shasta averaged $\frac{200}{x}$ mph for x hours before she had lunch. How many miles did she drive in the first 3 hours after lunch assuming that she continued to average $\frac{200}{x}$ mph?

b) If a bathtub can be filled in x minutes, then the rate at which it is filling is $\frac{1}{x}$ tub/min. How much of the tub is filled in 10 minutes?

Solution

a) Because $R \cdot T = D$, the distance she traveled after lunch is the product of the rate and time:

$$\frac{200}{x}\,\frac{\text{mi}}{\cancel{\text{hr}}} \cdot 3\,\cancel{\text{hr}} = \frac{600}{x}\,\text{mi}$$

b) Because $R \cdot T = W$, the work completed is the product of the rate and time:

$$\frac{1}{x}\,\frac{\text{tub}}{\cancel{\text{min}}} \cdot 10\,\cancel{\text{min}} = \frac{10}{x}\,\text{tub}$$

WARM-UPS

True or false? Explain your answer.

1. $\frac{2}{3} \cdot \frac{5}{3} = \frac{10}{9}$.
2. The product of $\frac{x-7}{3}$ and $\frac{6}{7-x}$ is -2.
3. Dividing by 2 is equivalent to multiplying by $\frac{1}{2}$.
4. $3 \div x = \frac{1}{3} \cdot x$ for any nonzero number x.
5. Factoring polynomials is essential in multiplying rational expressions.
6. One-half of one-fourth is one-sixth.
7. One-half divided by three is three-halves.
8. The quotient of $(839 - 487)$ and $(487 - 839)$ is -1.
9. $\frac{a}{3} \div 3 = \frac{a}{9}$ for any value of a.
10. $\frac{a}{b} \cdot \frac{b}{a} = 1$ for any nonzero values of a and b.

7.2 EXERCISES

Reading and Writing *After reading this section, write out the answers to these questions. Use complete sentences.*

1. How do you multiply rational numbers?
2. How do you multiply rational expressions?
3. What can be done to simplify the process of multiplying rational numbers or rational expressions?
4. How do you divide rational numbers or rational expressions?

Perform the indicated operation. See Example 1.

5. $\frac{8}{15} \cdot \frac{35}{24}$
6. $\frac{3}{4} \cdot \frac{8}{21}$
7. $\frac{12}{17} \cdot \frac{51}{10}$
8. $\frac{25}{48} \cdot \frac{56}{35}$
9. $24 \cdot \frac{7}{20}$
10. $\frac{3}{10} \cdot 35$

Perform the indicated operation. See Example 2.

11. $\frac{5a}{12b} \cdot \frac{3ab}{55a}$
12. $\frac{3m}{7p} \cdot \frac{35p}{6mp}$
13. $\frac{-2x^6}{7a^5} \cdot \frac{21a^2}{6x}$
14. $\frac{5z^3w}{-9y^3} \cdot \frac{-6y^5}{20z^9}$
15. $\frac{15t^3y^5}{20w^7} \cdot 24t^5w^3y^2$
16. $22x^2y^3z \cdot \frac{6x^5}{33y^3z^4}$

Perform the indicated operation. See Example 3.

17. $\frac{3a + 3b}{15} \cdot \frac{10a}{a^2 - b^2}$
18. $\frac{b^3 + b}{5} \cdot \frac{10}{b^2 + b}$
19. $(x^2 - 6x + 9) \cdot \frac{3}{x - 3}$
20. $\frac{12}{4x + 10} \cdot (4x^2 + 20x + 25)$
21. $\frac{16a + 8}{5a^2 + 5} \cdot \frac{2a^2 + a - 1}{4a^2 - 1}$
22. $\frac{6x - 18}{2x^2 - 5x - 3} \cdot \frac{4x^2 + 4x + 1}{6x + 3}$

Perform the indicated operation. See Example 4.

23. $12 \div \frac{2}{5}$
24. $32 \div \frac{1}{4}$
25. $\frac{5}{7} \div \frac{15}{14}$
26. $\frac{3}{4} \div \frac{15}{2}$
27. $\frac{40}{3} \div 12$
28. $\frac{22}{9} \div 9$

Perform the indicated operation. See Example 5.

29. $\frac{5x^2}{3} \div \frac{10x}{21}$
30. $\frac{4u^2}{3v} \div \frac{14u}{15v^6}$
31. $\frac{8m^3}{n^4} \div (12mn^2)$
32. $\frac{2p^4}{3q^3} \div (4pq^5)$
33. $\frac{y - 6}{2} \div \frac{6 - y}{6}$
34. $\frac{4 - a}{5} \div \frac{a^2 - 16}{3}$

35. $\dfrac{x^2 + 4x + 4}{8} \div \dfrac{(x + 2)^3}{16}$

36. $\dfrac{a^2 + 2a + 1}{3} \div \dfrac{a^2 - 1}{a}$

37. $\dfrac{t^2 + 3t - 10}{t^2 - 25} \div (4t - 8)$

38. $\dfrac{w^2 - 7w + 12}{w^2 - 4w} \div (w^2 - 9)$

39. $(2x^2 - 3x - 5) \div \dfrac{2x - 5}{x - 1}$

40. $(6y^2 - y - 2) \div \dfrac{2y + 1}{3y - 2}$

Perform the indicated operation. See Example 6.

41. $\dfrac{\frac{x - 2y}{5}}{\frac{1}{10}}$

42. $\dfrac{\frac{3m + 6n}{8}}{\frac{3}{4}}$

43. $\dfrac{\frac{x^2 - 4}{12}}{\frac{x - 2}{6}}$

44. $\dfrac{\frac{6a^2 + 6}{5}}{\frac{6a + 6}{5}}$

45. $\dfrac{\frac{x^2 + 9}{3}}{5}$

46. $\dfrac{1}{\frac{a - 3}{4}}$

47. $\dfrac{\frac{x^2 - y^2}{x - y}}{9}$

48. $\dfrac{x^2 + 6x + 8}{\frac{x + 2}{x + 1}}$

Perform the indicated operation.

49. $\dfrac{x - 1}{3} \cdot \dfrac{9}{1 - x}$

50. $\dfrac{2x - 2y}{3} \cdot \dfrac{1}{y - x}$

51. $\dfrac{3a + 3b}{a} \cdot \dfrac{1}{3}$

52. $\dfrac{a - b}{2b - 2a} \cdot \dfrac{2}{5}$

53. $\dfrac{\frac{b}{a}}{\frac{1}{2}}$

54. $\dfrac{\frac{2g}{3h}}{\frac{1}{h}}$

55. $\dfrac{6y}{3} \div (2x)$

56. $\dfrac{8x}{9} \div (18x)$

57. $\dfrac{a^3b^4}{-2ab^2} \cdot \dfrac{a^5b^7}{ab}$

58. $\dfrac{-2a^2}{3a^2} \cdot \dfrac{20a}{15a^3}$

59. $\dfrac{2mn^4}{6mn^2} \div \dfrac{3m^5n^7}{m^2n^4}$

60. $\dfrac{rt^2}{rt^2} \div \dfrac{rt^2}{r^3t^2}$

61. $\dfrac{3x^2 + 16x + 5}{x} \cdot \dfrac{x^2}{9x^2 - 1}$

62. $\dfrac{x^2 + 6x + 5}{x} \cdot \dfrac{x^4}{3x + 3}$

63. $\dfrac{a^2 - 2a + 4}{a^2 - 4} \cdot \dfrac{(a + 2)^3}{2a + 4}$

64. $\dfrac{w^2 - 1}{(w - 1)^2} \cdot \dfrac{w - 1}{w^2 + 2w + 1}$

65. $\dfrac{2x^2 + 19x - 10}{x^2 - 100} \div \dfrac{4x^2 - 1}{2x^2 - 19x - 10}$

66. $\dfrac{x^3 - 1}{x^2 + 1} \div \dfrac{9x^2 + 9x + 9}{x^2 - x}$

67. $\dfrac{9 + 6m + m^2}{9 - 6m + m^2} \cdot \dfrac{m^2 - 9}{m^2 + mk + 3m + 3k}$

68. $\dfrac{3x + 3w + bx + bw}{x^2 - w^2} \cdot \dfrac{6 - 2b}{9 - b^2}$

Solve each problem. Answers could be rational expressions. Be sure to give your answer with appropriate units. See Example 7.

69. ***Distance.*** Florence averaged $\frac{26.2}{x}$ mph for the x hours in which she ran the Boston Marathon. If she ran at that same rate for $\frac{1}{2}$ hour in the Manchac Fun Run, then how many miles did she run at Manchac?

70. ***Work.*** Henry sold 120 magazine subscriptions in $x + 2$ days. If he sold at the same rate for another week, then how many magazines did he sell in the extra week?

71. ***Area of a rectangle.*** If the length of a rectangular flag is x meters and its width is $\frac{5}{x}$ meters, then what is the area of the rectangle?

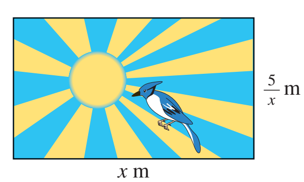

FIGURE FOR EXERCISE 71

72. ***Area of a triangle.*** If the base of a triangle is $8x + 16$ yards and its height is $\frac{1}{x + 2}$ yards, then what is the area of the triangle?

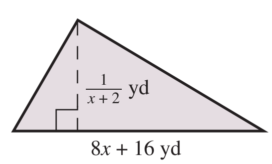

FIGURE FOR EXERCISE 72

GETTING MORE INVOLVED

73. ***Discussion.*** Evaluate each expression.

a) One-half of $\frac{1}{4}$ **b)** One-third of 4

c) One-half of $\frac{4x}{3}$ **d)** One-half of $\frac{3x}{2}$

74. ***Exploration.*** Let $R = \frac{6x^2 + 23x + 20}{24x^2 + 29x - 4}$ and $H = \frac{2x + 5}{8x - 1}$.

a) Find R when $x = 2$ and $x = 3$. Find H when $x = 2$ and $x = 3$.

b) How are these values of R and H related and why?

7.3 FINDING THE LEAST COMMON DENOMINATOR

In this section

- Building Up the Denominator
- Finding the Least Common Denominator
- Converting to the LCD

Every rational expression can be written in infinitely many equivalent forms. Because we can add or subtract only fractions with identical denominators, we must be able to change the denominator of a fraction. You have already learned how to change the denominator of a fraction by reducing. In this section you will learn the opposite of reducing, which is called **building up the denominator.**

Building Up the Denominator

To convert the fraction $\frac{2}{3}$ into an equivalent fraction with a denominator of 21, we factor 21 as $21 = 3 \cdot 7$. Because $\frac{2}{3}$ already has a 3 in the denominator, multiply the numerator and denominator of $\frac{2}{3}$ by the missing factor 7 to get a denominator of 21:

$$\frac{2}{3} = \frac{2}{3} \cdot \frac{7}{7} = \frac{14}{21}$$

For rational expressions the process is the same. To convert the rational expression

$$\frac{5}{x + 3}$$

into an equivalent rational expression with a denominator of $x^2 - x - 12$, first factor $x^2 - x - 12$:

$$x^2 - x - 12 = (x + 3)(x - 4)$$

From the factorization we can see that the denominator $x + 3$ needs only a factor of $x - 4$ to have the required denominator. So multiply the numerator and denominator by the missing factor $x - 4$:

$$\frac{5}{x + 3} = \frac{5}{x + 3} \cdot \frac{x - 4}{x - 4} = \frac{5x - 20}{x^2 - x - 12}$$

EXAMPLE 1 **Building up the denominator**

Build each rational expression into an equivalent rational expression with the indicated denominator.

a) $3 = \frac{?}{12}$ **b)** $\frac{3}{w} = \frac{?}{wx}$ **c)** $\frac{2}{3y^3} = \frac{?}{12y^8}$