**61.** $y = (x - 3)^2 - 1$

**62.** $y = x^2 - 2x - 3$

**63.** $y = |x + 3| + 1$

**64.** $f(x) = -2x + 4$

**65.** $y = \sqrt{x} - 3$

**66.** $y = 2|x|$

**67.** $y = 3x - 5$

**68.** $g(x) = (x + 2)^2$

**69.** $y = -x^2 + 4x - 4$

**70.** $y = -2|x - 1| + 4$

## GRAPHING CALCULATOR EXERCISES

**71.** Graph the function $f(x) = \sqrt{x^2}$ and explain what this graph illustrates.

**72.** Graph the function $f(x) = \frac{1}{x}$ and state the domain and range.

**73.** Graph $y = x^2$, $y = \frac{1}{2}x^2$, and $y = 2x^2$ on the same coordinate system. What can you say about the graph of $y = kx^2$?

**74.** Graph $y = x^2$, $y = x^2 + 2$, and $y = x^2 - 3$ on the same screen. What can you say about the position of $y = x^2 + k$ relative to $y = x^2$.

**75.** Graph $y = x^2$, $y = (x + 5)^2$, and $y = (x - 2)^2$ on the same screen. What can you say about the position of $y = (x - k)^2$ relative to $y = x^2$.

**76.** You can graph the relation $x = y^2$ by graphing the two functions $y = \sqrt{x}$ and $y = -\sqrt{x}$. Try it and explain why this works.

**77.** Graph $y = (x - 3)^2$, $y = |x - 3|$, and $y = \sqrt{x - 3}$ on the same coordinate system. How does the graph of $y = f(x - k)$ compare to the graph of $y = f(x)$?

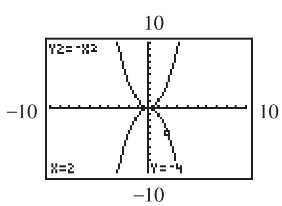
# 11.2 TRANSFORMATIONS OF GRAPHS

**In this section**

- Reflecting
- Translating
- Stretching and Shrinking
- Graphing Parabolas

**calculator close-up**

With a graphing calculator, you can quickly see the result of modifying the formula for a function. If you have a graphing calculator, use it to graph the functions in the examples. Experimenting with it will help you to understand the ideas in this section.




We can discover what the graph of almost any function looks like if we plot enough points. However, it is helpful to know something about a graph so that we do not have to plot very many points. In this section we will learn how one graph can be transformed into another by modifying the formula that defines the function.

## Reflecting

Consider the graphs of $f(x) = x^2$ and $g(x) = -x^2$ shown in Fig. 11.10. Notice that the graph of $g$ is a mirror image of the graph of $f$. For any value of $x$ we compute the $y$-coordinate of an ordered pair of $f$ by squaring $x$. For an ordered pair of $g$ we square first and then find the opposite because of the order of operations. This gives a correspondence between the ordered pairs of $f$ and the ordered pairs of $g$. For every ordered pair on the graph of $f$ there is a corresponding ordered pair directly below it on the graph of $g$, and these ordered pairs are the same distance from the $x$-axis. We say that the graph of $g$ is obtained by reflecting the graph of $f$ in the $x$-axis or that $g$ is a reflection of the graph of $f$.

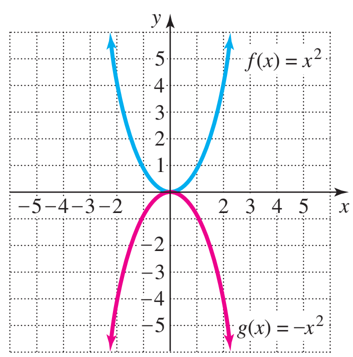



FIGURE 11.10

**Reflection**

The graph of $y = -f(x)$ is a **reflection** in the $x$-axis of the graph of $y = f(x)$.

**EXAMPLE 1** **Reflection**

Sketch the graphs of each pair of functions on the same coordinate system.

**a)** $f(x) = \sqrt{x}$, $g(x) = -\sqrt{x}$

**b)** $f(x) = |x|$, $g(x) = -|x|$

**Solution**

In each case the graph of $g$ is a reflection of the graph of $f$. Recall that we graphed the square root function and the absolute value function in the last section. Figures 11.11 and 11.12 show the graphs for these functions.

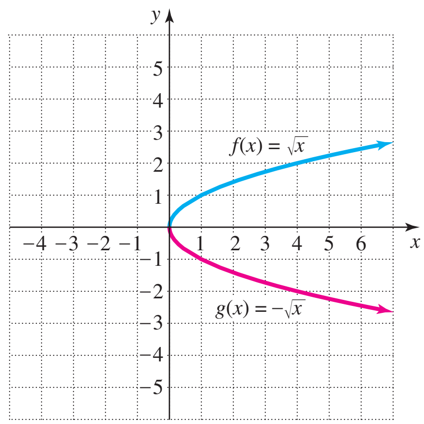



FIGURE 11.11

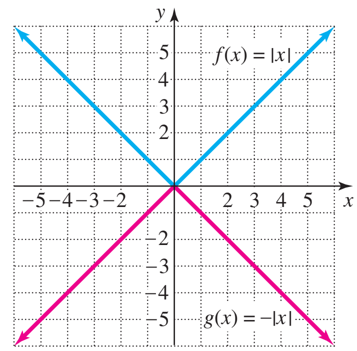



FIGURE 11.12

## Translating

Consider the graphs of the functions $f(x) = \sqrt{x}$, $g(x) = \sqrt{x} + 2$, and $h(x) = \sqrt{x} - 6$ shown in Fig. 11.13. In the expression $\sqrt{x} + 2$, adding 2 is the last operation to perform. So every point on the graph of $g$ is exactly two units above a corresponding point on the graph of $f$, and $g$ has the same shape as the graph of $f$. Every point on the graph of $h$ is exactly six units below a corresponding point on the graph of $f$. The graph of $g$ is an upward translation of the graph of $f$, and the graph of $h$ is a downward translation of the graph of $f$.

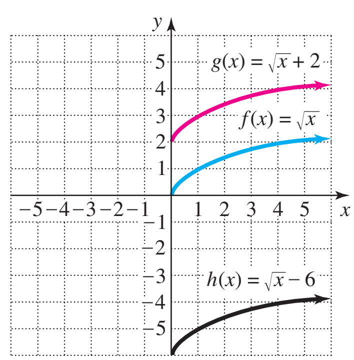



FIGURE 11.13

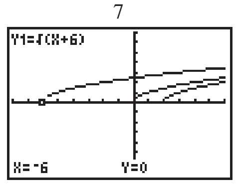
### Translating Upward or Downward

If $k > 0$, then the graph of $y = f(x) + k$ is an **upward translation** of the graph of $y = f(x)$ and the graph of $y = f(x) - k$ is a **downward translation** of the graph of $y = f(x)$.

Consider the graphs of $f(x) = \sqrt{x}$, $g(x) = \sqrt{x - 2}$, and $h(x) = \sqrt{x + 6}$ shown in Fig. 11.14. In the expression $\sqrt{x - 2}$ subtracting 2 is the first operation to perform. So every point on the graph of $g$ is exactly two units to the right of a corresponding point on the graph of $f$. (We must start with a larger value of $x$ to get the

**calculator close-up**

Note that for a translation of six units to the left, $x + 6$ must be written in parentheses on a graphing calculator.




same $y$-coordinate because we first subtract 2.) Every point on the graph of $h$ is exactly six units to the left of a corresponding point on the graph of $f$.

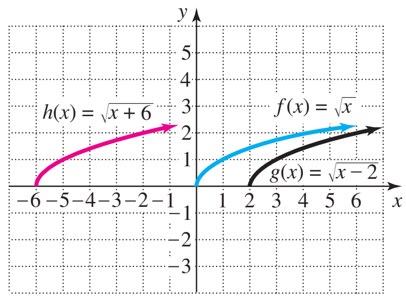



**FIGURE 11.14**

### Translating to the Right or Left

If $h > 0$, then the graph of $y = f(x - h)$ is a **translation to the right** of the graph of $y = f(x)$, and the graph of $y = f(x + h)$ is a **translation to the left** of the graph of $y = f(x)$.

## EXAMPLE 2

**Translation**

Sketch the graph of each function.

**a)** $f(x) = |x| - 6$ **b)** $f(x) = (x - 2)^2$

**c)** $f(x) = |x + 3|$

### Solution

**a)** The graph of $f(x) = |x| - 6$ is a translation six units downward of the familiar graph of $f(x) = |x|$. Calculate a few ordered pairs to get an accurate graph. The pairs $(0, -6)$, $(1, -5)$, and $(-1, -5)$ are on the graph shown in Fig. 11.15.

**b)** The graph of $f(x) = (x - 2)^2$ is a translation two units to the right of the familiar graph of $f(x) = x^2$. Calculate a few ordered pairs to get an accurate graph. The pairs $(2, 0)$, $(0, 4)$, and $(4, 4)$ are on the graph shown in Fig. 11.16.

**c)** The graph of $f(x) = |x + 3|$ is a translation three units to the left of the familiar graph of $f(x) = |x|$. The pairs $(0, 3)$, $(-3, 0)$, and $(-6, 3)$ are on the graph shown in Fig. 11.17.

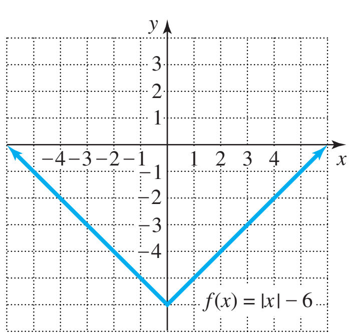



**FIGURE 11.15**

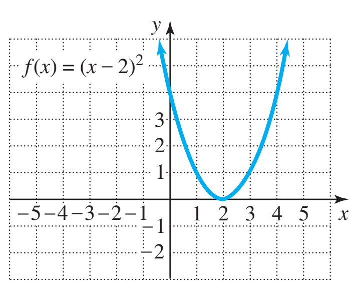



**FIGURE 11.16**

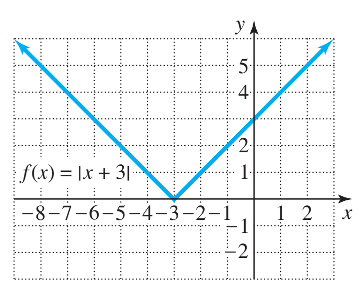


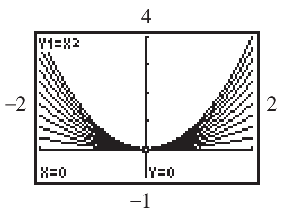

**FIGURE 11.17**

## Stretching and Shrinking

Consider the graphs of $f(x) = x^2$, $g(x) = 2x^2$, and $h(x) = \frac{1}{2}x^2$ shown in Fig. 11.18. Every point on $g(x) = 2x^2$ corresponds to a point directly below on the graph of $f(x) = x^2$. The $y$-coordinate on $g$ is exactly twice as large as the corresponding $y$-coordinate on $f$. This situation occurs because in the expression $2x^2$, multiplying by 2 is the last operation performed. Every point on $h$ corresponds to a point directly above on $f$, where the $y$-coordinate on $h$ is half as large as the $y$-coordinate on $f$. The factor 2 has stretched the graph of $f$ to form the graph of $g$, and the factor $\frac{1}{2}$ has shrunk the graph of $f$ to form the graph of $h$.

**calculator close-up**

A typical graphing calculator can draw 10 curves on the same screen. On this screen there are the curves $y = 0.1x^2$, $y = 0.2x^2$, and so on, through $y = x^2$.



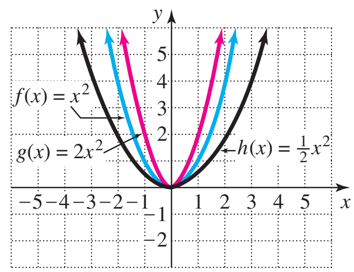




**FIGURE 11.18**

> **Stretching and Shrinking**
>
> If $a > 1$, then the graph of $y = af(x)$ is obtained by **stretching** the graph of $y = f(x)$. If $0 < a < 1$, then the graph of $y = af(x)$ is obtained by **shrinking** the graph of $y = f(x)$.

Note that the last operation to be performed in stretching or shrinking is multiplication by $a$. Whereas the function $g(x) = 2\sqrt{x}$ is obtained by stretching $f(x) = \sqrt{x}$ by a factor of 2, $h(x) = \sqrt{2x}$ is not.

**EXAMPLE 3** **Stretching and shrinking**

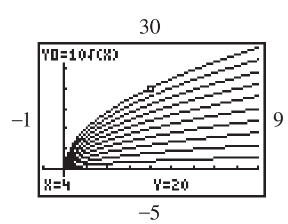
Graph the functions $f(x) = \sqrt{x}$, $g(x) = 2\sqrt{x}$, and $h(x) = \frac{1}{2}\sqrt{x}$ on the same coordinate system.

### Solution

The graph of $g$ is obtained by stretching the graph of $f$, and the graph of $h$ is obtained by shrinking the graph of $f$. The graph of $f$ includes the points (0, 0), (1, 1), and (4, 2). The graph of $g$ includes the points (0, 0), (1, 2), and (4, 4). The graph of $h$ includes the points (0, 0), (1, 0.5), and (4, 1). The graphs are shown in Fig. 11.19.

**calculator close-up**

The following calculator screen shows the curves $y = \sqrt{x}$, $y = 2\sqrt{x}$, $y = 3\sqrt{x}$, and so on, through $y = 10\sqrt{x}$.



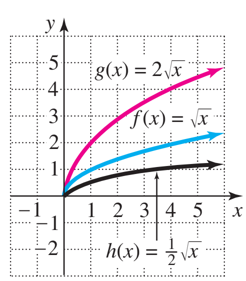




**FIGURE 11.19**

In the next example we graph a function that results from a combination of translating, stretching, and reflecting.

**EXAMPLE 4** **Translating, stretching, and reflecting**

Graph the function $y = -2\sqrt{x - 3}$.

### Solution

The graph of $y = \sqrt{x - 3}$ is a translation three units to the right of the graph of $y = \sqrt{x}$. The graph of $y = 2\sqrt{x - 3}$ is obtained from $y = \sqrt{x - 3}$ by stretching by a factor of 2. The graph of $y = -2\sqrt{x - 3}$ is the mirror image of the graph of $y = 2\sqrt{x - 3}$. All of these graphs are shown in Fig. 11.20.

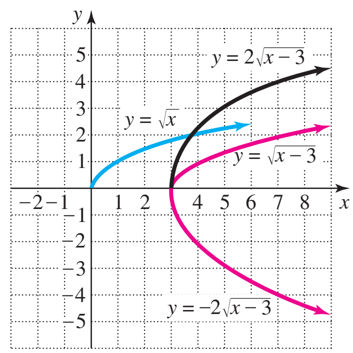
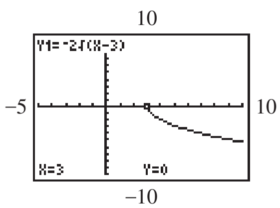



**FIGURE 11.20**

**calculator close-up**

You can check Example 4 by graphing $y = -2\sqrt{x - 3}$ with a graphing calculator.




## Graphing Parabolas

The graph of $y = x^2$ is a parabola with vertex (0, 0). The graph of $y = a(x - h)^2 + k$ is a transformation of $y = x^2$ and is also a parabola. It opens upward if $a > 0$ and downward if $a < 0$. Its vertex is $(h, k)$. To graph a parabola in the form $y = a(x - h)^2 + k$, determine its vertex and a couple of points near the vertex.

**EXAMPLE 5** **Graphing the parabola $y = a(x - h)^2 + k$**

Graph $y = -2(x + 3)^2 + 4$.

### Solution

Because $x + 3 = x - (-3)$, we have $h = -3$. The vertex is $(-3, 4)$. Since $a = -2$, the parabola opens downward. The points $(-2, 2)$ and $(-4, 2)$ also satisfy the equation. Sketch a parabola through these points as shown in Fig. 11.21.

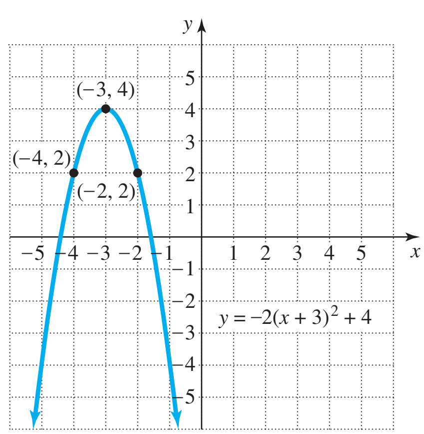



**FIGURE 11.21**

## WARM-UPS

**True or false? Explain your answer.**

1. The graph of $f(x) = (-x)^2$ is a reflection in the $x$-axis of the graph of $g(x) = x^2$.
2. The graph of $f(x) = -2$ is a reflection in the $x$-axis of the graph of $f(x) = 2$.
3. The graph of $f(x) = x + 3$ lies three units to the left of the graph of $f(x) = x$.
4. The graph of $y = |x - 3|$ lies three units to the left of the graph of $y = |x|$.
5. The graph of $y = |x| - 3$ lies three units below the graph of $y = |x|$.
6. The graph of $y = -2x^2$ can be obtained by stretching and reflecting the graph of $y = x^2$.
7. The graph of $f(x) = (x - 2)^2$ is symmetric about the $y$-axis.
8. For each point on the graph of $y = \sqrt{x/9}$ there is a corresponding point on $y = \sqrt{x}$ that has a $y$-coordinate three times as large.
9. The graph of $y = \sqrt{x - 3} + 5$ has the same shape as the graph of $y = \sqrt{x}$.
10. The graph of $y = -(x + 2)^2 - 7$ can be obtained by moving $y = x^2$ two units to the left and down seven units and then reflecting in the $x$-axis.

## 11.2 EXERCISES

***Reading and Writing*** *After reading this section, write out the answers to these questions. Use complete sentences.*

1. What is a reflection in the $x$-axis of a graph?
2. What is an upward translation of a graph?
3. What is a downward translation of a graph?
4. What is a translation to the right of a graph?
5. What is a translation to the left of a graph?
6. What is stretching and shrinking of a graph?

*Sketch the graphs of each pair of functions on the same coordinate system. See Example 1.*

7. $f(x) = \sqrt{2x}$, $g(x) = -\sqrt{2x}$
8. $y = x$, $y = -x$

**9.** $f(x) = x^2 + 1,$
$g(x) = -(x^2 + 1)$

**10.** $f(x) = |x| + 1,$
$g(x) = -|x| - 1$

**11.** $y = \sqrt{x - 2},$
$y = -\sqrt{x - 2}$

**12.** $y = |x - 1|,$
$y = -|x - 1|$

**13.** $f(x) = x - 3,$
$g(x) = 3 - x$

**14.** $f(x) = x^2 - 2,$
$g(x) = 2 - x^2$

*Use the ideas of translation to graph each function. See Example 2.*

**15.** $f(x) = x^2 - 4$

**16.** $f(x) = x^2 + 2$

**17.** $y = x + 3$

**18.** $y = x - 1$

**19.** $f(x) = (x - 3)^2$

**20.** $f(x) = (x + 1)^2$

**21.** $y = \sqrt{x} + 1$

**22.** $y = \sqrt{x} - 3$

**23.** $f(x) = |x + 2|$

**24.** $f(x) = |x - 4|$

**25.** $y = |x| + 2$

**26.** $y = |x| - 4$

**27.** $f(x) = \sqrt{x - 1}$

**28.** $f(x) = \sqrt{x + 6}$

*Use the ideas of stretching and shrinking to graph each function. See Example 3.*

**29.** $f(x) = 3x^2$

**30.** $f(x) = \frac{1}{3}x^2$

**31.** $y = \frac{1}{5}x$

**32.** $y = 5x$

**33.** $f(x) = 3\sqrt{x}$

**34.** $f(x) = \frac{1}{3}\sqrt{x}$

**35.** $y = \frac{1}{4}|x|$

**36.** $y = 4|x|$

*Sketch the graph of each function. See Example 4.*

**37.** $y = \sqrt{x - 2} + 1$

**38.** $y = -\sqrt{x + 3}$

**39.** $y = -|x + 3|$

**40.** $y = |x - 2| + 1$

**41.** $f(x) = (x + 3)^2 - 5$

**42.** $f(x) = -2x^2$

**43.** $y = -\sqrt{x + 1} - 2$

**44.** $y = -3\sqrt{x + 4} + 6$

**45.** $y = -2|x - 3| + 4$

**46.** $y = 3|x - 1| + 2$

**47.** $y = -2x + 3$

**48.** $y = 3x - 1$

*Find the vertex and graph each parabola. See Example 5.*

**49.** $y = 2(x + 3)^2 + 1$

**50.** $y = 2(x + 1)^2 - 2$

**51.** $y = -2(x - 4)^2 + 2$

**52.** $y = -2(x - 1)^2 + 3$

**53.** $y = -3(x - 1)^2 + 6$

**54.** $y = 3(x + 2)^2 - 6$

*Match each function with its graph a–h.*

**55.** $y = 2 + \sqrt{x}$

**56.** $y = \sqrt{2 + x}$

**57.** $y = 2\sqrt{x}$

**58.** $y = \sqrt{\dfrac{x}{2}}$

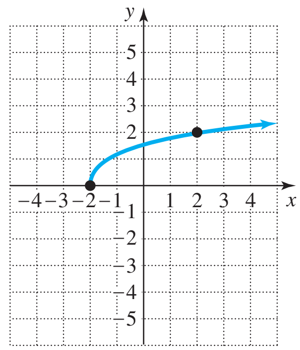
**59.** $y = \dfrac{1}{2}\sqrt{x}$

**60.** $y = 2 + \sqrt{x - 2}$

**61.** $y = -2\sqrt{x}$

**62.** $y = \sqrt{-x}$

**a)**

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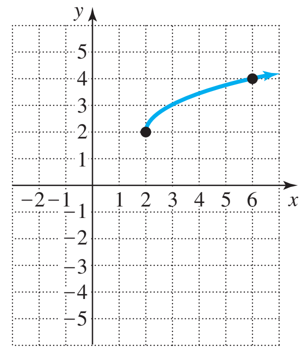


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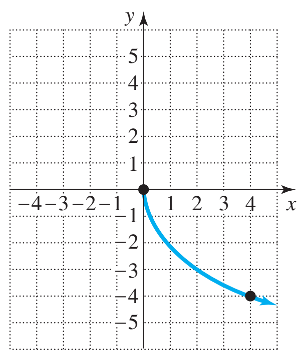


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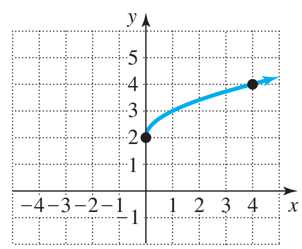
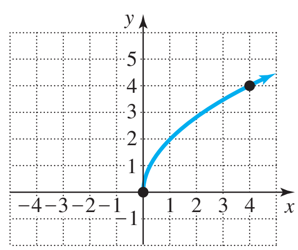
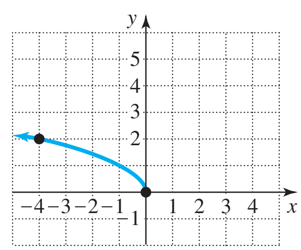


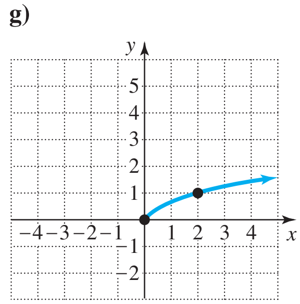

**e)**



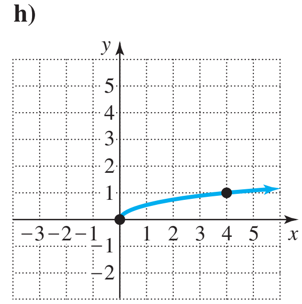

**f)**




**g)**




**h)**




## GETTING MORE INVOLVED

**63.** If the graph of $y = x^2$ is translated eight units upward, then what is the equation of the curve at that location?

**64.** If the graph of $y = x^2$ is translated six units to the right, then what is the equation of the curve at that location?

**65.** If the graph of $y = \sqrt{x}$ is translated five units to the left, then what is the equation of the curve at that location?

**66.** If the graph of $y = \sqrt{x}$ is translated four units downward, then what is the equation of the curve at that location?

**67.** If the graph of $y = |x|$ is translated three units to the left and then five units upward, then what is the equation of the curve at that location?

**68.** If the graph of $y = |x|$ is translated four units downward and then nine units to the right, then what is the equation of the curve at that location?

### GRAPHING CALCULATOR EXERCISES

**69.** Graph $f(x) = |x|$ and $g(x) = |x - 20| + 30$ on the same screen of your calculator. What transformations will transform the graph of $f$ into the graph of $g$?

**70.** Graph $f(x) = (x + 3)^2$, $g(x) = x^2 + 3^2$, and $h(x) = x^2 + 6x + 9$ on the same screen of your calculator.

**a)** Which two of these functions has the same graph? Why are they the same?

**b)** Is it true that $(x + 3)^2 = x^2 + 9$ for all real numbers $x$?

**c)** Describe each graph in terms of a transformation of the graph of $y = x^2$.

---

## 11.3 COMBINING FUNCTIONS

**In this section**

- Basic Operations with Functions
- Composition

In this section you will learn how to combine functions to obtain new functions.

### Basic Operations with Functions

An entrepreneur plans to rent a stand at a farmers market for \$25 per day to sell strawberries. If she buys $x$ flats of berries for \$5 per flat and sells them for \$9 per flat, then her daily cost in dollars can be written as a function of $x$:

$$C(x) = 5x + 25$$

Assuming she sells as many flats as she buys, her revenue in dollars is also a function of $x$:

$$R(x) = 9x$$

Because profit is revenue minus cost, we can find a function for the profit by subtracting the functions for cost and revenue:

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= 9x - (5x + 25) \\ &= 4x - 25 \end{aligned}$$

The function $P(x) = 4x - 25$ expresses the daily profit as a function of $x$. Since $P(6) = -1$ and $P(7) = 3$, the profit is negative if 6 or fewer flats are sold and positive if 7 or more flats are sold.

In the example of the entrepreneur we subtracted two functions to find a new function. In other cases we may use addition, multiplication, or division to combine two functions. For any two given functions we can define the sum, difference, product, and quotient functions as follows.