38. ***Three-digit number.*** The sum of the digits of a three-digit number is 11. If the digits are reversed, the new number is 46 more than five times the old number. If the hundreds digit plus twice the tens digit is equal to the units digit, then what is the number?

39. ***Working overtime.*** To make ends meet, Ms. Farnsby works three jobs. Her total income last year was \$48,000. Her income from teaching was just \$6000 more than her income from house painting. Royalties from her textbook sales were one-seventh of the total money she received from teaching and house painting. How much did she make from each source last year?

40. ***Pocket change.*** Harry has \$2.25 in nickels, dimes, and quarters. If he had twice as many nickels, half as many dimes, and the same number of quarters, he would have \$2.50. If he has 27 coins altogether, then how many of each does he have?

GETTING MORE INVOLVED

41. ***Exploration.*** Draw diagrams showing the possible ways to position three planes in three-dimensional space.

42. ***Discussion.*** Make up a system of three linear equations in three variables for which the solution set is $\{(0, 0, 0)\}$. A system with this solution set is called a *homogeneous* system. Why do you think it is given that name?

43. ***Cooperative learning.*** Working in groups, do parts (a)–(d) below. Then write a report on your findings.

a) Find values of a, b, and c so that the graph of $y = ax^2 + bx + c$ goes through the points $(-1, -2)$, $(1, 0)$, and $(2, 7)$.

b) Arbitrarily select three ordered pairs and find the equation of the parabola that goes through the three points.

c) Could more than one parabola pass through three given points? Give reasons for your answer.

d) Explain how to pick three points for which no parabola passes through all of them.

8.4 SOLVING LINEAR SYSTEMS USING MATRICES

In this section

- Matrices
- The Augmented Matrix
- The Gaussian Elimination Method
- Inconsistent and Dependent Equations

You solved linear systems in two variables by substitution and addition in Sections 8.1 and 8.2. Those methods are done differently on each system. In this section you will learn the Gaussian elimination method, which is related to the addition method. The Gaussian elimination method is performed in the same way on every system. We first need to introduce some new terminology.

Matrices

A **matrix** is a rectangular array of numbers. The **rows** of a matrix run horizontally, and the **columns** of a matrix run vertically. A matrix with m rows and n columns has **order** $m \times n$ (read "m by n"). Each number in a matrix is called an **element** or **entry** of the matrix.

EXAMPLE 1 **Order of a matrix**

Determine the order of each matrix.

a) $\begin{bmatrix} -1 & 2 \\ 5 & \sqrt{2} \\ 0 & 3 \end{bmatrix}$ **b)** $\begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix}$ **c)** $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -1 & 0 & 2 \end{bmatrix}$ **d)** $\begin{bmatrix} 1 & 3 & 6 \end{bmatrix}$

Solution

Because matrix (a) has 3 rows and 2 columns, its order is 3×2. Matrix (b) is a 2×2 matrix, matrix (c) is a 3×3 matrix, and matrix (d) is a 1×3 matrix. ■

study tip

As soon as possible after class, find a quiet place and work on your homework. The longer you wait, the harder it is to remember what happened in class.

The Augmented Matrix

The solution to a system of linear equations such as

$$\begin{aligned} x - 2y &= -5 \\ 3x + y &= 6 \end{aligned}$$

depends on the coefficients of x and y and the constants on the right-hand side of the equation. The matrix of coefficients for this system is the 2×2 matrix

$$\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}.$$

If we insert the constants from the right-hand side of the system into the matrix of coefficients, we get the 2×3 matrix

$$\left[\begin{array}{cc|c} 1 & -2 & -5 \\ 3 & 1 & 6 \end{array}\right].$$

We use a vertical line between the coefficients and the constants to represent the equal signs. This matrix is the **augmented matrix** of the system. Two systems of linear equations are **equivalent** if they have the same solution set. Two augmented matrices are **equivalent** if the systems they represent are equivalent.

EXAMPLE 2 **Writing the augmented matrix**

Write the augmented matrix for each system of equations.

a) $$\begin{aligned} 3x - 5y &= 7 \\ x + y &= 4 \end{aligned}$$

b) $$\begin{aligned} x + y - z &= 5 \\ 2x \quad + z &= 3 \\ 2x - y + 4z &= 0 \end{aligned}$$

c) $$\begin{aligned} x + y \quad &= 1 \\ y + z &= 6 \\ z &= -5 \end{aligned}$$

Solution

a) $\left[\begin{array}{cc|c} 3 & -5 & 7 \\ 1 & 1 & 4 \end{array}\right]$ **b)** $\left[\begin{array}{ccc|c} 1 & 1 & -1 & 5 \\ 2 & 0 & 1 & 3 \\ 2 & -1 & 4 & 0 \end{array}\right]$ **c)** $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 1 & -5 \end{array}\right]$ ■

EXAMPLE 3 **Writing the system**

Write the system of equations represented by each augmented matrix.

a) $\left[\begin{array}{cc|c} 1 & 4 & -2 \\ 1 & -1 & 3 \end{array}\right]$ **b)** $\left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 1 \end{array}\right]$ **c)** $\left[\begin{array}{ccc|c} 2 & 3 & 4 & 6 \\ -1 & 0 & 5 & -2 \\ 1 & -2 & 3 & 1 \end{array}\right]$

Solution

a) Use the first two numbers in each row as the coefficients of x and y and the last number as the constant to get the following system:

$$\begin{aligned} x + 4y &= -2 \\ x - y &= 3 \end{aligned}$$

b) Use the first two numbers in each row as the coefficients of x and y and the last number as the constant to get the following system:

$$\begin{aligned} x &= 5 \\ y &= 1 \end{aligned}$$

c) Use the first three numbers in each row as the coefficients of x, y, and z and the last number as the constant to get the following system:

$$\begin{aligned} 2x + 3y + 4z &= 6 \\ -x \quad + 5z &= -2 \\ x - 2y + 3z &= 1 \end{aligned}$$

■

study tip

When doing homework or taking notes, use a pencil with an eraser. Everyone makes mistakes. If you get a problem wrong, don't start over. Check your work for errors and use the eraser. It is better to find out where you went wrong than simply to get the right answer.

The Gaussian Elimination Method

When we solve a single equation, we write simpler and simpler equivalent equations to get an equation whose solution is obvious. In the **Gaussian elimination method** we write simpler and simpler equivalent augmented matrices until we get an augmented matrix (like the one in Example 3(b)) in which the solution to the corresponding system is obvious.

Because each row of an augmented matrix represents an equation, we can perform the **row operations** on the augmented matrix. These row operations, which follow, correspond to the usual operations with equations used in the addition method.

Row Operations

The following row operations on an augmented matrix give an equivalent augmented matrix:

1. Interchange two rows of the matrix.
2. Multiply every element in a row by a nonzero real number.
3. Add to a row a multiple of another row.

In the Gaussian elimination method our goal is to use row operations to obtain an augmented matrix that has ones on the **diagonal** in its matrix of coefficients and zeros elsewhere:

$$\left[\begin{array}{cc|c} 1 & 0 & a \\ 0 & 1 & b \end{array}\right]$$

The system corresponding to this augmented matrix is $x = a$ and $y = b$. So the solution set to the system is $\{(a, b)\}$.

EXAMPLE 4 **Gaussian elimination with two equations in two variables**

Use the Gaussian elimination method to solve the system:

$$\begin{aligned} x - 3y &= 11 \\ 2x + y &= 1 \end{aligned}$$

study tip

Be active in class. Don't be embarrassed to ask questions or answer questions. You can often learn more from a wrong answer than from a right one. Your instructor knows that you are not yet an expert in algebra. Instructors love active classrooms and will not think less of you for speaking out.

Solution

Start with the augmented matrix:

$$\left[\begin{array}{cc|c} 1 & -3 & 11 \\ 2 & 1 & 1 \end{array}\right]$$

Multiply row 1 (R_1) by -2 and add the result to row 2 (R_2). So R_2 is replaced with $-2R_1 + R_2$. In symbols $-2R_1 + R_2 \rightarrow R_2$. Read the arrow as "replaces." Because $-2R_1 = [-2, 6, -22]$ and $R_2 = [2, 1, 1]$, $-2R_1 + R_2 = [0, 7, -21]$. Note that the coefficient of x in the second equation is now 0. We get the following matrix:

$$\left[\begin{array}{cc|c} 1 & -3 & 11 \\ 0 & 7 & -21 \end{array}\right] \quad -2R_1 + R_2 \rightarrow R_2$$

Multiply each element of row 2 by $\frac{1}{7}$ (in symbols, $\frac{1}{7}R_2 \to R_2$):

$$\left[\begin{array}{rr|r} 1 & -3 & 11 \\ 0 & 1 & -3 \end{array}\right] \quad \frac{1}{7}R_2 \to R_2$$

Multiply row 2 by 3 and add the result to row 1. Because $3R_2 = [0, 3, -9]$ and $R_1 = [1, -3, 11]$, $3R_2 + R_1 = [1, 0, 2]$. Note that the coefficient of y in the first equation is now 0. We get the following matrix:

$$\left[\begin{array}{rr|r} 1 & 0 & 2 \\ 0 & 1 & -3 \end{array}\right] \quad 3R_2 + R_1 \to R_1$$

This augmented matrix represents the system $x = 2$ and $y = -3$. So the solution set to the system is $\{(2, -3)\}$. Check in the original system.

The augmented matrix in Example 4 started off with a 1 in the first position on the diagonal. If that position contains a nonzero number other than 1, we can divide the first row by that number. This division might cause fractions to appear in the augmented matrix as shown in the next example.

EXAMPLE 5 **Gaussian elimination involving fractions**

Use the Gaussian elimination method to solve the system

$$2x - 3y = 8$$
$$3x + 2y = -1$$

Solution

Start with the augmented matrix and multiply the first row by $\frac{1}{2}$ (or divide it by 2):

$$\left[\begin{array}{rr|r} 2 & -3 & 8 \\ 3 & 2 & -1 \end{array}\right]$$

$$\left[\begin{array}{rr|r} 1 & -\frac{3}{2} & 4 \\ 3 & 2 & -1 \end{array}\right] \quad \frac{1}{2}R_1 \to R_1$$

Now multiply the first row by -3 and add the result onto the second row:

$$\left[\begin{array}{rr|r} 1 & -\frac{3}{2} & 4 \\ 0 & \frac{13}{2} & -13 \end{array}\right] \quad -3R_1 + R_2 \to R_2$$

Multiply the second row by $\frac{2}{13}$:

$$\left[\begin{array}{rr|r} 1 & -\frac{3}{2} & 4 \\ 0 & 1 & -2 \end{array}\right] \quad \frac{2}{13}R_2 \to R_2$$

Multiply the second row by $\frac{3}{2}$ and add the result onto the first row:

$$\left[\begin{array}{rr|r} 1 & 0 & 1 \\ 0 & 1 & -2 \end{array}\right] \quad \frac{3}{2}R_2 + R_1 \to R_1$$

The final augmented matrix represents the system $x = 1$ and $y = -2$. So the solution set is $\{(1, -2)\}$.

In the next example we use the row operations on the augmented matrix of a system of three linear equations in three variables.

EXAMPLE 6 **Gaussian elimination with three equations in three variables**

Use the Gaussian elimination method to solve the following system:

$$\begin{aligned} 2x - y + z &= -3 \\ x + y - z &= 6 \\ 3x - y - z &= 4 \end{aligned}$$

Solution

Start with the augmented matrix and interchange the first and second rows to get a 1 in the upper left position in the matrix:

$$\left[\begin{array}{rrr|r} 2 & -1 & 1 & -3 \\ 1 & 1 & -1 & 6 \\ 3 & -1 & -1 & 4 \end{array}\right] \quad \text{The augmented matrix}$$

$$\left[\begin{array}{rrr|r} 1 & 1 & -1 & 6 \\ 2 & -1 & 1 & -3 \\ 3 & -1 & -1 & 4 \end{array}\right] \quad R_1 \leftrightarrow R_2$$

Now multiply the first row by -2 and add the result onto the second row. Multiply the first row by -3 and add the result onto the third row. These two steps eliminate the variable x from the second and third rows:

$$\left[\begin{array}{rrr|r} 1 & 1 & -1 & 6 \\ 0 & -3 & 3 & -15 \\ 0 & -4 & 2 & -14 \end{array}\right] \quad \begin{array}{l} -2R_1 + R_2 \to R_2 \\ -3R_1 + R_3 \to R_3 \end{array}$$

Multiply the second row by $-\frac{1}{3}$ to get 1 in the second position on the diagonal:

$$\left[\begin{array}{rrr|r} 1 & 1 & -1 & 6 \\ 0 & 1 & -1 & 5 \\ 0 & -4 & 2 & -14 \end{array}\right] \quad -\tfrac{1}{3}R_2 \to R_2$$

Use the second row to eliminate the variable y from the first and third rows:

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 5 \\ 0 & 0 & -2 & 6 \end{array}\right] \quad \begin{array}{l} -1R_2 + R_1 \to R_1 \\ \\ 4R_2 + R_3 \to R_3 \end{array}$$

Multiply the third row by $-\frac{1}{2}$ to get a 1 in the third position on the diagonal:

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 5 \\ 0 & 0 & 1 & -3 \end{array}\right] \quad -\tfrac{1}{2}R_3 \to R_3$$

helpful hint

It is not necessary to perform the row operations in exactly the same order as is shown in Example 6. As long as you use the legitimate row operations and get to the final form, you will get the solution to the system. Of course, you must double check your arithmetic at every step if you want to be successful at Gaussian elimination.

Use the third row to eliminate the variable z from the second row:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{array}\right] \quad R_3 + R_2 \to R_2$$

This last augmented matrix represents the system $x = 1$, $y = 2$, and $z = -3$. So the solution set to the system is $\{(1, 2, -3)\}$.

Gaussian elimination may not be as easy to perform on some systems as it was in Example 6. Fractions may appear early in the process, making the computations more difficult. However, computers and even graphing calculators can perform row operations on matrices. So problems with computation can be overcome. The systems that we solve in this section will not be that complicated, but they will increase your understanding of Gaussian elimination.

Inconsistent and Dependent Equations

Inconsistent and dependent equations are easily recognized in using the Gaussian elimination method.

EXAMPLE 7 **Gaussian elimination with an inconsistent system**

Solve the system:

$$\begin{aligned} x - y &= 1 \\ -3x + 3y &= 4 \end{aligned}$$

Solution

Start with the augmented matrix:

$$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ -3 & 3 & 4 \end{array}\right]$$

Multiply row 1 by 3 and add the result to row 2. We get the following matrix:

$$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 0 & 7 \end{array}\right] \quad 3R_1 + R_2 \to R_2$$

The second row of the augmented matrix corresponds to the equation $0 = 7$. So the equations are inconsistent, and there is no solution to the system.

helpful hint

The point of Example 7 is to recognize an inconsistent system with Gaussian elimination. We could also observe that -3 times the first equation yields

$$-3x + 3y = -3,$$

which is inconsistent with

$$-3x + 3y = 4.$$

EXAMPLE 8 **Gaussian elimination with a dependent system**

Solve the system:

$$\begin{aligned} 3x + y &= 1 \\ 6x + 2y &= 2 \end{aligned}$$

Solution

Start with the augmented matrix:

$$\left[\begin{array}{cc|c} 3 & 1 & 1 \\ 6 & 2 & 2 \end{array}\right]$$

Multiply row 1 by -2 and add the result to row 2. We get the following matrix:

$$\left[\begin{array}{cc|c} 3 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right] \quad -2R_1 + R_2 \rightarrow R_2$$

In the second row of the augmented matrix we have the equation $0 = 0$. So the equations are dependent. Every ordered pair that satisfies the first equation satisfies both equations. The solution set is $\{(x, y) \mid 3x + y = 1\}$.

WARM-UPS

True or false? Explain your answer.

Statements 1–7 refer to the following matrices:

a) $\left[\begin{array}{cc|c} 1 & 3 & 5 \\ -1 & -3 & 2 \end{array}\right]$ **b)** $\left[\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & 0 & 7 \end{array}\right]$

c) $\left[\begin{array}{cc|c} -1 & 2 & -3 \\ 2 & -4 & 3 \end{array}\right]$ **d)** $\left[\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & 0 & 0 \end{array}\right]$

1. The augmented matrix for $x + 3y = 5$ and $-x - 3y = 2$ is matrix (a).
2. The augmented matrix for $2y - x = -3$ and $2x - 4y = 3$ is matrix (c).
3. Matrix (a) is equivalent to matrix (b).
4. Matrix (c) is equivalent to matrix (d).
5. The system corresponding to matrix (b) is inconsistent.
6. The system corresponding to matrix (c) is dependent.
7. The system corresponding to matrix (d) is independent.
8. The augmented matrix for a system of two linear equations in two unknowns is a 2×2 matrix.
9. The notation $2R_1 + R_3 \rightarrow R_3$ means to replace R_3 by $2R_1 + R_3$.
10. The notation $R_1 \leftrightarrow R_2$ means to replace R_2 by R_1.

8.4 EXERCISES

Reading and Writing *After reading this section, write out the answers to these questions. Use complete sentences.*

1. What is a matrix?
2. What is the difference between a row and a column of a matrix?
3. What is the order of a matrix?
4. What is an element of a matrix?
5. What is an augmented matrix?
6. What is the goal of Gaussian elimination?

Determine the order of each matrix. See Example 1.

7. $\begin{bmatrix} 5 & 0 \\ -2 & 3 \end{bmatrix}$ **8.** $\begin{bmatrix} 1 & 3 & 6 \\ -7 & 0 & 2 \end{bmatrix}$ **9.** $\begin{bmatrix} a & c \\ 0 & d \\ 3 & w \end{bmatrix}$

10. $\begin{bmatrix} 0 & a & b \\ 5 & 7 & -8 \\ a & b & 2 \end{bmatrix}$ **11.** $\begin{bmatrix} -\sqrt{3} \\ \pi \\ \frac{1}{2} \end{bmatrix}$ **12.** $[3 \quad 0 \quad 4]$

Write the augmented matrix for each system of equations. See Example 2.

13. $\begin{aligned} 2x - 3y &= 9 \\ -3x + y &= -1 \end{aligned}$

14. $\begin{aligned} x - y &= 4 \\ 2x + y &= 3 \end{aligned}$

15. $\begin{aligned} x - y + z &= 1 \\ x + y - 2z &= 3 \\ y - 3z &= 4 \end{aligned}$

16. $\begin{aligned} x + y &= 2 \\ y - 3z &= 5 \\ -3x + 2z &= 8 \end{aligned}$

Write the system of equations represented by each augmented matrix. See Example 3.

17. $\left[\begin{array}{rr|r} 5 & 1 & -1 \\ 2 & -3 & 0 \end{array}\right]$

18. $\left[\begin{array}{rr|r} 1 & 0 & 4 \\ 0 & 1 & -3 \end{array}\right]$

19. $\left[\begin{array}{rrr|r} 1 & 0 & 0 & 6 \\ -1 & 0 & 1 & -3 \\ 1 & 1 & 0 & 1 \end{array}\right]$

20. $\left[\begin{array}{rrr|r} 1 & 0 & 4 & 3 \\ 0 & 2 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{array}\right]$

Determine the row operation that was used to convert each given augmented matrix into the equivalent augmented matrix that follows it. See Example 4.

21. $\left[\begin{array}{rr|r} 3 & 2 & 12 \\ 1 & -1 & -1 \end{array}\right], \left[\begin{array}{rr|r} 1 & -1 & -1 \\ 3 & 2 & 12 \end{array}\right]$

22. $\left[\begin{array}{rr|r} 1 & -1 & -1 \\ 3 & 2 & 12 \end{array}\right], \left[\begin{array}{rr|r} 1 & -1 & -1 \\ 0 & 5 & 15 \end{array}\right]$

23. $\left[\begin{array}{rr|r} 1 & -1 & -1 \\ 0 & 5 & 15 \end{array}\right], \left[\begin{array}{rr|r} 1 & -1 & -1 \\ 0 & 1 & 3 \end{array}\right]$

24. $\left[\begin{array}{rr|r} 1 & -1 & -1 \\ 0 & 1 & 3 \end{array}\right], \left[\begin{array}{rr|r} 1 & 0 & 2 \\ 0 & 1 & 3 \end{array}\right]$

Solve each system using the Gaussian elimination method. See Examples 4–8.

25. $\begin{aligned} x + y &= 3 \\ -3x + y &= -1 \end{aligned}$

26. $\begin{aligned} x - y &= -1 \\ 2x - y &= 2 \end{aligned}$

27. $\begin{aligned} 2x - y &= 3 \\ x + y &= 9 \end{aligned}$

28. $\begin{aligned} 3x - 4y &= -1 \\ x - y &= 0 \end{aligned}$

29. $\begin{aligned} 3x - y &= 4 \\ 2x + y &= 1 \end{aligned}$

30. $\begin{aligned} 2x - y &= -3 \\ 3x + y &= -2 \end{aligned}$

31. $\begin{aligned} 6x - 7y &= 0 \\ 2x + y &= 20 \end{aligned}$

32. $\begin{aligned} 2x + y &= 11 \\ 2x - y &= 1 \end{aligned}$

33. $\begin{aligned} 2x - 3y &= 4 \\ -2x + 3y &= 5 \end{aligned}$

34. $\begin{aligned} x - 3y &= 8 \\ 2x - 6y &= 1 \end{aligned}$

35. $\begin{aligned} x + 2y &= 1 \\ 3x + 6y &= 3 \end{aligned}$

36. $\begin{aligned} 2x - 3y &= 1 \\ -6x + 9y &= -3 \end{aligned}$

37. $\begin{aligned} x + y + z &= 6 \\ x - y + z &= 2 \\ 2y - z &= 1 \end{aligned}$

38. $\begin{aligned} x - y - z &= 0 \\ -x - y + z &= -4 \\ -x + y - z &= -2 \end{aligned}$

39. $\begin{aligned} 2x + y + z &= 4 \\ x + y - z &= 1 \\ x - y + 2z &= 2 \end{aligned}$

40. $\begin{aligned} 3x - y &= 1 \\ x + y + z &= 4 \\ x + 2z &= 3 \end{aligned}$

41. $\begin{aligned} 2x - y + z &= 0 \\ x + y - 3z &= 3 \\ x - y + z &= -1 \end{aligned}$

42. $\begin{aligned} x - y - z &= 0 \\ -x - y + 2z &= -1 \\ -x + y - 2z &= -3 \end{aligned}$

43. $\begin{aligned} -x + 3y + z &= 0 \\ x - y - 4z &= -3 \\ x + y + 2z &= 3 \end{aligned}$

44. $\begin{aligned} -x + z &= -2 \\ 2x - y &= 5 \\ y + 3z &= 9 \end{aligned}$

45. $\begin{aligned} x - y + z &= 1 \\ 2x - 2y + 2z &= 2 \\ -3x + 3y - 3z &= -3 \end{aligned}$

46. $\begin{aligned} 4x - 2y + 2z &= 2 \\ 2x - y + z &= 1 \\ -2x + y - z &= -1 \end{aligned}$

47. $\begin{aligned} x + y - z &= 2 \\ 2x - y + z &= 1 \\ 3x + 3y - 3z &= 8 \end{aligned}$

48. $\begin{aligned} x + y + z &= 5 \\ x - y - z &= 8 \\ -x + y + z &= 2 \end{aligned}$

GETTING MORE INVOLVED

49. ***Cooperative learning.*** Write a step-by-step procedure for solving any system of two linear equations in two variables by the Gaussian elimination method. Have a classmate evaluate your procedure by using it to solve a system.

50. ***Cooperative learning.*** Repeat Exercise 49 for a system of three linear equations in three variables.