11.1 GRAPHS OF FUNCTIONS AND RELATIONS

In this section

- Linear and Constant Functions
- Absolute Value Functions
- Quadratic Functions
- Square-Root Functions
- Graphs of Relations
- Vertical-Line Test

Functions were introduced in Section 4.6. In this section we will study the graphs of several types of functions. We graphed linear functions in Chapter 4 and quadratic functions in Chapter 10, but for completeness we will review them here.

Linear and Constant Functions

Linear functions get their name from the fact that their graphs are straight lines.

Linear Function

A **linear function** is a function of the form

$$f(x) = mx + b,$$

where m and b are real numbers with $m \neq 0$.

The graph of the linear function $f(x) = mx + b$ is exactly the same as the graph of the linear equation $y = mx + b$. If $m = 0$, then we get $f(x) = b$, which is called a **constant function.** If $m = 1$ and $b = 0$, then we get the function $f(x) = x$, which is called the **identity function.** When we graph a function given in function notation, we usually label the vertical axis as $f(x)$ rather than y.

EXAMPLE 1

Graphing a constant function

Graph $f(x) = 3$ and state the domain and range.

Solution

The graph of $f(x) = 3$ is the same as the graph of $y = 3$, which is the horizontal line in Fig. 11.1. Since any real number can be used for x in $f(x) = 3$ and since the line in Fig. 11.1 extends without bounds to the left and right, the domain is the set of all real numbers, $(-\infty, \infty)$. Since the only y-coordinate for $f(x) = 3$ is 3, the range is $\{3\}$.

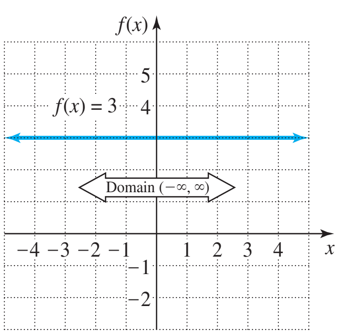


FIGURE 11.1

The domain and range of a function can be determined from the formula or the graph. However, the graph is usually very helpful for understanding domain and range.

EXAMPLE 2

Graphing a linear function

Graph the function $f(x) = 3x - 4$ and state the domain and range.

Solution

The y-intercept is $(0, -4)$ and the slope of the line is 3. We can use the y-intercept and the slope to draw the graph in Fig. 11.2. Since any real number can be used for x in $f(x) = 3x - 4$, and since the line in Fig. 11.2 extends without bounds to the left and right, the domain is the set of all real numbers, $(-\infty, \infty)$. Since the graph extends without bounds upward and downward, the range is the set of all real numbers, $(-\infty, \infty)$.

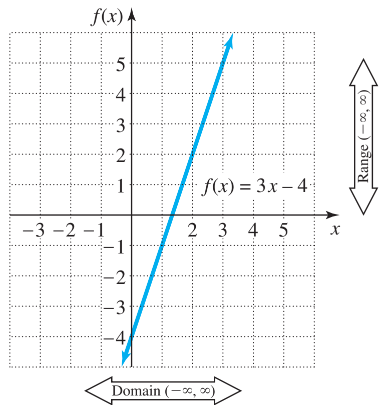


FIGURE 11.2

Absolute Value Functions

The equation $y = |x|$ defines a function because every value of x determines a unique value of y. We call this function the absolute value function.

Absolute Value Function

The **absolute value function** is the function defined by

$$f(x) = |x|.$$

To graph the absolute value function, we simply plot enough ordered pairs of the function to see what the graph looks like.

EXAMPLE 3

The absolute value function

Graph $f(x) = |x|$ and state the domain and range.

Solution

To graph this function, we find points that satisfy the equation $f(x) = |x|$.

x	-2	-1	0	1	2
$f(x) = \|x\|$	2	1	0	1	2

helpful hint

The most important feature of an absolute value function is its V-shape. If we had plotted only points in the first quadrant, we would not have seen the V-shape. So for an absolute value function we always plot enough points to see the V-shape.

Plotting these points, we see that they lie along the V-shaped graph shown in Fig. 11.3. Since any real number can be used for x in $f(x) = |x|$ and since the graph extends without bounds to the left and right, the domain is $(-\infty, \infty)$. Because the graph does not go below the x-axis and because $|x|$ is never negative, the range is the set of nonnegative real numbers, $[0, \infty)$.

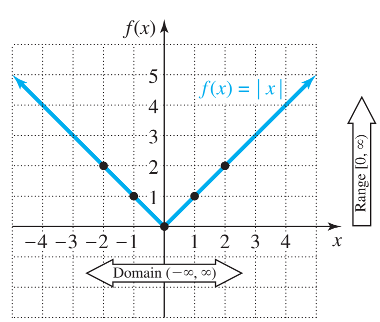


FIGURE 11.3

Many functions involving absolute value have graphs that are V-shaped, as in Fig. 11.3. To graph functions involving absolute value, we must choose points that determine the correct shape and location of the V-shaped graph.

EXAMPLE 4

Other functions involving absolute value

Graph each function and state the domain and range.

a) $f(x) = |x| - 2$ **b)** $g(x) = |2x - 6|$

Solution

a) Choose values for x and find $f(x)$.

x	-2	-1	0	1	2
$f(x) = \|x\| - 2$	0	-1	-2	-1	0

Plot these points and draw a V-shaped graph through them as shown in Fig. 11.4. The domain is $(-\infty, \infty)$, and the range is $[-2, \infty)$.

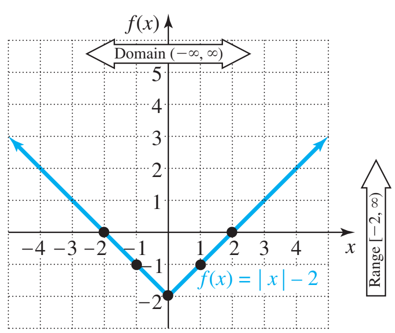


FIGURE 11.4

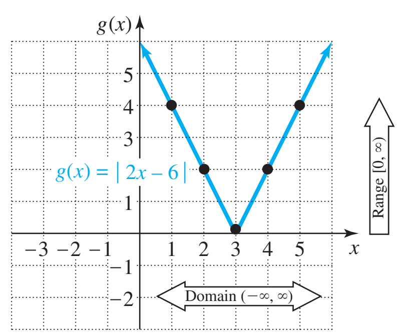


FIGURE 11.5

b) Make a table of values for x and $g(x)$.

x	1	2	3	4	5
$g(x) = \|2x - 6\|$	4	2	0	2	4

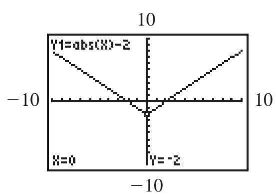
Draw the graph as shown in Fig. 11.5. The domain is $(-\infty, \infty)$, and the range is $[0, \infty)$.

calculator close-up

To check Example 4(a) set

$y_1 = \text{abs}(x) - 2$

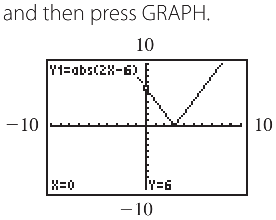
and then press GRAPH.



To check Example 4(b) set

$y_2 = \text{abs}(2x - 6)$

and then press GRAPH.



Quadratic Functions

In Chapter 10 we learned that the graph of any quadratic function is a parabola, which opens upward or downward. The vertex of a parabola is the lowest point on a parabola that opens upward or the highest point on a parabola that opens downward. Parabolas will be discussed again when we study conic sections in Chapter 13.

Quadratic Function

A **quadratic function** is a function of the form

$$f(x) = ax^2 + bx + c,$$

where a, b, and c are real numbers, with $a \neq 0$.

EXAMPLE 5

A quadratic function

Graph the function $g(x) = 4 - x^2$ and state the domain and range.

Solution

We plot enough points to get the correct shape of the graph.

x	-2	-1	0	1	2
$g(x) = 4 - x^2$	0	3	4	3	0

See Fig. 11.6 for the graph. The domain is $(-\infty, \infty)$. From the graph we see that the largest y-coordinate is 4. So the range is $(-\infty, 4]$.

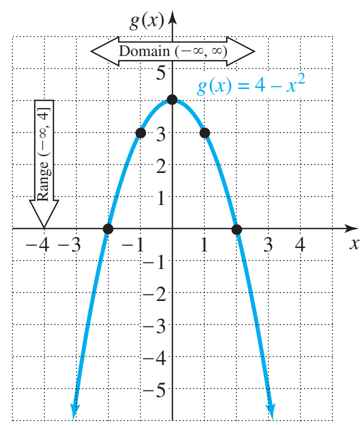


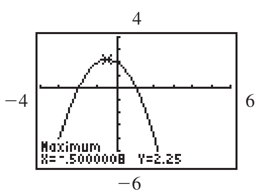
FIGURE 11.6

calculator close-up

You can find the vertex of a parabola with a calculator. For example, graph

$$y = -x^2 - x + 2.$$

Then use the maximum feature, which is found in the CALC menu. For the left bound pick a point to the left of the vertex; for the right bound pick a point to the right of the vertex; and for the guess pick a point near the vertex.



Square-Root Functions

Functions involving square roots typically have graphs that look like half a parabola.

Square-Root Function

The **square-root function** is the function defined by

$$f(x) = \sqrt{x}.$$

EXAMPLE 6

Square-root functions

Graph each equation and state the domain and range.

a) $y = \sqrt{x}$ **b)** $y = \sqrt{x + 3}$

Solution

a) The graph of the equation $y = \sqrt{x}$ and the graph of the function $f(x) = \sqrt{x}$ are the same. Because $\sqrt{x}$ is a real number only if $x \geq 0$, the domain of this function is the set of nonnegative real numbers. The following ordered pairs are on the graph:

x	0	1	4	9
$y = \sqrt{x}$	0	1	2	3

The graph goes through these ordered pairs as shown in Fig. 11.7. Note that x is chosen from the nonnegative numbers. The domain is $[0, \infty)$ and the range is $[0, \infty)$.

b) Note that $\sqrt{x+3}$ is a real number only if $x + 3 \geq 0$, or $x \geq -3$. So we make a table of ordered pairs in which $x \geq -3$:

x	-3	-2	1	6
$y = \sqrt{x+3}$	0	1	2	3

The graph goes through these ordered pairs as shown in Fig. 11.8. The domain is $[-3, \infty)$ and the range is $[0, \infty)$.

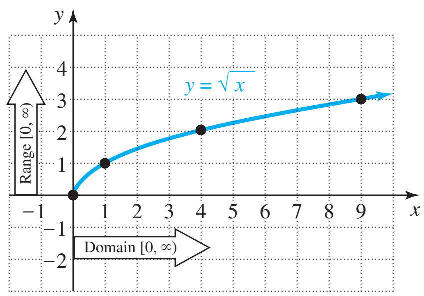


FIGURE 11.7

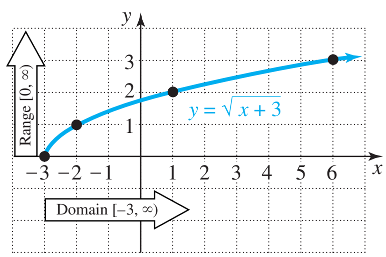


FIGURE 11.8

Graphs of Relations

A function is a set of ordered pairs in which no two have the same first coordinate and different second coordinates. A **relation** is any set of ordered pairs. Relations and functions can be defined by equations. For example, the set of ordered pairs that satisfies $x = y^2$ is a relation. The equation $x = y^2$ does not define y as a function of x because ordered pairs such as $(4, 2)$ and $(4, -2)$ satisfy $x = y^2$.

The domain of a relation is the set of x-coordinates of the ordered pairs and the range of a relation is the set of y-coordinates. In the next example we graph the relation $x = y^2$ by simply plotting enough points to see the shape of the graph.

EXAMPLE 7 **The graph of a relation**

Graph $x = y^2$ and state the domain and range.

Solution

Because the equation $x = y^2$ expresses x in terms of y, it is easier to choose the y-coordinate first and then find the x-coordinate.

$x = y^2$	4	1	0	1	4
y	-2	-1	0	1	2

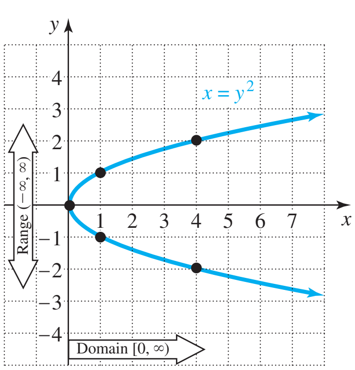


FIGURE 11.9

Figure 11.9 shows the graph. The domain is $[0, \infty)$ and the range is $(-\infty, \infty)$.

Vertical-Line Test

Every graph is the graph of a relation, because a relation is any set of ordered pairs. We did not use the term "relation" when we discussed the vertical line test in Section 4.6. So we restate the vertical line test here using that term.

Vertical-Line Test

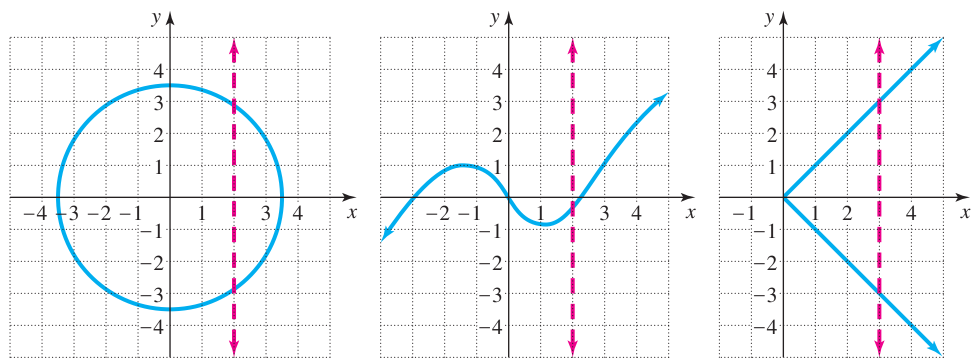
If it is possible to draw a vertical line that crosses the graph of a relation two or more times, then the graph is not the graph of a function.

If there is a vertical line that crosses a graph twice (or more), then we have two points (or more) with the same x-coordinate and different y-coordinates, and so the graph is not the graph of a function. If you mentally consider every possible vertical line and none of them crosses the graph more than once, then you can conclude that the graph is the graph of a function.

EXAMPLE 8 **Using the vertical-line test**

Which of the following graphs are graphs of functions?

a) **b)** **c)**



Solution

Neither (a) nor (c) is the graph of a function, since we can draw vertical lines that cross these graphs twice. The graph (b) is the graph of a function, since no vertical line crosses it twice.

WARM-UPS

True or false? Explain your answer.

1. The graph of a function is a picture of all ordered pairs of the function.
2. The graph of every linear function is a straight line.
3. The absolute value function has a V-shaped graph.
4. The domain of $f(x) = \frac{1}{x}$ is $(-\infty, \infty)$.
5. The graph of a quadratic function is a parabola.
6. The range of any quadratic function is $(-\infty, \infty)$.
7. The y-axis and the $f(x)$-axis are the same.
8. The domain of $x = y^2$ is $[0, \infty)$.
9. The domain of $f(x) = \sqrt{x - 1}$ is $(1, \infty)$.
10. The domain of any quadratic function is $(-\infty, \infty)$.

11.1 EXERCISES

Reading and Writing *After reading this section, write out the answers to these questions. Use complete sentences.*

1. What is a linear function?
2. What is a constant function?
3. What is the graph of a constant function?
4. What shape is the graph of an absolute value function?
5. What is the graph of quadratic function called?
6. How can you tell at a glance if a graph is the graph of a function?

Graph each function and state its domain and range. See Examples 1 and 2.

7. $h(x) = -2$

8. $f(x) = 4$

9. $f(x) = 2x - 1$

10. $g(x) = x + 2$

11. $g(x) = \frac{1}{2}x + 2$

12. $h(x) = \frac{2}{3}x - 4$

13. $y = -\frac{2}{3}x + 3$

14. $y = -\frac{3}{4}x + 4$

15. $y = -0.3x + 6.5$

16. $y = 0.25x - 0.5$

Graph each absolute value function and state its domain and range. See Examples 3 and 4.

17. $f(x) = |x| + 1$

18. $g(x) = |x| - 3$

19. $h(x) = |x + 1|$

20. $f(x) = |x - 2|$

21. $g(x) = |3x|$

22. $h(x) = |-2x|$

23. $f(x) = |2x - 1|$

24. $y = |2x - 3|$

25. $f(x) = |x - 2| + 1$

26. $y = |x - 1| + 2$

Graph each quadratic function and state its domain and range. See Example 5.

27. $g(x) = x^2 + 2$

28. $f(x) = x^2 - 4$

29. $f(x) = 2x^2$

30. $h(x) = -3x^2$

31. $y = 6 - x^2$

32. $y = -2x^2 + 3$

Graph each square-root function and state its domain and range. See Example 6.

33. $g(x) = 2\sqrt{x}$

34. $g(x) = \sqrt{x} - 1$

35. $f(x) = \sqrt{x - 1}$

36. $f(x) = \sqrt{x + 1}$

37. $h(x) = -\sqrt{x}$

38. $h(x) = -\sqrt{x - 1}$

39. $y = \sqrt{x} + 2$

40. $y = 2\sqrt{x} + 1$

Graph each relation and state its domain and range. See Example 7.

41. $x = |y|$

42. $x = -|y|$

43. $x = -y^2$

44. $x = 1 - y^2$

45. $x = 5$

46. $x = -3$

47. $x + 9 = y^2$

48. $x + 3 = |y|$

49. $x = \sqrt{y}$

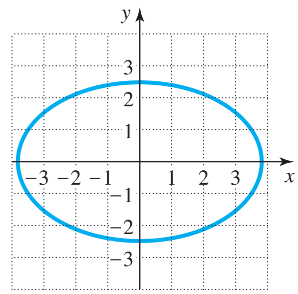
50. $x = -\sqrt{y}$

51. $x = (y - 1)^2$

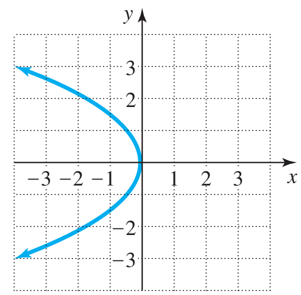
52. $x = (y + 2)^2$

Each of the following graphs is the graph of a relation. Use the vertical-line test on each graph to determine whether y is a function of x. See Example 8.

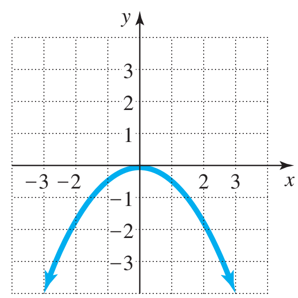
53.



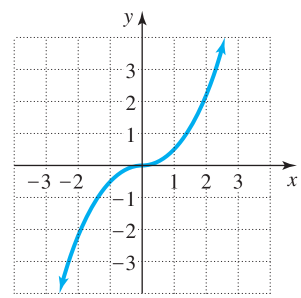
54.



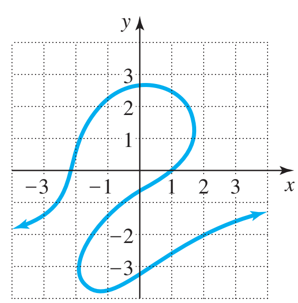
55.



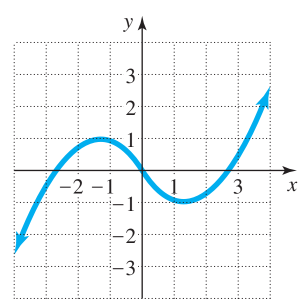
56.



57.



58.



Graph each function and state the domain and range.

59. $f(x) = 1 - |x|$

60. $h(x) = \sqrt{x - 3}$

61. $y = (x - 3)^2 - 1$

62. $y = x^2 - 2x - 3$

63. $y = |x + 3| + 1$

64. $f(x) = -2x + 4$

65. $y = \sqrt{x} - 3$

66. $y = 2|x|$

67. $y = 3x - 5$

68. $g(x) = (x + 2)^2$

69. $y = -x^2 + 4x - 4$

70. $y = -2|x - 1| + 4$

GRAPHING CALCULATOR EXERCISES

71. Graph the function $f(x) = \sqrt{x^2}$ and explain what this graph illustrates.

72. Graph the function $f(x) = \frac{1}{x}$ and state the domain and range.

73. Graph $y = x^2$, $y = \frac{1}{2}x^2$, and $y = 2x^2$ on the same coordinate system. What can you say about the graph of $y = kx^2$?

74. Graph $y = x^2$, $y = x^2 + 2$, and $y = x^2 - 3$ on the same screen. What can you say about the position of $y = x^2 + k$ relative to $y = x^2$.

75. Graph $y = x^2$, $y = (x + 5)^2$, and $y = (x - 2)^2$ on the same screen. What can you say about the position of $y = (x - k)^2$ relative to $y = x^2$.

76. You can graph the relation $x = y^2$ by graphing the two functions $y = \sqrt{x}$ and $y = -\sqrt{x}$. Try it and explain why this works.

77. Graph $y = (x - 3)^2$, $y = |x - 3|$, and $y = \sqrt{x - 3}$ on the same coordinate system. How does the graph of $y = f(x - k)$ compare to the graph of $y = f(x)$?