

## 8.1

## SOLVING SYSTEMS BY GRAPHING AND SUBSTITUTION

## In this section

- Solving a System by Graphing
- Independent, Inconsistent, and Dependent Equations
- Solving by Substitution
- Applications

In Chapter 4 we studied linear equations in two variables, but we have usually considered only one equation at a time. In this chapter we will see problems that involve more than one equation. Any collection of two or more equations is called a **system** of equations. If the equations of a system involve two variables, then the set of ordered pairs that satisfy all of the equations is the **solution set of the system**. In this section we solve systems of linear equations in two variables and use systems to solve problems.

## Solving a System by Graphing

Because the graph of each linear equation is a line, points that satisfy both equations lie on both lines. For some systems these points can be found by graphing.

## EXAMPLE 1

## A system with only one solution

Solve the system by graphing:

$$y = x + 2$$

$$x + y = 4$$

## Solution

First write the equations in slope-intercept form:

$$y = x + 2$$

$$y = -x + 4$$

Use the y-intercept and the slope to graph each line. The graph of the system is shown in Fig. 8.1. From the graph it appears that these lines intersect at (1, 3). To be certain, we can check that (1, 3) satisfies both equations. Let  $x = 1$  and  $y = 3$  in  $y = x + 2$  to get

$$3 = 1 + 2.$$

Let  $x = 1$  and  $y = 3$  in  $x + y = 4$  to get

$$1 + 3 = 4.$$

Because (1, 3) satisfies both equations, the solution set to the system is  $\{(1, 3)\}$ .

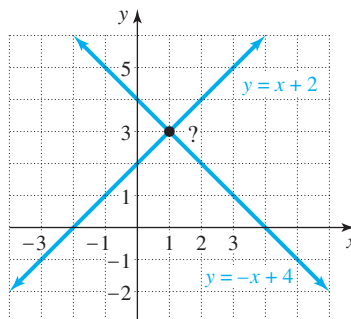


FIGURE 8.1

The graphs of the equations in the next example are parallel lines, and there is no point of intersection.

calculator

close-up

To check Example 1, graph

$$y_1 = x + 2$$

and

$$y_2 = -x + 4.$$

From the CALC menu, choose intersect to have the calculator locate the point of intersection of the two lines. After choosing intersect, you must indicate which two lines you want to intersect and then guess the point of intersection.

**EXAMPLE 2** A system with no solution

Solve the system by graphing:

$$2x - 3y = 6$$

$$3y - 2x = 3$$

**Solution**

First write each equation in slope-intercept form:

$$2x - 3y = 6$$

$$3y - 2x = 3$$

$$-3y = -2x + 6$$

$$3y = 2x + 3$$

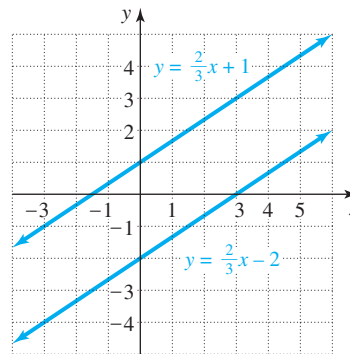
$$y = \frac{2}{3}x - 2$$

$$y = \frac{2}{3}x + 1$$

The graph of the system is shown in Fig. 8.2. Because the two lines in Fig. 8.2 are parallel, there is no ordered pair that satisfies both equations. The solution set to the system is the empty set,  $\emptyset$ .

**study tip**

Relax and don't worry about grades. If you are doing everything you can and should be doing, then there is no reason to worry. If you are neglecting your homework and skipping class, then you should be worried.

**FIGURE 8.2**

The equations in the next example look different, but their graphs are the same straight line.

**EXAMPLE 3** A system with infinitely many solutions

Solve the system by graphing:

$$2(y + 2) = x$$

$$x - 2y = 4$$

**Solution**

Write each equation in slope-intercept form:

$$2(y + 2) = x$$

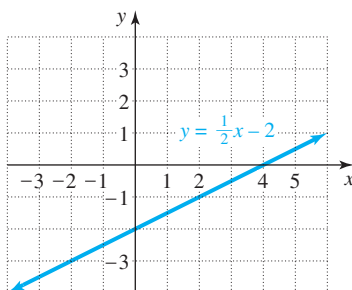
$$x - 2y = 4$$

$$2y + 4 = x$$

$$-2y = -x + 4$$

$$y = \frac{1}{2}x - 2$$

$$y = \frac{1}{2}x - 2$$

**FIGURE 8.3**

Because the equations have the same slope-intercept form, the original equations are equivalent. Their graphs are the same straight line as shown in Fig. 8.3. Every point on the line satisfies both equations of the system. There are infinitely many points in the solution set. The solution set is  $\{(x, y) \mid x - 2y = 4\}$ .

## Independent, Inconsistent, and Dependent Equations

Our first three examples illustrate the three possible ways in which two lines can be positioned in a plane. In Example 1 the lines intersect in a single point. In this case we say that the equations are **independent** or the system is independent. If the two lines are parallel, as in Example 2, then there is no solution to the system, and the equations are **inconsistent** or the system is inconsistent. If the two equations of a system are equivalent, as in Example 3, the equations are **dependent** or the system is dependent. Figure 8.4 shows the types of graphs that correspond to independent, inconsistent, and dependent systems.

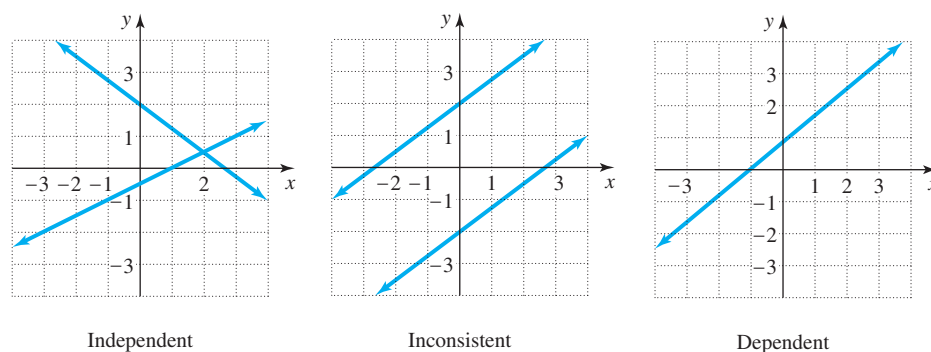


FIGURE 8.4

### study tip

Correct answers often have more than one form. If your answer to an exercise doesn't agree with the one in the back of this text, try to determine if it is simply a different form of the answer. For example,  $\frac{1}{2}x$  and  $x/2$  look different but they are equivalent expressions.

## Solving by Substitution

Solving a system by graphing is certainly limited by the accuracy of the graph. If the lines intersect at a point whose coordinates are not integers, then it is difficult to determine those coordinates from the graph. The method of solving a system by **substitution** does not depend on a graph and is totally accurate. For substitution we replace a variable in one equation with an equivalent expression obtained from the other equation. Our intention in this substitution step is to eliminate a variable and to give us an equation involving only one variable.

### EXAMPLE 4 An independent system solved by substitution

Solve the system by substitution:

$$\begin{aligned} 2x + 3y &= 8 \\ y &= -2x + 6 \end{aligned}$$

#### Solution

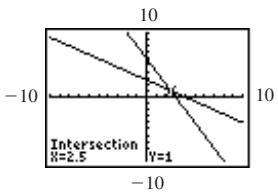
Since  $y = -2x + 6$  we can replace  $y$  in  $2x + 3y = 8$  by  $-2x + 6$ :

$$\begin{aligned} 2x + 3y &= 8 \\ 2x + 3(-2x + 6) &= 8 && \text{Substitute } -2x + 6 \text{ for } y. \\ 2x - 6x + 18 &= 8 \\ -4x &= -10 \\ x &= \frac{5}{2} \end{aligned}$$

**calculator**

**close-up**

To check Example 4, graph  
 $y_1 = (8 - 2x)/3$   
 and  
 $y_2 = -2x + 6$ .  
 From the CALC menu, choose intersect to have the calculator locate the point of intersection of the two lines.



To find  $y$ , we let  $x = \frac{5}{2}$  in the equation  $y = -2x + 6$ :

$$y = -2\left(\frac{5}{2}\right) + 6 = -5 + 6 = 1$$

The next step is to check  $x = \frac{5}{2}$  and  $y = 1$  in each equation. If  $x = \frac{5}{2}$  and  $y = 1$  in  $2x + 3y = 8$ , we get

$$2\left(\frac{5}{2}\right) + 3(1) = 8.$$

If  $x = \frac{5}{2}$  and  $y = 1$  in  $y + 2x = 6$ , we get

$$1 + 2\left(\frac{5}{2}\right) = 6.$$

Because both of these equations are true, the solution set to the system is  $\left\{\left(\frac{5}{2}, 1\right)\right\}$ . The equations of this system are independent. ■

### EXAMPLE 5

#### An inconsistent system solved by substitution

Solve by substitution:

$$x - 2y = 3$$

$$2x - 4y = 7$$

#### Solution

Solve the first equation for  $x$  to get  $x = 2y + 3$ . Substitute  $2y + 3$  for  $x$  in the second equation:

$$2x - 4y = 7$$

$$2(2y + 3) - 4y = 7$$

$$4y + 6 - 4y = 7$$

$$6 = 7$$

Because  $6 = 7$  is incorrect no matter what values are chosen for  $x$  and  $y$ , there is no solution to this system of equations. The equations are inconsistent. To check, we write each equation in slope-intercept form:

$$x - 2y = 3$$

$$2x - 4y = 7$$

$$-2y = -x + 3$$

$$-4y = -2x + 7$$

$$y = \frac{1}{2}x - \frac{3}{2}$$

$$y = \frac{1}{2}x - \frac{7}{4}$$

The graphs of these equations are parallel lines with different  $y$ -intercepts. The solution set to the system is the empty set,  $\emptyset$ . ■

### EXAMPLE 6

#### A dependent system solved by substitution

Solve by substitution:

$$2x + 3y = 5 + x + 4y$$

$$y = x - 5$$

**helpful hint**

The purpose of Example 6 is to show what happens when a dependent system is solved by substitution. If we had first written the first equation in slope-intercept form, we would have known that the equations are dependent and would not have done substitution.

**Solution**

Substitute  $y = x - 5$  into the first equation:

$$2x + 3(x - 5) = 5 + x + 4(x - 5)$$

$$2x + 3x - 15 = 5 + x + 4x - 20$$

$$5x - 15 = 5x - 15$$

Because the last equation is an identity, any ordered pair that satisfies  $y = x - 5$  will also satisfy  $2x + 3y = 5 + x + 4y$ . The equations of this system are dependent. The solution set to the system is the set of all points that satisfy  $y = x - 5$ . We write the solution set in set notation as

$$\{(x, y) \mid y = x - 5\}.$$

We can verify this result by writing  $2x + 3y = 5 + x + 4y$  in slope-intercept form:

$$2x + 3y = 5 + x + 4y$$

$$3y = -x + 5 + 4y$$

$$-y = -x + 5$$

$$y = x - 5$$

Because this slope-intercept form is identical to the slope-intercept form of the other equation, they are two equations that look different for the same straight line. ■

If a system is dependent, then an identity will result after the substitution. If the system is inconsistent, then an inconsistent equation will result after the substitution. The strategy for solving an independent system by substitution can be summarized as follows.

**The Substitution Method**

1. Solve one of the equations for one variable in terms of the other.
2. Substitute into the other equation to get an equation in one variable.
3. Solve for the remaining variable (if possible).
4. Insert the value just found into one of the original equations to find the value of the other variable.
5. Check the two values in both equations.

**study tip**

When working a test, scan the problems and pick out the ones that are the easiest for you. Do them first. Save the harder problems until the last.

**Applications**

Many of the problems that we solved in previous chapters involved more than one unknown quantity. To solve them, we wrote expressions for all of the unknowns in terms of one variable. Now we can solve problems involving two unknowns by using two variables and writing a system of equations.

**EXAMPLE 7****Perimeter of a rectangle**

The length of a rectangular swimming pool is twice the width. If the perimeter is 120 feet, then what are the length and width?

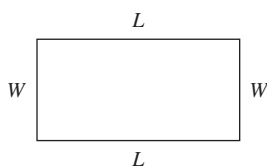


FIGURE 8.5

**Solution**

Draw a diagram as shown in Fig. 8.5. If  $L$  represents the length and  $W$  represents the width, then we can write the following system.

$$\begin{aligned}L &= 2W \\ 2L + 2W &= 120\end{aligned}$$

Since  $L = 2W$ , we can replace  $L$  in  $2L + 2W = 120$  with  $2W$ :

$$\begin{aligned}2(2W) + 2W &= 120 \\ 4W + 2W &= 120 \\ 6W &= 120 \\ W &= 20\end{aligned}$$

So the width is 20 feet and the length is  $2(20)$  or 40 feet. ■

**EXAMPLE 8****helpful hint**

In Chapter 2 we would have done Example 8 with one variable by letting  $x$  represent the amount invested at 10% and  $20,000 - x$  represent the amount invested at 12%.

**Tale of two investments**

Belinda had \$20,000 to invest. She invested part of it at 10% and the remainder at 12%. If her income from the two investments was \$2160, then how much did she invest at each rate?

**Solution**

Let  $x$  be the amount invested at 10% and  $y$  be the amount invested at 12%. We can summarize all of the given information in a table:

	Amount	Rate	Interest
First investment	$x$	10%	$0.10x$
Second investment	$y$	12%	$0.12y$

We can write one equation about the amounts invested and another about the interest from the investments:

$$\begin{aligned}x + y &= 20,000 && \text{Total amount invested} \\ 0.10x + 0.12y &= 2160 && \text{Total interest}\end{aligned}$$

Solve the first equation for  $x$  to get  $x = 20,000 - y$ . Substitute  $20,000 - y$  for  $x$  in the second equation:

$$\begin{aligned}0.10x + 0.12y &= 2160 \\ 0.10(20,000 - y) + 0.12y &= 2160 && \text{Replace } x \text{ by } 20,000 - y. \\ 2000 - 0.10y + 0.12y &= 2160 && \text{Solve for } y. \\ 0.02y &= 160 \\ y &= 8000 \\ x &= 12,000 && \text{Because } x = 20,000 - y\end{aligned}$$

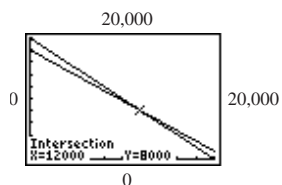
To check this answer, find 10% of \$12,000 and 12% of \$8000:

$$\begin{aligned}0.10(12,000) &= 1,200 \\ 0.12(8000) &= 960\end{aligned}$$

Because  $\$1200 + \$960 = \$2160$  and  $\$8000 + \$12,000 = \$20,000$ , we can be certain that Belinda invested \$12,000 at 10% and \$8000 at 12%. ■

**calculator****close-up**

To check Example 8, graph  $y_1 = 20,000 - x$  and  $y_2 = (2160 - 0.1x)/0.12$ . The viewing window needs to be large enough to contain the point of intersection. Use the intersection feature to find the point of intersection.



## MATH AT WORK

$$x^2 + (x+1)^2 = 52$$

Accountants work with both people and numbers. When Maria L. Manning, an auditor at Deloitte & Touche LLP, is preparing for an audit, she studies the numbers on the balance sheet and income statement, comparing the current fiscal year to the prior one. The purpose of an independent audit is to give investors a realistic view of a company's finances. To determine that a company's financial statements are fairly stated in accordance with the General Accepted Accounting Procedures (GAAP), she first interviews the comptroller to get the story behind the numbers. Typical questions she could ask are: How productive was your year? Are there any new products? What was behind the big stories in the newspapers? Then Ms. Manning and members of the audit team test the financial statement in detail, closely examining accounts relating to unusual losses or profits.



ACCOUNTANT

Ms. Manning is responsible for both manufacturing and mutual fund companies. At a manufacturing company, accounts receivable and inventory are two key components of an audit. For example, to test inventory, Ms. Manning visits a company's warehouse and physically counts all the items for sale to verify a company's assets. For a mutual fund company the audit team pays close attention to current events, for they indirectly affect the financial industry.

In Exercises 55 and 56 of this section you will work problems that involve one aspect of cost accounting: calculating the amount of taxes and bonuses paid by a company.

## WARM - U P S

**True or false? Explain your answer.**

1. The ordered pair  $(1, 2)$  is in the solution set to the equation  $2x + y = 4$ .
2. The ordered pair  $(1, 2)$  satisfies  $2x + y = 4$  and  $3x - y = 6$ .
3. The ordered pair  $(2, 3)$  satisfies  $4x - y = 5$  and  $4x - y = -5$ .
4. If two distinct straight lines in the coordinate plane are not parallel, then they intersect in exactly one point.
5. The substitution method is used to eliminate a variable.
6. No ordered pair satisfies  $y = 3x - 5$  and  $y = 3x + 1$ .
7. The equations  $y = 3x - 6$  and  $y = 2x + 4$  are independent.
8. The equations  $y = 2x + 7$  and  $y = 2x + 8$  are inconsistent.
9. The graphs of dependent equations are the same.
10. The graphs of independent linear equations intersect at exactly one point.

## 8.1 EXERCISES

**Reading and Writing** After reading this section, write out the answers to these questions. Use complete sentences.

- How do we solve a system of linear equations by graphing?
- How can you determine whether a system has no solution by graphing?
- What is the major disadvantage to solving a system by graphing?
- How do we solve systems by substitution?
- How can you identify an inconsistent system when solving by substitution?
- How can you identify a dependent system when solving by substitution?

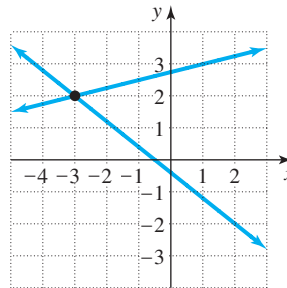
Solve each system by graphing. See Examples 1–3.

- |   |   |
|---|---|
| 7. $y = 2x$<br>$y = -x + 3$                 | 8. $y = x - 3$<br>$y = -x + 1$              |
| 9. $y = 2x - 1$<br>$2y = x - 2$             | 10. $y = 2x + 1$<br>$x + y = -2$            |
| 11. $y = x - 3$<br>$x - 2y = 4$             | 12. $y = -3x$<br>$x + y = 2$                |
| 13. $2y - 2x = 2$<br>$2y - 2x = 6$          | 14. $3y - 3x = 9$<br>$x - y = 1$            |
| 15. $y = -\frac{1}{2}x + 4$<br>$x + 2y = 8$ | 16. $2x - 3y = 6$<br>$y = \frac{2}{3}x - 2$ |

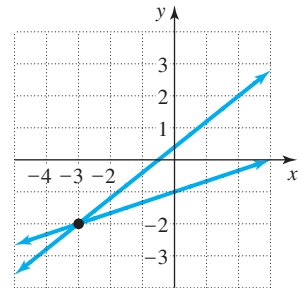
The graphs of the following systems are given in (a) through (d). Match each system with the correct graph.

- |                                     |                                      |
|-------------------------------------|--------------------------------------|
| 17. $5x + 4y = 7$<br>$x - 3y = 9$   | 18. $3x - 5y = -9$<br>$5x - 6y = -8$ |
| 19. $4x - 5y = -2$<br>$3y - x = -3$ | 20. $4x + 5y = -2$<br>$4y - x = 11$  |

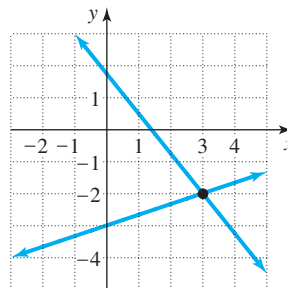
a)



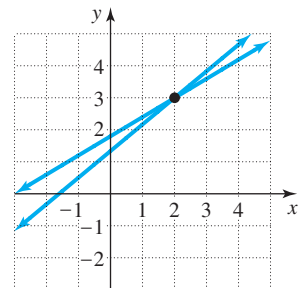
b)



c)



d)



Solve each system by the substitution method. Determine whether the equations are independent, dependent, or inconsistent. See Examples 4–6.

- |   |   |
|---|---|
| 21. $y = x - 5$<br>$2x - 5y = 1$                        | 22. $y = x + 4$<br>$3y - 5x = 6$            |
| 23. $x = 2y - 7$<br>$3x + 2y = -5$                      | 24. $x = y + 3$<br>$3x - 2y = 4$            |
| 25. $x - y = 5$<br>$2x = 2y + 14$                       | 26. $2x - y = 3$<br>$2y = 4x - 6$           |
| 27. $y = 2x - 5$<br>$y + 1 = 2(x - 2)$                  | 28. $3x - 6y = 5$<br>$2y = 4x - 6$          |
| 29. $2x + y = 9$<br>$2x - 5y = 15$                      | 30. $3y - x = 0$<br>$x - 4y = -2$           |
| 31. $x - y = 0$<br>$2x + 3y = 35$                       | 32. $2y = x + 6$<br>$-3x + 2y = -2$         |
| 33. $x + y = 40$<br>$0.1x + 0.08y = 3.5$                | 34. $x - y = 10$<br>$0.2x + 0.05y = 7$      |
| 35. $y = 2x - 30$<br>$\frac{1}{5}x - \frac{1}{2}y = -1$ | 36. $3x - 5y = 4$<br>$y = \frac{3}{4}x - 2$ |



$$\begin{aligned} 37. \quad x + y &= 4 \\ x - y &= 5 \end{aligned}$$

$$\begin{aligned} 38. \quad y &= 2x - 3 \\ y &= 3x - 3 \end{aligned}$$

$$\begin{aligned} 39. \quad 2x - y &= 4 \\ 2x - y &= 3 \end{aligned}$$

$$\begin{aligned} 40. \quad y &= 3(x - 4) \\ 3x - y &= 12 \end{aligned}$$

$$\begin{aligned} 41. \quad 3(y - 1) &= 2(x - 3) \\ 3y - 2x &= -3 \end{aligned}$$

$$\begin{aligned} 42. \quad y &= 3x \\ y &= 3x + 1 \end{aligned}$$

$$\begin{aligned} 43. \quad x - y &= -0.375 \\ 1.5x - 3y &= -2.25 \end{aligned}$$

$$\begin{aligned} 44. \quad y - 2x &= 1.875 \\ 2.5y - 3.5x &= 11.8125 \end{aligned}$$

In Exercises 45–58, write a system of two equations in two unknowns for each problem. Solve each system by substitution. See Examples 7 and 8.

45. **Perimeter of a rectangle.** The length of a rectangular swimming pool is 15 feet longer than the width. If the perimeter is 82 feet, then what are the length and width?

46. **Household income.** Alkena and Hsu together earn \$84,326 per year. If Alkena earns \$12,468 more per year than Hsu, then how much does each of them earn per year?

47. **Different interest rates.** Mrs. Brighton invested \$30,000 and received a total of \$2,300 in interest. If she invested part of the money at 10% and the remainder at 5%, then how much did she invest at each rate?

48. **Different growth rates.** The combined population of Marysville and Springfield was 25,000 in 1990. By 1995 the population of Marysville had increased by 10%, while Springfield had increased by 9%. If the total population increased by 2380 people, then what was the population of each city in 1990?

49. **Finding numbers.** The sum of two numbers is 2, and their difference is 26. Find the numbers.

50. **Finding more numbers.** The sum of two numbers is  $-16$ , and their difference is 8. Find the numbers.

51. **Toasters and vacations.** During one week a land developer gave away Florida vacation coupons or toasters to 100 potential customers who listened to a sales presentation. It costs the developer \$6 for a toaster and \$24 for a Florida vacation coupon. If his bill for prizes that week was \$708, then how many of each prize did he give away?

52. **Ticket sales.** Tickets for a concert were sold to adults for \$3 and to students for \$2. If the total receipts were

\$824 and twice as many adult tickets as student tickets were sold, then how many of each were sold?

53. **Corporate taxes.** According to Bruce Harrell, CPA, the amount of federal income tax for a class C corporation is deductible on the Louisiana state tax return, and the amount of state income tax for a class C corporation is deductible on the federal tax return. So for a state tax rate of 5% and a federal tax rate of 30%, we have

$$\text{state tax} = 0.05(\text{taxable income} - \text{federal tax})$$

and

$$\text{federal tax} = 0.30(\text{taxable income} - \text{state tax}).$$

Find the amounts of state and federal income tax for a class C corporation that has a taxable income of \$100,000.

54. **More taxes.** Use information given in Exercise 53 to find the amounts of state and federal income tax for a class C corporation that has a taxable income of \$300,000. Use a state tax rate of 6% and a federal tax rate of 40%.

55. **Cost accounting.** The problems presented in this exercise and the next are encountered in cost accounting. A company has agreed to distribute 20% of its net income  $N$  to its employees as a bonus;  $B = 0.20N$ . If the company has income of \$120,000 before the bonus, the bonus  $B$  is deducted from the \$120,000 as an expense to determine net income;  $N = 120,000 - B$ . Solve the system of two equations in  $N$  and  $B$  to find the amount of the bonus.

56. **Bonus and taxes.** A company has an income of \$100,000 before paying taxes and a bonus. The bonus  $B$  is to be 20% of the income after deducting income taxes  $T$  but before deducting the bonus. So

$$B = 0.20(100,000 - T).$$

Because the bonus is a deductible expense, the amount of income tax  $T$  at a 40% rate is 40% of the income after deducting the bonus. So

$$T = 0.40(100,000 - B).$$

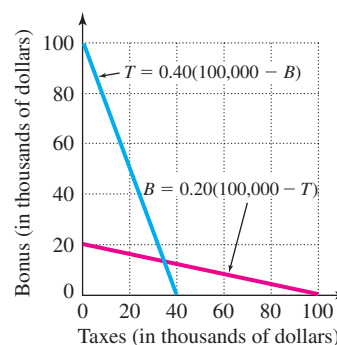


FIGURE FOR EXERCISE 56

- a) Use the accompanying graph to estimate the values of  $T$  and  $B$  that satisfy both equations.  
 b) Solve the system algebraically to find the bonus and the amount of tax.

57. **Textbook case.** The accompanying graph shows the cost of producing textbooks and the revenue from the sale of those textbooks.

- a) What is the cost of producing 10,000 textbooks?  
 b) What is the revenue when 10,000 textbooks are sold?  
 c) For what number of textbooks is the cost equal to the revenue?  
 d) The cost of producing zero textbooks is called the *fixed cost*. Find the fixed cost.

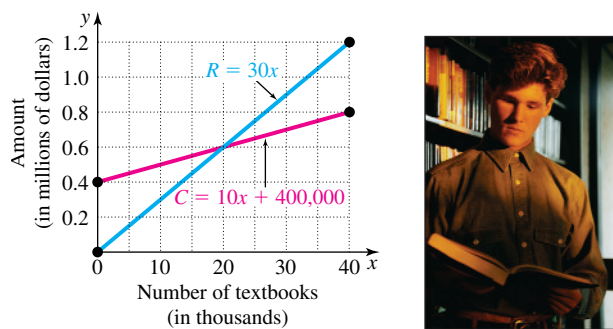


FIGURE FOR EXERCISE 57

58. **Free market.** The function  $S = 5000 + 200x$  and  $D = 9500 - 100x$  express the supply  $S$  and the demand  $D$ , respectively, for a popular compact disk brand as a function of its price  $x$  (in dollars).

- a) Graph the functions on the same coordinate system.  
 b) What happens to the supply as the price increases?  
 c) What happens to the demand as the price increases?  
 d) The price at which supply and demand are equal is called the *equilibrium price*. What is the equilibrium price?

### GETTING MORE INVOLVED

59. **Discussion.** Which of the following equations is not equivalent to  $2x - 3y = 6$ ?

- a)  $3y - 2x = 6$       b)  $y = \frac{2}{3}x - 2$   
 c)  $x = \frac{3}{2}y + 3$       d)  $2(x - 5) = 3y - 4$

60. **Discussion.** Which of the following equations is inconsistent with the equation  $3x + 4y = 8$ ?

- a)  $y = \frac{3}{4}x + 2$       b)  $6x + 8y = 16$   
 c)  $y = -\frac{3}{4}x + 8$       d)  $3x - 4y = 8$



### GRAPHING CALCULATOR EXERCISES

61. Solve each system by graphing each pair of equations on a graphing calculator and using the trace feature or intersect feature to estimate the point of intersection. Find the coordinates of the intersection to the nearest tenth.

- a)  $y = 3.5x - 7.2$       b)  $2.3x - 4.1y = 3.3$   
 $y = -2.3x + 9.1$        $3.4x + 9.2y = 1.3$

## In this section

- The Addition Method
- Equations Involving Fractions or Decimals
- Applications

## 8.2 THE ADDITION METHOD

In Section 8.1 you used substitution to eliminate a variable in a system of equations. In this section we see another method for eliminating a variable in a system of equations.

### The Addition Method

In the **addition method** we eliminate a variable by adding the equations.

### EXAMPLE 1

#### An independent system solved by addition

Solve the system by the addition method:

$$\begin{aligned} 3x - 5y &= -9 \\ 4x + 5y &= 23 \end{aligned}$$