11.6 SYNTHETIC DIVISION AND THE FACTOR THEOREM

In this section

- Synthetic Division
- The Factor Theorem
- Solving Polynomial Equations

In this section we study functions defined by polynomials and learn to solve some higher-degree polynomial equations.

Synthetic Division

When dividing a polynomial by a binomial of the form $x - c$, we can use **synthetic division** to speed up the process. For synthetic division we write only the essential parts of ordinary division. For example, to divide $x^3 - 5x^2 + 4x - 3$ by $x - 2$, we write only the coefficients of the dividend 1, -5, 4, and -3 in order of descending exponents. From the divisor $x - 2$ we use 2 and start with the following arrangement:

$$\begin{array}{r|rrrr} 2 & 1 & -5 & 4 & -3 \\ \hline \end{array} \qquad (1 \cdot x^3 - 5x^2 + 4x - 3) \div (x - 2)$$

Next we bring the first coefficient, 1, straight down:

$$\begin{array}{r|rrrr} 2 & 1 & -5 & 4 & -3 \\ & \downarrow & \text{Bring down} & & \\ \hline & 1 & & & \end{array}$$

We then multiply the 1 by the 2 from the divisor, place the answer under the -5, and then add that column. Using 2 for $x - 2$ allows us to add the column rather than subtract as in ordinary division:

$$\begin{array}{r|rrrr} 2 & 1 & -5 & 4 & -3 \\ & & 2 & \text{Add} & \\ \hline \text{Multiply} & 1 & -3 & & \end{array}$$

We then repeat the multiply-and-add step for each of the remaining columns:

$$\begin{array}{r|rrrr} 2 & 1 & -5 & 4 & -3 \\ & & 2 & -6 & -4 \\ \hline \text{Multiply} & 1 & -3 & -2 & -7 \end{array} \quad \leftarrow \text{Remainder}$$

$\underbrace{1 \quad -3 \quad -2}_{\text{Quotient}}$

From the bottom row we can read the quotient and remainder. Since the degree of the quotient is one less than the degree of the dividend, the quotient is $1x^2 - 3x - 2$. The remainder is -7.

The strategy for getting the quotient $Q(x)$ and remainder R by synthetic division can be stated as follows.

Strategy for Using Synthetic Division

1. List the coefficients of the polynomial (the dividend).
2. Be sure to include zeros for any missing terms in the dividend.
3. For dividing by $x - c$, place c to the left.
4. Bring the first coefficient down.
5. Multiply by c and add for each column.
6. Read $Q(x)$ and R from the bottom row.

CAUTION Synthetic division is used only for dividing a polynomial by the binomial $x - c$, where c is a constant. If the binomial is $x - 7$, then $c = 7$. For the binomial $x + 7$ we have $x + 7 = x - (-7)$ and $c = -7$.

EXAMPLE 1 **Using synthetic division**

Find the quotient and remainder when $2x^4 - 5x^2 + 6x - 9$ is divided by $x + 2$.

Solution

Since $x + 2 = x - (-2)$, we use -2 for the divisor. Because x^3 is missing in the dividend, use a zero for the coefficient of x^3:

$$\begin{array}{r|rrrrr}
-2 & 2 & 0 & -5 & 6 & -9 \\
 & & -4 & 8 & -6 & 0 \\
\hline
 & 2 & -4 & 3 & 0 & -9
\end{array}$$

Multiply; Add

$\leftarrow 2x^4 + 0 \cdot x^3 - 5x^2 + 6x - 9$

$\leftarrow$ Quotient and remainder

Because the degree of the dividend is 4, the degree of the quotient is 3. The quotient is $2x^3 - 4x^2 + 3x$, and the remainder is -9. ■

The Factor Theorem

Consider the polynomial function

$$P(x) = x^2 + 2x - 15.$$

The values of x for which $P(x) = 0$ are called the **zeros** or **roots** of the function. We can find the zeros of the function by solving the equation $P(x) = 0$:

$$x^2 + 2x - 15 = 0$$
$$(x + 5)(x - 3) = 0$$
$$x + 5 = 0 \quad \text{or} \quad x - 3 = 0$$
$$x = -5 \quad \text{or} \quad x = 3$$

helpful hint

Note that the zeros of the polynomial function are factors of the constant term 15.

Because $x + 5$ is a factor of $x^2 + 2x - 15$, -5 is a solution to the equation $x^2 + 2x - 15 = 0$ and a zero of the function $P(x) = x^2 + 2x - 15$. We can check that -5 is a zero of $P(x) = x^2 + 2x - 15$ as follows:

$$\begin{aligned} P(-5) &= (-5)^2 + 2(-5) - 15 \\ &= 25 - 10 - 15 \\ &= 0 \end{aligned}$$

Because $x - 3$ is a factor of the polynomial, 3 is also a solution to the equation $x^2 + 2x - 15 = 0$ and a zero of the polynomial function. Check that $P(3) = 0$:

$$\begin{aligned} P(3) &= 3^2 + 2 \cdot 3 - 15 \\ &= 9 + 6 - 15 \\ &= 0 \end{aligned}$$

Every linear factor of the polynomial corresponds to a zero of the polynomial function, and every zero of the polynomial function corresponds to a linear factor.

Now suppose $P(x)$ represents an arbitrary polynomial. If $x - c$ is a factor of the polynomial $P(x)$, then c is a solution to the equation $P(x) = 0$, and so $P(c) = 0$. If we divide $P(x)$ by $x - c$ and the remainder is 0, we must have

$$P(x) = (x - c)(\text{quotient}).$$ Dividend equals the divisor times the quotient.

If the remainder is 0, then $x - c$ is a factor of $P(x)$.

The **factor theorem** summarizes these ideas.

The Factor Theorem

The following statements are equivalent for any polynomial $P(x)$.

1. The remainder is zero when $P(x)$ is divided by $x - c$.
2. $x - c$ is a factor of $P(x)$.
3. c is a solution to $P(x) = 0$.
4. c is a zero of the function $P(x)$, or $P(c) = 0$.

To say that statements are equivalent means that the truth of any one of them implies that the others are true.

According to the factor theorem, if we want to determine whether a given number c is a zero of a polynomial function, we can divide the polynomial by $x - c$. The remainder is zero if and only if c is a zero of the polynomial function. The quickest way to divide by $x - c$ is to use synthetic division.

EXAMPLE 2

calculator close-up

You can perform the multiply-and-add steps for synthetic division with a graphing calculator as shown here.

```
                 1
Ans*2+-3
                -1
Ans*2+5
                 3
Ans*2+-2
                 4
```

Using the factor theorem

Use synthetic division to determine whether 2 is a zero of

$$P(x) = x^3 - 3x^2 + 5x - 2.$$

Solution

By the factor theorem, 2 is a zero of the function if and only if the remainder is zero when $P(x)$ is divided by $x - 2$. We can use synthetic division to determine the remainder. If we divide by $x - 2$, we use 2 on the left in synthetic division along with the coefficients $1, -3, 5, -2$ from the polynomial:

$$\begin{array}{r|rrrr} 2 & 1 & -3 & 5 & -2 \\ & & 2 & -2 & 6 \\ \hline & 1 & -1 & 3 & 4 \end{array}$$

Because the remainder is 4, 2 is not a zero of the function. ■

EXAMPLE 3

Using the factor theorem

Use synthetic division to determine whether -4 is a solution to the equation $2x^4 - 28x^2 + 14x - 8 = 0$.

Solution

By the factor theorem, -4 is a solution to the equation if and only if the remainder is zero when $P(x)$ is divided by $x + 4$. When dividing by $x + 4$, we use -4 in the synthetic division:

$$\begin{array}{r|rrrrr} -4 & 2 & 0 & -28 & 14 & -8 \\ & & -8 & 32 & -16 & 8 \\ \hline & 2 & -8 & 4 & -2 & 0 \end{array}$$

Because the remainder is zero, -4 is a solution to $2x^4 - 28x^2 + 14x - 8 = 0$. ■

In the next example we use the factor theorem to determine whether a given binomial is a factor of a polynomial.

EXAMPLE 4

Using the factor theorem

Use synthetic division to determine whether $x + 4$ is a factor of $x^3 + 3x^2 + 16$.

Solution

According to the factor theorem, $x + 4$ is a factor of $x^3 + 3x^2 + 16$ if and only if the remainder is zero when the polynomial is divided by $x + 4$. Use synthetic division to determine the remainder:

$$\begin{array}{r|rrrr} -4 & 1 & 3 & 0 & 16 \\ & & -4 & 4 & -16 \\ \hline & 1 & -1 & 4 & 0 \end{array}$$

Because the remainder is zero, $x + 4$ is a factor, and the polynomial can be written as

$$x^3 + 3x^2 + 16 = (x + 4)(x^2 - x + 4).$$

Because $x^2 - x + 4$ is a prime polynomial, the factoring is complete. ■

study tip

Stay calm and confident. Take breaks when you study. Get 6 to 8 hours of sleep every night and keep reminding yourself that working hard all of the semester will really pay off.

Solving Polynomial Equations

The techniques used to solve polynomial equations of degree 3 or higher are not as straightforward as those used to solve linear equations and quadratic equations. The next example shows how the factor theorem can be used to solve a third-degree polynomial equation.

EXAMPLE 5

Solving a third-degree equation

Suppose the equation $x^3 - 4x^2 - 17x + 60 = 0$ is known to have a solution that is an integer between -3 and 3 inclusive. Find the solution set.

Solution

Because one of the numbers $-3, -2, -1, 0, 1, 2,$ and 3 is a solution to the equation, we can use synthetic division with these numbers until we discover which one is a solution. We arbitrarily select 1 to try first:

$$\begin{array}{r|rrrr} 1 & 1 & -4 & -17 & 60 \\ & & 1 & -3 & -20 \\ \hline & 1 & -3 & -20 & 40 \end{array}$$

Because the remainder is 40, 1 is not a solution to the equation. Next try 2:

$$\begin{array}{r|rrrr} 2 & 1 & -4 & -17 & 60 \\ & & 2 & -4 & -42 \\ \hline & 1 & -2 & -21 & 18 \end{array}$$

Because the remainder is not zero, 2 is not a solution to the equation. Next try 3:

$$\begin{array}{r|rrrr} 3 & 1 & -4 & -17 & 60 \\ & & 3 & -3 & -60 \\ \hline & 1 & -1 & -20 & 0 \end{array}$$

helpful hint

How did we know where to find a solution to the equation in Example 5? One way to get a good idea of where the solutions are is to graph

$$y = x^3 - 4x^2 - 17x + 60.$$

Every x-intercept on this graph corresponds to a solution to the equation.

The remainder is zero, so 3 is a solution to the equation, and $x - 3$ is a factor of the polynomial. (If 3 had not produced a remainder of zero, then we would have tried -3, -2, -1, and 0.) The other factor is the quotient, $x^2 - x - 20$.

$$x^3 - 4x^2 - 17x + 60 = 0$$

$$(x - 3)(x^2 - x - 20) = 0 \quad \text{Use the results of synthetic division to factor.}$$

$$(x - 3)(x - 5)(x + 4) = 0 \quad \text{Factor completely.}$$

$$x - 3 = 0 \quad \text{or} \quad x - 5 = 0 \quad \text{or} \quad x + 4 = 0$$

$$x = 3 \quad \text{or} \quad x = 5 \quad \text{or} \quad x = -4$$

Check each of these solutions in the original equation. The solution set is $\{3, 5, -4\}$.

WARM-UPS

True or false? Explain your answers.

1. To divide $x^3 - 4x^2 - 3$ by $x - 5$, use 5 in the synthetic division.
2. To divide $5x^4 - x^3 + x - 2$ by $x + 7$, use -7 in the synthetic division.
3. The number 2 is a zero of $P(x) = 3x^3 - 5x^2 - 2x + 2$.
4. If $x^3 - 8$ is divided by $x - 2$, then $R = 0$.
5. If $R = 0$ when $x^4 - 1$ is divided by $x - a$, then $x - a$ is a factor of $x^4 - 1$.
6. If -2 satisfies $x^4 + 8x = 0$, then $x + 2$ is a factor of $x^4 + 8x$.
7. The binomial $x - 1$ is a factor of $x^{35} - 3x^{24} + 2x^{18}$.
8. The binomial $x + 1$ is a factor of $x^3 - 3x^2 + x + 5$.
9. If $x^3 - 5x + 4$ is divided by $x - 1$, then $R = 0$.
10. If $R = 0$ when $P(x) = x^3 - 5x - 2$ is divided by $x + 2$, then $P(-2) = 0$.

11.6 EXERCISES

Reading and Writing *After reading this section, write out the answers to these questions. Use complete sentences.*

1. What is a zero of a function?
2. What is a root of a function?
3. What does it mean that statements are equivalent?
4. What is the quickest way to divide a polynomial by $x - c$?
5. If the remainder is zero when you divide $P(x)$ by $x - c$, then what can you say about $P(c)$?
6. What are two ways to determine whether c is a zero of a polynomial?

Use synthetic division to find the quotient and remainder when the first polynomial is divided by the second. See Example 1.

7. $x^3 - 5x^2 + 6x - 3, \quad x - 2$
8. $x^3 + 6x^2 - 3x - 5, \quad x - 3$
9. $2x^2 - 4x + 5, \quad x + 1$
10. $3x^2 - 7x + 4, \quad x + 2$
11. $3x^4 - 15x^2 + 7x - 9, \quad x - 3$
12. $-2x^4 + 3x^2 - 5, \quad x - 2$

13. $x^5 - 1, \quad x - 1$
14. $x^6 - 1, \quad x + 1$
15. $x^3 - 5x + 6, \quad x + 2$
16. $x^3 - 3x - 7, \quad x - 4$
17. $2.3x^2 - 0.14x + 0.6, \quad x - 0.32$
18. $1.6x^2 - 3.5x + 4.7, \quad x + 1.8$

Determine whether each given value of x is a zero of the given function. See Example 2.

19. $x = 1, \quad P(x) = x^3 - x^2 + x - 1$
20. $x = -2, \quad P(x) = -2x^3 - 5x^2 + 3x + 10$
21. $x = -3, \quad P(x) = -x^4 - 3x^3 - 2x^2 + 18$
22. $x = 4, \quad P(x) = x^4 - x^2 - 8x - 16$
23. $x = 2, \quad P(x) = 2x^3 - 4x^2 - 5x + 9$
24. $x = -3, \quad P(x) = x^3 + 5x^2 + 2x + 1$

Use synthetic division to determine whether each given value of x is a solution to the given equation. See Example 3.

25. $x = -3, \quad x^3 + 5x^2 + 2x - 12 = 0$
26. $x = -5, \quad x^2 - 3x - 40 = 0$
27. $x = -2, \quad x^4 + 3x^3 - 5x^2 - 10x + 5 = 0$
28. $x = -3, \quad -x^3 - 4x^2 + x + 12 = 0$
29. $x = 4, \quad -2x^4 + 30x^2 + 5x + 12 = 0$
30. $x = 6, \quad x^4 + x^3 - 40x^2 - 72 = 0$
31. $x = 3, \quad 0.8x^2 - 0.3x - 6.3 = 0$
32. $x = 5, \quad 6.2x^2 - 28.2x - 41.7 = 0$

Use synthetic division to determine whether the first polynomial is a factor of the second. If it is, then factor the polynomial completely. See Example 4.

33. $x - 3, \quad x^3 - 6x - 9$
34. $x + 2, \quad x^3 - 6x - 4$
35. $x + 5, \quad x^3 + 9x^2 + 23x + 15$
36. $x - 3, \quad x^4 - 9x^2 + x - 7$
37. $x - 2, \quad x^3 - 8x^2 + 4x - 6$
38. $x + 5, \quad x^3 + 125$
39. $x + 1, \quad x^4 + x^3 - 8x - 8$
40. $x - 2, \quad x^3 - 6x^2 + 12x - 8$
41. $x - 0.5, \quad 2x^3 - 3x^2 - 11x + 6$
42. $x - \frac{1}{3}, \quad 3x^3 - 10x^2 - 27x + 10$

Solve each equation, given that at least one of the solutions to each equation is an integer between −5 and 5. See Example 5.

43. $x^3 - 13x + 12 = 0$
44. $x^3 + 2x^2 - 5x - 6 = 0$
45. $2x^3 - 9x^2 + 7x + 6 = 0$
46. $6x^3 + 13x^2 - 4 = 0$
47. $2x^3 - 3x^2 - 50x - 24 = 0$
48. $x^3 - 7x^2 + 2x + 40 = 0$
49. $x^3 + 5x^2 + 3x - 9 = 0$
50. $x^3 + 6x^2 + 12x + 8 = 0$
51. $x^4 - 4x^3 + 3x^2 + 4x - 4 = 0$
52. $x^4 + x^3 - 7x^2 - x + 6 = 0$

GETTING MORE INVOLVED

53. ***Exploration.*** We can find the zeros of a polynomial function by solving a polynomial equation. We can also work backward to find a polynomial function that has given zeros.

a) Write a first-degree polynomial function whose zero is −2.
b) Write a second-degree polynomial function whose zeros are 5 and −5.
c) Write a third-degree polynomial function whose zeros are 1, −3, and 4.
d) Is there a polynomial function with any given number of zeros? What is its degree?

GRAPHING CALCULATOR EXERCISES

54. The x-coordinate of each x-intercept on the graph of a polynomial function is a zero of the polynomial function. Find the zeros of each function from its graph. Use synthetic division to check that the zeros found on your calculator really are zeros of the function.

a) $P(x) = x^3 - 2x^2 - 5x + 6$
b) $P(x) = 12x^3 - 20x^2 + x + 3$

55. With a graphing calculator an equation can be solved without the kind of hint that was given for Exercises 43–52. Solve each of the following equations by examining the graph of a corresponding function. Use synthetic division to check.

a) $x^3 - 4x^2 - 7x + 10 = 0$
b) $8x^3 - 20x^2 - 18x + 45 = 0$